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Time Warp Edit Distance with Stiffness Adjustment for Time Series Matching (Version 3 : August 2007)

Pierre-François Marteau¹

Abstract-- In a way similar to the *string-to-string correction problem* we address time series similarity in the light of a *time-series-to-time-series-correction problem* for which the similarity between two time series is measured as the minimum cost sequence of "*edit operations*" needed to transform one time series into another. To define the "*edit operations*" we use the paradigm of a graphical editing process and end up with a dynamic programming algorithm that we call Time Warp Edit Distance (TWED). TWED is slightly different in form from Dynamic Time Warping, Longest Common Subsequence or Edit Distance with Real Penalty algorithms. In particular, it highlights a parameter which drives a kind of *stiffness* of the elastic measure along the time axis. We show that the similarity provided by TWED is a metric potentially useful in time series retrieval applications since it could benefit from the triangular inequality property to speed up the retrieval process while tuning the parameters of the elastic measure. In that context, a lower bound is derived to relate the matching of time series into down sampled representation spaces to the matching into the original space. Empiric quality of the TWED distance is evaluated on a simple classification task. Compared to Edit Distance, Dynamic Time Warping, Longest Common Subsequence and Edit Distance with Real Penalty, TWED has proven to be quite effective on the considered experimental task.

Index Terms-- Pattern Recognition, Time Series, Algorithms, Similarity Measures.

I. INTRODUCTION

More and more computer applications are faced with the problem of searching for time series within large datasets which are close to a given query element under some similarity criteria. Among numerous examples, we find financial and stock data analysis [31], moving objects

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identification [4], astronomy [23], medicine [17], meteorology, data mining [1] etc.

All these applications embed time series in a representation space and exploit some similarity measure defined for this space. Given a specific dataset and task, similarity measures are not equivalent. They basically fall basically into three categories:

- Non elastic metrics such as Lp-norms that do not support time shifting such as Euclidian Distance (ED) and Correlation,
- Elastic similarity measures that tolerate time shifting but are not metrics such as Dynamic Time Warping (DTW) [28], [27] or Longest Common Subsequence (LCSS) [6], [29].
- Elastic metrics that tolerate time shifting such as Edit distance with Real Penalty (ERP) [5].

When considering time series information retrieval, working in a metric space can be appealing since a lot of data structures (essentially tree based structures) and algorithms (partitioning, pivoting, etc.) have been optimized and made available for indexing and retrieving objects efficiently in metric spaces: see [3] for a review. All these structures and algorithms take advantage of the triangle inequality which allows for efficient pruning of a large number of time series which are too far from the query. For some non-metric measures all these data structures can still be used if a lower bounding approximation which is a metric is available. Such a lower bound exists for both LCSS and DTW as detailed in [29].

In this paper we address the case of elastic metrics, namely elastic similarity measures that jointly exploit time shifting and possess all the properties of a distance, in particular the triangle inequality. Our contribution is basically four folded:

- The first contribution of this paper is the proposal of a new elastic metric which we call TWED (“Time Warp Edit Distances”). This contribution has to be placed in the perspective of former works that seek to combine L_p -norms with the edit distance, in particular in the light of the ERP distance [5] that can support local time shifting while being a metric. Other elastic similarity measures that belong to the Dynamic Time Warping category are not metrics since they do not satisfy the triangle inequality. Part II of the paper motivates the need for triangle inequality to process time series in a data compression context based on a down sampling perspective.
- The second contribution is related to the introduction of a parameter we call *stiffness* which drives the *elasticity* of TWED, placing this kind of distance in between the Euclidian distance (somehow a distance with ‘*infinite stiffness*’) and DTW (somehow a similarity measure with no ‘*stiffness*’ at all). One of the differences between TWED and former similarity measures is the use of time stamp differences between compared samples as part of the local matching costs. The motivation for such a characteristic is also given in part II of the paper.
- The third contribution proposes a lower bound for the TWED measure which allows relating the evaluation of the matching of two time series into down sampled representation spaces to the evaluation of their matching into their original representation spaces.
- The fourth contribution of the paper is an empiric evaluation of the quality of TWED based on a simple classification experiment that provides some highlights on the effectiveness of TWED compared to the Euclidian Distance (ED), DTW, LCSS and ERP. The influence of the *stiffness* parameter on classification error rates is also analyzed.

The paper is organized as follows. Section II addresses the motivation aspects. Section III presents shortly the main relevant founding works about elastic distances for time series matching. Section IV details the definition and implementation of the Time Warp Edit Distance with stiffness adjustment that is proposed in this paper. Section V details a lower bounding procedure we suggest to speed up range queries processing. Section VI reports a classification experiments that shows the empirical effectiveness of TWED comparatively to the Euclidian distance and other classical elastic measures. Section VII concludes the paper and proposes some perspectives.

II. MOTIVATION FOR A SIMILARITY MEASURE THAT VERIFIES THE TRIANGLE INEQUALITY AND TAKES INTO ACCOUNT TIME STAMP DIFFERENCES

The motivation for using time stamps is related to the way we want to model the elasticity of the measure. Coping with index differences between two match samples has been successfully used to improve elastic measures such as Dynamic time Warping or Longest Common Subsequence measures. The idea is to limit the elasticity of the measure by using a threshold: if the index difference between two samples is lower than the value of the threshold, then the matching is allowed, otherwise it is forbidden. This binary decision might in some cases limit the effectiveness of the measure. Keeping in mind the mechanical analogy of a spring, instead of a threshold we suggest using the range of the sample index difference to linearly penalize the matching of samples for which the index values are too far and to favour the matching of samples for which the index values are close. In the case where time series are sampled using non uniform

or varying sampling rates one can benefit from time stamps instead of sample indices since this approach helps avoid resampling the data.

The second motivation for defining a measure that exploits time stamps while verifying the triangle inequality is two folds: first it provides an effective solution to compare approximated representations of time series not necessarily using uniform down-sampling methods; second, it establishes a useful relationship between the matching performed in the down sampled space and the matching performed in the original space.

Approximation of multi dimensional discrete curves has been widely studied essentially to speed up data processing required by resource demanding applications. Among other approaches, polygonal approximation of discrete curves has been quite popular recently. The problem can be informally stated as follows: given a digitized curve X of $N \geq 2$ ordered samples, find K (in general $K \ll N$) dominant samples among them that define a sequence of connected segments which most closely approximate the original curve. This problem is known as the *min- ϵ* problem. Numerous algorithms have been proposed for more than thirty years to solve efficiently this optimisation problem. Most of them belong either to graph-theoretic, dynamic programming or to heuristic approaches. See for instance [10] [24] [14] [21] among others for details.

Such approaches can be used to adaptively down sample time series. For instance, in [21] polygonal curves approximations have been used to down sample optimally gesture signals and in [21] an elastic matching procedure has been proposed to compare two time series with a linear time complexity. For these approaches, a down sampled time series is a reduced sequence of tuples (sample, time stamps) that corresponds to the end extremities of the polygonal segments. The sampling rate for such down sampled time series is not uniform in general.

Down sampling time series can be used to drastically reduce the dimension of the space in which we could potentially process the time series. Nevertheless one difficulty emerges: how can down sampled time series using non uniform (e.g. varying) sampling frequencies be compared? Not taking into account the occurring time of the samples could introduce discrepancies between the original space and the down sampled space. For instance, phase or frequency information is potentially lost or at least damaged as well as the slope of spikes.

In this context, the triangle inequality is also of great importance since it maintains distance relations between the original space and the down sampled space. Let X and Y be two time series in the original space and \tilde{X} and \tilde{Y} their down sampled counter parts. Let δ be a measure for which the triangle inequality holds. Then the following inequalities hold:

$$\delta(\tilde{X}, \tilde{Y}) \leq \delta(\tilde{X}, X) + \delta(X, Y) + \delta(Y, \tilde{Y})$$

$$\delta(X, Y) \leq \delta(X, \tilde{X}) + \delta(\tilde{X}, \tilde{Y}) + \delta(Y, \tilde{Y})$$

leading to:

$$|\delta(X, Y) - \delta(\tilde{X}, \tilde{Y})| \leq \delta(X, \tilde{X}) + \delta(Y, \tilde{Y})$$

In the case where $\delta(X, \tilde{X})$ and $\delta(Y, \tilde{Y})$ are maintained small by the similarity measure, $\delta(X, Y)$ and $\delta(\tilde{X}, \tilde{Y})$ are comparable and the following inequality gives a lower bound to $\delta(X, Y)$ measure:

$$\delta(\tilde{X}, \tilde{Y}) - \delta(X, \tilde{X}) - \delta(Y, \tilde{Y}) \leq \delta(X, Y) \quad (1)$$

This lower bound which can be used to significantly speed up the time series information retrieval process since a pruning strategy can be proposed in the down sampled space. We will get back to this issue in section V.

III. ELASTIC SIMILARITY IN THE LIGHT OF THE SYMBOLIC EDIT DISTANCE

In this section we present succinctly the main elastic measures developed in the literature, from founding work to more recent studies.

The Levenshtein Distance (LD) proposed in 1966 [16], also known as the edit distance, is the smallest number of insertions, deletions, and substitutions required to change one string into another. For more than thirty years, the ideas behind LD have been largely reused and extended by various research communities. The main contributions are rapidly reviewed below. In 1974 Wagner and Fisher [30] developed a computationally efficient algorithm to calculate LD in $O(n.m)$ using dynamic programming [2]. Meanwhile Dynamic Time Warping, which shares many similarities with LD despite the fact that it is not a metric, was proposed in 1970 [28] and 1971 [27] to align speech utterances, namely time series, with time shift tolerances. The Longest Common Subsequence (LCSS) similarity measure initially defined for string matching [9] has also been adapted for time series matching [6][29]. Recently, a lot of fruitful research dealing with DTW and LCSS has been carried out to propose efficient computation and pruning strategies that are required to process massive data [29][11][32]. Some work has also been conducted to provide the ‘triangle inequality’ to DTW: the Edit Distance with Real Penalty (ERP) [5] has been proposed as an edit distance based metric for time series matching with time shift tolerance. The edit distance principle has been also proposed to develop match 1D-point-patterns (ascending lists of real values) [18][19]. The measure proposed to match 1D-Point Patterns (PPM) is shown to be a metric that can be extended to the multidimensional case, at the price of a non polynomial complexity. Shortly, hereinafter, we present shortly hereinafter DTW, ERP and LCSS in the light of the edit distance and develop the TWED metrics as an alternative

to ERP.

A. Definitions

Let U be the set of finite time series: $U = \{A_1^p / p \in N^+\} \cup \{\Omega\}$, where Ω is the empty time series (with null length), A_1^p is a time series with discrete time index varying between 1 and p .

Let A be a finite discrete time series. Let a'_i be the i^{th} sample of time series A . We will consider that $a'_i \in S \times T$ where $S \subset R^d$ with $d \geq 1$ embeds the multidimensional space variables and $T \subset R$ embeds the time stamp variable, so that we can write $a'_i = (a_i, t_{a_i})$ where $a_i \in S$ and $t_{a_i} \in T$, with the condition that $t_{a_i} > t_{a_j}$ whenever $i > j$ (time stamp are strictly increasing in the sequence of samples).

A_i^j with $i \leq j$ is the sub time series consisting of the i^{th} through the j^{th} samples (inclusive) of A . So $A_i^j = a'_i a'_{i+1} \dots a'_j$. $|A|$ denotes the length (the number of samples) of A . Λ denotes the null sample. A_i^j with $i > j$ is the null time series noted Ω .

