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# Constructions of Grassmannian Simplices

Jean Creignou

13th March 2007

## Abstract

In this article an explicit method (relying on representation theory) to construct packings in Grassmannian space is presented. Infinite families of configurations having only one non-trivial set of principal angles are found using 2-transitive groups. These packings are proved to reach the simplex bound and are therefore optimal w.r.t. the chordal distance. The construction is illustrated by an example on  $\mathfrak{S}_n$ . Then some natural extends and consequences of this situation are given.

**Keywords :** Explicit construction, Grassmannian packing, Space-time coding, Simplex bound, Wireless communication.

## 1 Introduction

The problem of finding ‘good’ configurations of  $m$ -dimensional subspaces in  $\mathbb{R}^n$  or  $\mathbb{C}^n$  has various applications, especially in the field of Information Theory dealing with multiple-antennas transmission. The use of multiple antennas instead of one enables an higher rate of transmission but moves the problem from spherical codes to Grassmannian codes. The success of multiple antennas communications depends on the ability to design efficient codes with fast encoding and decoding algorithms. Basic facts concerning multiple-antennas communication are explained in many articles, namely [17, 16, 15, 18] and their references. For the sake of brevity and coherence, we will start on the following problem : given  $N$ ,  $m$  and  $n$  such that  $m \leq \frac{n}{2}$ , find ‘good’ configurations of  $N$  subspaces of dimension  $m$  in  $\mathbb{C}^n$ .

In this paper we present a method to construct such configurations using representation theory. We first recall all the bases concerning Grassmannian spaces and representation theory in § 2 and 3 respectively. The method used with 2-transitive groups leads to very special configurations which are in a way optimal. Section 4 gives a detailed formulation of this result and its proof. We then give an explicit example and say some words about the classification of 2-transitive groups in § 5. We end with some remarks in § 6 and conclude.

A skilled reader who already well knows the theory of Grassmannian packings and representation theory can directly jump to §4. Otherwise the paper is almost self contained.

## 2 Distances in Grassmannian spaces

In this section we recall from [14] the essential background about Grassmannian spaces used in the sequel. We introduce Grassmannian spaces, define distances between elements and recall the expression of the simplex bound. We also state the general idea behind our constructions.

Let us start with the definition.

**Definition 1** *The Grassmannian space  $\mathcal{G}_{m,n}$  is the set of all  $m$ -dimensional vector subspaces of  $\mathbb{C}^n$ .*

Before giving the different kind of distances between the  $m$ -dimensional subspaces, we need to introduce the notion of principal angles.

**Definition 2** *For any  $\mathcal{P}, \mathcal{Q}$  elements of  $\mathcal{G}_{m,n}$  (i.e. two  $m$ -dimensional subspaces in  $\mathbb{C}^n$ ). Let<sup>1</sup>*

$$x_1, y_1 := \arg \max_{\substack{x \in \mathcal{P}, y \in \mathcal{Q} \\ \|x\| = \|y\| = 1}} |\langle x, y \rangle|.$$

*Then by induction, we constrain  $x_i$  (resp.  $y_i$ ) to be a unit vector orthogonal to  $\{x_1, \dots, x_{i-1}\}$  (resp.  $\{y_1, \dots, y_{i-1}\}$ ) and let*

$$x_i, y_i := \arg \max_{\substack{x \in \mathcal{P}, y \in \mathcal{Q} \\ \text{constrained}}} |\langle x, y \rangle|.$$

*By definition the principal angles are the values  $\theta_i \in [0, \frac{\pi}{2}]$  such that  $\cos(\theta_i) = |\langle x_i, y_i \rangle|$ .*

Now we can begin to define a distance. The problem is that there is no canonical choice (the appropriate general data is the set of principal angles)! We give four definitions:

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<sup>1</sup>By “arg max” we mean the arguments (any) which allow the following function to reach it’s maximum.

$$d_\infty(\mathcal{P}, \mathcal{Q}) := \max_i \theta_i$$

$$d_g(\mathcal{P}, \mathcal{Q}) := \sqrt{\sum_i \theta_i^2}$$

$$d_c(\mathcal{P}, \mathcal{Q}) := \sqrt{\sum_i \sin^2(\theta_i)}$$

$$\tilde{d}(\mathcal{P}, \mathcal{Q}) := \prod_i \sin(\theta_i)$$

The chordal distance  $d_c$  is usually preferred to the infinite distance  $d_\infty$  and the geodesic distance  $d_g$ , for reasons of smoothness [14]. The fourth one  $\tilde{d}$  is not a distance in the mathematical sense, but it is the value used for estimating the performance of a wireless communication. Indeed the condition  $\tilde{d} \neq 0$  is linked to the full-diversity property of the constellations associated to the Grassmannian packing.

The chordal distance can also be expressed in an easy way using Trace and projection matrices [14]. Let  $\Pi_{\mathcal{P}}$  and  $\Pi_{\mathcal{Q}}$  be the projection matrices on  $\mathcal{P}$  and  $\mathcal{Q}$  then,

$$d_c^2 = \frac{1}{2} \|\Pi_{\mathcal{P}} - \Pi_{\mathcal{Q}}\|_2^2 = \text{Trace}(\Pi_{\mathcal{P}}) - \text{Trace}(\Pi_{\mathcal{P}}\Pi_{\mathcal{Q}}). \quad (1)$$

A way to obtain this formula is to express the projection matrices in term of the  $x_i$ 's and  $y_j$ 's used to define the principal angles.

In this paper we deal mainly with the chordal distance  $d_c$ , but some main results (such as Propositions 15 and 16) extend to other definitions.

Another advantage of considering the chordal distance is that with this metric the Grassmannian space can be embedded in a sphere [14]. Then one can easily deduce bounds on codes in Grassmannian spaces from bounds in spherical codes. We recall here the simplex bound on Grassmannian configuration (obtained by this very way). This bound was stated for real Grassmannian spaces but extends itself in an easy way to the complex case<sup>2</sup>.

**Lemma 3** *For any configuration of  $N$  subspaces of dimension  $m$  in  $\mathbb{C}^n$ , the following inequality holds :*

$$d_c^2 \leq \frac{m(n-m)}{n} \frac{N}{N-1} \quad (2)$$

$$\text{equality requiring } N \leq \binom{n+1}{2}.$$

---

<sup>2</sup>Indeed any complex configuration can be embedded into a real space doubling  $m$ ,  $n$  and  $d_c^2$ . After simplifications one obtains the same bound.

The equality can occur only if the distance between each pair of distinct elements is the same. As an extension of the definition from the spherical case we use the term *simplicial* for such a configuration. One particular case is when there is only one non-trivial set of principal angles between any pair of  $m$ -dimensional planes. This second assertion is in fact much stronger, we will then use the term *strongly simplicial* to be more precise.

