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# Time Warp Edit Distance with Stiffness Adjustment for Time Series Matching

(Version 4: February 2007)

P.F. MARTEAU

**Abstract--** In a way similar to the string-to-string correction problem we address time series similarity in the light of a time-series-to-time-series-correction problem for which the similarity between two time series is measured as the minimum cost sequence of "edit operations" needed to transform one time series into another. To define the "edit operations" we use the paradigm of a graphical editing process and end up with a dynamic programming algorithm that we call Time Warp Edit Distance (TWED). TWED is slightly different in form from Dynamic Time Warping, Longest Common Subsequence or Edit Distance with Real Penalty algorithms. In particular, it highlights a parameter which drives a kind of stiffness of the elastic measure along the time axis. We show that the similarity provided by TWED is a metric potentially useful in time series retrieval applications since it could benefit from the triangular inequality property to speed up the retrieval process while tuning the parameters of the elastic measure. In that context, a lower bound is derived to relate the matching of time series into down sampled representation spaces to the matching into the original space. Empiric quality of the TWED distance is evaluated on a simple classification task. Compared to Edit Distance, Dynamic Time Warping, Longest Common Subsequence and Edit Distance with Real Penalty, TWED has proven to be quite effective on the considered experimental task.

**Index Terms--** Pattern Recognition, Time Series, Algorithms, Similarity Measures.

## I. INTRODUCTION

More and more computer applications are faced with the problem of searching for time series within large datasets which are close to a given query element under some similarity criteria. Among numerous examples, we find financial and stock data analysis [37], moving objects identification [4], astronomy [28], medicine [19], meteorology, data mining [1], etc.

All these applications embed time series in a representation space and exploit some similarity measure defined for this space. Similarity measures fall basically into three categories:

- Non elastic metrics such as  $L_p$ -norms that do not support time shifting such as Euclidian Distance (ED) and Correlation,

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- Elastic similarity measures that tolerate time shifting but are not metrics such as Dynamic Time Warping (DTW) [34], [32] or Longest Common Subsequence (LCSS) [6], [35].
- Elastic metrics that tolerate time shifting such as Edit distance with Real Penalty (ERP) [5].

When considering time series information retrieval, working in a metric space can be appealing since a lot of data structures (essentially tree based structures) and algorithms (partitioning, pivoting, etc.) have been optimized and made available for indexing and retrieving objects efficiently in metric spaces: see [3] for a review. All these structures and algorithms take advantage of the triangle inequality which allows for efficient pruning of a large number of time series which are too far from the query. For some non-metric measures all these data structures can still be used if a lower bounding approximation which is a metric is available. Such a lower bound exists for both LCSS and DTW as detailed in [35].

In this paper we address the case of elastic metrics, namely elastic similarity measures that jointly exploit time shifting and possess all the properties of a distance, in particular the triangle inequality. Our contribution is basically four folded:

- The first contribution of this paper is the proposal of new elastic metric which we call TWED (“Time Warp Edit Distances”). This contribution has to be placed in the perspective of former works that seek to combine *L<sub>p</sub>-norms* with the edit distance, in particular in the light of the ERP distance [5] that can support local time shifting while being a metric. Other elastic similarity measures that belong to the Dynamic Time Warping category are not metrics since they do not satisfy the triangle inequality. Part II of the paper motivates the need for triangle inequality to process time series in a data
-

compression context based on a down sampling perspective.

- The second contribution is related to the introduction of a parameter we call *stiffness* which controls the *elasticity* of TWED, placing this kind of distance in between the Euclidian distance (somehow a distance with '*infinite stiffness*') and DTW (somehow a similarity measure with no '*stiffness*' at all). One of the differences between TWED and former similarity measures is the use of time stamp differences between compared samples as part of the local matching costs. The motivation for such a characteristic is also given in part II of the paper.
- The third contribution proposes a lower bound for the TWED measure which allows relating the evaluation of the matching of two time series into down sampled representation spaces to the evaluation of their matching into their original representation spaces.
- The fourth contribution of the paper is an empiric evaluation of the quality of TWED based on a simple classification experiment that provides some highlights on the effectiveness of TWED compared to the Euclidian Distance (ED), DTW, LCSS and ERP. The influence of the *stiffness* parameter on classification error rates is also analyzed.

The paper is organized as follows. Section II addresses the motivation aspects. Section III presents shortly the main relevant founding works about elastic distances for time series matching. Section IV details the definition and implementation of the Time Warp Edit Distance with stiffness adjustment that is proposed in this paper. Section V details a lower bounding procedure we suggest to speed up range queries processing. Section VI reports a classification experiments that shows the empirical effectiveness of TWED comparatively to the Euclidian distance and other classical elastic measures. Section VII concludes the paper and proposes some

perspectives.

## II. MOTIVATION FOR A SIMILARITY MEASURE THAT VERIFIES THE TRIANGLE INEQUALITY AND TAKES INTO ACCOUNT TIMESTAMP DIFFERENCES

The use of elasticity theory to model the behaviour of non-rigid curves, surfaces, and solids as function of time has given rise to a lot of applications in medical image analysis, vision or computer graphics (see [33] [26] for surveys). These models are fundamentally dynamic and unify the description of shape and the description of motion. In another hand, elastic distances have been proposed to define similarity measures that are tolerant to object deformations, in particular stretching or shrinking. If analogy exists between these general purpose distances and physical models of deformable objects, we cannot push the analogy too far since the physics law that should govern the matching of object are not systematically known. Our motivation for using time stamps (or sample indices) is related to the way we want to control the elasticity of the measure. Differences of indices between match samples have been successfully used to improve elastic measures such as Dynamic time Warping [32] or Longest Common Subsequence measures [11]. The general idea is to limit the elasticity of the measure by using a threshold: if the index difference between two samples that are candidates for a match is lower than the value of the threshold, then the matching is allowed, otherwise it is forbidden. This binary decision might in some cases limit the effectiveness of the measure. Keeping in mind the mechanical analogy of a spring (for which the deformation effort is proportional to the stretching or the shrinking), instead of using a threshold we suggest using the range of the sample index difference to linearly penalize the matching of samples for which the index values are too far and to favor the matching of samples for which the index values are close. In the case where time series are sampled using non uniform or varying sampling rates one can benefit from time stamps instead

of sample indices since this approach does not require resampling of the data.

The second motivation for defining a measure that exploits time stamps (or sample indices) while verifying the triangle inequality is two folds: first it provides an effective solution to compare approximated representations of time series not necessarily using uniform down-sampling methods; second, it establishes a useful relationship between the matching performed in the down sampled space and the matching performed in the original space.

Approximation of multi dimensional discrete curves has been widely studied essentially to speed up data processing required by resource demanding applications. Among other approaches, polygonal approximation of discrete curves has been quite popular recently. The problem can be informally stated as follows: given a digitized curve  $X$  of  $N \geq 2$  ordered samples, find  $K$  (in general  $K \ll N$ ) dominant samples among them that define a sequence of piecewise linear segments which most closely approximate the original curve. This problem is known as the *min- $\epsilon$*  problem. Numerous algorithms have been proposed for more than thirty years to solve efficiently this optimization problem. Most of them belong either to graph-theoretic, dynamic programming or to heuristic approaches. See for instance [12] [29] [16] [23] among others for details.

Such approaches can be used to adaptively down sample time series. For instance, in [23] polygonal curves approximations have been used to down sample optimally gesture signals and in [23] an elastic matching procedure has been proposed to compare two time series with a linear time complexity. For these approaches, a down sampled time series is a reduced sequence of tuples (sample, time stamps) that corresponds to the end extremities of the polygonal segments. The sampling rate for such down sampled time series is not uniform in general.

Down sampling time series can be used to drastically reduce the dimension of the space in which we could potentially process the time series. Nevertheless one difficulty emerges: how can

we compare down sampled time series using non uniform (e.g. varying) sampling frequencies? Not taking into account the occurring time of the samples could introduce discrepancies between the original space and the down sampled space. For instance, phase or frequency information is potentially lost or at least damaged as well as the slope of spikes.

In this context, the triangle inequality is also of great importance since it maintains distance relations between the original space and the down sampled space. Let  $X$  and  $Y$  be two time series in the original space and  $\tilde{X}$  and  $\tilde{Y}$  their down sampled counter parts. If  $\delta$  is a measure for which the triangle inequality holds, then we have:  $|\delta(X, Y) - \delta(\tilde{X}, \tilde{Y})| \leq \delta(X, \tilde{X}) + \delta(Y, \tilde{Y})$

In the case where  $\delta(X, \tilde{X})$  and  $\delta(Y, \tilde{Y})$  are maintained small by the similarity measure,  $\delta(X, Y)$  and  $\delta(\tilde{X}, \tilde{Y})$  are comparable and the following inequality gives an exploitable lower bound to  $\delta(X, Y)$  measure:

$$\delta(\tilde{X}, \tilde{Y}) - \delta(X, \tilde{X}) - \delta(Y, \tilde{Y}) \leq \delta(X, Y) \quad (1)$$

This lower bound can be used to significantly speed up the time series information retrieval process since a pruning strategy can be proposed in the down sampled space. We will get back to this issue in section V.

### III. ELASTIC SIMILARITY IN THE LIGHT OF THE SYMBOLIC EDIT DISTANCE

In this section we present succinctly the main elastic measures developed in the literature, from founding work to more recent studies.

The Levenshtein Distance (LD) proposed in 1966 [18], also known as the edit distance, is the smallest number of insertions, deletions, and substitutions required to change one string into another. For more than thirty years, the ideas behind LD have been largely reused and extended by various research communities. The main contributions are rapidly reviewed below. In 1974

Wagner and Fisher [36] developed a computationally efficient algorithm to calculate LD in  $O(n.m)$  using dynamic programming [2]. Meanwhile Dynamic Time Warping, which shares many similarities with LD despite the fact that it is not a metric, was proposed in 1970 [34] and 1971 [32] to align speech utterances, namely time series, with time shift tolerances. The Longest Common Subsequence (LCSS) similarity measure initially defined for string matching [11] has also been adapted for time series matching [6][35]. Recently, a lot of fruitful research dealing with DTW and LCSS has been carried out to propose efficient computation and pruning strategies that are required to process massive data [35][13][38]. Some work has also been conducted to provide the ‘triangle inequality’ to DTW: the Edit Distance with Real Penalty (ERP) [5] has been proposed as an edit distance based metric for time series matching with time shift tolerance. The edit distance principle has been also proposed to develop match 1D-point-patterns (ascending lists of real values) [20][21]. The measure proposed to match 1D-Point Patterns (PPM) is shown to be a metric that can be extended to the multidimensional case, at the price of a non polynomial complexity. Shortly, hereinafter, we present shortly hereinafter DTW, ERP and LCSS in the light of the edit distance and develop the TWED metrics as an alternative to ERP.