An edit operation is a pair $(a', b') \neq (\Lambda, \Lambda)$ of time series samples, written $a' \rightarrow b'$. Time series B results from the application of the edit operation $a \rightarrow b$ into time series A , written $A \Rightarrow B$ via $a' \rightarrow b'$, if $A = \sigma a' \tau$ and $B = \sigma b' \tau$ for some time series σ and τ . We call $a' \rightarrow b'$ a match operation if $a' \neq \Lambda$ and $b' \neq \Lambda$, a delete operation if $b' = \Lambda$, an insert operation if $a' = \Lambda$.

Similarly to the edit distance defined for string [9], we define $\delta(A, B)$ the similarity between any two time series A and B of finite length, respectively p and q as:

$$\delta(A_1^p, B_1^q) = \text{Min} \begin{cases} \delta(A_1^{p-1}, B_1^q) + \Gamma(a'_p \rightarrow \Lambda) & \text{delete} \\ \delta(A_1^{p-1}, B_1^{q-1}) + \Gamma(a'_p \rightarrow b'_q) & \text{match} \\ \delta(A_1^p, B_1^{q-1}) + \Gamma(\Lambda \rightarrow b'_q) & \text{insert} \end{cases} \quad (2)$$

Where $p \geq 1, q \geq 1$ and Γ is an arbitrary cost function which assigns to each edit operation $a' \rightarrow b'$ a nonnegative real number $\Gamma(a' \rightarrow b')$.

The recursion is initialized by setting:

$$\begin{aligned} \delta(A_1^0, B_1^0) &= 0 \\ \delta(A_1^0, B_1^j) &= \infty \\ \delta(A_1^i, B_1^0) &= \infty \end{aligned}$$

Dynamic Time Warping (DTW) and Edit Distance with Real penalties (ERP), 1D Point Pattern matching (PPM) and Longest Common Subsequence (LCSS) are special cases of the previous definitions:

B. The DTW special case

The DTW similarity measure [28][27] δ_{DTW} is defined according to the previous notations as:

$$\delta_{DTW}(A_1^p, B_1^q) = d_{LP}(a_p, b_q) + \text{Min} \begin{cases} \delta_{DTW}(A_1^{p-1}, B_1^q) \\ \delta_{DTW}(A_1^{p-1}, B_1^{q-1}) \\ \delta_{DTW}(A_1^p, B_1^{q-1}) \end{cases} \quad (3)$$

where $d_{LP}(x, y)$ is the L_p norm of vector $x-y$ in R^d ,

and so for DTW, $\Gamma(a'_p \rightarrow \Lambda_q) = \Gamma(\Lambda_p \rightarrow b'_q) = \Gamma(a'_p \rightarrow b'_q) = d_{LP}(a_p, b_q)$

We notice that the time stamp values are not used so that the costs of each edit operation involve vectors a and b in S instead of vectors a' and b' in $S \times T$. One of the main restrictions of δ_{DTW} is that it does not comply with the triangle inequality as shown by the following example:

$$\begin{aligned}
 A_1^1 &= [1]; \quad B_1^2 = [1,2]; \quad C_1^3 = [1,2,2](*) \\
 \delta_{DTW}(A_1^1, B_1^2) &= 1; \quad \delta_{DTW}(B_1^2, C_1^3) = 0; \quad \delta_{DTW}(A_1^1, C_1^3) = 2 \\
 \text{and thus : } \delta_{DTW}(A_1^1, C_1^3) &> \delta_{DTW}(A_1^1, B_1^2) + \delta_{DTW}(B_1^2, C_1^3)
 \end{aligned}$$

(*) *ID time series with no stamp value given*

C. The ERP special case

$$\delta_{ERP}(A_1^p, B_1^q) = \text{Min} \begin{cases} \delta_{ERP}(A_1^{p-1}, B_1^q) + \Gamma(a'_p \rightarrow \Lambda) \\ \delta_{ERP}(A_1^{p-1}, B_1^{q-1}) + \Gamma(a'_p \rightarrow b'_q) \\ \delta_{ERP}(A_1^p, B_1^{q-1}) + \Gamma(\Lambda \rightarrow b'_q) \end{cases} \quad (4)$$

$$\begin{aligned}
 \Gamma(a'_p \rightarrow \Lambda) &= d_{LP}(a_p, g) \\
 \text{with } \Gamma(a'_p \rightarrow b'_q) &= d_{LP}(a_p, b_q) \\
 \Gamma(\Lambda \rightarrow b'_q) &= d_{LP}(g, b_q)
 \end{aligned}$$

and g a constant in S .

where $d_{LP}(x, y)$ is the LP norm of vector $x-y$ in S . Note that the time stamp coordinate is not taken into account such that δ_{ERP} is a distance on S but not on $S \times T$.

We notice here again that the time stamp values are not used such that the costs of each edit operation involve vectors a and b in R^d instead of vectors a' and b' in R^{d+1} .

According to the authors of ERP [5], the constant g should be set to 0 for some intuitive

geometric interpretation and in order to preserve the mean value of the transformed time series when adding gap samples.

D. The LCSS special case

The Longest Common Subsequence (LCSS) similarity measure has been first defined for string matching purposes [9] and then extended for times series [6][29]. LCSS is recursively defined in [29] as follows:

$$LCSS_{\varepsilon,\delta}(A_1^p, B_1^q) = \begin{cases} 0 & \text{if } p < 1 \text{ or } q < 1, \\ 1 + LCSS_{\varepsilon,\delta}(A_1^{p-1}, B_1^{q-1}) & \text{if } d_{LP}(a_p, b_q) < \varepsilon \text{ and } |p - q| < \delta, \\ \text{Max}\{LCSS_{\varepsilon,\delta}(A_1^{p-1}, B_1^q), LCSS_{\varepsilon,\delta}(A_1^p, B_1^{q-1})\} & \text{otherwise} \end{cases} \quad (5)$$

For LCSS the match reward is 1, while no reward is offered for *insert* or *delete* operations.

The LCSS measure is transposed into a normalized dissimilarity measure $D_{\varepsilon,\delta}$ which is close in its formal structure to the ERP measure:

$$D_{\varepsilon,\delta} = 1 - \frac{LCSS_{\varepsilon,\delta}(A_1^p, B_1^q)}{\text{Min}\{p, q\}} \quad (6)$$

E. The 1D PPM special case

For Point-Pattern matching problems [18], A_1^p and B_1^q are *1D* ascending lists of real values.

$$\delta_{PPM}(A_1^p, B_1^q) = \text{Min} \begin{cases} \delta_{PPM}(A_1^{p-1}, B_1^q) + \Gamma(a'_p \rightarrow \Lambda) \\ \delta_{PPM}(A_1^{p-1}, B_1^{q-1}) + \Gamma(a'_p \rightarrow b'_q) \\ \delta_{PPM}(A_1^p, B_1^{q-1}) + \Gamma(\Lambda \rightarrow b'_q) \end{cases} \quad (7)$$

$$\begin{aligned} \Gamma(a'_p \rightarrow \Lambda) &= a_p - a_{p-1} \\ \text{with } \Gamma(a'_p \rightarrow b'_q) &= |(a_p - a_{p-1}) - (b_q - b_{q-1})| && \text{if } p > 1 \text{ and } q > 1 \\ \Gamma(\Lambda \rightarrow b'_q) &= b_q - b_{q-1} \end{aligned}$$

The author [18] shows that δ_{PPM} is a metric that calculates the minimum amount of space needed to delete or insert between pairs of points to convert one point-pattern into another. One can notice that if we consider successive increments instead of the initial values, δ_{PPM} coincides with the δ_{ERP} applied to the lists of positive increments.

IV. THE TWED DISTANCE

We propose an alternative to the definition of the edit operations for time series alignment leading to the definition of the new similarity measure TWED. To understand the semantic associated to the edit operations for TWED, we reconsider the editing analogy with strings and suggest some differences. The edit distance between two strings is defined as the minimal transformation cost allowing for the transformation of the first string into the second one. For string edition, a transformation is a finite sequence of edit operations whose associated cost is the sum over the sequence of edit operations of the elementary costs Γ associated to each edit operation.

For time series we are seeking a sequence of edit operations allowing for the transformation of two time series simultaneously in order to make them superimposed with a minimal cost. If we use a graphical editor paradigm, we can imagine a $2D$ representation of time series for which the horizontal axis represents the time scale or the time stamp coordinate and the vertical axis represents a spatial coordinate scale displaying the projection of the $d-1$ spatial coordinates of the samples onto a $1D$ scale. The graphical editor we imagine allows for the editing of two time series A and B using three elementary edit operations depicted in Fig. *1a*, *1.b* and *1.c*.

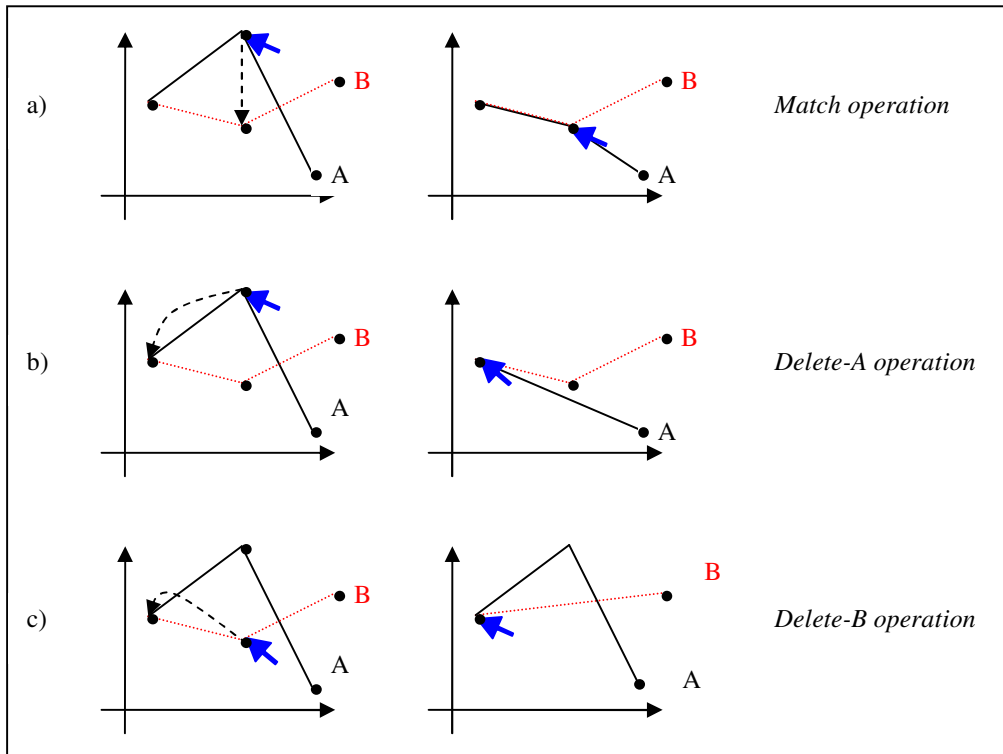


Fig 1: The edit operations in the graphical editor paradigm.

Instead of the classical *delete*, *insert* and *match* operations, we introduce *delete-A*, *delete-B* and *match* operations as follows:

i) The *delete-A* (delete inside the first time series) operation (Fig. 1.b) consists in clicking on the dot representing the sample in A to delete (a'_i) and dragging and dropping this dot onto the previous sample dot (a'_{i-1}). We suggest that the editing effort or cost associated with this delete operation be proportional to the length of vector ($a'_i - a'_{i-1}$) to which we add a constant penalty $\lambda \geq 0$.

ii) The *delete-B* (delete inside the second time series) operation (Fig. 1.c) consists in clicking on the dot representing the sample in B to delete (b'_i) and dragging and dropping this dot onto the previous sample dot (b'_{i-1}). Here again we suggest that the editing effort or cost associated with this delete operation be proportional to the length of vector ($b'_i - b'_{i-1}$) to which we add a constant penalty $\lambda \geq 0$.