In this article we give an explicit construction of *strongly simplicial* configurations in the Grassmannian space. For that purpose we use the following general idea given in [14] to construct codes. Let  $\mathcal{P}$  be an  $m$ -dimensional subspace of  $\mathbb{C}^n$  and  $G$  a (finite) group acting on  $\mathcal{G}_{m,n}$ . The orbit of  $\mathcal{P}$  under the group  $G$  give us a configuration in  $\mathcal{G}_{m,n}$ . The number of elements in this configuration is  $|G/H|$  where  $H$  is the subgroup of  $G$  which stabilizes  $\mathcal{P}$ . The problem then is to find groups and elements which give interesting configurations. The choices of these parameters in this article are based on representation theory.

We are now done with the bases of Grassmannian spaces. The next section recalls important results of representation theory used in the sequel and sets some notation.

### 3 Representation theory

Our constructions are based on finite group representation theory. The main tool used in this article is the projection on isotypic spaces. Some good references on the subject are [4,5,6,7,10,11] in which one can find more explanations, proofs and formulas related to representation theory. We give here a brief summary of results and definitions for completeness<sup>3</sup>.

**Definition 4** *Let  $G$  be a group. A (complex) matrix representation of  $G$  is a morphism,  $\rho : G \rightarrow GL_n(\mathbb{C})$ . The dimension of the matrices is called the dimension or the degree of the representation. The function  $\chi_\rho : G \rightarrow \mathbb{C}$  defined by  $\chi_\rho(g) = \text{Trace}(\rho(g))$  is the character associated to  $\rho$ . One can remark that the dimension equals  $\chi_\rho(1)$ .*

**Definition 5** *Two representations  $\rho, \rho'$  are said to be equivalent if and only if there exists an invertible matrix  $U$  such that  $\forall g \in G, U \cdot \rho(g) \cdot U^{-1} = \rho'(g)$ . Two equivalent representations have equal characters, and are often identified. Furthermore the matrices of a representation can always be supposed unitary. As a consequence  $\chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$ . We may also point out that characters are constant on conjugacy classes.*

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<sup>3</sup>But we strongly advise to read an introduction to the subject.

**Definition 6** A representation is called reducible if there exists a proper subspace  $W$  ( $0 \subsetneq W \subsetneq \mathbb{C}^n$ ) such that  $\forall g \in G, \rho(g).W \subset W$ . It is irreducible otherwise. A character associated to an irreducible representation is called irreducible. A complex representation can be decomposed as a direct sum of irreducible components (Maschke Theorem [5, 6]). Its character is then equal to the sum of the irreducible characters associated to irreducible components.

**Notations 7** For an element  $g \in G$  we denote by  $C_{Gg}$  its conjugacy class in  $G$ . If  $\chi$  is a character of  $G$ ,  $\chi(C_{Gg})$  means the value of  $\chi$  on this conjugacy class<sup>4</sup>.

**Proposition 8** There are as many (non-equivalent) irreducible characters of  $G$  as conjugacy classes in  $G$ . If  $\{\chi_i : i = 1, \dots, t\}$  is the set of irreducible characters, we have the following properties :

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} ,$$

$$\frac{1}{|G|} \sum_{i=1}^t \chi_i(C_{Gg}) \overline{\chi_i(C_{Gh})} = \begin{cases} |C_{Gg}| & \text{if } C_{Gg} = C_{Gh} \\ 0 & \text{if } C_{Gg} \neq C_{Gh} \end{cases} . \quad (3)$$

These two formulas are called orthogonality relation. Their proofs (based on Schur's lemma) can be found in any aforementioned reference. A direct consequence is that the set of irreducible characters form an orthogonal basis of the space of class-functions<sup>5</sup> with respect to the following inner product

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

**Definition 9** Let  $\rho : H \rightarrow GL_n(\mathbb{C})$  a reducible complex representation and  $\chi_H$  an irreducible character. Suppose that we have the following decomposition in irreducible subspaces :  $\mathbb{C}^n = V_1 + \dots + V_\ell$ . The isotypic subspace associated to  $\chi_H$  is the direct sum of all the  $V_i$ 's whose associated character is equal to  $\chi_H$ . In other words we have gathered together all the equivalent irreducible representations (associated to  $\chi_H$ ) contained in  $\rho$ .

**Proposition 10** Let  $\rho : H \rightarrow GL_n(\mathbb{C})$  be a reducible representation and  $\chi_H$  an irreducible character. The isotypic space  $W$  associated to  $\chi_H$  has this projection matrix :

$$\Pi_W = \frac{\chi_H(1)}{|H|} \sum_{h \in H} \overline{\chi_H(h)} \rho(h). \quad (4)$$

---

<sup>4</sup>This article deal with irreducible characters on  $G$  and  $H$ . Be careful that  $\chi_H(C_{Gh})$  has no meaning ! So it is important to know on which group we consider characters and conjugacy classes!

<sup>5</sup>A class function is a function which is constant on each conjugacy classe.

*Especially the definition of  $W$  does not depend on the decomposition  $\mathbb{C}^n = V_1 + \dots + V_\ell$  in irreducible subspaces.*

This proposition is again a consequence of Schur's lemma see [10] for further details.

The relations (3) can be generalised [5] (chp. 1). From this generalisation one can prove directly the following :

**Lemma 11** *Let  $H$  be a group,  $\chi_H$  an irreducible<sup>6</sup> character of  $H$  and  $\chi_1, \chi_2$  any characters (or class functions) then*

$$\sum_{h_1, h_2 \in H} \overline{\chi_1(h_1)\chi_2(h_2)\chi_H(h_1h_2)} = |H|^2 \frac{\langle \chi_1, \chi_H \rangle \langle \chi_2, \chi_H \rangle}{\chi_H(1)}. \quad (5)$$

Now we can state clearly the main result of this paper.

## 4 The result

The main result of this paper is the existence of an explicit method to construct some *strongly simplicial* packings in Grassmannian spaces which reach the simplex bound and are therefore optimal w.r.t. the chordal distance. This section is devoted to the statement and demonstration of this theorem.

### 4.1 The main theorem

**Theorem 12** *Let  $G$  a group and  $H$  a subgroup such that  $G$  act 2-transitively by left multiplication on  $G/H$ . Suppose furthermore that we dispose of an irreducible representation of the group  $G$ ,  $\rho_G : G \rightarrow GL_n(\mathbb{C})$  which is reducible when restricted to  $H$ . Under these hypothesis let  $W$  be a direct sum of  $H$ -isotypic subspaces. Then the orbit of  $W$  under  $G$  form a strongly simplicial<sup>7</sup> Grassmannian configuration reaching the simplex bound. The parameters are  $N = |G/H|$ ;  $n = \dim(\rho_G)$  and  $m$  equals the dimension of the direct sum of isotypic subspaces.*

A more concrete description of the configuration is  $\{\rho_G(g)W : g \in G\}$  where the projection matrix on  $W$  is

$$\Pi_W = \sum_{i=1}^s \frac{\chi_i(1)}{|H|} \sum_{h \in H} \overline{\chi_i(h)} \rho_G(h).$$

---

<sup>6</sup>If  $\chi_H$  is reducible the formula has a trivial generalization by linearity of the left term.