#### A. Definitions

Let  $U$  be the set of finite time series:  $U = \{A_1^p / p \in N\}$ .  $A_1^p$  is a time series with discrete time index varying between 1 and  $p$ . We note  $\Omega$  the empty time series (with null length) and by convention  $A_1^0 = \Omega$  so that  $\Omega$  is member of set  $U$ .

Let  $A$  be a finite discrete time series. Let  $a'_i$  be the  $i^{th}$  sample of time series  $A$ . We will consider that  $a'_i \in S \times T$  where  $S \subset R^d$  with  $d \geq 1$  embeds the multidimensional space variables



and  $T \subset R$  embeds the time stamp variable, so that we can write  $a'_i = (a_i, t_{a_i})$  where  $a_i \in S$  and  $t_{a_i} \in T$ , with the condition that  $t_{a_i} > t_{a_j}$  whenever  $i > j$  (time stamp are strictly increasing in the sequence of samples).

$A_i^j$  with  $i \leq j$  is the sub time series consisting of the  $i^{\text{th}}$  through the  $j^{\text{th}}$  samples (inclusive) of  $A$ . So  $A_i^j = a'_i a'_{i+1} \dots a'_j$ .  $|A|$  denotes the length (the number of samples) of  $A$ .  $\Lambda$  denotes the null sample.  $A_i^j$  with  $i > j$  is the null time series noted  $\Omega$ .

An edit operation is a pair  $(a', b') \neq (\Lambda, \Lambda)$  of time series samples, written  $a' \rightarrow b'$ . Time series  $B$  results from the application of the edit operation  $a \rightarrow b$  into time series  $A$ , written  $A \Rightarrow B$  via  $a' \rightarrow b'$ , if  $A = \sigma a' \tau$  and  $B = \sigma b' \tau$  for some time series  $\sigma$  and  $\tau$ . We call  $a' \rightarrow b'$  a match operation if  $a' \neq \Lambda$  and  $b' \neq \Lambda$ , a delete operation if  $b' = \Lambda$ , an insert operation if  $a' = \Lambda$ .

Similarly to the edit distance defined for string [11], we define  $\delta(A, B)$  the similarity between any two time series  $A$  and  $B$  of finite length, respectively  $p$  and  $q$  as:

$$\delta(A_1^p, B_1^q) = \text{Min} \begin{cases} \delta(A_1^{p-1}, B_1^q) + \Gamma(a'_p \rightarrow \Lambda) & \text{delete} \\ \delta(A_1^{p-1}, B_1^{q-1}) + \Gamma(a'_p \rightarrow b'_q) & \text{match} \\ \delta(A_1^p, B_1^{q-1}) + \Gamma(\Lambda \rightarrow b'_q) & \text{insert} \end{cases} \quad (2)$$

Where  $p \geq 1, q \geq 1$  and  $\Gamma$  is an arbitrary cost function which assigns to each edit operation  $a' \rightarrow b'$  a nonnegative real number  $\Gamma(a' \rightarrow b')$ .

The recursion is initialized by setting:

$$\begin{aligned} \delta(A_1^0, B_1^0) &= 0 \\ \delta(A_1^0, B_1^j) &= \infty \\ \delta(A_1^i, B_1^0) &= \infty \end{aligned}$$

Dynamic Time Warping (DTW) and Edit Distance with Real penalties (ERP), 1D Point Pattern matching (PPM) and Longest Common Subsequence (LCSS) are special cases of the previous definitions:

### B. The DTW special case

The DTW similarity measure [34][32]  $\delta_{DTW}$  is defined according to the previous notations as:

$$\delta_{DTW}(A_1^p, B_1^q) = d_{LP}(a_p, b_q) + \text{Min} \begin{cases} \delta_{DTW}(A_1^{p-1}, B_1^q) \\ \delta_{DTW}(A_1^{p-1}, B_1^{q-1}) \\ \delta_{DTW}(A_1^p, B_1^{q-1}) \end{cases} \quad (3)$$

where  $d_{LP}(x, y)$  is the  $LP$  norm of vector  $x-y$  in  $R^d$ ,

and so for DTW,  $\Gamma(a'_p \rightarrow \Lambda_q) = \Gamma(\Lambda_p \rightarrow b'_q) = \Gamma(a'_p \rightarrow b'_q) = d_{LP}(a_p, b_q)$

We notice that the time stamp values are not used so that the costs of each edit operation involve vectors  $a$  and  $b$  in  $S$  instead of vectors  $a'$  and  $b'$  in  $S \times T$ . One of the main restrictions of  $\delta_{DTW}$  is that it does not comply with the triangle inequality as shown by the following example:

$$\begin{aligned} A_1^1 &= [1]; B_1^2 = [1,2]; C_1^3 = [1,2,2](*) \\ \delta_{DTW}(A_1^1, B_1^2) &= 1; \delta_{DTW}(B_1^2, C_1^3) = 0; \delta_{DTW}(A_1^1, C_1^3) = 2 \\ \text{and thus : } \delta_{DTW}(A_1^1, C_1^3) &> \delta_{DTW}(A_1^1, B_1^2) + \delta_{DTW}(B_1^2, C_1^3) \end{aligned}$$

(\*) 1D time series with no stamp value given

### C. The ERP special case

$$\delta_{ERP}(A_1^p, B_1^q) = \text{Min} \begin{cases} \delta_{ERP}(A_1^{p-1}, B_1^q) + \Gamma(a'_p \rightarrow \Lambda) \\ \delta_{ERP}(A_1^{p-1}, B_1^{q-1}) + \Gamma(a'_p \rightarrow b'_q) \\ \delta_{ERP}(A_1^p, B_1^{q-1}) + \Gamma(\Lambda \rightarrow b'_q) \end{cases} \quad (4)$$

$$\begin{aligned} \Gamma(a'_p \rightarrow \Lambda) &= d_{LP}(a_p, g) \\ \text{with } \Gamma(a'_p \rightarrow b'_q) &= d_{LP}(a_p, b_q) \\ \Gamma(\Lambda \rightarrow b'_q) &= d_{LP}(g, b_q) \end{aligned}$$

and  $g$  a constant in  $S$ .

where  $d_{LP}(x, y)$  is the  $LP$  norm of vector  $x-y$  in  $S$ . Note that the time stamp coordinate is not taken into account such that  $\delta_{ERP}$  is a distance on  $S$  but not on  $S \times T$ .

We notice here again that the time stamp values are not used such that the costs of each edit operation involve vectors  $a$  and  $b$  in  $R^d$  instead of vectors  $a'$  and  $b'$  in  $R^{d+1}$ .

According to the authors of ERP [5], the constant  $g$  should be set to 0 for some intuitive geometric interpretation and in order to preserve the mean value of the transformed time series when adding gap samples.

#### D. The LCSS special case

The Longest Common Subsequence (LCSS) similarity measure has been first defined for string matching purposes [11] and then extended for times series [6][35]. LCSS is recursively defined in [35] as follows:

$$LCSS_{\epsilon, \delta}(A_1^p, B_1^q) = \begin{cases} 0 & \text{if } p < 1 \text{ or } q < 1, \\ 1 + LCSS_{\epsilon, \delta}(A_1^{p-1}, B_1^{q-1}) & \text{if } d_{LP}(a_p, b_q) < \epsilon \text{ and } |p - q| < \delta, \\ \text{Max}\{LCSS_{\epsilon, \delta}(A_1^{p-1}, B_1^q), LCSS_{\epsilon, \delta}(A_1^p, B_1^{q-1})\} & \text{otherwise} \end{cases} \quad (5)$$

For LCSS the match reward is 1, while no reward is offered for *insert* or *delete* operations.

The LCSS measure is transposed into a normalized dissimilarity measure  $D_{\epsilon, \delta}$  which is close in its formal structure to the ERP measure:

$$D_{\epsilon, \delta} = 1 - \frac{LCSS_{\epsilon, \delta}(A_1^p, B_1^q)}{\text{Min}\{p, q\}} \quad (6)$$

### E. The 1D PPM special case

For Point-Pattern matching problems [20],  $A_1^p$  and  $B_1^q$  are 1D ascending lists of real values.

$$\delta_{PPM}(A_1^p, B_1^q) = \text{Min} \begin{cases} \delta_{PPM}(A_1^{p-1}, B_1^q) + \Gamma(a'_p \rightarrow \Lambda) \\ \delta_{PPM}(A_1^{p-1}, B_1^{q-1}) + \Gamma(a'_p \rightarrow b'_q) \\ \delta_{PPM}(A_1^p, B_1^{q-1}) + \Gamma(\Lambda \rightarrow b'_q) \end{cases} \quad (7)$$

$$\begin{aligned} \Gamma(a'_p \rightarrow \Lambda) &= a_p - a_{p-1} \\ \text{with } \Gamma(a'_p \rightarrow b'_q) &= |(a_p - a_{p-1}) - (b_q - b_{q-1})| \quad \text{if } p > 1 \text{ and } q > 1 \\ \Gamma(\Lambda \rightarrow b'_q) &= b_q - b_{q-1} \end{aligned}$$

The author [20] shows that  $\delta_{PPM}$  is a metric that calculates the minimum amount of space needed to delete or insert between pairs of points to convert one point-pattern into another. One can notice that if we consider successive increments instead of the initial values,  $\delta_{PPM}$  coincides with the  $\delta_{ERP}$  applied to the lists of positive increments.