Due to sampling rate variations or process variability one can face the situation where in time series data, one event could be registered many times or only few times when recording different utterances; this would justify the deletion cost to be proportional to the distance to the previous sample. Nevertheless 'outlier' samples (e.g. spurious data points) deletion cannot be covered by this argument. According to TWED, the deletion cost for such sample depends on the previous sample in the time series, and we do not have satisfactory argument to justify it. The other elastic measures (DTW, ERP, LCSS) do not offer better justification for the deletion cost of 'outliers'.

iii) The *match* operation (Fig. 1.a) consists in clicking on the dot representing the sample (a'_i) to match and then dragging and dropping this dot onto the graphic position corresponding to the matching sample (b'_j). We can suggest that the editing effort or cost associated with the *match* operation be proportional to the length of vector ($b'_j - a'_i$).

This provides the basis for the TWED distance we propose:

$$\delta_{TWED}(A_1^p, B_1^q) = \text{Min} \begin{cases} \delta_{TWED}(A_1^{p-1}, B_1^q) + d(a'_p, a'_{p-1}) + \lambda & \text{delete - A} \\ \delta_{TWED}(A_1^{p-1}, B_1^{q-1}) + d(a'_p, b'_q) & \text{match} \\ \delta_{TWED}(A_1^p, B_1^{q-1}) + d(b'_{q-1}, b'_q) + \lambda & \text{delete - B} \end{cases} \quad (8)$$

The recursion is initialized setting:

$$\begin{aligned} \delta_{TWED}(A_1^0, B_1^0) &= 0 \\ \delta_{TWED}(A_1^0, B_1^j) &= \infty \\ \delta_{TWED}(A_1^i, B_1^0) &= \infty \\ \text{with } a'_0 = b'_0 &= 0 \quad \text{by convention.} \end{aligned}$$

It is interesting to note that the penalties for *delete-A* or *delete-B* operations are similar to those proposed in the δ_{PPM} measure if we do not consider the time stamps coordinate and address the matching of *ID* monotone increasing time series.

Furthermore, using the graphical editor paradigm, we define the time series matching game as follows: two time series, *A* and *B*, are displayed on the graphic. The goal consists in editing *A* and *B* in order to completely superimpose the two curves.

The editing process is performed left to right: if *i* is an index on the samples of *A* and *j* on the samples of *B*, then the process is initialized setting $i=j=1$. A match operation will increment *i* and *j* simultaneously: $i \leftarrow i+1, j \leftarrow j+1$. A *delete-A* operation will increment *i* only: $i \leftarrow i+1$. A *delete-B* operation will increment *j* only: $j \leftarrow j+1$.

According to the above mentioned constraint, once a sample a'_i in *A* has been processed using either a *match* or a *delete-A* operation, it is impossible to edit it again and so it is for former samples a'_r, r in $\{1, \dots, i-1\}$. Similarly, once sample b'_j in *B* has been used either in a *match* or in an *delete-B* operation it is impossible to use former samples b'_r, r in $\{1..j\}$ for future match or insertion operations. Therefore, according to this game, the editing process provides a sequence of edit operations as well as ordered pairs of indices (i, j) where *i* is an index in *A* and *j* an index in *B*. In other words, the process provides an ordered sequence of triplets (op_k, i_k, j_k) where op_k is

the k^{th} edit operation selected, and i_k and j_k are the values of the index in A and B respectively when the edit operation is performed. A partial order can be defined on the triplets as follows:

$$(op_{k1}, i_{k1}, j_{k1}) < (op_{k2}, i_{k2}, j_{k2}) \text{ iff } i_{k1} \leq i_{k2} \text{ and } j_{k1} \leq j_{k2} \text{ and either } i_{k1} \neq i_{k2} \text{ or } j_{k1} \neq j_{k2}.$$

Since for each step of the editing game, one of the indices is increased by one while the other is either incremented by one or remains unchanged, all the triplets in the output editing sequence are ordered in increasing order.

Supposing that the game editing process has provided a sequence of edit operations up to i_k and j_k index values, if the sub sequences $\bar{A}_1^{i_{k-1}}$ ($\bar{A}_1^{i_{k-1}}$ refers to the sequence obtained from A after the first $k-1$ edit operations) and $\bar{B}_1^{j_{k-1}}$ are not superimposed, then, as there exists no possibility to process former samples to superimpose them the game process cannot be successful.

It is easy to show that δ_{TWED} as defined in eq. (4) provides a successful sequence of editing operations at a minimal global cost for all pairs of time series in U^2 .

Theorem 1: δ_{TWED} is a distance on the set of finite discrete time series U :

P1: $\delta_{TWED}(A, B) \geq 0$ for any finite discrete time series A and B ,

P2: $\delta_{TWED}(A, B) = 0$ iff $A = B$ for any finite discrete time series A and B ,

P3: $\delta_{TWED}(A, B) = \delta_{TWED}(B, A)$ for any finite discrete time series A and B ,

P4: $\delta_{TWED}(A, B) \leq \delta_{TWED}(A, C) + \delta_{TWED}(C, B)$ for any finite discrete time series A, B and C .

A sketch of the proofs for this theorem is given in APPENDIX I.

A. Providing 'stiffness' into TWED

Going back to the graphical editor game we have envisaged that the penalty or cost associated with each edit operation should be proportional to the mouse pointer displacement involved during the edition. If we separate the spatial displacement in S from the temporal displacement in T then we have to consider a spatial penalty that could be handled by a distance measured in S and a temporal penalty more or less proportional to some distance measured in T . By doing this, we could parameterize a distance in between the Euclidian Distance, which is characterized with a kind of 'infinite stiffness', and DTW which is characterized by a 'null stiffness. In practice, we can choose $d(a', b') = d_{LP}(a, b) + \gamma \cdot d_{LP}(t_a, t_b)$ where γ is a non negative constant which characterizes the *stiffness* of TWED elastic measures. Notice that $\gamma > 0$ is required for TWED to be a distance. If $\gamma = 0$ then TWED will be a distance on S but not on $S \times T$.

The final formulation of TWED is as follows:

$$\delta_{TWED}(A_1^p, B_1^q) = \text{Min} \begin{cases} \delta_{TWED}(A_1^{p-1}, B_1^q) + d_{LP}(a_p, a_{p-1}) + \gamma \cdot d_{LP}(t_{a_p}, t_{a_{p-1}}) + \lambda \\ \delta_{TWED}(A_1^{p-1}, B_1^{q-1}) + d_{LP}(a_p, b_q) + \gamma \cdot d_{LP}(t_{a_p}, t_{b_{q1}}) \\ \delta_{TWED}(A_1^p, B_1^{q-1}) + d_{LP}(b_q, b_{q-1}) + \gamma \cdot d_{LP}(t_{b_q}, t_{b_{q-1}}) + \lambda \end{cases}$$

The iterative implementation of TWED is depicted in *Fig.2*.

B. Algorithmic complexity of TWED

The time complexity of TWED is the same as DTW and ERP, namely $O(p.q)$, where p and q are the lengths of the two time series being matched. The space complexity is also the same as DTW i.e. $O(p.q)$, but as well as the ERP distance, the costs $\gamma(a' \rightarrow \Lambda)$ and $\gamma(\Lambda \rightarrow b')$ can be tabulated to speed up the calculation leading to an extra space complexity of $O(p+q)$ for δ_{TWED} .

```

float TWED(float A[1..n], float timeStampsA[1..n],
           float B[1..m], float timeStampsB[1..m],
           float lambda, float nu) {

  declare int DTW[0..n,0..m];
  declare int i, j;
  declare float cost;
  declare float A[0] :=0, timeStampsA[0] :=0;
  declare float B[0] :=0, timeStampsB[0] :=0;

  for i := 1 to m
    TWED[0,i] := infinity;
  for i := 1 to n
    TWED[i,0] := infinity;
  TWED[0,0] := 0;

  for i := 1 to n {
    for j := 1 to m {
      cost:= L1Dist(A[i],B[j]); // Distance-L1
      DTW[i,j] := minimum(
        // insertion
        DTW[i-1,j ] + L1Dist(A[i-1], A[i])+
          nu*(timeStampsA(i)- timeStampsA(i-1))+lambda,
        // deletion
        DTW[i ,j-1] + L1Dist(B[j-1], B[j])+
          nu*(timeStampsB(j)- timeStampsB(j-1))+lambda,
        // match
        DTW[i-1,j-1] + L1Dist(A[i],B[j])+
          nu*|timeStampsA(i)- timeStampsB(j)| );
    } // End for j
  } // End for i

  Cost = TWED[n,m];
  Return; }

```

Fig 2: Iterative implementation of the TWED distance.

C. Some properties of TWED

Let us use $\delta_{\lambda,\gamma}$ instead of δ_{TWED} to refer to the TWED measure and let us use D_{L1} to refer to the *L1-distance*.

Lemma 1: $\delta_{\lambda,\gamma}$ is upper bounded by D_{L1} .

$\forall \lambda \geq 0, \gamma > 0 \quad \forall X, Y \in U^2 \quad \delta_{\lambda,\gamma}(X, Y) \leq D_{L1}(X, Y)$, whenever X and Y have the same length.

Proof: let us consider the sequence of editing operations consisting in m match operations, where m is the length of the X and Y . This sequence has a cost equal to the *L1-distance* between the two time series X and Y . Since $\delta_{\lambda,\gamma}$ is equal to the cost of the optimal sequences of edit operations, the result follows. \square

Lemma 2: $\delta_{\lambda,\gamma}$ is an increasing function of λ and γ :

$$\forall \lambda \geq 0, \gamma > 0 \quad \forall \lambda' \geq \lambda \quad \forall \gamma' \geq \gamma \quad \forall X, Y \in U^2 \quad \delta_{\lambda,\gamma}(X, Y) \leq \delta_{\lambda',\gamma'}(X, Y)$$

Proof: Let us consider one of the optimal sequences of editing operations evaluated with the tuple (λ, γ) with minimal cost equal to $\delta_{\lambda,\gamma}(X, Y)$. If we keep this sequence of editing operation while replacing (λ', γ') with (λ, γ) in all the elementary operation costs we get a cost for this sequence that is lower than $\delta_{\lambda',\gamma'}(X, Y)$ but greater than the cost of the optimal sequences $\delta_{\lambda,\gamma}(X, Y)$ evaluated using (λ, γ) . The result follows. \square

V. BOUNDING THE TWED MEASURE

In this section we get back to our second motivation about defining a measure that exploits time stamps while verifying the triangle inequality and shows how piecewise constant approximations (PWCA) with few segments of time series can be used to improve the efficiency of range queries. Various methods exist to get polygonal curve approximations of time series, in particular heuristic [7][8][10][12], near optimal [14][21] or optimal [24] solutions. Most of them can be adapted to provide PCWA approximation of time series.

We define $\bar{A}_1^{p,r}$ a PWCA of time series A_1^p containing $r-1 \geq 0$ constant segments. This approximation can be obtained using any kind of solution (from heuristic to optimal solutions), let say the optimal solution similar to the one proposed in [24]. $\bar{A}_1^{p,r}$ and A_1^p have the same number of samples, namely p . Let \tilde{A}_1^r be the time series composed with the r segment extremities of $\bar{A}_1^{p,r}$. \tilde{A}_1^r contains r samples. Let us similarly define $\bar{B}_1^{p,r'}$ and $\tilde{B}_1^{r'}$ from time series B_1^p .