<sup>7</sup>We recall that by this we mean that there is only one non-trivial set of principal angles between any pair of distinct elements.

**Remarks 13** • *This construction leads to explicit expression of the projection matrices on every subspaces of the configuration when the representations of  $G$  and the characters of  $H$  are known. This is the case for some infinite families of 2-transitive groups but alas not all of them.*

- *We omit the morphism  $\rho_G$  in the group action. So we write  $gW$  instead of  $\rho_G(g)W$ . This convention greatly lightens the writing (and thinking).*
- *Without the condition of 2-transitivity on  $G$  and  $H$  the construction described in Theorem 12 remain quite interesting. This general construction can be used in conjunction with Proposition 16 or extended to find good codes in Grassmannian spaces (see Appendices).*
- *The condition  $G$  is 2-transitive on  $G/H$  is equivalent to  $|H \backslash G/H| = 2$  (the short proof follows). It is this less common statement which naturally arises when dealing with our constructions. This also means that there is only two double cosets so  $G = H \sqcup HgH$  (where  $g$  is any element not in  $H$ ).*

**Proof :** [Last remark] Let us consider the (transitive) action of  $G$  on  $G/H$ . The condition  $|H \backslash G/H| = 2$  is equivalent to say that a point stabiliser ( $H$  which stabilise  $1H$ ) acts transitively on the other elements. This assertion is tantamount to say that the action of  $G$  on  $G/H$  is 2-transitive.

The next subsection deals with the proof of the main theorem.

## 4.2 The proof

In this section we prove Theorem 12. For the sake of brevity we do not rewrite all the hypotheses of the theorem in each lemma and proposition but those are supposed to be satisfied. We have three different statements to prove :

1. there are  $N = |G/H|$  elements in the configuration ;
2. the configuration is *strongly simplicial* ;
3. the simplex bound is reached.

They correspond to Lemma 14, Proposition 15 and Proposition 17 respectively.

1. “there are  $N = |G/H|$  elements in the configuration”

**Lemma 14** *Let  $G$  be a two transitive group on  $G/H$  then  $H$  is a maximal subgroup. Consequently the construction of Theorem 12 has exactly  $|G/H|$  distinct elements.*

**Proof :**

- Let  $B$  be a subgroup of  $G$  such that  $H \subsetneq B \subsetneq G$ . Then



- if  $g \in B$  then  $BgB = B$ ,
- if  $g \notin B$  then  $BgB \supset G \setminus H$ .

But then  $|B| + |G \setminus H| > |G|$  so there is only one double class namely  $G$  and then  $G = B$  which is a contradiction. So  $H$  is maximal.

- The elements  $g \in G$  such that  $g.W = W$  form a proper subgroup of  $G$  which contains  $H$  (the isotypic space  $W$  is naturally  $H$ -stable). So  $H = \{g \in G : gW = W\}$  and the orbit of  $W$  has  $|G/H|$  elements.

2. “the configuration is *strongly simplicial*”

**Proposition 15** *The constructed packings in Theorem 12 are strongly simplicial (i.e. they have only one non-trivial set of principal angles).*

**Proof :** We know that action of  $G$  on  $G/H$  is double transitive. That is, if  $g_1 \neq g_2 \pmod H$  and  $g'_1 \neq g'_2 \pmod H$  are four elements of  $G/H$ , there exists  $g \in G$  such that  $gg_1 = g'_1 \pmod H$  and  $gg_2 = g'_2 \pmod H$ .

Now any unitary transformation preserves the set of principal angles between two planes. Take two pairs of different planes in our configuration  $(g_1W, g_2W)$  and  $(g'_1W, g'_2W)$ , then choosing  $g \in G$  as above and remembering that the representation matrices are unitary we get that :

$$\begin{aligned} \text{Angles}(g_1W, g_2W) &= \text{Angles}(gg_1W, gg_2W) \\ &= \text{Angles}(g'_1W, g'_2W) \end{aligned}$$

So the set of principal angles between any pair of distinct planes is the same.

This was a fast and simple proof. In fact a much stronger statement is true. The proof of the following proposition which involves heavy expressions can be found in appendices.

**Proposition 16** *Let  $H \subset G$  be any subgroup of  $G$  and  $\Pi_E, \Pi_{\bar{E}}$  be two projections matrices on subspaces defined has before by any two set of irreducible character i.e. two sums of isotopic subspaces (they may be equal, disjoint,...). Then the set of principal angles between  $\Pi_E$  and  $\rho(g)\Pi_{\bar{E}}\rho(g^{-1})$  only depends of the class  $HgH$ . The number of different sets of principal angles between orbits of  $\Pi_E$  and  $\Pi_{\bar{E}}$  is then bounded by  $|H \setminus G/H|$ .*

3) “the simplex bound is reached”

**Proposition 17** *All the constructed packings in Theorem 12 reach the simplex bound.*

The proof of this proposition require some tougher work. We need three lemmas. The first is just another expression of the value of the chordal distance in our constructions. The two others deal with sums related to character theory.

**Lemma 18** *Let  $W$  be the direct sum of isotypic subspaces, associated to the irreducible characters (on  $H$ ) :  $\chi_1, \dots, \chi_s$ . Then the square value of the chordal distance between  $W$  and  $gW$  is*

$$d_c^2 = \sum_{i=1}^s \lambda_i \chi_i(1) - \frac{1}{|H|^2} \sum_{h_1, h_2 \in H} E(h_1) E(h_2) \chi_G(h_1 g h_2 g^{-1}) \quad (6)$$

where

$$E(h) := \left( \sum_{i=1}^s \lambda_i \chi_i(1) \overline{\chi_i(h)} \right)$$

and  $\lambda_i$  is the multiplicity of  $\chi_i$  in the decomposition of

$$\chi_\rho \downarrow_H^G = \sum_{i=1}^r \lambda_i \chi_i$$

**Proof :** The projection matrices on  $W$  and on  $gW$  are

$$\Pi_W = \sum_{i=1}^s \frac{\chi_i(1)}{|H|} \sum_{h \in H} \overline{\chi_i(h)} \rho(h),$$

$$\Pi_{gW} = \sum_{i=1}^s \frac{\chi_i(1)}{|H|} \sum_{h \in H} \overline{\chi_i(h)} \rho(ghg^{-1}),$$

The result follows from these expressions, formula (1) for chordal distance and orthogonality relations (3).

We now give two character formulas. We also need to introduce further notations.