### F. Symbolic sequence alignment with affine gap penalty

In biomolecular sequences (DNA, RNA, or amino acid sequences), high sequence similarity usually implies significant functional or structural similarity. The basic mutational processes behind the evolution of such sequences are substitutions, insertions and deletions, the latter two giving rise to gaps. Various similarity models based on dynamic programming have been developed by the bioinformatics community. Among them we highlight the affine gap model [10] [8] that extends the Needleman-Wunsch algorithm [27]. The originality of this model is to penalized gap sequences according to the affine equation  $\gamma(g) = -d - (g - 1)e$ , where  $g$  is the length of the gap,  $d$  is the *open-gap* penalty and  $e$  is the *gap-extension* penalty. The recursion is given in equation (8):

$$\begin{aligned}
 M(A_1^p, B_1^q) &= \Gamma(a_p \rightarrow b'_q) + \text{Max} \begin{cases} I_x(A_1^{p-1}, B_1^{q-1}) \\ M(A_1^{p-1}, B_1^{q-1}) \\ I_y(A_1^{p-1}, B_1^{q-1}) \end{cases} \\
 I_x(A_1^p, B_1^q) &= \text{Max} \begin{cases} M(A_1^{p-1}, B_1^q) - d \\ I_x(A_1^{p-1}, B_1^q) - e \end{cases} \\
 I_x(A_1^p, B_1^q) &= \text{Max} \begin{cases} M(A_1^p, B_1^{q-1}) - d \\ I_x(A_1^p, B_1^{q-1}) - e \end{cases}
 \end{aligned} \tag{8}$$

Here  $\Gamma(a_p \rightarrow b_q)$  is an integer value either positive when  $a_p$  and  $b_q$  are similar symbols, or negative when  $a_p$  and  $b_q$  are dissimilar symbols,  $M(A_1^p, B_1^q)$  is the best score up to  $(a_p, b_q)$ , given that  $a_p$  is aligned to  $b_q$ ,  $I_x(A_1^p, B_1^q)$  is the best score up to  $(a_p, b_q)$ , given that  $a_p$  is aligned to a gap, and  $I_y(A_1^p, B_1^q)$  is the best score up to  $(a_p, b_q)$ , given that  $b_q$  is aligned to a gap. The previous recursions are initialized as follows:

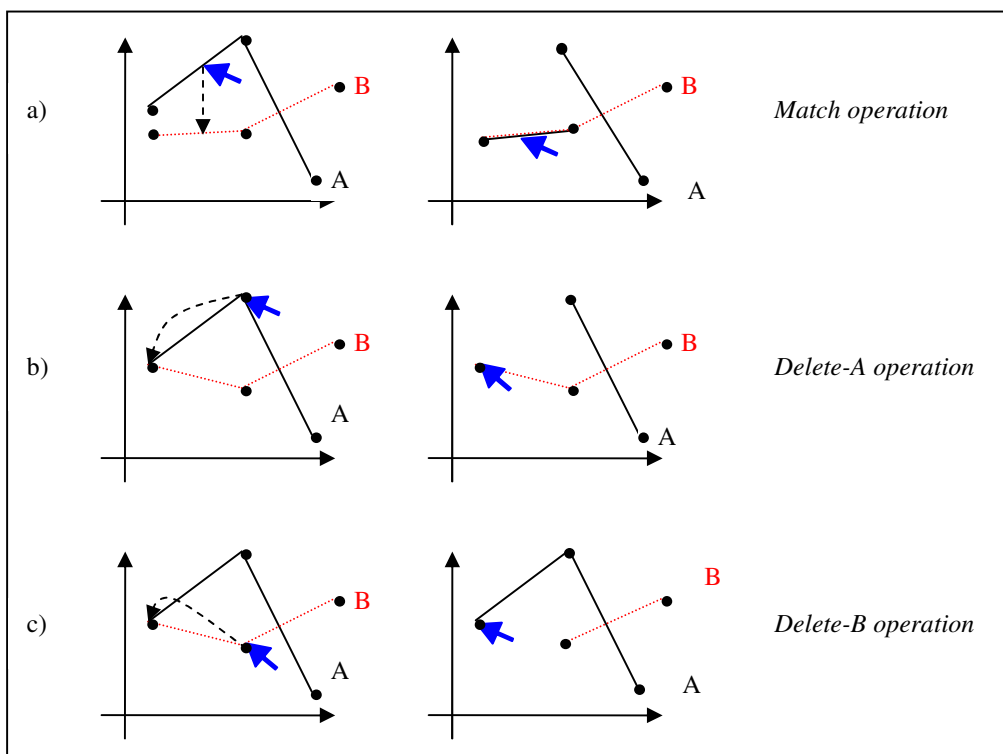
$$\begin{aligned}
 M(A_1^0, B_1^0) &= 0, \quad I_x(A_1^0, B_1^0) = I_y(A_1^0, B_1^0) = -\infty \\
 M(A_1^i, B_1^0) &= I_x(A_1^i, B_1^0) = -d - (i-1).e, \quad I_y(A_1^i, B_1^0) = -\infty, \text{ for } i = 1, \dots, p \\
 M(A_1^0, B_1^j) &= I_y(A_1^0, B_1^j) = -d - (j-1).e, \quad I_x(A_1^0, B_1^j) = -\infty, \text{ for } j = 1, \dots, q
 \end{aligned}$$

#### IV. THE TWED DISTANCE

We propose an alternative to the definition of the edit operations for time series alignment leading to the definition of the new similarity measure TWED. To understand the semantic associated to the edit operations for TWED, we reconsider the editing analogy with strings and suggest some differences. The edit distance between two strings is defined as the minimal transformation cost allowing for the transformation of the first string into the second one. For string edition, a transformation is a finite sequence of edit operations whose associated cost is the sum over the sequence of edit operations of the elementary costs  $\Gamma$  associated to each edit

operation.

For discrete time series we are seeking a sequence of edit operations allowing for the transformation of two time series simultaneously in order to make them superimposed with a minimal cost. If we use a graphical editor paradigm, we can imagine a  $2D$  representation of time series for which the horizontal axis represents the time scale or the time stamp coordinate and the vertical axis represents a spatial coordinate scale displaying the projection of the  $d-1$  spatial coordinates of the samples onto a  $1D$  scale. In this display, discrete time series are considered as a sequence of linear segments between successive samples. The graphical editor we imagine allows for the editing of two time series  $A$  and  $B$  using three elementary edit operations depicted in Fig. 1.a, 1.b and 1.c.



**Fig 1: The edit operations in the graphical editor paradigm.**

Instead of the classical *delete*, *insert* and *match* operations, we introduce *delete-A*, *delete-B* and *match* operations as follows:

i) The *delete-A* (delete inside the first time series) operation (Fig. 1.b) consists in clicking on the dot representing the sample in *A* to delete ( $a'_i$ ) and dragging and dropping this dot onto the previous sample dot ( $a'_{i-1}$ ). We suggest that the editing effort or cost associated with this delete operation be proportional to the length of vector ( $a'_i - a'_{i-1}$ ) to which we add a constant penalty  $\lambda \geq 0$ .

ii) The *delete-B* (delete inside the second time series) operation (Fig. 1.c) consists in clicking on the dot representing the sample in *B* to delete ( $b'_i$ ) and dragging and dropping this dot onto the previous sample dot ( $b'_{i-1}$ ). Here again we suggest that the editing effort or cost associated with this delete operation be proportional to the length of vector ( $b'_i - b'_{i-1}$ ) to which we add a constant penalty  $\lambda \geq 0$ .

Due to sampling rate variations or process variability one can face the situation where in time series data, one event could be registered many times or only few times when recording different utterances; this would justify the deletion cost to be proportional to the distance to the previous sample. Nevertheless 'outlier' samples (e.g. spurious data points) deletion cannot be covered by this argument. According to TWED, the deletion cost for such sample depends on the previous sample in the time series, and we do not have satisfactory argument to justify it. The other elastic measures (DTW, ERP, LCSS) do not offer better justification for the deletion cost of 'outliers'.

iii) The *match* operation (Fig. 1.a) consists in clicking on the segment to match in the first time series ( $a'_{i-1} a'_i$ ) then dragging and dropping this segment onto the graphic position corresponding to the matching segment ( $b'_{j-1} b'_j$ ) in the second time series. We can suggest that

the editing effort or cost associated with the *match* operation be proportional to the sum of the lengths of the two vectors  $(b'_j - a'_i)$  and  $(b'_{j-1} - a'_{i-1})$ .

This provides the basis for the TWED distance we propose

$$\delta_{\lambda,\gamma}(A_1^p, B_1^q) = \text{Min} \begin{cases} \delta_{\lambda,\gamma}(A_1^{p-1}, B_1^q) + d(a'_p, a'_{p-1}) + \lambda & \text{delete - A} \\ \delta_{\lambda,\gamma}(A_1^{p-1}, B_1^{q-1}) + d(a'_p, b'_q) + d(a'_{p-1}, b'_{q-1}) & \text{match} \\ \delta_{\lambda,\gamma}(A_1^p, B_1^{q-1}) + d(b'_{q-1}, b'_q) + \lambda & \text{delete - B} \end{cases} \quad (9)$$

$$\begin{aligned} \delta_{TWED}(A_1^0, B_1^0) &= 0 \\ \delta_{TWED}(A_1^0, B_1^j) &= \infty \\ \delta_{TWED}(A_1^i, B_1^0) &= \infty \\ \text{with } a'_0 &= b'_0 = 0 \quad \text{by convention.} \end{aligned}$$

The recursion is initialized setting:

It is interesting to note that the penalties for *delete-A* or *delete-B* operations are similar to those proposed in the  $\delta_{PPM}$  measure if we do not consider the time stamps coordinate and address the matching of *ID* monotone increasing time series.

Furthermore, using the graphical editor paradigm, we define the time series matching game as follows: two time series, *A* and *B*, are displayed on the graphic. The goal consists in editing *A* and *B* in order to completely superimpose the two curves.

The editing process is performed left to right: if *i* is an index on the segments of *A* and *j* on the segments of *B*, then the process is initialized setting  $i=j=1$ . A match operation will increment *i* and *j* simultaneously:  $i \leftarrow i+1, j \leftarrow j+1$ . A *delete-A* operation will increment *i* only:  $i \leftarrow i+1$ . A *delete-B* operation will increment *j* only:  $j \leftarrow j+1$ .