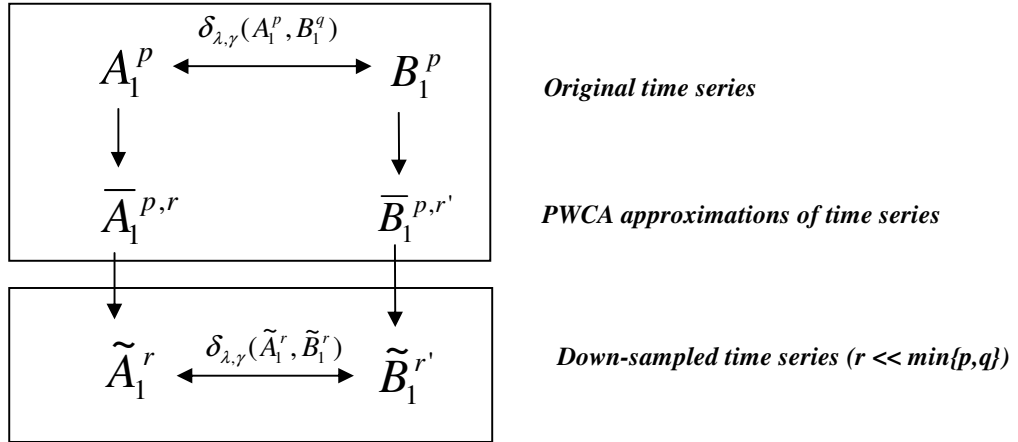


Fig 2: Linking the matching of time series in the original space to the matching in the down-sampled space.

Lemma 3: $\forall \lambda \geq 0, \gamma > 0, \forall r \in [1; p], \forall X_1^p \in U \quad \delta_{\lambda, \gamma}(\bar{X}_1^{p,r}, \tilde{X}_1^r) \leq (p-r)(\lambda + \gamma \Delta T)$, where ΔT is the time difference average between two successive samples inside the piecewise constant segments of the approximation.

The proof for this lemma is straightforward: let us consider the sequence of operations consisting in r match operations for the end extremities of the piecewise constant segments and $(p-r)$ delete operations for the set of samples in $\bar{X}_1^{p,r}$ that are not end extremities of the piecewise constant segments. In this sequence each match operations has a null cost, and each delete operation has a λ fixed penalty and a penalty proportional to the time stamps difference between two successive samples $\gamma \cdot (\text{timeStamps}(i) - \text{timeStamps}(i-1))$. Then, the cost for this sequence of editing operations is $(p-r)(\lambda + \gamma \Delta T)$. Finally the optimal sequence of editing operation as a cost $\delta_{\lambda, \gamma}(\bar{X}_1^{p,r}, \tilde{X}_1^r)$ lower or equal to $(p-r)(\lambda + \gamma \Delta T)$. \square

From these previous lemmas we can bound the matching of two time series evaluated into the original space with the matching of their approximations evaluated into the down sampled space as follows:

$\forall \lambda \geq 0, \gamma > 0, \forall A_1^p, B_1^q \in U^2, \forall r \in [1; p[, \forall r' \in [1; q[$ we have :

$$\delta_{\lambda, \gamma}(\bar{A}_1^{p,r}, A_1^p) \leq D_{L1}(\bar{A}_1^{p,r}, A_1^p); \text{ since } \bar{A}_1^{p,r} \text{ and } A_1^p \text{ have the same length lemma 1 applies}$$

$$\delta_{\lambda, \gamma}(\bar{A}_1^{p,r}, \tilde{A}_1^r) \leq (p-r)(\lambda + \gamma \Delta T); \quad \text{lemma 3}$$

$$\delta_{\lambda, \gamma}(\bar{B}_1^{q,r'}, B_1^q) \leq D_{L1}(\bar{B}_1^{q,r'}, B_1^q); \text{ since } \bar{B}_1^{q,r'} \text{ and } B_1^q \text{ have the same length lemma 1 applies}$$

$$\delta_{\lambda, \gamma}(\bar{B}_1^{q,r'}, \tilde{B}_1^{r'}) \leq (q-r')(\lambda + \gamma \Delta T); \quad \text{lemma 3}$$

From these inequalities and the triangle inequality verified by $\delta_{\lambda, \gamma}$ we get:

$$\delta_{\lambda, \gamma}(\tilde{A}_1^r, A_1^p) \leq (p-r)(\lambda + \gamma \Delta T) + D_{L1}(\bar{A}_1^{p,r}, A_1^p)$$

$$\delta_{\lambda, \gamma}(\tilde{B}_1^{r'}, B_1^q) \leq (q-r')(\lambda + \gamma \Delta T) + D_{L1}(\bar{B}_1^{q,r'}, B_1^q)$$

And since $\delta_{\lambda, \gamma}$ verifies the triangle inequality we have:

$$\begin{aligned} \delta_{\lambda, \gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) &\leq \delta_{\lambda, \gamma}(\tilde{A}_1^r, A_1^p) + \delta_{\lambda, \gamma}(A_1^p, B_1^q) + \delta_{\lambda, \gamma}(B_1^q, \tilde{B}_1^{r'}) \\ &\leq (\lambda + \gamma \Delta T)(p + q - r - r') + D_{L1}(\bar{A}_1^{p,r}, A_1^p) + D_{L1}(\bar{B}_1^{q,r'}, B_1^q) + \delta_{\lambda, \gamma}(A_1^p, B_1^q) \end{aligned} \quad (9)$$

$$\begin{aligned} \delta_{\lambda, \gamma}(A_1^p, B_1^q) &\leq \delta_{\lambda, \gamma}(A_1^p, \tilde{A}_1^r) + \delta_{\lambda, \gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) + \delta_{\lambda, \gamma}(\tilde{B}_1^{r'}, B_1^q) \\ &\leq (\lambda + \gamma \Delta T)(p + q - r - r') + D_{L1}(\bar{A}_1^{p,r}, A_1^p) + D_{L1}(\bar{B}_1^{q,r'}, B_1^q) + \delta_{\lambda, \gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) \end{aligned}$$

Leading to:

$$\left| \delta_{\lambda, \gamma}(A_1^p, B_1^q) - \delta_{\lambda, \gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) \right| \leq (\lambda + \gamma \Delta T)(p + q - r - r') + D_{L1}(\bar{A}_1^{p,r}, A_1^p) + D_{L1}(\bar{B}_1^{q,r'}, B_1^q)$$

This shows that $\delta_{\lambda, \gamma}(A_1^p, B_1^q)$ and $\delta_{\lambda, \gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'})$ are potentially close when two conditions are satisfied:

1. The PWCA approximations of A and B are close to the original time series in the sense of the $L1$ -distance. This should be ensured by the optimal solution of the $min-\varepsilon$ problem using piecewise constant segments whenever the number of segments r is not too small.

2. $\lambda + \gamma \cdot \Delta T$ is small comparatively to $(p+q-r-r')$.

Hence we get the following lower bounds that can be considered tight if the two previous conditions are satisfied:

$$\begin{aligned}
 & \forall \lambda \geq 0, \gamma > 0, \forall \lambda' \geq \lambda \forall \gamma' \geq \gamma, \forall A_1^p, B_1^q \in U^2, \forall r \in [1, p[, \forall r' \in [1, q[, \\
 & \delta_{\lambda, \gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) - (\lambda + \gamma \cdot \Delta T)(p + q - r - r') - D_{L1}(\bar{A}_1^{p,r}, A_1^p) - D_{L1}(\bar{B}_1^{q,r'}, B_1^q) \quad (10) \\
 & \leq \delta_{\lambda, \gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) - \delta_{\lambda, \gamma}(\tilde{A}_1^r, A_1^p) - \delta_{\lambda, \gamma}(\tilde{B}_1^{r'}, B_1^p) \\
 & \leq \delta_{\lambda, \gamma}(A_1^p, B_1^q) \leq \delta_{\lambda, \gamma}(A_1^p, B_1^q) \text{ since lemma 2 applies.}
 \end{aligned}$$

This last inequality is potentially still useful to design fast and dirty filters dedicated to range query searching for applications for which λ' and γ' cannot be small enough, while λ and γ can be set up small.

For range query search, if R is the radius of the range query and A_1^p the center of the query ball, then B_1^q is outside the search range if one of the following conditions is verified:

$$\begin{aligned}
 & \delta_{\lambda, \gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'}) > R + (\lambda + \gamma \cdot \Delta T)(p + q - r - r') + D_{L1}(\bar{A}_1^{p,r}, A_1^p) + D_{L1}(\bar{B}_1^{q,r'}, B_1^q) \quad (11) \\
 & > R + \delta_{\lambda, \gamma}(\tilde{A}_1^r, A_1^p) + \delta_{\lambda, \gamma}(\tilde{B}_1^{r'}, B_1^p)
 \end{aligned}$$

For time series information retrieval applications, equations (10) and (11) are potentially useful. If $\delta_{\lambda,\gamma}(\tilde{A}_1^r, A_1^p)$ and $\delta_{\lambda,\gamma}(\tilde{B}_1^{r'}, B_1^p)$ are pre-computed during the indexing phase, the tighter bound can be used. Otherwise the second bound can be evaluated during the retrieval phase through the computation of *L1-distances*. If $r = r' = (1/K) \min(p, q)$ the complexity for evaluating $\delta_{\lambda,\gamma}(\tilde{A}_1^r, \tilde{B}_1^{r'})$ is lower than $O(p \cdot q / K^2)$.

VI. EXPERIMENTATIONS

To evaluate empirically the effectiveness of the TWED distance comparatively to other metrics or similarity measures, we address a simple classification task experiment. The classification task we have considered consists in assigning one of the possible categories to an unknown time series for the following data sets:

- 20 data sets available at UCR repository [13],
- the PWM1 and PWM2 data sets we have constructed with the intention of causing some difficulties either to Dynamic Time Warping pseudo distances or ERP distance. These two data sets are detailed in Appendix 2.

For each dataset, a training subset is defined as well as a testing subset. The classification is based on the simple nearest neighbor decision rule: first we select a training data set containing time series for which the correct category is known. To assign a category to an unknown time series selected from a testing data set (different from the train set), we select its nearest neighbor (in the sense of a distance or similarity measure) within the training data set, then, assign the

category associated to its nearest neighbor.

Given a dataset, we adapt the *stiffness* parameter as follows: we use the **training dataset** to select the '*best stiffness*' (γ) value as well as the best λ value, namely the ones leading to the minimal error rate **on the training data**, according to a leave-one-out procedure (that consists in selecting iteratively one time series into the **training set** and then in considering it as a test against the remaining time series within the **training set** itself).

Finally, the **testing dataset** is used to evaluate the final error rate (reported in Tab.1 and Tab.2) with the best γ and λ values estimated on the **training set**. This leads to OTWED, the optimized versions of TWED.

The same procedure is used to set up the parameters defined for the other parametric measures, i.e. ODTW and LCSS.

Tab.1 and *Tab.2* show the results obtained for the tested methods, e.g. Euclidian Distance on the original time series, optimized DTW with best warping windows (ODTW) as defined in [25], classical DTW (DTW) with no warping window, Longest Common Subsequence (LCSS) as defined in [29], Edit distance with Real Penalty (ERP) as defined in [5] and OTWED. In *Tab.1* and *Fig.3* the time series are not preprocessed, while in *Tab.2* and *Fig 4*. time series are down sampled using an optimal Piecewise Constant Approximation procedure similar as the one described in [24] for polygonal approximation. In this last experiment, each down sampled time series has exactly 50% less samples than the original time series. The sampling rate for the down sampled time series is indeed varying since the size of the constant segments used to approximate

the time series are not constant in general.