**Notations 19** *For any set  $S$ , the formal sum of all elements in  $S$  is written  $\widehat{S}$ . For a character  $\chi(\widehat{S})$  means  $\sum_{s \in S} \chi(s)$ . As a consequence if  $\chi_G$  is a character of  $G$  then  $\chi(\widehat{C}_{Gh_1}) = |C_{Gh_1}| \chi(h_1)$ . We use  $\mathcal{Z}_{Gh}$  to denote the set  $\{g \in G : gh = hg\}$ . In particular  $|\mathcal{Z}_{Gh}| |C_{Gh}| = |G|$ .*

**Lemma 20** *For any irreducible character  $\chi_G$  on  $G$ ,*

$$\chi_G \left( \widehat{C}_{Gh_1} \widehat{C}_{Gh_2} \right) = \frac{|G|^2 \chi_G(h_1) \chi_G(h_2)}{|\mathcal{Z}_{Gh_1}| |\mathcal{Z}_{Gh_2}| \chi_G(1)} \quad (7)$$

**Proof :** The following relation can be found in [6], chapter 30. If  $C_1, \dots, C_\ell$  are all conjugacy classes in  $G$  and  $Z_1, \dots, Z_\ell$  their centralizer then

$$\widehat{C}_i \widehat{C}_j = \sum_{k=1}^{\ell} a_{ijk} \widehat{C}_k$$

where

$$a_{ijk} = \frac{|G|}{|Z_i||Z_j|} \sum_{\chi \text{ irred.}} \frac{\chi(C_i)\chi(C_j)\overline{\chi(C_k)}}{\chi(1)}$$

So

$$\chi_G(\widehat{C}_{Gh_1} \widehat{C}_{Gh_2}) = \sum_{k=1}^{\ell} \frac{|G|}{|Z_{Gh_1}||Z_{Gh_2}|} \sum_{\chi \text{ irred.}} \frac{\chi(C_{Gh_1})\chi(C_{Gh_2})\overline{\chi(C_k)}}{\chi(1)} \chi_G(\widehat{C}_k)$$

$$\chi_G(\widehat{C}_{Gh_1} \widehat{C}_{Gh_2}) = \frac{|G|}{|Z_{Gh_1}||Z_{Gh_2}|} \sum_{\chi \text{ irred.}} \frac{\chi(h_1)\chi(h_2)}{\chi(1)} \sum_{k=1}^{\ell} \overline{\chi(C_k)} \chi_G(\widehat{C}_k)$$

Since  $\sum_{k=1}^{\ell} \overline{\chi(C_k)} \chi_G(\widehat{C}_k) = \begin{cases} 0 & \text{if } \chi \neq \chi_G \\ |G| & \text{if } \chi = \chi_G \end{cases}$  from first orthogonality relation (3), the formula is proved.

**Lemma 21** For any irreducible character  $\chi_G$  on  $G$ ,

$$\sum_{g \in G} \chi_G(h_1 g h_2 g^{-1}) = \frac{|G| \chi_G(h_1) \chi_G(h_2)}{\chi_G(1)}. \quad (8)$$

**Proof :**

$$\sum_{g \in G} \chi_G(h_1 g h_2 g^{-1}) = \frac{|G|}{|C_{Gh_2}|} \chi(h_1 \widehat{C}_{Gh_2}) \quad (9)$$

but also

$$\sum_{g \in G} \chi_G(h_1 g h_2 g^{-1}) = \sum_{g \in G} \chi_G(g^{-1} h_1 g h_2)$$

so

$$\sum_{g \in G} \chi_G(h_1 g h_2 g^{-1}) = \frac{|G|}{|C_{Gh_1}|} \chi(\widehat{C}_{Gh_1} h_2) \quad (10)$$

Now (9) and (10) imply that  $\sum_{g \in G} \chi_G(h_1 g h_2 g^{-1})$  does not depend of  $h_1 \in C_{Gh_1}$  or  $h_2 \in C_{Gh_2}$ .

So computing

$$\sum_{\substack{h_1 \in C_G h_1 \\ h_2 \in C_G h_2}} \sum_{g \in G} \chi_G (h_1 g h_2 g^{-1})$$

gives the formula :

$$\sum_{g \in G} \chi_G (h_1 g h_2 g^{-1}) = \frac{|G|}{|C_G h_1| |C_G h_2|} \chi(\widehat{C}_G h_1 \widehat{C}_G h_2) \quad (11)$$

Now this equation (11) together with (7) gives the result.

We can now prove Proposition 17.

**Proof :** [Proposition 17] Without loss of generality we assume that  $W$  is the isotypic subspace associated to  $\chi_1, \dots, \chi_s$  where  $\chi_G \downarrow_H = \sum_{i=1}^r \lambda_i \chi_i$  with  $r > s$ .

Let

$$B := \sum_{g \in G} \sum_{h_1, h_2 \in H} E(h_1) E(h_2) \chi_G (h_1 g h_2 g^{-1})$$

Then,

$$B = \sum_{h_1, h_2 \in H} E(h_1) E(h_2) \sum_{g \in G} \chi_G (h_1 g h_2 g^{-1})$$

Using formula (8) (lemma 21) we have,

$$B = \sum_{h_1, h_2 \in H} E(h_1) E(h_2) \frac{|G| \chi_G(h_1) \chi_G(h_2)}{\chi_G(1)}$$

Now reminding that  $E(h) = \sum_{i=1}^s \chi_i(1) \overline{\chi_i(h)}$  and  $\chi_G \downarrow_H = \sum_{i=1}^r \lambda_i \chi_i$  we can apply the orthogonality relation (3) (Proposition 8) to get

$$B = \frac{|G| |H|^2 (\sum \lambda_i \chi_i(1))^2}{\chi_G(1)}$$

Now just sum each side of (6) for all  $g \in G$ . Call  $m := \sum_{i=1}^s \lambda_i \chi_i(1)$ , and  $n := \chi_G(1)$ .

$$|G \setminus H| d_c^2 = |G| m - \frac{1}{|H|^2} B$$

so

$$|G \setminus H| d_c^2 = |G| m - \frac{|G| m^2}{n}$$

and then

$$d_c^2 = \frac{|G|}{|G \setminus H|} \left( m - \frac{m^2}{n} \right)$$

If  $N := |G|/|H|$  then

$$d_c^2 = \frac{N}{N-1} \frac{m(n-m)}{n}$$

Looking at the definitions of  $m, n$  and  $N$  we have exactly a packing of  $N$  elements in  $\mathcal{G}_{m,n}$  that reach the simplex bound.

## 5 Examples

The next question is “Do such groups and subgroups exist”. We are interested in groups  $G$  and subgroups  $H$  such that the action of  $G$  on  $G/H$  by left multiplication is 2-transitive. One can give an easy example by taking  $G = \mathfrak{S}_n$  and  $H = \mathfrak{S}_{n-1}$ . This example is developed in the first subsection. We recall the main families of 2-transitive groups in the second subsection.