According to the above mentioned constraint, once a segment *i* in *A* has been processed using either a *match* or a *delete-A* operation, it is impossible to edit it again and so it is for former segments *r* in  $\{1, \dots, i-1\}$ . Similarly, once a segment *j* in *B* has been used either in a *match* or in an



*delete-B* operation it is impossible to use former samples  $r$  in  $\{1..j\}$  for future match or insertion operations. Therefore, according to this game, the editing process provides a sequence of edit operations as well as ordered pairs of indices  $(i,j)$  where  $i$  is an index in the sequence of segments of  $A$  and  $j$  an index in the sequence of segments of  $B$ . In other words, the process provides an ordered sequence of triplets  $(op_k, i_k, j_k)$  where  $op_k$  is the  $k^{th}$  edit operation selected, and  $i_k$  and  $j_k$  are the values of the index in  $A$  and  $B$  respectively when the edit operation is performed. A partial order can be defined on the triplets as follows:

$$(op_{k_1}, i_{k_1}, j_{k_1}) < (op_{k_2}, i_{k_2}, j_{k_2}) \text{ iff } i_{k_1} \leq i_{k_2} \text{ and } j_{k_1} \leq j_{k_2} \text{ and either } i_{k_1} \neq i_{k_2} \text{ or } j_{k_1} \neq j_{k_2}.$$

Since for each step of the editing game, one of the indices is increased by one while the other is either incremented by one or remains unchanged, all the triplets in the output editing sequence are ordered in increasing order.

Supposing that the game editing process has provided a sequence of edit operations up to  $i_k$  and  $j_k$  index values, if the sub sequences  $\bar{A}_1^{i_{k-1}}$  ( $\bar{A}_1^{i_{k-1}}$  refers to the sequence obtained from  $A$  after the first  $k-1$  edit operations) and  $\bar{B}_1^{j_{k-1}}$  are not superimposed, then, as there exists no possibility to process former samples to superimpose them the game process cannot be successful.

It is easy to show that  $\delta_{\lambda,\gamma}$  as defined in eq. (9) provides a successful sequence of editing operations at a minimal global cost for all pairs of time series in  $U^2$ .

**Proposition I:**  $\delta_{\lambda,\gamma}$  is a distance on the set of finite discrete time series  $U$ :

P1:  $\delta_{\lambda,\gamma}(A, B) \geq 0$  for any finite discrete time series  $A$  and  $B$ ,

P2:  $\delta_{\lambda,\gamma}(A, B) = 0$  iff  $A = B$  for any finite discrete time series  $A$  and  $B$ ,

P3:  $\delta_{\lambda,\gamma}(A, B) = \delta_{\lambda,\gamma}(B, A)$  for any finite discrete time series  $A$  and  $B$ ,

P4:  $\delta_{\lambda,\gamma}(A, B) \leq \delta_{\lambda,\gamma}(A, C) + \delta_{\lambda,\gamma}(C, B)$  for any finite discrete time series  $A, B$  and  $C$ .

Proof of Proposition 1 is given in [24].

#### A. Providing 'stiffness' into $\delta_{\lambda,\gamma}$

Going back to the graphical editor game we have envisaged that the penalty or cost associated with each edit operation should be proportional to the mouse pointer displacement involved during the edition. If we separate the spatial displacement in  $S$  from the temporal displacement in  $T$  then we have to consider a spatial penalty that could be handled by a distance measured in  $S$  and a temporal penalty more or less proportional to some distance measured in  $T$ . By doing this, we could parameterize a distance in between the Euclidian Distance, which is characterized with a kind of 'infinite stiffness', and DTW which is characterized by a 'null stiffness. In practice, we can choose  $d(a', b') = d_{LP}(a, b) + \gamma \cdot d_{LP}(t_a, t_b)$  where  $\gamma$  is a non negative constant which characterizes the *stiffness* of  $\delta_{\lambda,\gamma}$  elastic measures. Notice that  $\gamma > 0$  is required for  $\delta_{\lambda,\gamma}$  to be a distance. If  $\gamma = 0$  then  $\delta_{\lambda,\gamma}$  will be a distance on  $S$  but not on  $S \times T$ .

The final formulation of  $\delta_{\lambda,\gamma}$  is as follows:

$$\delta_{\lambda,\gamma}(A_1^p, B_1^q) = \text{Min} \begin{cases} \delta_{\lambda,\gamma}(A_1^{p-1}, B_1^q) + d_{LP}(a_p, a_{p-1}) + \gamma \cdot d_{LP}(t_{a_p}, t_{a_{p-1}}) + \lambda \\ \delta_{\lambda,\gamma}(A_1^{p-1}, B_1^{q-1}) + d_{LP}(a_p, b_q) + \gamma \cdot d_{LP}(t_{a_p}, t_{b_q}) + d_{LP}(a_{p-1}, b_{q-1}) + \gamma \cdot d_{LP}(t_{a_{p-1}}, t_{b_{q-1}}) \\ \delta_{\lambda,\gamma}(A_1^p, B_1^{q-1}) + d_{LP}(b_q, b_{q-1}) + \gamma \cdot d_{LP}(t_{b_q}, t_{b_{q-1}}) + \lambda \end{cases} \quad (10)$$

Some analogy can be found between the parameters  $\lambda$  and  $\gamma$  of  $\delta_{\lambda,\gamma}$  and parameters  $e$  and  $d$  of the affine model defined for symbolic sequence matching (see section III.F). Nevertheless, some

major differences exist: the penalties in  $\delta_{\lambda,\gamma}$  are, for one part, proportional to the time stamp difference between matching, deleted or inserted samples. A constant penalty is added for the two deletion operations that correspond to gaps. Conversely, the affine model proposes a penalty proportional to the gap length corresponding to series of successive insertions or deletions, with a constant penalty for the first operation in the sequence.

The iterative implementation of  $\delta_{\lambda,\gamma}$  using the  $Lp$  metrics to evaluate the distance between two samples is depicted in *Fig.2*.

### *B. Algorithmic complexity of $\delta_{\lambda,\gamma}$*

The time complexity of  $\delta_{\lambda,\gamma}$  is the same as DTW and ERP, namely  $O(p.q)$ , where  $p$  and  $q$  are the lengths of the two time series being matched. The space complexity is also the same as DTW i.e.  $O(p.q)$ , but as well as the ERP distance, the costs  $\gamma(a' \rightarrow \Lambda)$  and  $\gamma(\Lambda \rightarrow b')$  can be tabulated to speed up the calculation leading to an extra space complexity of  $O(p+q)$  for  $\delta_{TWED}$ .

```

float TWED(float A[1..n], float timeStampsA[1..n],
           float B[1..m], float timeStampsB[1..m],
           float lambda, float nu) {

  declare int DTW[0..n,0..m];
  declare int i, j;
  declare float cost;
  declare float A[0] :=0, timeStampsA[0] :=0;
  declare float B[0] :=0, timeStampsB[0] :=0;

  for i := 1 to m
    TWED[0,i] := infinity;
  for i := 1 to n
    TWED[i,0] := infinity;
  TWED[0,0] := 0;

  for i := 1 to n {
    for j := 1 to m {
      cost:= LpDist(A[i],B[j]); // Distance-L1
      DTW[i,j] := minimum(
        // insertion
        DTW[i-1,j ] + LpDist(A[i-1], A[i])+
          nu*(timeStampsA(i)- timeStampsA(i-1))+lambda,
        // deletion
        DTW[i ,j-1] + LpDist(B[j-1], B[j])+
          nu*(timeStampsB(j)- timeStampsB(j-1))+lambda,
        // match
        DTW[i-1,j-1] + LpDist(A[i],B[j])+
          nu*|timeStampsA(i)- timeStampsB(j)| );
    } // End for j
  } // End for i

  Cost = TWED[n,m];
  Return; }

```

**Fig 2: Iterative implementation of the TWED distance.**

### C. Some properties of TWED

**Proposition 2:**  $\delta_{\lambda,\gamma}$  is upper bounded by twice the distance  $D_{LP}$ .

$\forall \lambda \geq 0, \gamma > 0 \quad \forall X, Y \in U^2, \quad \delta_{\lambda,\gamma}(X, Y) \leq 2 \cdot D_{LP}(X, Y)$ , whenever  $X$  and  $Y$  have the same length.

Proof: let us consider the sequence of editing operations consisting in  $m$  match operations, where  $m$  is the length of the  $X$  and  $Y$ . This sequence has a cost equal to twice the  $LP$ -distance

between the two time series  $X$  and  $Y$ . Since  $\delta_{\lambda,\gamma}$  is equal to the cost of the optimal sequences of edit operations, the result follows.  $\square$

**Proposition 3:**  $\delta_{\lambda,\gamma}$  is an increasing function of  $\lambda$  and  $\gamma$  :

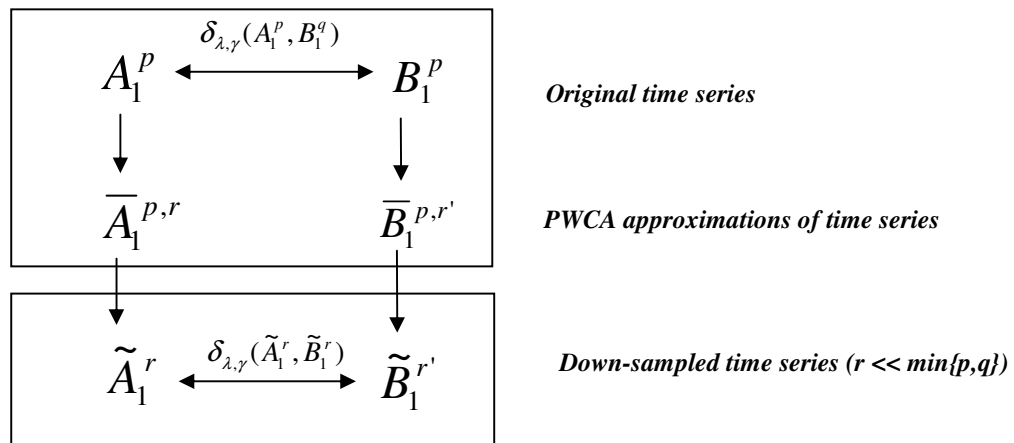
$$\forall \lambda \geq 0, \gamma > 0 \quad \forall \lambda' \geq \lambda \quad \forall \gamma' \geq \gamma \quad \forall X, Y \in U^2 \quad \delta_{\lambda,\gamma}(X, Y) \leq \delta_{\lambda',\gamma'}(X, Y)$$