For parameterized measures, best values are selected on the training data in order to minimize the error rate estimated on the training data. More precisely the settings are as follows:

- ODTW: the best corridor value is selected for each dataset among the set $\{0, \max\{p, q\}\}$ such as to minimize the classification errors estimated on the **training data**. If different corridor values lead to the minimal error rate estimated on the training data then the **lowest corridor value is selected**.
- LCSS: the best δ and ε values are selected for each dataset respectively among the sets $\{n, n/2, n/4, \dots, n/2^k\}$, with $n = \max\{p, q\}$ and $n/2^k \leq .5 < n/2^{k+1}$, and $\{20, 20/2, 20/4, \dots, 20/2^k\}$, with $20/2^k \leq 1e^{-2} < 20/2^{k+1}$ such as to minimize the classification errors estimated on the **training data**. If different (δ, ε) values lead to the minimal error rate estimated on the training data then the pairs having the **highest δ value are selected first, then the pair with the highest ε value is finally selected**.
- OTWED: for our experiment, ‘*stiffness value*’ (γ) is selected into $\{1e10^{-5}, 1e10^{-4}, 1e10^{-3}, 1e10^{-2}, e10^{-1}, 1\}$ and λ is selected into $\{0, .25, .5, .75, 1.0\}$. The γ and λ parameter values are selected for each dataset such as to minimize the classification errors estimated on the **training data**. If different (γ, λ) values lead to the minimal

error rate estimated on the training data then the pairs containing the **highest γ value are selected first, then the pairs with the highest λ value** is finally selected.

For ERP and OTWED we used the *L1-norm*, while the *L2-norm* has been implemented in DTW and ODTW as reported in [25]. The gap value used in ERP has been set to 0 as suggested by the authors [5].

Finally, as time is not explicitly given for these datasets, we used the index value of the samples as the time stamps for the whole experiment. This leads to the following implementations of TWED:

$$\delta_{TWED}(A_1^p, B_1^q) = \text{Min} \begin{cases} \delta_{TWED}(A_1^{p-1}, B_1^q) + |a_p, a_{p-1}| + \gamma + \lambda \\ \delta_{TWED}(A_1^{p-1}, B_1^{q-1}) + |a_p, b_q| + \gamma \cdot |p - q| \\ \delta_{TWED}(A_1^p, B_1^{q-1}) + |b_{q-1}, b_q| + \gamma + \lambda \end{cases}$$

This experiment shows that the TWED distance is effective on the considered task comparatively to ED, DTW, ODTW, ERP and LCSS measures since it exhibits in average the lowest error rates on the testing data as shown in *Tab. 1* and *Fig. 3*. The gain in average is relatively significant: 3.9% against ODTW, 4.8% against LCSS, 5% against ERP, 10.8% against ED and 11% against DTW.

The same experiment carried on down sampled time series (*Tab. 2* and *Fig. 4*) shows that error rates drop more than twice time faster for ED, DTW, ERP, LCSS, ODTW (>7%) than for

TWED (3.5%). In that experimental context coping with time stamps when matching not uniformly down sampled time series seems to be quite effective.

Dataset	Nbr of classes Size of testing set	1-NN ED	1-NN ODTW	1-NN DTW	1-NN LCSS	1-NN ERP	1-NN OTWED
Synthetic Control	61300	0.12	0.017	0.007	0.047	0.036	0.033
Gun-Point	21150	0.087	0.087	0.093	0.013	0.04	0.007
CBF	3900	0.148	0.004	0.003	0.009	0.003	0.007
Face (all)	1411690	0.286	0.192	0.192	0.201	0.202	0.192
OSU Leaf	6242	0.483	0.384	0.409	0.202	0.397	0.219
Swedish Leaf	15625	0.213	0.157	0.210	0.117	0.12	0.102
50Words	50455	0.369	0.242	0.310	0.213	0.281	0.189
Trace	41100	0.24	0.01	0.0	0.02	0.17	0.03
Two Patterns	444000	0.09	0.0015	0.0	0.0	0.0	0.001
Wafer	216174	0.005	0.005	0.020	0	0.009	0.004
Face (four)	488	0.216	0.114	0.170	0.068	0.102	0.045
Lighting2	2161	0.246	0.131	0.131	0.18	0.148	0.131
Lighting7	7173	0.425	0.288	0.274	0.452	0.301	0.247
ECG	21100	0.12	0.12	0.23	0.1	0.13	0.09
Adiac	371391	0.389	0.391	0.396	0.425	0.378	0.366
Yoga	0213000	0.170	0.155	0.164	0.137	0.147	0.132
Fish	71175	0.267	0.233	0.267	0.091	0.12	0.057
Coffee	228	0.25	0.179	0.179	0.214	0.25	0.25
OliveOil	4130	0.133	0.167	0.133	0.8	0.167	0.133
Beef	5130	0.467	0.467	0.5	0.533	0.5	0.567
PWM1 ¹	60600	0.12	0.033	0.245	0.038	0.42	0.003
PWM2 ²	30300	0.34	0.34	0.31	0.053	0.047	0.047
<i>MEAN</i>		0.238	0.169	0.240	0.178	0.180	0.13
<i>STD</i>		0.127	0.138	0.132	0.205	0.146	0.14

TABLE 1: COMPARATIVE STUDY USING THE UCR DATASETS [13]: CLASSIFICATION ERROR RATE OBTAINED USING THE FIRST NEAR NEIGHBOR CLASSIFICATION RULE FOR ED, DTW, ODTW, LCSS, ERP, AND OTWED DISTANCES

¹ PWM1 is a synthetic adhoc dataset built to beat DTW and ERP distances (see annexe 2 for details)

² PWM2 is a synthetic adhoc dataset built to beat ODTW and DTW distances (see annexe 2 for details)

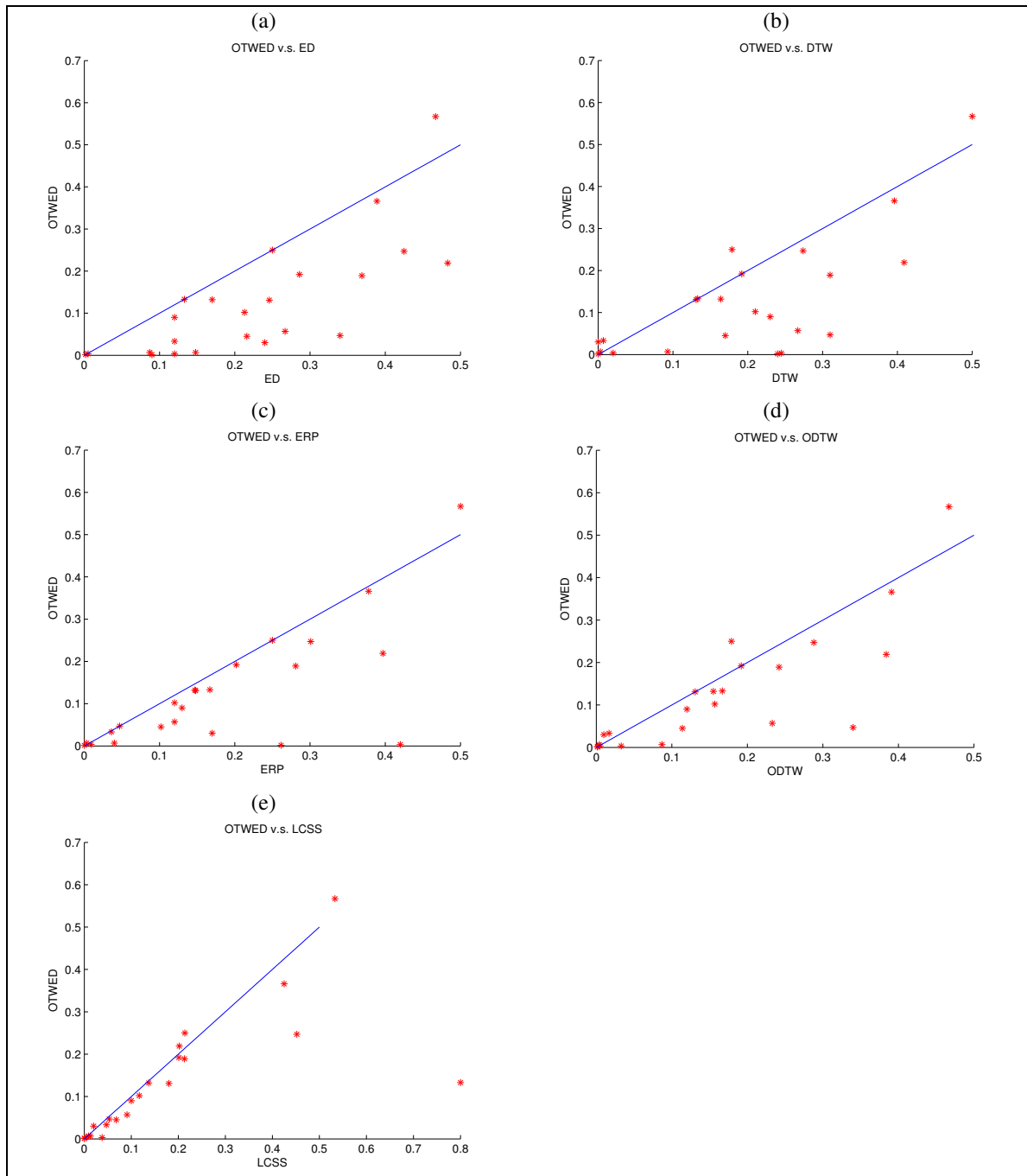


Fig. 3: Comparison of distance pairs (δ_x, δ_y) . The x and y axes show the error rates for the two compared distances. The straight line has a slope of 1.0 and dots correspond to the error rate for the selected distance pair and tested data sets. A dot below (resp. above) the straight line indicates that distance δ_y has a lower (resp. higher) error rate than distance δ_x . Plot (a) shows OTWED v.s. ED, plot (b) shows OTWED v.s. DTW, plot (c) shows OTWED v.s. ERP, plot (d) shows OTWED v.s. ODTW, plot (e) shows OTWED v.s. LCSS.

Dataset	Nbr of classes Size of testing set	1-NN ED	1-NN ODTW	1-NN DTW	1-NN LCSS	1-NN ERP	1-NN OTWED
Synthetic Control	61300	0.233	0.173	0.177	0.243	0.22	0.147
Gun-Point	21150	0.14	0.113	0.067	0.027	0.047	0.047
CBF	31900	0.24	0.027	0.017	0.03	0.028	0.034
Face (all)	1411690	0.482	0.273	0.292	0.336	0.346	0.26
OSU Leaf	61242	0.541	0.455	0.43	0.393	0.475	0.376
Swedish Leaf	151625	0.932	0.323	0.322	0.288	0.291	0.162
50Words	50455	0.327	0.303	0.369	0.251	0.323	0.2
Trace	41100	0.07	0	0	0	0.06	0.01
Two Patterns	414000	0.593	0	0	0.104	0.013	0.008
Wafer	216174	0.025	0.014	0.022	0.018	0.013	0.004
Face (four)	488	0.432	0.239	0.216	0.295	0.261	0.136
Lighting2	2161	0.263	0.098	0.115	0.148	0.115	0.148
Lighting7	7173	0.521	0.315	0.342	0.427	0.26	0.301
ECG	21100	0.18	0.25	0.26	0.32	0.2	0.12
Adiac	371391	0.527	0.486	0.483	0.448	0.496	0.437
Yoga	0213000	0.204	0.166	0.171	0.188	0.189	0.153
Fish	71175	0.371	0.354	0.354	0.189	0.28	0.194
Coffee	228	0.179	0.143	0.179	0.214	0.25	0.214
OliveOil	4130	0.567	0.167	0.167	0.333	0.333	0.133
Beef	5130	0.533	0.533	0.5	0.5	0.5	0.167
PWM1 ³	60600	0.407	0.41	0.632	0.4	0.623	0.135
PWM2 ⁴	30300	0.4	0.46	0.387	0.5	0.247	0.253
<i>MEAN</i>		0.371	0.241	0.250	0.257	0.253	0.165
<i>STD</i>		0.211	0.165	0.178	0.157	0.169	0.113