### 5.1 The symmetric group

Taking  $G = \mathfrak{S}_n$  and  $H = \mathfrak{S}_{n-1}$  (so  $G/H \approx \{1, \dots, n\}$ ), the action of  $G$  is 2-transitive on  $G/H$ . The irreducible representations of  $\mathfrak{S}_n$  are well known (see [9]). We give in Table 1 the required information about them and develop an example in parallel.

### 5.2 The classification

Now we are eager to know which groups fulfill the condition  $|H \backslash G/H| = 2$ . As we have seen any such group verifies that  $G$  has a 2-transitive action on  $G/H$ . But reciprocally for any 2-transitive group acting on a set  $\Omega$ , choose  $H$  to be any point stabiliser ; then  $|H \backslash G/H| = 2$ . To be precise we remark that nothing says that the action of  $G$  on  $G/H$  is faithful. In the case where the action is not faithful the groups  $G$  (and  $H$ ) may be rewritten  $G := G_1 \times N$  and  $H := H_1 \times N$  where the action of  $G_1$  on  $G_1/H_1 \approx G/H$  is 2 transitive and faithful.

We are mainly interested in finding all 2-transitive groups. Fortunately these groups have been classified see [2, 3]. To summarize, there are eight types of infinite families of 2-transitive groups and some ‘sporadic’ such groups. Among the eight families four are quite easy to describe:

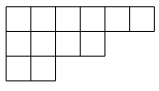
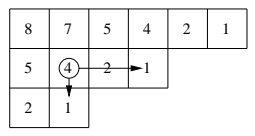
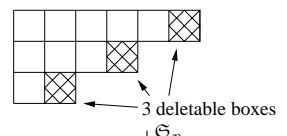
1. The alternating groups  $\mathfrak{A}_n$  acting on the set  $\{1, \dots, n\}$  ( $n - 2$  transitive).
2. The symmetric groups  $\mathfrak{S}_n$  acting on the set  $\{1, \dots, n\}$  ( $n$  transitive).
3. Affine groups. Let  $V$  be the vector space  $(\mathbb{F}_q)^d$ . Affine groups have this shape :  $G := V \rtimes G_0$  where  $G_0$  is a subgroup<sup>8</sup> of  $\Gamma L_d(\mathbb{F}_q)$ . The 2-transitive groups are obtained for any  $G_0$  which verify one of the following conditions :

- $SL_d(\mathbb{F}_q) \leq G_0 \leq \Gamma L_d(\mathbb{F}_q)$ ,
- $Sp_d(\mathbb{F}_q) \leq G_0 \leq \Gamma L_d(\mathbb{F}_q)$  where  $d = 2m$ ,

---

<sup>8</sup> $\Gamma L_d(\mathbb{F}_q)$  is the group acting on  $V$  generated by  $GL_d(\mathbb{F}_q)$  and all field automorphisms,  $\sigma : \mathbb{F}_q \rightarrow \mathbb{F}_q$  acting component-wise on elements of  $V$ .

Table 1: The example of  $\mathfrak{S}_n$

Facts	Example
<ul style="list-style-type: none"> <li>• There is an irreducible representation of <math>\mathfrak{S}_n</math> for each partition <math>\lambda = [\lambda_1, \dots, \lambda_k]</math> of <math>n</math> (<i>i.e.</i> a decreasing sequence of integers <math>[\lambda_1, \dots, \lambda_k]</math> whose sum is <math>n</math>).</li> <li>• One can associate a diagram to a partition in the following way. Draw <math>\lambda_1</math> box on the first line, <math>\lambda_2</math> on the second... As in the example.</li> <li>• The hook length of a box is the sum of the number of boxes under it (in the same column) and at its right (in the same line) plus one. In the example, we have filled each box with the length of the associated hook.</li> <li>• The dimension of the representation associated to <math>\lambda</math> is given by <math>\frac{n!}{z}</math> where <math>z</math> is the product of the hook length of every box.</li> <li>• The branching rule (see [9]) states that when restricted to <math>\mathfrak{S}_{n-1}</math> the irreducible character associated to <math>\lambda</math> decomposes itself as <math>\chi_\lambda = \chi_{\mu^{(1)}} + \dots + \chi_{\mu^{(l)}}</math> where each <math>\mu^{(i)}</math> is obtained from <math>\lambda</math> by deleting a ‘corner’ box.</li> <li>• Now we have everything needed to compute the parameters obtained. Let us choose the isotypic space associated to <math>\chi_{[5,4,2]}</math>. Its dimension can be computed with the hook formula and have a configuration with the following parameters.</li> </ul>	<p><math>n:=12</math></p> <p><math>\lambda := [6, 4, 2]</math></p>   <p><math>\dim(\chi_\lambda) = 2673</math></p>  <p>3 deletable boxes</p> <p><math>\chi_{[6,4,2]} \downarrow_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} = \chi_{[5,4,2]} + \chi_{[6,3,2]} + \chi_{[6,4,1]}</math></p> <p><math>N = 12, n = 2673, m = 990</math>  so <math>d_c^2 = 680</math>  (reaching the Simplex bound)</p>

- $G_0 = G_2(2^m)$ .

There is also a finite number of special cases with dimensions 2, 4 or 6.

4. Projective groups. Any group  $G$  with  $\text{PSL}_d(\mathbb{F}_q) \leq G \leq \text{P}\Gamma\text{L}_d(\mathbb{F}_q)$  acting on lines of  $(\mathbb{F}_q)^d$ .

There are also four other families coming from groups of Lie type. Describing the groups and their action in detail is not the aim of this article, see the references for more details.

5. Symplectic groups.  $G = \text{Sp}_d(\mathbb{F}_2)$  ( $d = 2m$ ) acting on subsets of transvections. The degree is  $2^{d-1}(2^d - 1)$  or  $2^{d-1}(2^d + 1)$ .
6. The unitary projective groups.  $\text{PSU}_3(q) \leq G \leq \text{P}\Gamma\text{U}_3(q)$  acting on isotropic lines of a quadratic form or on points of a  $S(2, q + 1, q^3 + 1)$  Steiner system.
7. Suzuki groups  $Sz(q)$  ( $q = 2^{2m+1}$ ) acting on points of a  $S(3, q + 1, q^2 + 1)$  Steiner system.
8. Ree groups  $R(q)$  ( $q = 3^{2m+1}$ ) acting on the points of a  $S(2, q + 1, q^3 + 1)$  Steiner system.

There are also some ‘sporadic’ 2-transitive groups with peculiar actions. Among them are<sup>9</sup> (cited with their degree) :  $(M_{11}, 11)$ ,  $(M_{11}, 12)$ ,  $(M_{22}, 22)$ ,  $(M_{23}, 23)$ ,  $(M_{24}, 24)$ ,  $(A_7, 15)$ ,  $(HS, 176)$ ,  $(Co_3, 276)$ .