Proof: Let us consider one of the optimal sequences of editing operations evaluated with the tuple  $(\lambda, \gamma)$  with minimal cost equal to  $\delta_{\lambda,\gamma}(X, Y)$ . If we keep this sequence of editing operation while replacing  $(\lambda', \gamma')$  with  $(\lambda, \gamma)$  in all the elementary operation costs we get a cost for this sequence that is lower than  $\delta_{\lambda',\gamma'}(X, Y)$  but greater than the cost of the optimal sequence  $\delta_{\lambda,\gamma}(X, Y)$  evaluated using  $(\lambda, \gamma)$ . The result follows.  $\square$

## V. BOUNDING THE TWED MEASURE

In this section we get back to our second motivation about defining a measure that exploits time stamps while verifying the triangle inequality. We show how piecewise constant approximations (PWCA) with few segments of time series can be used to improve the efficiency of range queries. Various methods exist to get polygonal curve approximations of time series, in particular heuristic [7][9][12][14], near optimal [16][23] or optimal [29] solutions. Most of them can be adapted to provide PCWA approximation of time series.

We define  $\bar{A}_1^{p,r}$  a PWCA of time series  $A_1^p$  containing  $r-1 \geq 0$  constant segments. This approximation can be obtained using any kind of solution (from heuristic to optimal solutions), let say the optimal solution similar to the one proposed in [29].  $\bar{A}_1^{p,r}$  and  $A_1^p$  have the same number of samples, namely  $p$ . Let  $\tilde{A}_1^r$  be the time series composed with the  $r$  segment extremities of  $\bar{A}_1^{p,r}$ .  $\tilde{A}_1^r$  contains  $r$  samples. Let us similarly define  $\bar{B}_1^{p,r'}$  and  $\tilde{B}_1^{r'}$  from time series  $B_1^p$ .



**Fig 2: Linking the matching of time series in the original space to the matching in the down-sampled space.**

**Proposition 4:**

$\forall \lambda \geq 0, \gamma > 0, \forall r \in [1; p[, \forall X_1^p \in U \quad \delta_{\lambda, \gamma}(\bar{X}_1^{p,r}, \tilde{X}_1^r) \leq \lambda \cdot (p - r) + \gamma \cdot \Delta T(2 \cdot p - r)$ , where  $\Delta T$  is the time difference average between two successive samples inside the piecewise constant segments of the approximation.

The proof for this lemma is straightforward: let us consider the sequence of operations consisting in  $r$  match operations for the end extremities of the piecewise constant segments and  $(p-r)$  delete operations for the set of samples in  $\bar{X}_1^{p,r}$  that are not end extremities of the piecewise constant segments. In this sequence each match operations has in average the cost  $(p/r) \cdot \gamma \cdot \Delta T$ , and each delete operation has a  $\lambda$  fixed penalty and a penalty proportional to the time stamps difference between two successive samples  $\gamma \cdot (\text{timeStamps}(i) - \text{timeStamps}(i-1))$ . Then, the cost for this sequence of editing operations is  $(p - r)(\lambda + \gamma \cdot \Delta T) + p \cdot \gamma \cdot \Delta T$ . Finally the optimal sequence of editing operations has a cost  $\delta_{\lambda, \gamma}(\bar{X}_1^{p,r}, \tilde{X}_1^r)$  lower or equal to  $\lambda \cdot (p - r) + \gamma \cdot \Delta T(2 \cdot p - r)$ .  $\square$

From these previous lemmas we can upper-bound the matching of two time series evaluated into the original space with the matching of their approximations evaluated into the down sampled space as follows:

$\forall \lambda \geq 0, \gamma > 0, \forall A_1^p, B_1^q \in U^2, \forall r \in [1; p[, \forall r' \in [1; q[$  we have :

$$\delta_{\lambda, \gamma}(\bar{A}_1^{p,r}, A_1^p) \leq 2 \cdot D_{LP}(\bar{A}_1^{p,r}, A_1^p); \text{ since } \bar{A}_1^{p,r} \text{ and } A_1^p \text{ have the same length lemma 1 applies}$$

$$\delta_{\lambda, \gamma}(\bar{A}_1^{p,r}, \tilde{A}_1^r) \leq \lambda \cdot (p - r) + \gamma \cdot \Delta T(2 \cdot p - r); \quad \text{Proposition 3}$$

$$\delta_{\lambda, \gamma}(\bar{B}_1^{q,r'}, B_1^q) \leq 2 \cdot D_{LP}(\bar{B}_1^{q,r'}, B_1^q); \text{ since } \bar{B}_1^{q,r'} \text{ and } B_1^q \text{ have the same length lemma 1 applies}$$

$$\delta_{\lambda, \gamma}(\bar{B}_1^{q,r'}, \tilde{B}_1^{r'}) \leq \lambda \cdot (q - r') + \gamma \cdot \Delta T(2 \cdot q - r'); \quad \text{Proposition 3}$$

From these inequalities and the triangle inequality verified by  $\delta_{\lambda, \gamma}$  we get:

$$\begin{aligned}\delta_{\lambda,\gamma}(\tilde{A}_1^r, A_1^p) &\leq \lambda \cdot (p - r) + \gamma \cdot \Delta T(2 \cdot p - r) + 2 \cdot D_{LP}(\bar{A}_1^{p,r}, A_1^p) \\ \delta_{\lambda,\gamma}(\tilde{B}_1^{r'}, B_1^q) &\leq \lambda \cdot (q - r') + \gamma \cdot \Delta T(2 \cdot q - r') + 2 \cdot D_{LP}(\bar{B}_1^{q,r'}, B_1^q)\end{aligned}$$

And since  $\delta_{\lambda,\gamma}$  verifies the triangle inequality we have:

$$\begin{aligned}\delta_{\lambda,\gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) &\leq \delta_{\lambda,\gamma}(\tilde{A}_1^r, A_1^p) + \delta_{\lambda,\gamma}(A_1^p, B_1^q) + \delta_{\lambda,\gamma}(B_1^q, \tilde{B}_1^{r'}) \\ &\leq \lambda \cdot (p + q - r - r') + \gamma \cdot \Delta T(2 \cdot (p + q) - r - r') + \quad ( \\ &\quad 2 \cdot D_{LP}(\bar{A}_1^{p,r}, A_1^p) + 2 \cdot D_{LP}(\bar{B}_1^{q,r'}, B_1^q) + \delta_{\lambda,\gamma}(A_1^p, B_1^q)\end{aligned}$$

$$\begin{aligned}\delta_{\lambda,\gamma}(A_1^p, B_1^q) &\leq \delta_{\lambda,\gamma}(A_1^p, \tilde{A}_1^r) + \delta_{\lambda,\gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) + \delta_{\lambda,\gamma}(\tilde{B}_1^{r'}, B_1^q) \\ &\leq \lambda \cdot (p + q - r - r') + \gamma \cdot \Delta T(2 \cdot (p + q) - r - r') + \\ &\quad 2 \cdot D_{LP}(\bar{A}_1^{p,r}, A_1^p) + 2 \cdot D_{LP}(\bar{B}_1^{q,r'}, B_1^q) + \delta_{\lambda,\gamma}(A_1^p, B_1^q)\end{aligned}$$

Leading to:

$$\begin{aligned}\left| \delta_{\lambda,\gamma}(A_1^p, B_1^q) - \delta_{\lambda,\gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) \right| &\leq \lambda \cdot (p + q - r - r') + \gamma \cdot \Delta T(2 \cdot (p + q) - r - r') + \\ &\quad 2 \cdot D_{LP}(\bar{A}_1^{p,r}, A_1^p) + 2 \cdot D_{LP}(\bar{B}_1^{q,r'}, B_1^q)\end{aligned}$$

This shows that  $\delta_{\lambda,\gamma}(A_1^p, B_1^q)$  and  $\delta_{\lambda,\gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'})$  are potentially close when two conditions are satisfied:

1. The PWCA approximations of  $A$  and  $B$  are close to the original time series in the sense of the  $LP$ -distance. This should be ensured by the optimal solution of the  $min-\varepsilon$  problem using piecewise constant segments whenever the number of segments  $r$  is not too small.
2.  $\lambda$  and  $\gamma \cdot \Delta T$  are small comparatively to  $2 \cdot (p + q) - r - r'$

Hence we get the following lower bounds that can be considered tight if the two previous conditions are satisfied:



$$\begin{aligned}
 & \forall \lambda \geq 0, \gamma > 0, \forall \lambda' \geq \lambda \forall \gamma' \geq \gamma, \forall A_1^p, B_1^q \in U^2, \forall r \in [1, p[, \forall r' \in [1, q[, \\
 & \delta_{\lambda, \gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) - \lambda \cdot (p + q - r - r') + \gamma \cdot \Delta T(2 \cdot (p + q) - r - r') - 2 \cdot D_{LP}(\bar{A}_1^{p, r}, A_1^p) - 2 \cdot D_{LP}(\bar{B}_1^{q, r'}, B_1^q) \\
 & \leq \delta_{\lambda, \gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) - \delta_{\lambda, \gamma}(\tilde{A}_1^r, A_1^p) - \delta_{\lambda, \gamma}(\tilde{B}_1^{r'}, B_1^p) \\
 & \leq \delta_{\lambda, \gamma}(A_1^p, B_1^q) \leq \delta_{\lambda', \gamma'}(A_1^p, B_1^q) \text{ since Proposition 3 applies.} \tag{12}
 \end{aligned}$$

This last inequality is potentially still useful to design fast and dirty filters dedicated to range query searching for applications for which  $\lambda'$  and  $\gamma'$  cannot be small enough, while  $\lambda$  and  $\gamma$  can be set up small.