TABLE 2: COMPARATIVE STUDY USING THE UCR DATASETS [13]: CLASSIFICATION ERROR RATE OBTAINED USING THE FIRST NEAR NEIGHBOR CLASSIFICATION RULE FOR ED, DTW, ODTW, LCSS, ERP, AND OTWED DISTANCES

³ PWM1 is a synthetic adhoc dataset built to beat DTW and ERP distances (see annexe 2 for details)

⁴ PWM2 is a synthetic adhoc dataset built to beat ODTW and DTW distances (see annexe 2 for details)

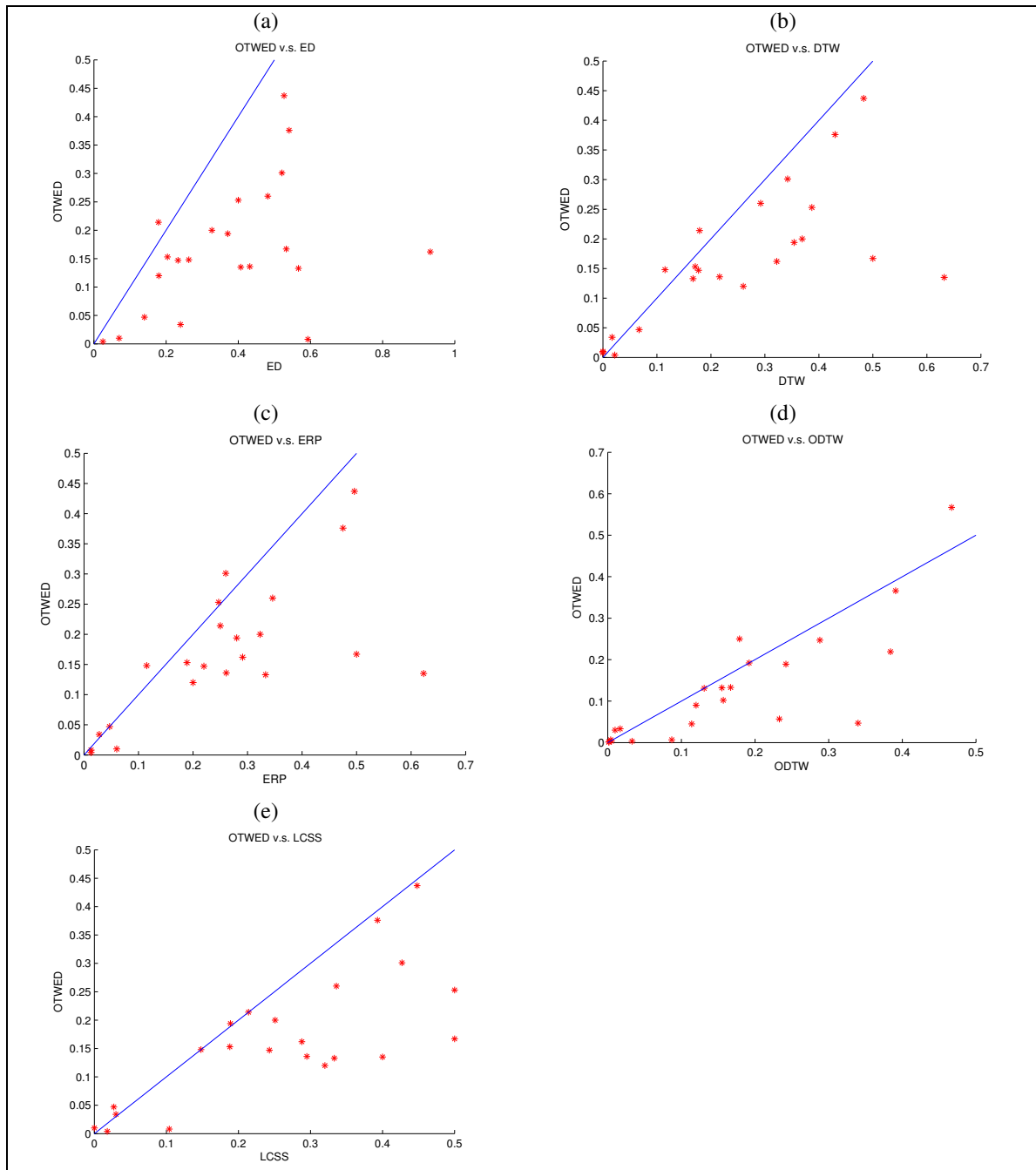


Fig. 4: Comparison of distance pairs (δ_x, δ_y) when times series have been down sampled . The x and y axes show the error rates for the two compared distances. The straight line has a slope of 1.0 and dots correspond to the error rate for the selected distance pair and tested data sets. A dot below (resp. above) the straight line indicates that distance δ_y has a lower (resp. higher) error rate than distance δ_x . Plot (a) shows OTWED v.s. ED, plot (b) shows OTWED v.s. DTW, plot (c) shows OTWED v.s. ERP, plot (d) shows OTWED v.s. ODTW, plot (e) shows OTWED v.s. LCSS.

VII. CONCLUSION

From a graphical curve editing perspective and from earlier work on symbolic edit distance and dynamic time warping we have developed an elastic similarity measure called TWED to match time series with some time shifting tolerance. We have proved that the TWED measure is a metric, and as such TWED can be used complementarily with methods developed for searching in metric spaces as potential solutions for time series searching and retrieval applications when time shift tolerance is concerned. The originality of TWED, comparatively to similar elastic measures, apart from the way insertions and deletions are managed, lies in the introduction of a parameter which drives the '*stiffness*' of the measure thus placing TWED in between the Euclidian distances (infinite stiffness) and the DTW similarity measure (null stiffness). Moreover TWED involved a second parameter which defines a constant penalty for insert or delete operations, similarly to the edit distance defined for string matching. These two parameters can be straightforwardly optimized for each application or dataset as far as training data are available.

Furthermore, a procedure has been sketched to lower bound the TWED metric. This procedure consists in approximating the time series using polygonal or piecewise constant approximations. It takes benefits from the triangle inequality to relate the TWED measure evaluated on the approximated representations of time series to the TWED measure evaluated on the original time series. The computational cost reduction of TWED when evaluated in the approximated representation space is quadratic with the compression rate of the approximation. Nevertheless, this kind of lower bound has no linear complexity. Experimentation to evaluate the effectiveness of this procedure is a perspective as well as the search for a lower bound whose complexity is

effectively linear and that could be efficiently used in conjunction with the proposed lower bounding procedure.

The empirical quality of the distance has been evaluated on a classification experiment based on the first near neighbor classification rule for 22 different datasets. Globally, on this experiment, TWED performs in average significantly better than the Euclidian distance and Dynamic Time Warp measure and slightly better than the Longest Common Subsequence measure, the Edit Distance with Real Penalty and the Dynamic Time Warping measure with optimized search corridor size. When the classification experiment is applied on down sampled time series, TWED shows to be more robust than the other tested measures. This is mainly because the times series are not uniformly sampled in this experiment and in that case it is relevant coping with time stamps.

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APPENDIX 1

Let U be the set of finite discrete time series: $U = \{A_1^p / p \in \mathbb{N}^+\} \cup \{\Omega\}$, where Ω is the empty time series (with null length). Let A_1^p be a time series with discrete time index varying between 1 and p . Let a'_i be the i^{th} sample of time series A . We will consider that $a'_i \in S \times T$ where $S \subset \mathbb{R}^d$ with $d \geq 1$ embeds the multidimensional space variables and $T \subset \mathbb{R}$ embeds the time stamp variable, so that we can write $a'_i = (a_i, t_{a_i})$ where $a_i \in S$ and $t_{a_i} \in T$, with the condition that $t_{a_i} > t_{a_j}$ whenever $i > j$ (time stamp are strictly increasing in the sequence of samples).

Let define δ_{TWED} as:

$$\delta_{TWED}(A_1^p, B_1^q) = \text{Min} \begin{cases} \delta_{TWED}(A_1^{p-1}, B_1^q) + d(a'_p, a'_{p-1}) + \lambda \\ \delta_{TWED}(A_1^{p-1}, B_1^{q-1}) + d(a'_p, b'_q) \\ \delta_{TWED}(A_1^p, B_1^{q-1}) + d(b'_{q-1}, b'_q) + \lambda \end{cases}$$

where d is any distance on \mathbb{R}^{d+1} . In practice, we will choose $d(a', b') = d_{LP}(a, b) + \gamma d_{LP}(t_a, t_b)$ where γ is a parameter which characterizes the *stiffness* of the elastic distance δ_{TWED} , and λ any positive constant element in \mathbb{R}^{d+1} .

The recursion is initialized setting:

$$\begin{aligned}\delta_{TWED}(A_1^0, B_1^0) &= 0 \\ \delta_{TWED}(A_1^0, B_1^j) &= \sum_{k=1}^j d(b'_k, b'_{k-1}), j \in \{1, \dots, q\} \\ \delta_{TWED}(A_1^i, B_1^0) &= \sum_{k=1}^i d(a'_k, a'_{k-1}), i \in \{1, \dots, p\} \\ &\text{with } a'_0 = b'_0 = 0 \text{ by convention.}\end{aligned}$$

Proof of theorem 1: δ_{TWED} is a distance on the set U of finite discrete time series:

P1: non-negativity

For all (A_1^p, B_1^q) in $U \times U$ let $m=p+q$. Non-negativity of δ_{TWED_0} is proved by induction on m .

P1 is true for $m=0$ by definition of δ_{TWED} and the induction hypothesis holds.

Suppose *P1* is true for all $m \in \{0, \dots, n-1\}$ for some $n > 0$. Then for all (A_1^p, B_1^q) in $U \times U$ such that $m = n$, as $\delta_{TWED}(A_1^{p-1}, B_1^q)$, $\delta_{TWED}(A_1^{p-1}, B_1^{q-1})$ and $\delta_{TWED}(A_1^p, B_1^{q-1})$ are assumed positive and as the non-negativity of distance d holds, $\delta_{TWED}(A_1^p, B_1^q)$ is necessary non-negative, showing that *P1* is true for all $m \in \{0, \dots, n\}$. By induction, *P1* holds for all $m \in \mathbb{N}$. \square

P2: identity of indiscernibles

For all (A_1^p, B_1^q) in $U \times U$, if $A_1^p = B_1^q$ then $p=q$ and $\forall i \in \{1, \dots, p\}$, $a'_i = b'_i$. It is easy to show by

induction on p that $0 \leq \delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^p, B_1^p) \leq \sum_{i=1}^p d(a'_i, b'_i) = 0$ leading to

$$\delta_{TWED}(A_1^p, B_1^q) = 0.$$

Now consider the backward proposition *P'2*: $\delta_{TWED}(A_1^p, B_1^q) = 0 \Rightarrow A_1^p = B_1^q$. *P'2* is proved by

induction on $m=p+q$.

Suppose $P2$ is true for all $m \in \{0, \dots, n-1\}$ for some $n > 0$. Then for all (A_1^p, B_1^q) in $U \times U$ such that

$m = n$ and $\delta_{TWED}(A_1^p, B_1^q) = 0$ we have necessarily

$$\delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^{p-1}, B_1^{q-1}) + d(a'_p, b'_q).$$

We verify that the cases where $\delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^p, B_1^{q-1}) + d(b'_q, b'_{q-1}) + \lambda$ or

$$\delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^{p-1}, B_1^q) + d(a'_p, a'_{p-1}) + \lambda$$

are impossible since $d(b'_q, b'_{q-1})$ and $d(a'_p, a'_{p-1})$ are strictly positive (the reason being that time stamps are strictly increasing). Thus,

$\delta_{TWED}(A_1^{p-1}, B_1^{q-1}) = 0$ and $d(a'_p, b'_q) = 0$ leading to $A_1^{p-1} = B_1^{q-1}$ and $a'_p = b'_q$. Finally $p=q$ and

necessarily $A_1^p = B_1^q \square$.