Alas the use of this classification is partially theoretical. Indeed the character tables are not known for all these 2-transitive groups  $G$  and their associated subgroup  $H$ . But they are known for some infinite families (as for example  $\text{PGL}_2(\mathbb{F}_q)$  or  $\text{PSL}_2(\mathbb{F}_q)$  and their upper triangular subgroups<sup>10</sup> or simply  $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ ) and then they give birth to explicit infinite families of optimal simplicial configurations.

## 6 Remarks

The results of this article have some direct consequences. They imply some inequalities involving character degree and group order. The result also extends to unions of configurations obtained from the same irreducible character of  $G$ . This leads to configurations with only 3 non-zero distances. We also state a method to extend optimal simplicial configurations in a very easy way. This method enlarges our list of simplicial configuration and enlight ou judgement on ‘new’ configurations.

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<sup>9</sup>This list is not exhaustive. For more precision look at references

<sup>10</sup>All the packings obtained by the use of these two groups are given in the appendices.

## 6.1 Inequalities between character degrees and group orders

### 6.1.1 On simplicial configurations

The first inequality we present here comes from the simplicial configuration we have obtained. So let  $G$  be a 2-transitive group and  $H$  any point stabiliser. Then we have a simplicial configuration. This in particular implies (see [14] §5) that  $N \leq \binom{n+1}{2}$  so any character  $\chi$  of the group  $G$  is either irreducible when restricted to  $H$  or

$$\frac{|G|}{|H|} \leq \binom{\dim(\chi) + 1}{2}.$$

Yet, looking at some known values, this inequality seems weak.

### 6.1.2 On normal subgroups

Trying the general construction with any normal subgroup  $H \subset G$  gives an interesting result. Let  $\chi_G \downarrow_H^G = \sum_i \lambda_i \chi_i$  be the decomposition of the restriction to  $H$  of an irreducible character  $\chi_G$ . Let  $W$  be the isotypic space associated to  $\chi_1$ . Using the extended distance formula (6), and formula (5), we find that the distance between the two spaces  $W$  and  $gW$  is

$$\lambda_1 \chi_1(1) (1 - \langle \chi_1^g, \chi_1 \rangle)$$

where  $\forall h \in H$ ,  $\chi_1^g(h) = \chi_1(g^{-1}.h.g)$ . So the different spaces obtained are orthogonal.

- Suppose first that  $H$  is also maximal then we deduce the following lemma :

**Lemma 22** *Let  $H$  be a maximal subgroup of  $G$  which is also normal, then every irreducible character  $\chi$  of  $G$  is either irreducible when restricted to  $H$  or it splits  $\chi = \sum_i \lambda_i \chi_i$  but then*

$$\forall i, \lambda_i \chi_i(1) \cdot |G/H| \leq \chi(1),$$

and

$$\forall g \in G \setminus H, \chi_i^g \neq \chi_i.$$

- We can now extend this lemma easily to non-maximal subgroups.

**Lemma 23** *Let  $H$  be a normal subgroup of  $G$ , then every irreducible character  $\chi$  of  $G$  is either irreducible when restricted to  $H$  or it splits  $\chi = \sum_i \lambda_i \chi_i$  but then*

$$\forall i, \lambda_i \chi_i(1) |G/H| \leq \chi(1) \aleph_i$$

where  $\aleph_i = |\{g \in G/H / \chi_i^g = \chi_i\}|$ .

This case cannot be merged with the case of simplicial configurations. Indeed a point stabiliser subgroup in a 2-transitive group cannot be a normal subgroup (except in trivial cases).



## 6.2 Unions of configurations

One may be interested in the following extension. Assume that the irreducible character  $\chi_G$  is reducible when restricted to  $H$  i.e.  $\chi_G \downarrow_H^G = \sum_{i=1}^s \lambda_i \cdot \chi_i$  and that we have found subsets  $S_1, \dots, S_t$  such that

$$\forall j \in \llbracket 1, \dots, t \rrbracket, \sum_{i \in S_j} \lambda_i \cdot \chi_i(1) = m$$

for a fixed  $m$ . Otherwise stated we have similar configurations obtained by our method applied to the same irreducible character of  $G$  but with different (disjoint) character decomposition. In this case taking the union of all these configurations we have a packing of  $t \cdot N$  planes of dimension  $m$  in  $\mathbb{C}^n$  and the minimal distance can be computed in a similar way as Theorem 12 (cf appendices) and equals  $d_c^2 = \frac{N}{N-1} \frac{m(n-m-\frac{n}{N})}{n}$ .

To carry on the example of section 5.1 one can observe that the two components  $\chi_{[5,4,2]}$  and  $\chi_{[6,3,2]}$  in the decomposition of  $\chi_{[6,4,2]} \downarrow_{\mathfrak{S}_{11}}^{\mathfrak{S}_{12}}$  have the same dimension (990). So if we take the union of the two configurations we get a configuration of  $N = 24$  planes of dimension  $m = 990$  in  $\mathbb{C}^{2673}$  with minimal distance  $d_c^2 = \frac{13970}{23} \approx 607.4$  (for a value of 650.4 of the simplex bound). This may give not so bad configurations especially if we have a lot of subspaces with the same dimension (for the new value of minimal distance does not depend on how many sets are joined).

## 6.3 An easy extension

Thinking in term of projection matrices one can have the idea to use Kronecker product ( $\otimes$ ). As a first try we can make the product of all matrices of a configuration with the identity matrix of rank  $k : I_k$ .

$$\{I_k \otimes \Pi_{g.W} : g \in G/H\}$$

One can easily see that this multiplies  $m, n, d_c^2$  by  $k$  and keeps  $N$  invariant. If one applies this trick to any optimal simplicial configuration, the new chordal distance is then  $k d_c^2 = \frac{km(kn-km)}{kn}$  and so also reaches the simplex bound.

We can say that for all  $N, m, n$  any simplicial configuration with parameters  $N, k.m, k.n$  ( $k$  a positive integer) is not really a new one. For a fixed  $N$ , to be sure that a configuration is not a trivial extension of another one, we can compute the ratio  $n/m$ .

*Digression :* This idea can be extended to any couple of configuration obtained in any way. Let  $\{\Pi_i : i \in \llbracket 1, \dots, N_1 \rrbracket\}$  and  $\{\Pi'_j : j \in \llbracket 1, \dots, N_2 \rrbracket\}$  be two configurations in  $G_{m_1, n_1}$  and  $G_{m_2, n_2}$  respectively with squared minimal chordal distances equal to  $d_{c_1}^2$  and  $d_{c_2}^2$ . Then  $\{\Pi_i \otimes \Pi'_j : i \in \llbracket 1, \dots, N_1 \rrbracket, j \in \llbracket 1, \dots, N_2 \rrbracket\}$  is a configuration in  $G_{m_1 m_2, n_1 n_2}$  with squared minimal distance equaling  $\min(m_2 d_{c_1}^2, m_1 d_{c_2}^2)$ .