For range query search, if  $R$  is the radius of the range query and  $A_1^p$  the center of the query ball, then  $B_1^q$  is outside the search range if one of the following conditions is verified:

$$\begin{aligned}
 \delta_{\lambda, \gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) & > R + \lambda \cdot (p + q - r - r') + \gamma \cdot \Delta T(2 \cdot (p + q) - r - r') + 2 \cdot D_{LP}(\bar{A}_1^{p, r}, A_1^p) + 2 \cdot D_{LP}(\bar{B}_1^{q, r'}, B_1^q) \\
 & > R + \delta_{\lambda, \gamma}(\tilde{A}_1^r, A_1^p) + \delta_{\lambda, \gamma}(\tilde{B}_1^{r'}, B_1^p) \tag{13}
 \end{aligned}$$

For time series information retrieval applications, equations (11) and (12) are potentially useful. If  $\delta_{\lambda, \gamma}(\tilde{A}_1^r, A_1^p)$  and  $\delta_{\lambda, \gamma}(\tilde{B}_1^{r'}, B_1^p)$  are pre-computed during the indexing phase, the tighter bound can be used. Otherwise the second bound can be evaluated during the retrieval phase through the computation of *LI-distances*. If  $r = r' = (1/K) \cdot \min(p, q)$  the complexity for evaluating  $\delta_{\lambda, \gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'})$  is lower than  $O(p \cdot q / K^2)$ .

## VI. EXPERIMENTATIONS

To evaluate empirically the effectiveness of the TWED distance comparatively to other metrics or similarity measures, we address a simple classification task experiment. The classification task we have considered consists in assigning one of the possible categories to an unknown time series for the 20 data sets available at UCR repository [15].

For each dataset, a training subset is defined as well as a testing subset. The classification is based on the simple nearest neighbor decision rule: first we select a training data set containing time series for which the correct category is known. To assign a category to an unknown time series selected from a testing data set (different from the train set), we select its nearest neighbor (in the sense of a distance or similarity measure) within the training data set, then, assign the category associated to its nearest neighbor.

Given a dataset, we adapt the *stiffness* parameter as follows: we use the **training dataset** to select the '*best stiffness*' ( $\gamma$ ) value as well as the best  $\lambda$  value, namely the ones leading to the minimal error rate **on the training data**, according to a leave-one-out procedure (that consists in selecting iteratively one time series into the **training set** and then in considering it as a test against the remaining time series within the **training set** itself).

Finally, the **testing dataset** is used to evaluate the final error rate (reported in Tab.1 and Tab.2) with the best  $\gamma$  and  $\lambda$  values estimated on the **training set**. This leads to OTWED, the optimized versions of TWED.

The same procedure is used to set up the parameters defined for the other parametric measures, i.e. ODTW and LCSS.

*Tab.1* and *Tab.2* show the results obtained for the tested methods, e.g. Euclidian Distance on the original time series, optimized DTW with best warping windows (ODTW) as defined in [30], classical DTW (DTW) with no warping window, Longest Common Subsequence (LCSS) as defined in [35], Edit distance with Real Penalty (ERP) as defined in [5] and OTWED. In *Tab.1* and *Fig.3* the time series are not preprocessed, while in *Tab.2* and *Fig 4*. time series are down sampled using an optimal Piecewise Constant Approximation procedure similar as the one described in [29] for polygonal approximation. In this last experiment, each down sampled time

series has exactly 50% less samples than the original time series. The sampling rate for the down sampled time series is indeed varying since the size of the constant segments used to approximate the time series are not constant in general.

For parameterized measures, best values are selected on the training data in order to minimize the error rate estimated on the training data. More precisely the settings are as follows:

- ODTW: the best corridor value is selected for each dataset among the set  $\{0, \max\{p, q\}\}$  such as to minimize the classification errors estimated on the **training data**. If different corridor values lead to the minimal error rate estimated on the training data then the **lowest corridor value is selected**.
- LCSS: the best  $\delta$  and  $\varepsilon$  values are selected for each dataset respectively among the sets  $\{n, n/2, n/4, \dots, n/2^k\}$ , with  $n = \max\{p, q\}$  and  $n/2^k \leq .5 < n/2^{k+1}$ , and  $\{20, 20/2, 20/4, \dots, 20/2^k\}$ , with  $20/2^k \leq 1e^{-2} < 20/2^{k+1}$  such as to minimize the classification errors estimated on the **training data**. If different  $(\delta, \varepsilon)$  values lead to the minimal error rate estimated on the training data then the pairs having the **highest  $\delta$  value are selected first, then the pair with the highest  $\varepsilon$  value** is finally selected.
- OTWED: for our experiment, ‘stiffness value’ ( $\gamma$ ) is selected into  $\{1e10^{-5}, 1e10^{-4}, 1e10^{-3}, 1e10^{-2}, e10^{-1}, 1\}$  and  $\lambda$  is selected into  $\{0, .25, .5, .75, 1.0\}$ . The  $\gamma$  and  $\lambda$  parameter values are selected for each dataset such as to minimize the classification errors estimated on the **training data**. If different  $(\gamma, \lambda)$  values lead to the minimal error rate estimated on the training data then the pairs containing the **highest  $\gamma$  value are selected first, then the pairs with the highest  $\lambda$  value** is finally

selected.

For ERP and OTWED we used the  $L1$ -norm, while the  $L2$ -norm has been implemented in DTW and ODTW as reported in [30]. The gap value used in ERP has been set to the distance between the deleted or inserted sample and 0 as suggested by the authors [5].

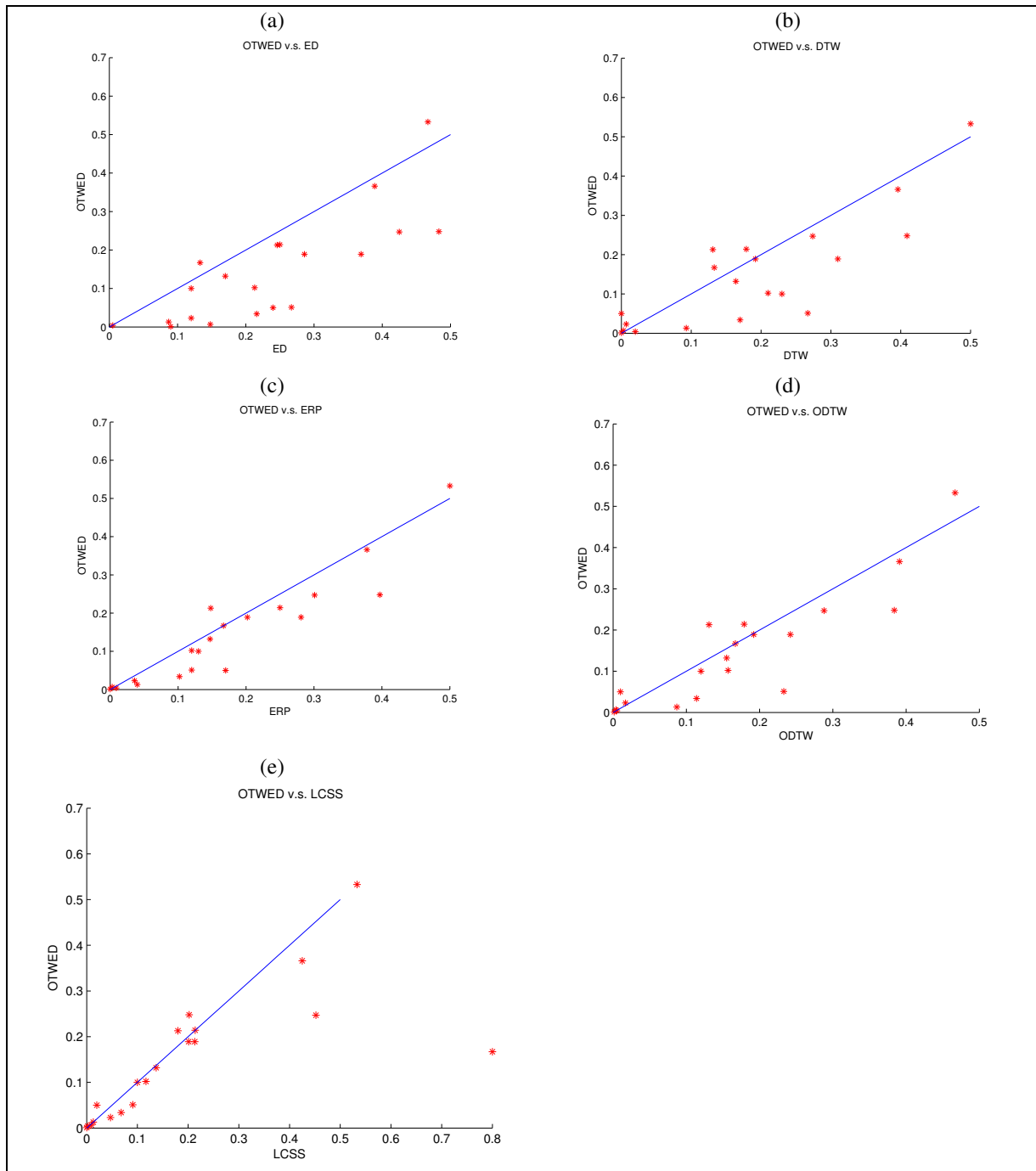
Finally, as time is not explicitly given for these datasets, we used the index value of the samples as the time stamps for the whole experiment.

This experiment shows that the TWED distance is effective on the considered task comparatively to ED, DTW, ODTW, ERP and LCSS measures since it exhibits in average the lowest error rates on the testing data as shown in *Tab. 1* and *Fig. 3*. The gain in average is relatively significant: 2.5% against ODTW, 4.7% against LCSS, 3% against ERP, 9.4% against ED and 8.8% against DTW.

The same experiment carried out on down sampled time series (*Tab. 2* and *Fig. 4*) shows that error rates drop more than twice time faster for ED, DTW, ERP, LCSS, ODTW (>7%) than for TWED (2.1%). In that experimental context, coping with time stamps when match not uniformly down sampled time series is quite effective.