P3: Symmetry

Proof: Since the distance d on the sample space $S \times T$ is symmetric, it is easy to show that

$\delta_{TWED}(A_1^p, B_1^q)$ is symmetric for all (A_1^p, B_1^q) in $U \times U$ by induction on $m=p+q$. \square

Lemma 4: For all A_1^p in U , $d(a'_{p-1}, a'_p) + \lambda \geq \delta_{TWED}(A_1^{p-1}, A_1^p)$

Proof: For all A_1^p in U ,

$$\begin{aligned} d(a'_{p-1}, a'_p) + \lambda &= 0 + d(a'_{p-1}, a'_p) + \lambda = \delta_{TWED}(A_1^{p-1}, A_1^{p-1}) + d(a'_{p-1}, a'_p) + \lambda \\ &= \delta_{TWED}(A_1^{p-1}, A_1^{p-1}) + \Gamma(a'_p \rightarrow \Lambda) \geq \delta_{TWED}(A_1^p, A_1^{p-1}) \square \end{aligned}$$

(P4): Triangle inequality

For all (A_1^p, B_1^q, C_1^r) in $U \times U \times U$, $\delta_{TWED}(A_1^p, C_1^r) \leq \delta_{TWED}(A_1^p, B_1^q) + \delta_{TWED}(B_1^q, C_1^r)$.

Proof: We will prove $P4$ by induction on $m=p+q+r$.

$P4$ is true for $m=0$ since $\delta_{TWED}(\Omega, \Omega) = 0 \leq \delta_{TWED}(\Omega, \Omega) + \delta_{TWED}(\Omega, \Omega)$ and the induction hypothesis holds.

(H4): Suppose $P4$ is true for all $m \in \{0, \dots, n-1\}$ for some $n > 0$. Let

$\Sigma = \delta_{TWED}(A_1^p, B_1^q) + \delta_{TWED}(B_1^q, C_1^r)$. Then for all (A_1^p, B_1^q, C_1^r) in $U \times U \times U$ such that $m=n$, we have basically 9 different cases to explore for the decomposition of $\delta_{TWED}(A_1^p, B_1^q)$ and

$\delta_{TWED}(B_1^q, C_1^r)$:

$$1^{\text{st}} \text{ Case: if } \begin{cases} \delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^{p-1}, B_1^{q-1}) + \Gamma(a'_p \rightarrow b'_q) \\ \delta_{TWED}(B_1^q, C_1^r) = \delta_{TWED}(B_1^{q-1}, C_1^{r-1}) + \Gamma(b'_q \rightarrow c'_r) \end{cases}, \text{ then:}$$

$$\begin{aligned} \Sigma &\geq \delta_{TWED}(A_1^{p-1}, B_1^{q-1}) + \delta_{TWED}(B_1^{q-1}, C_1^{r-1}) + d(a'_p, b'_q) + d(b'_q, c'_r) \\ &\geq \delta_{TWED}(A_1^{p-1}, C_1^{r-1}) + d(a'_p, c'_r) \stackrel{\text{def}}{\geq} \delta_{TWED}(A_1^p, C_1^r) \text{ since (H4) applies.} \end{aligned}$$

$$2^{\text{nd}} \text{ Case: if } \begin{cases} \delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^p, B_1^{q-1}) + \Gamma(\Lambda \rightarrow b'_q) \\ \delta_{TWED}(B_1^q, C_1^r) = \delta_{TWED}(B_1^{q-1}, C_1^{r-1}) + \Gamma(b'_q \rightarrow c'_r) \end{cases}, \text{ then:}$$

$$\Sigma = \delta_{TWED}(A_1^p, B_1^{q-1}) + d(b'_q, b'_{q-1}) + \lambda + \delta_{TWED}(B_1^q, C_1^r)$$

since (lemma 4) $d(b'_q, b'_{q-1}) + \lambda \geq \delta_{TWED}(B_1^{q-1}, B_1^q)$ we have

$$\begin{aligned} \Sigma &\geq \delta_{TWED}(A_1^p, B_1^{q-1}) + \delta_{TWED}(B_1^{q-1}, B_1^q) + \delta_{TWED}(B_1^q, C_1^r) \\ &\geq \delta_{TWED}(A_1^p, B_1^{q-1}) + \delta_{TWED}(B_1^{q-1}, C_1^r) \geq \delta_{TWED}(A_1^p, C_1^r) \text{ since (H4) applies twice.} \end{aligned}$$

$$3^{\text{rd}} \text{ Case: if } \begin{cases} \delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^{p-1}, B_1^q) + \Gamma(a'_p \rightarrow \Lambda) \\ \delta_{TWED}(B_1^q, C_1^r) = \delta_{TWED}(B_1^{q-1}, C_1^{r-1}) + \Gamma(b'_q \rightarrow c'_r) \end{cases}, \text{ then:}$$

$$\Sigma = \delta_{TWED}(A_1^{p-1}, B_1^q) + d(a'_p, a'_{p-1}) + \lambda + \delta_{TWED}(B_1^q, C_1^r)$$

$$\begin{aligned} \Sigma &\geq \delta_{TWED}(A_1^{p-1}, B_1^q) + \delta_{TWED}(A_1^p, A_1^{p-1}) + \delta_{TWED}(B_1^q, C_1^r) && \text{(lemma 1)} \\ &\geq \delta_{TWED}(A_1^p, A_1^{p-1}) + \delta_{TWED}(A_1^{p-1}, C_1^r) && \text{(H4) applies} \\ &\geq \delta_{TWED}(A_1^p, C_1^r) && \text{(H4) applies.} \end{aligned}$$

$$4^{\text{th}} \text{ Case: if } \begin{cases} \delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^p, B_1^{q-1}) + \Gamma(\Lambda \rightarrow b'_q) \\ \delta_{TWED}(B_1^q, C_1^r) = \delta_{TWED}(B_1^q, C_1^{r-1}) + \Gamma(\Lambda \rightarrow c'_r) \end{cases}, \text{ then:}$$

$$\Sigma = \delta_{TWED_0}(A_1^p, B_1^{q-1}) + d(b'_q, b'_{q-1}) + \lambda + \delta_{TWED}(B_1^q, C_1^{r-1}) + d(c'_r, c'_{r-1}) + \lambda$$

since (lemma 4) $d(b'_q, b'_{q-1}) + \lambda \geq \delta_{TWED}(B_1^{q-1}, B_1^q)$ and

$d(c'_r, c'_{r-1}) + \lambda \geq \delta_{TWED}(C_1^{r-1}, C_1^r)$ we have:

$$\begin{aligned} \Sigma &\geq \delta_{TWED}(A_1^p, B_1^{q-1}) + \delta_{TWED}(B_1^{q-1}, B_1^q) + \delta_{TWED}(B_1^q, C_1^{r-1}) + \delta_{TWED}(C_1^{r-1}, C_1^r) \\ &\geq \delta_{TWED}(A_1^p, B_1^{q-1}) + \delta_{TWED}(B_1^{q-1}, C_1^{r-1}) + \delta_{TWED}(C_1^{r-1}, C_1^r) && \text{(H4) applies} \\ &\geq \delta_{TWED}(A_1^p, B_1^{q-1}) + \delta_{TWED}(B_1^{q-1}, C_1^r) \geq \delta_{TWED}(A_1^p, C_1^r) && \text{(H4) applies twice.} \end{aligned}$$

$$5^{\text{th}} \text{ Case: if } \begin{cases} \delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^{p-1}, B_1^{q-1}) + \Gamma(a'_p \rightarrow b'_q) \\ \delta_{TWED}(B_1^q, C_1^r) = \delta_{TWED}(B_1^q, C_1^{r-1}) + \Gamma(\Lambda \rightarrow c'_r) \end{cases}, \text{ then:}$$

$$\begin{aligned} \Sigma &= \delta_{TWED}(A_1^p, B_1^q) + \delta_{TWED}(B_1^q, C_1^{r-1}) + d(c'_r, c'_{r-1}) + \lambda \\ &\geq \delta_{TWED}(A_1^p, B_1^q) + \delta_{TWED}(B_1^q, C_1^{r-1}) + \delta_{TWED}(C_1^{r-1}, C_1^r) \quad (\text{lemma1}) \\ &\geq \delta_{TWED}(A_1^p, C_1^{r-1}) + \delta_{TWED}(C_1^{r-1}, C_1^r) \geq \delta_{TWED}(A_1^p, C_1^r) \quad (H4) \text{ applies twice.} \end{aligned}$$

$$6^{\text{th}} \text{ Case: if } \begin{cases} \delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^{p-1}, B_1^q) + \Gamma(a'_p \rightarrow \Lambda) \\ \delta_{TWED}(B_1^q, C_1^r) = \delta_{TWED}(B_1^q, C_1^{r-1}) + \Gamma(\Lambda \rightarrow c'_r) \end{cases}, \text{ then:}$$

$$\begin{aligned} \Sigma &= \delta_{TWED}(A_1^p, B_1^q) + \delta_{TWED}(B_1^q, C_1^{r-1}) + d(c'_r, c'_{r-1}) + \lambda \\ &\geq \delta_{TWED}(A_1^p, B_1^q) + \delta_{TWED}(B_1^q, C_1^{r-1}) + \delta_{TWED}(C_1^{r-1}, C_1^r) \quad (\text{lemma1}) \\ &\geq \delta_{TWED}(A_1^p, C_1^{r-1}) + \delta_{TWED}(C_1^{r-1}, C_1^r) \geq \delta_{TWED}(A_1^p, C_1^r) \quad (H4) \text{ applies twice.} \end{aligned}$$

$$7^{\text{th}} \text{ Case: if } \begin{cases} \delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^{p-1}, B_1^{q-1}) + \Gamma(a'_p \rightarrow b'_q) \\ \delta_{TWED}(B_1^q, C_1^r) = \delta_{TWED}(B_1^{q-1}, C_1^r) + \Gamma(b'_q \rightarrow \Lambda) \end{cases}, \text{ then:}$$

$$\begin{aligned} \Sigma &= \delta_{TWED}(A_1^p, B_1^q) + \delta_{TWED}(B_1^{q-1}, C_1^r) + d(b'_q, b'_{q-1}) + \lambda \\ &\geq \delta_{TWED}(A_1^p, B_1^q) + \delta_{TWED}(B_1^{q-1}, C_1^r) + \delta_{TWED}(B_1^q, B_1^{q-1}) \quad (\text{lemma1}) \\ &\geq \delta_{TWED}(A_1^p, B_1^{q-1}) + \delta_{TWED}(B_1^{q-1}, C_1^r) \geq \delta_{TWED}(A_1^p, C_1^r) \quad (H4) \text{ applies twice.} \end{aligned}$$

$$8^{\text{th}} \text{ Case: if } \begin{cases} \delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^{p-1}, B_1^q) + \Gamma(a'_p \rightarrow \Lambda) \\ \delta_{TWED}(B_1^q, C_1^r) = \delta_{TWED}(B_1^{q-1}, C_1^r) + \Gamma(b'_q \rightarrow \Lambda) \end{cases}, \text{ then:}$$

$$\begin{aligned}
 \Sigma &= \delta_{TWED}(A_1^{p-1}, B_1^q) + d(a'_p, a'_{p-1}) + \lambda + \delta_{TWED}(B_1^q, C_1^r) \\
 &\geq \delta_{TWED}(A_1^{p-1}, B_1^q) + \delta_{TWED}(A_1^p, A_1^{p-1}) + \delta_{TWED}(B_1^q, C_1^r) \quad (\text{lemma 1}) \\
 &\geq \delta_{TWED}(A_1^p, A_1^{p-1}) + \delta_{TWED}(A_1^{p-1}, C_1^r) \geq \delta_{TWED}(A_1^p, C_1^r) \quad (H4) \text{ applies twice.}
 \end{aligned}$$

9th Case: if $\begin{cases} \delta_{TWED}(A_1^p, B_1^q) = \delta_{TWED}(A_1^p, B_1^{q-1}) + \Gamma(\Lambda \rightarrow b'_q) \\ \delta_{TWED}(B_1^q, C_1^r) = \delta_{TWED}(B_1^{q-1}, C_1^r) + \Gamma(b'_q \rightarrow \Lambda) \end{cases}$, then:

$$\begin{aligned}
 \Sigma &\geq \delta_{TWED}(A_1^p, B_1^{q-1}) + \delta_{TWED}(B_1^{q-1}, C_1^r) \\
 &\geq \delta_{TWED}(A_1^p, C_1^r) \quad (H4) \text{ applies.}
 \end{aligned}$$

So property *P4* holds for all m in $\{0, \dots, n\}$. By induction *P4* holds for all m in N and so *P4* holds

for all (A_1^p, B_1^q, C_1^r) in $U \times U \times U$. \square

APPENDIX 2

THE PWM DATASETS

1. Motivation

The artificial ‘Pulse Width Modulation’ (PWM) datasets have been defined to demonstrate a weakness in dynamic time warping (DTW) pseudo distance.