## 7 Conclusion

In this article we have described an explicit method to construct Grassmannian packing using representation theory. We have shown how to use 2-transitive groups to get optimal simplicial configuration w.r.t. the chordal distance. With this construction the chordal distance arises in a natural way with formula (6). If our configurations perform well with this distance they have a less obvious behaviour regarding the pseudo-distance  $\tilde{d}$ . It may be also a challenging question to find an algebraic context in which the pseudo-distance  $\tilde{d}$  is privileged then try to prove the optimality of some configurations. This will provide optimal constellations for high SNR communications. Although we have focused on simplex configurations the method described in this article can be used with many group  $G$  and  $H$  and is quite promising and has many extension. We have already found optimal configurations of [12] by such an extension (see appendices). It shall also be noticed that the group structure may give fast decoding algorithm using group stabiliser.

## A Dependences of the principal angles

In this first appendix we will prove Proposition 16. As the principal angles only depends of the eigenvalues of

$$A := \Pi_E \rho(g) \Pi_{\tilde{E}} \rho(g^{-1})$$

We just have to prove that  $\text{Trace} A^k$  only depends of the conjugacy class  $HgH$ . But  $\text{Trace} A^k$  equals

$$\sum_{\substack{h_1, \dots, h_k \in H \\ \tilde{h}_1, \dots, \tilde{h}_k \in H}} E(h_1) \dots E(h_k) \tilde{E}(\tilde{h}_1) \dots \tilde{E}(\tilde{h}_k) \chi(h_1 g \tilde{h}_1 g^{-1} h_2 g \tilde{h}_2 g^{-1} \dots h_k g \tilde{h}_k g^{-1})$$

Let introduce conjugacy classes in  $H : \bigcup C_i = H$ , and  $h_i^*$  a family of classes representatives. Then  $\text{Trace} A^k$  equals

$$\sum_{\substack{C_1, \dots, C_k \in \mathcal{C}(H) \\ \tilde{C}_1, \dots, \tilde{C}_k \in \mathcal{C}(H)}} E(C_1) \dots E(C_k) \tilde{E}(\tilde{C}_1) \dots \tilde{E}(\tilde{C}_k) \left( \prod_{i=1}^k \frac{|C_i| |\tilde{C}_i|}{|G|^2} \right) \times \\ \sum_{\substack{c_1, \dots, c_k \in H \\ \tilde{c}_1, \dots, \tilde{c}_k \in H}} \chi \left( (c_1 h_1^* c_1^{-1}) g (\tilde{c}_1 \tilde{h}_1^* \tilde{c}_1^{-1}) g^{-1} (c_2 h_2^* c_2^{-1}) g \dots (c_k h_k^* c_k^{-1}) g (\tilde{c}_k \tilde{h}_k^* \tilde{c}_k^{-1}) g^{-1} \right)$$

We may content ourself to prove that the inner sum (on  $c_i$ 's in  $H$  ...) only depends of the double class  $HgH$ . Let introduce  $\lambda_i = c_1^{-1} c_i$  we can rewrite this sum as

$$\sum_{\substack{c_1, \lambda_2, \dots, \lambda_k \in H \\ \tilde{c}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_k \in H}} \chi \left( (c_1 h_1^* c_1^{-1}) g (\tilde{c}_1 \tilde{h}_1^* \tilde{c}_1^{-1}) g^{-1} (c_1 \lambda_2 h_2^* \lambda_2^{-1} c_1^{-1}) g \right. \\ \left. (\tilde{c}_1 \tilde{\lambda}_2 \tilde{h}_2^* \tilde{\lambda}_2^{-1} \tilde{c}_1^{-1}) g^{-1} \dots (\tilde{c}_1 \tilde{\lambda}_k \tilde{h}_k^* \tilde{\lambda}_k^{-1} \tilde{c}_1^{-1}) g^{-1} \right)$$

As  $\chi$  is invariant by cyclic shift we can rewrite the previous term :

$$\sum_{\substack{c_1, \lambda_2, \dots, \lambda_k \in H \\ \tilde{c}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_k \in H}} \chi \left( h_1^* (c_1^{-1} g \tilde{c}_1) \tilde{h}_1^* (\tilde{c}_1^{-1} g^{-1} c_1) \lambda_2 h_2^* \lambda_2^{-1} (c_1^{-1} g \tilde{c}_1) \tilde{\lambda}_2 \tilde{h}_2^* \tilde{\lambda}_2^{-1} \right. \\ \left. (\tilde{c}_1^{-1} g^{-1} \dots c_1) \lambda_k h_k^* \lambda_k^{-1} (c_1^{-1} g \tilde{c}_1) \tilde{\lambda}_k \tilde{h}_k^* \tilde{\lambda}_k^{-1} (\tilde{c}_1^{-1} g^{-1} c_1) \right)$$

If we do the following change of variable  $u = c_1^{-1} g \tilde{c}_1$  as  $c_1$  and  $\tilde{c}_1$  run through  $H$ , then  $u$  run  $\frac{|H|^2}{|HgH|}$  times through the double class  $HgH$ . So up to this factor the forementioned sum equals

$$\sum_{\substack{\lambda_2, \dots, \lambda_k \in H \\ \tilde{\lambda}_2, \dots, \tilde{\lambda}_k \in H \\ u \in HgH}} \chi \left( h_1^* u \tilde{h}_1^* u^{-1} \lambda_2 h_2^* \lambda_2^{-1} u \tilde{\lambda}_2 \tilde{h}_2^* \tilde{\lambda}_2^{-1} u^{-1} \dots \right. \\ \left. \dots u^{-1} \lambda_k h_k^* \lambda_k^{-1} u \tilde{\lambda}_k \tilde{h}_k^* \tilde{\lambda}_k^{-1} u^{-1} \right)$$

So  $\text{Trace}(A^k)$  does not depend of  $g$  but only of the double class  $HgH$ .

## B Minimal distance value in unions

We here hint how to prove the value of minimal distance when joining configurations (cf 6.2). Let  $G$  be a 2 transitive group and  $H$  a point stabiliser subgroup. Suppose that  $\chi_G$  is an irreducible character, that  $\chi_G \downarrow_H^G = \sum_{i=1}^s \lambda_i \cdot \chi_i$ , and that we have found subsets  $S_1, \dots, S_t$  such that

$$\forall j \in \llbracket 1, \dots, t \rrbracket, \sum_{i \in S_j} \lambda_i \cdot \chi_i(1) = m$$

for a fixed  $m$ .

We need to compute the distance value for any pair of  $m$  dimensional planes. Let  $W_1$  and  $W_2$  be the direct sum of isotopic spaces associated to  $S_1$  and  $S_2$ . Let us focus on the distance between  $W_1$  and  $g.W_2$ .