Dataset	Nbr of classes   Size of training testing set	1-NN ED	1-NN ODTW	1-NN DTW	1-NN LCSS	1-NN ERP	1-NN OTWED
Synthetic Control	6 300 300	0.12	0.017	<b>0.007</b>	0.047	0.036	0.023
Gun-Point	2 50 150	0.087	0.087	0.093	<b>0.013</b>	0.04	<b>0.013</b>
CBF	3 30 900	0.148	0.004	<b>0.003</b>	0.009	<b>0.003</b>	0.007
Face (all)	14 560 1690	0.286	0.192	0.192	0.201	0.202	<b>0.189</b>
OSU Leaf	6 200 242	0.483	0.384	0.409	<b>0.202</b>	0.397	0.248
Swedish Leaf	15 500 625	0.213	0.157	0.210	0.117	0.12	<b>0.102</b>
50Words	50 450 455	0.369	0.242	0.310	0.213	0.281	<b>0.189</b>
Trace	4 100 100	0.24	0.01	<b>0.0</b>	0.02	0.17	0.050
Two Patterns	4 1000 4000	0.09	0.0015	<b>0.0</b>	<b>0.0</b>	<b>0.0</b>	0.001
Wafer	2 1000 6174	0.005	0.005	0.020	<b>0</b>	0.009	0.004
Face (four)	4 2488	0.216	0.114	0.170	0.068	0.102	<b>0.034</b>
Lighting2	2 60 61	0.246	<b>0.131</b>	<b>0.131</b>	0.18	0.148	0.213
Lighting7	7 70 73	0.425	0.288	0.274	0.452	0.301	<b>0.247</b>
ECG	2 100 100	0.12	0.12	0.23	<b>0.100</b>	0.13	<b>0.100</b>
Adiac	37 390 391	0.389	0.391	0.396	0.425	0.378	<b>0.366</b>
Yoga	02 300 3000	0.170	0.155	0.164	0.137	0.147	<b>0.132</b>
Fish	7 1175 1175	0.267	0.233	0.267	0.091	0.12	<b>0.051</b>
Coffee	2 28 28	0.25	<b>0.179</b>	<b>0.179</b>	0.214	0.25	0.21
OliveOil	4 30 30	0.133	0.167	<b>0.133</b>	0.8	0.167	0.167
Beef	5 30 30	0.467	<b>0.467</b>	0.5	0.533	0.5	0.533
<i>MEAN</i>		0.240	0.167	0.232	0.191	0.175	<b>0.144</b>
<i>STD</i>		0.131	<b>0.136</b>	0.145	0.210	0.140	0.138

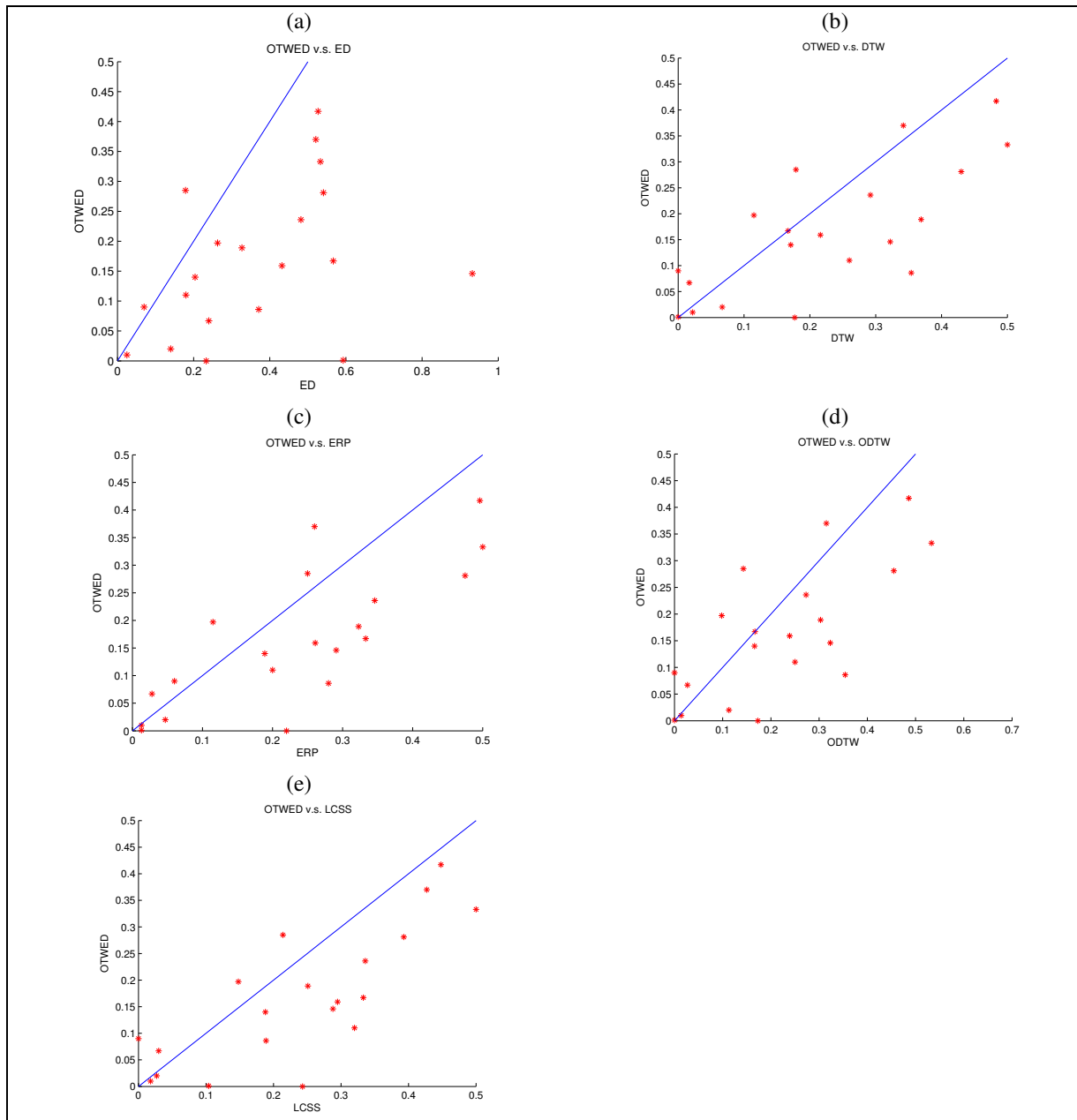
***TAB.1: COMPARATIVE STUDY USING THE UCR DATASETS [15]: CLASSIFICATION ERROR RATE OBTAINED USING THE FIRST NEAR NEIGHBOR CLASSIFICATION RULE FOR ED, DTW, ODTW, LCSS, ERP, AND OTWED DISTANCE***



**Fig. 3: Comparison of distance pairs  $(\delta_x, \delta_y)$ .** The x and y axes show the error rates for the two compared distances. The straight line has a slope of 1.0 and dots correspond to the error rate for the selected distance pair and tested data sets. A dot below (resp. above) the straight line indicates that distance  $\delta_y$  has a lower (resp. higher) error rate than distance  $\delta_x$ . Plot (a) shows OTWED v.s. ED, plot (b) shows OTWED v.s. DTW, plot (c) shows OTWED v.s. ERP, plot (d) shows OTWED v.s. ODTW, plot (e) shows OTWED v.s. LCSS.

Dataset	Nbr of classes   Size of testing set	1-NN ED	1-NN ODTW	1-NN DTW	1-NN LCSS	1-NN ERP	1-NN OTWED
Synthetic Control	6 300 300	0.233	0.173	0.177	0.243	0.22	0,000
Gun-Point	2 50 150	0.14	0.113	0.067	<b>0.027</b>	0.047	0,020
CBF	3 30 900	0.24	0.027	<b>0.017</b>	0.03	0.028	0,067
Face (all)	14 560 1690	0.482	0.273	0.292	0.336	0.346	0,236
OSU Leaf	6 200 242	0.541	0.455	0.43	0.393	0.475	0,281
Swedish Leaf	15 500 625	0.932	0.323	0.322	0.288	0.291	0,146
50Words	50 450 455	0.327	0.303	0.369	0.251	0.323	0,189
Trace	4 100 100	0.07	<b>0</b>	<b>0</b>	0	0.06	0,090
Two Patterns	4 1000 4000	0.593	<b>0</b>	<b>0</b>	0.104	0.013	0,001
Wafer	2 1000 6174	0.025	0.014	0.022	0.018	0.013	0,010
Face (four)	4 24 88	0.432	0.239	0.216	0.295	0.261	0,159
Lighting2	2 60 61	0.263	<b>0.098</b>	0.115	0.148	0.115	0,197
Lighting7	7 70 73	0.521	0.315	0.342	0.427	<b>0.26</b>	0,370
ECG	2 100 100	0.18	0.25	0.26	0.32	0.2	0,110
Adiac	37 390 391	0.527	0.486	0.483	0.448	0.496	0,417
Yoga	02 300 3000	0.204	0.166	0.171	0.188	0.189	0,140
Fish	7 175 175	0.371	0.354	0.354	<b>0.189</b>	0.28	0,086
Coffee	2 28 28	0.179	<b>0.143</b>	0.179	0.214	0.25	0,285
OliveOil	4 30 30	0.567	0.167	0.167	0.333	0.333	0,167
Beef	5 30 30	0.533	0.533	0.5	0.5	0.5	0,333
MEAN		0,37	0,22	0,22	0,24	0,24	0,165
STD		0,22	0,16	0,16	0,15	0,16	0,124

**TAB.2: COMPARATIVE STUDY USING THE UCR DATASETS [15]: CLASSIFICATION ERROR RATE OBTAINED USING THE FIRST NEAR NEIGHBOR CLASSIFICATION RULE ON DOWN SAMPLED TIME SERIES FOR ED, DTW, ODTW, LCSS, ERP, AND OTWED DISTANCE**



**Fig. 4: Comparison of distance pairs ( $\delta_x, \delta_y$ ) applied on down sampled times series. The x and y axes show the error rates for the two compared distances. The straight line has a slope of 1.0 and dots correspond to the error rate for the selected distance pair and tested data sets. A dot below (resp. above) the straight line indicates that distance  $\delta_y$  has a lower (resp. higher) error rate than distance  $\delta_x$ . Plot (a) shows OTWED v.s. ED, plot (b) shows OTWED v.s. DTW, plot (c) shows OTWED v.s. ERP, plot (d) shows OTWED v.s. ODTW, plot (e) shows OTWED v.s. LCSS.**



## VII. CONCLUSION

From a graphical curve editing perspective and from earlier work on symbolic edit distance and dynamic time warping we have developed an elastic similarity measure called TWED to match time series with some time shifting tolerance. We have proved that the TWED measure is a metric, and as such TWED can be used complementarily with methods developed for searching in metric spaces as potential solutions for time series searching and retrieval applications when time shift tolerance is concerned. The originality of TWED, comparatively to similar elastic measures, apart from the way insertions and deletions are managed, lies in the introduction of a parameter which drives the '*stiffness*' of the measure thus placing TWED in between the Euclidian distances (infinite stiffness) and the DTW similarity measure (null stiffness). Moreover TWED involved a second parameter which defines a constant penalty for insert or delete operations, similarly to the edit distance defined for string matching. These two parameters can be straightforwardly optimized for each application or dataset as far as training data are available.

Furthermore, a procedure has been sketched to lower bound the TWED metric. This procedure consists in approximating the time series using polygonal or piecewise constant approximations. It takes benefits from the triangle inequality to relate the TWED measure evaluated on the approximated representations of time series to the TWED measure evaluated on the original time series. The computational cost reduction of TWED when evaluated in the approximated representation space is quadratic with the compression rate of the approximation. Nevertheless, this kind of lower bound has no linear complexity. Experimentation to evaluate the effectiveness of this procedure is a perspective as well as the search for a lower bound whose complexity is

effectively linear and that could be efficiently used in conjunction with the proposed lower bounding procedure.

The empirical quality of the distance has been evaluated on a classification experiment based on the first near neighbor classification rule for 22 different datasets. Globally, on this experiment, TWED performs in average significantly better than the Euclidian distance and Dynamic Time Warp measure and slightly better than the Longest Common Subsequence measure, the Edit Distance with Real Penalty and the Dynamic Time Warping measure with optimized search corridor size. When the classification experiment is applied to down sampled time series, TWED shows to be more robust than the other tested measures. This is mainly because the times series are not uniformly sampled in this experiment and in that case it is relevant coping with time stamps.

## REFERENCES

- [1] R. Agrawal and R. Srikant. Mining sequential patterns IEEE Proc. of the 11th Int'l Conference on Data Engineering, pages 3–14, 1995
- [2] R. Bellman, Dynamic Programming, Princeton Univ. Press, 1957, New Jersey.
- [3] E. Chávez, G. Navarro, R. A. Baeza-Yates, José L. Marroquín, Searching in Metric Spaces, ACM Computing Surveys (CSUR), Volume 33 , Issue 3, P: 273 – 321, 2001
- [4] L. Chen., M. T , Ozsú., V. Oria, Robust and fast similarity search for moving object trajectories. *SIGMOD*, 2005.
- [5] L. Chen & R. Ng. On the marriage of Lp-norm and edit distance. In Proc. 30th Int'l Conf. on Very Large Data Bases, pp 792–801, 2004.
- [6] G. Das, D. Gunopulos, and H. Mannila. Finding Similar Time Series. In Proceedings of the Conference on Principles of Knowledge Discovery and Data Mining, 1997.
- [7] D.H., Douglas, T.K., Peucker, (1973). Algorithm for the reduction of the number of points required to represent a line or its caricature. *The Canadian Cartographer* 10 (2), 112–122.
- [8] R., Durbin, S., Eddy, A., Krogh, and G., Mitchinson. (1998). Biological sequence analysis. Probabilistic models of proteins and nucleic acids, Cambridge University Press. 356 pp, 1998.
- [9] E., Fink, B., Pratt. (2003). “Indexing of compressed time series”, *Data Mining in Time Series Databases*, pp.51-78, 2003.

- [10] Gotoh, O. (1982) "An improved algorithm for matching biological sequences." *J. Mol. Biol.* 162, 705-708.
- [11] D. S. Hirschberg. (1975). "A linear space algorithm for computing maximal common subsequences". *Communications of the ACM* 18 (6): 341–343, 1975.
- [12] E. J., Keogh, and M. J., Pazzani, A simple dimensionality reduction technique for fast similarity search in large time series databases, *4th Pacific-Asia Conference on Knowledge Discovery and Data Mining*, PAKDD, pp. 122–133, 2000.
- [13] E., J., Keogh and L., Wei and X., Xi and S.-H., Lee and M., Vlachos, LB\_Keogh Supports Exact Indexing of Shapes under Rotation Invariance with Arbitrary Representations and Distance Measures, *VLDB* 882-893 (2006)
- [14] E., J., Keogh, K., Chakrabarti, M. J., Pazzani, and S., Mehrotra. Dimensionality reduction for fast similarity search in large time series databases, *Journal of Knowledge and Information Systems*, 3(3):263–286, 2001.
- [15] E., J., Keogh, E., Xi, X., Wei, L. & Ratanamahatana, C. A. (2006). The UCR Time Series Classification/Clustering Homepage: [http://www.cs.ucr.edu/~eamonn/time\\_series\\_data/](http://www.cs.ucr.edu/~eamonn/time_series_data/)
- [16] A., Kolesnikov and P., Fränti. (2003) Reduced-search dynamic programming for approximation of polygonal curves *Pattern Recognition Letters* 24, p2243–2254, 2003.
- [17] A., Kolesnikov and P., Fränti, X., Wu. (2004). Multiresolution Polygonal Approximation of Digital Curves, *Proceedings of the 17th International Conference on Pattern Recognition (ICPR'04)*, 1051-4651, 2004.
- [18] V. I. Levenshtein, Binary codes capable of correcting deletions, insertions, and reversals, *Doklady Akademii Nauk SSSR*, 163(4):845-848, 1965 (Russian). English translation in *Soviet Physics Doklady*, 10(8):707-710, 1966.
- [19] J., Lin, E., Keogh, A., F., & H., Van Herie. Approximations to Magic: Finding Unusual Medical Time Series. In *proceedings of the 18<sup>th</sup> IEEE Int'l Symposium on Computer-Based Medical Systems*. June 23-24. Dublin, Ireland, 2005.
- [20] V. Mäkinen. Using Edit Distance in Point-Pattern Matching. In *Proc. 8th Symposium on String Processing and Information Retrieval (SPIRE 2001)*, IEEE CS Press, pp. 153-161, 2001.
- [21] V. Mäkinen. Parameterized approximate string matching and local-similarity-based point-pattern matching. PhD thesis. Department of Computer Science, University of Helsinki, Report A-2003-6, August 2003.
- [22] P.F., Marteau, S., Gibet, , (2005). Adaptive Sampling of Motion Trajectories for Discrete Task-Based Analysis and Synthesis of Gesture. In *LNCS/LNAI Proc. of Int. Gesture Workshop*, volume vol. of series, page pages, Vannes, France, month 2005. Publisher.
- [23] P.F., Marteau, G. Ménier. Adaptive Multiresolution and Dedicated Elastic Matching in Linear Time Complexity for Time Series Data Mining, *Sixth International conference on Intelligent Systems Design and Applications (ISDA 2006)*, Jinan Shandong, China, 16-18 October, 2006.
- [24] P.F., Marteau. Time Warp Edit Distance, Technical report N°: valoria/UBS.2008.1v, <http://hal.archives-ouvertes.fr/hal-00258669/fr/>.
- [25] M. Mongeau and D. Sankoff. Comparison of musical sequences, *Computers & the Humanities* 126, p. 161-175, 1990.

- [26] P. Moore and D. Molloy, (2007). A Survey of Computer-Based Deformable Models, Intern. Machine Vision and Image Processing Conference, IMVIP 2007, Ireland, pp.55-64, 2007.
- [27] Needleman, S.B., and C.D. Wunsch. 1970. A general method applicable to the search for similarities in the amino acid sequences of two proteins. *Journal of Molecular Biology* 48:443-453.
- [28] M.K, Ng, & , Z, Huang. Temporal data mining with a case study as astronomical data analysis. *Lecture Notes in Computer Sciences*. Springer. pp. 2-18, 1997.
- [29] Perez, J.C., Vidal, E., (1994). Optimum polygonal approximation of digitized curves, *Pattern Recognition Letters*, 15: 743-750.
- [30] C. A. Ratanamahatana E. Keogh. Making Time-series Classification More Accurate Using Learned Constraints, SDM, Orlando, USA, 2004
- [31] N., Saito. Local feature extraction and its application using a library of bases. PhD thesis, Yale University, 1994.
- [32] H. Sakoe and S. Chiba, "A dynamic programming approach to continuous speech recognition," in Proc. 7th Int. Congr. Acoust., Budapest, Hungary, Aug. 1971, pp. 65-68.
- [33] A., Singh, D., Terzopoulos and D., B., Goldgof, (1998). Deformable Models in Medical Image Analysis, IEEE Computer Society Press, Los Alamitos, CA, USA, 1998.
- [34] V. M. Velichko and N. G. Zagoruyko, "Automatic recognition of 200 words," *Int. J. Man-Machine Studies*, vol. 2, pp. 223-234, 1970.
- [35] Vlachos, M., Hadjieleftheriou, M., Gunopulos, D. & Keogh. E. (2003). Indexing Multi-Dimensional Time-Series with Support for Multiple Distance Measures. In the 9th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining. August 24 - 27, 2003. Washington, DC, USA. pp 216-225.
- [36] R. A. Wagner, Michael J. Fischer, The String-to-String Correction Problem, *Journal of the ACM (JACM)*, Volume 21 , Issue 1, P: 168 – 73, 1974.
- [37] H. Wu, B. Salzberg,, D. Zhang. Online event-driven subsequence matching over financial data streams. *SIGMOD*, 2004.
- [38] X. Xi, E. Keogh, L. Wei, A. Mafra-Neto (2007). Finding Motifs in Database of Shapes. SIAM International Conference on Data Mining (SDM'07).