If  $g \in H$  then  $g.W_2 = W_2$  and the two spaces  $W_1$  and  $W_2$  are orthogonal so the minimal distance between them is  $m$ . If  $g \notin H$  the chordal distance may take only one value by Proposition 16.

To get the value of chordal distance mentioned in 6.2 it suffices to use an analog of (6) with different characters and sum over  $g \in G$  as in the proof of Proposition 17. We get that

$$|H|m + |G \setminus H|d_c^2(W_1, gW_2) = |G|m - \frac{1}{|H|^2}B$$

where

$$B = \sum_{g \in G} \sum_{h_1, h_2 \in H} E_{S_1}(h_1)E_{S_2}(h_2)\chi_G(h_1gh_2g^{-1})$$

so

$$B = \frac{|G||H|^2 \left( \sum_{i \in S_1} \lambda_i \chi_i(1) \right) \left( \sum_{i \in S_2} \lambda_i \chi_i(1) \right)}{\chi_G(1)}$$

So, following the proof of Theorem 12, we found that

$$d_c^2(W_1, gW_2) = \frac{N}{N-1} \frac{m(n-m-\frac{n}{N})}{n}.$$

This value is smaller than  $d_c^2(W_1, W_2) = m$  or  $d_c^2(W_1, gW_1)$ . So it is therefore the minimal chordal distance.

## C Explicit parameters coming from $\mathrm{GL}_2(\mathbb{F}_q)$ and $\mathrm{SL}_2(\mathbb{F}_q)$

The construction obtained by the use of  $\mathrm{GL}_2(\mathbb{F}_q)$  and  $\mathrm{SL}_2(\mathbb{F}_q)$  have been fully studied. So we can give all the simplicial configurations coming from all characters of this groups. For all  $q$  such that  $\mathbb{F}_q$  exists (*i.e.*  $q$  is a power of a prime) we have Grassmannian configurations with the following parameters :

Table 2: Packings found

$$N := q + 1$$

$n$	$q + 1$	$q$	$q + 1$	$q$
$m$	1	1	2	$\frac{q-1}{2}$
$d_c^2$	1	$1 - \frac{1}{q^2}$	$2\frac{q-1}{q}$	$\frac{(q+1)^2(q-1)}{4q^2}$
$\bar{d}$	1	$1 - \frac{1}{q^2}$	$(\frac{q-1}{q})^2$	?

$n$	$q + 1$	$q + 1$	$\frac{q+1}{2}$	$q - 1$
$m$	$\frac{q-1}{2}$	$\frac{q+1}{2}$	1	$\frac{q-1}{2}$
$d_c^2$	$\frac{(q-1)(q+3)}{4 \cdot q}$	$\frac{(q+1)^2}{4 \cdot q}$	$\frac{q-1}{q}$	$\frac{q^2-1}{4 \cdot q}$
$\bar{d}$	?	?	$\frac{q-1}{q}$	?

Reminding section 6.3 we can state that the third column of first block is not useful because it is a consequence of the third one of the second block.

## D An optimal orthoplex configuration

In this last appendix we will show that a small variation of the general method give the optimal configurations discribed in [12]<sup>11</sup>. We recall the construction of the keystone groups :

Let  $U = \mathbb{F}_2^i$  and  $V = \mathbb{R}^n$  where  $n = 2^i$ . Consider  $\{e_u : u \in U\}$  a vector basis for  $V$  and  $E$  be the (extraspecial) subgroup of the orthogonal group  $\mathcal{O} = O(V)$  generated by

$$X(a) : e_u \mapsto e_{u+a}, \quad \text{and } Y(b) : e_u \mapsto (-1)^{b \cdot u} e_u, \quad u \in U.$$

The normalizer of  $L$  in  $\mathcal{O}$  is the (Clifford type) subgroup of  $\mathcal{O}$  generated<sup>12</sup> by :  $E, H, \tilde{H}_2, \widetilde{GL(V)}$  and  $\{d_M : M \text{ skew-symmetric}\}$ , where

- $H = \frac{1}{\sqrt{N}} [(-1)^{u \cdot v}]_{u, v \in V}$
- $\tilde{H}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes I_{2^{i-1}}$
- $\widetilde{GL(V)}$  is the group generated by the orthogonal transformations  $G_A : V \rightarrow V$  permuting coordinates :  $e_v \mapsto e_{Av}$
- $d_M$  is the diagonal matrix  $(-1)^{Q_M(v)}$  where  $Q_M$  is the quadratic form associated to the skew-symmetric matrix  $M$  i.e.  $Q_M(u+v) = Q_M(u) + Q_M(v) + uMv^T$ .

Remark that the group  $E$  is a representation of itself and its character  $\chi_E$  as value 0 for any element except  $\pm I$ .

Let  $S_0$  be any subgroup of  $E$  generated by  $-I$  and  $r$  independent order 2 element  $g_1, \dots, g_r$  of  $E$ . The restriction of  $\chi_E$  to  $S$  is equal to the sum of  $2^r$  distinct linear characters with multiplicity  $2^{i-r}$ .

Select one irreducible character of the above decomposition (say  $\chi$ ) and let  $\Pi_{S_0, \chi} = \frac{1}{|S_0|} \sum_{s \in S_0} \overline{\chi(s)} s$  be the projection matrix on the isotypic subspace  $W$ . Then the orbit of  $W$  under the action of  $L$  give back the construction of [12]. Indeed in the expression

$$\rho(g) \Pi_{S_0, \chi} \rho(g^{-1}) = \frac{1}{|S_0|} \sum_{s \in g S_0 g^{-1}} \overline{\chi(g^{-1} s g)} s$$

one can check that taking  $g \in E$  allow one to swap characters, taking  $g \in L$  allow to change the subgroup  $S_0$  (as  $L$  act transitively on such subgroups by conjugation).

Let now focus on the optimal case where  $r = 1$  (i.e.  $|S_0| = 4$ ) this may be a special case of the following fact :

<sup>11</sup>Warning : their notations for  $n$  and  $m$  are different from ours.

<sup>12</sup>Some of these elements are missing in [12], see [13].

**Proposition 24** *Let  $G$  be a matrix subgroup of  $U_n(\mathbb{C})$  such that only  $\pm I$  have a non-zero trace. Consider  $\mathcal{S}$  the set of distinct subgroup generated by  $-I$  and an order 2 element. Then the set of isotypic subspaces with projection matrices*

$$\frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \chi_1(s)s \quad \text{and} \quad \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} \chi_2(s)s \quad \text{for } S \in \mathcal{S}$$

*form a Grassmannian code in  $\mathcal{G}_{m,n}$  with  $m = \frac{n}{2}$  of cardinal  $2|\mathcal{S}|$  where only non-zero distances are  $m$  and  $m/2$ . This code reach the orthoplex bound when  $2|\mathcal{S}| \geq \frac{n(n+1)}{2}$ .*

This proposition can be easily proved by a calculus (using (6)) similar to previous demonstrations.

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