Basically, DTW does not penalize the matching of similar events having different time durations. For instance, the DTW similarity between any two time series chosen among the set given in Figure 3 below is the constant 0.

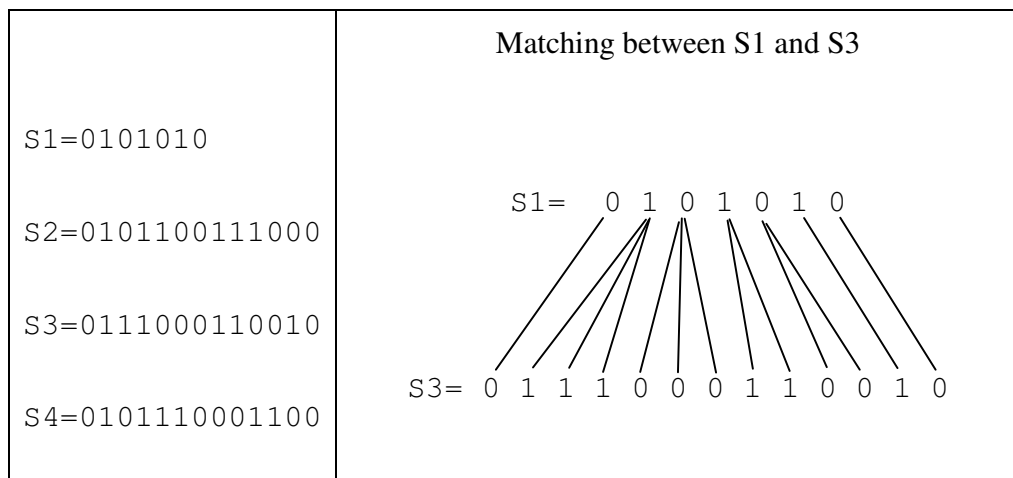


Fig. 1: Matching example between S1 and S3 time series

2. PWM Datasets

To demonstrate that this could affect recognition or classification of time series, the PWM datasets (PWM1 and PWM2) are defined having in mind a classification experiment with the same kind of conditions as those defined by E. Keogh & al. at UCR [13].

2.1 PWM1 dataset

The PWM1 dataset consists of artificial time series belonging to three categories. The considered task consists in classifying an input time series as one of three possible classes, '123' (a), '321' (b) or '132' (c). These classes are built as a sequence of three pulses having the same height but different widths: pulse '1' has the shortest width, while pulse '3' has the larger one and pulse '2' is in between.

To make it a little bit tougher, following Cylinder Bell Funnel artificial time series definition [26], random noise at four levels is added:

- a centered Gaussian noise with a standard deviation equal to 10% of the amplitude of each pulse is added,
- The onset time for each spike is selected uniformly inside a finite interval,
- The width of each spike is selected uniformly inside a finite interval that depends on its category ('1', '2' or '3').
- Time delays between two successive spikes are chosen uniformly inside a finite interval.

2.1.1 Time series definitions

Time series are generated according to their class as follows:

Class 123:

$$a(t) = \varepsilon(t) + \mathcal{X}_{[s_1, w_1]}(t) + \mathcal{X}_{[s_2, w_2]}(t) + \mathcal{X}_{[s_3, w_3]}(t)$$

Class 321:

$$a(t) = \varepsilon(t) + \chi_{[s_1, w_3]}(t) + \chi_{[s_2, w_2]}(t) + \chi_{[s_3, w_1]}(t)$$

Class 132:

$$a(t) = \varepsilon(t) + \chi_{[s_1, w_1]}(t) + \chi_{[s_2, w_3]}(t) + \chi_{[s_3, w_2]}(t)$$

where: $\varepsilon(t)$ is drawn from a standard normal distribution $N(0, 1/10)$,

w_1 is an integer drawn uniformly from $[4, 8]$,

w_2 is an integer drawn uniformly from $[10, 16]$,

w_3 is an integer drawn uniformly from $[20, 28]$,

s_1, s'_1, s''_1 are integers drawn uniformly from $[2, 18]$,

s_2 is an integer drawn uniformly from $[s_1 + w_1, s_1 + s'_1 + w_1]$,

s_3 is an integer drawn uniformly from $[s_2 + w_2, s_2 + s''_1 + w_2]$,

s'_2 is an integer drawn uniformly from $[s_1 + w_3, s_1 + s'_1 + w_3]$,

s'_3 is an integer drawn uniformly from $[s'_2 + s''_1 + w_2, s'_2 + s''_1 + w_2]$,

s''_3 is an integer drawn uniformly from $[s_2 + w_3, s_2 + s''_1 + w_3]$,

$$\chi_{[\alpha, \beta]}(t) = \begin{cases} 0 & t < \alpha \\ 1 & \alpha \leq t \leq \beta \\ 0 & t > \beta \end{cases}$$

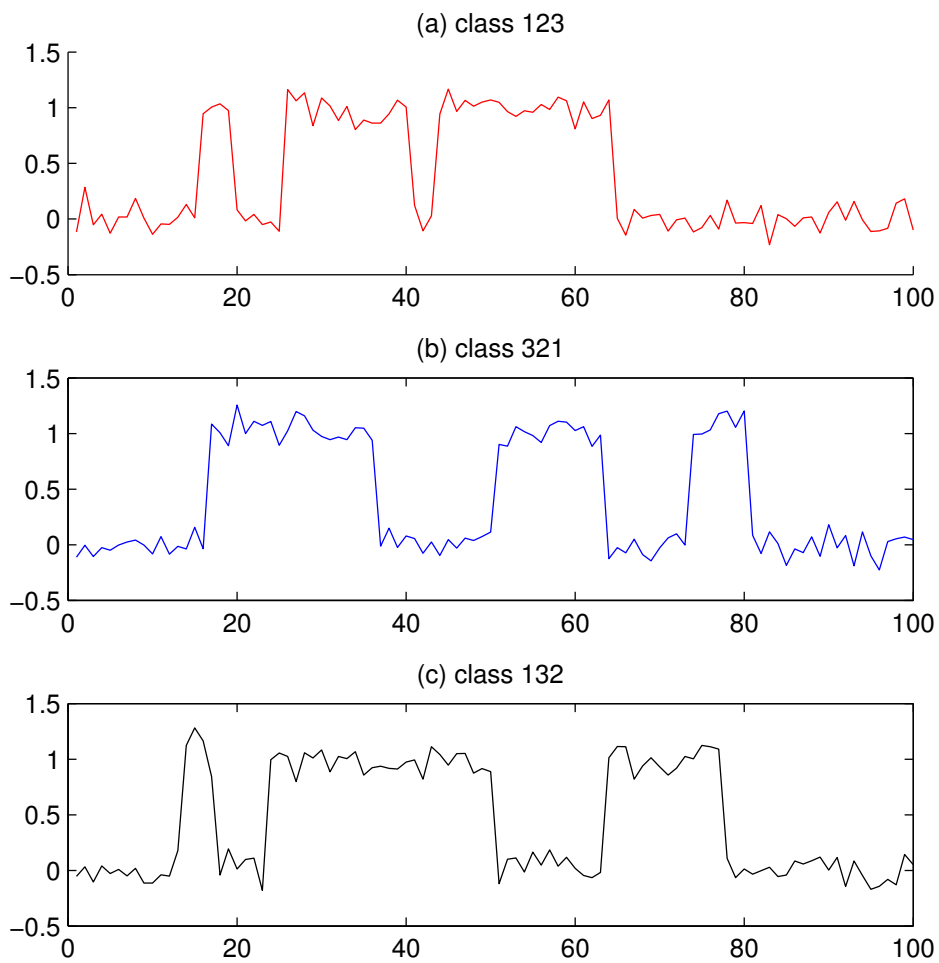


Figure 2: utterances for classes (a) 123 red, (b) 321 blue, (c) 132 black for the PWM1 dataset

2.2 The PWM2 dataset

The PWM2 dataset consists of artificial time series belonging to three categories. The considered task consists in classifying an input time series as one of three possible classes, ‘123’ (a), ‘321’ (b) or ‘132’ (c). These classes are built as a sequence of three subsequences of three pulses

having the same height but different widths: subsequence '1' is composed of the shortest width pulses, while subsequence '3' is composed of the largest width pulses and subsequence '2' is composed of in between width pulses.

To make it a little bit tougher, following Cylinder Bell Funnel artificial time series definition [26], random noise at four levels is added:

- a centered Gaussian noise with a standard deviation equal to 10% of the amplitude of each pulse is added,
- The onset time for each pulse is selected uniformly inside a finite interval,
- The width of each pulse is selected uniformly inside a finite interval that depends on its category ('1', '2' or '3').
- Time delays between two successive pulses are chosen uniformly inside finite intervals that depend on the subsequence class.

2.1.1 Time series definitions

Time series are generated according to their class. Each time series contains three subsequences. Each subsequence has a fixed length of 25 samples. A subsequence is defined using three parameters: the starting time stamps t_0 , the pulse width w and the pulse amplitude A . The subsequence $sq_{[t_0, w, A]}(t)$ is defined as follows:

$$sq_{[t_0, w, A]}(t) = \begin{cases} 0, & t < t_0 \\ A \cdot (\chi_{[t_0, w]} + \chi_{[t_1, w]} + \chi_{[t_2, w]}), & t_0 \leq t \leq t_0 + 25 \\ 0, & t > t_0 + 25 \end{cases}$$

where:

t_1 is an integer drawn uniformly from $[t_0 + w + 1, t_0 + w + 25 - 3 \cdot w - 1]$

t_2 is an integer drawn uniformly from $[t_1 + w + 1, t_1 + w + 25 - 2 \cdot w]$

$$\text{and } \chi_{[\alpha, \beta]}(t) = \begin{cases} 0 & t < \alpha \\ 1 & \alpha \leq t \leq \beta \\ 0 & t > \beta \end{cases}$$

Finally the three classes are defined as follows:

Class 123:

$$a(t) = -1 + \varepsilon(t) + sq_{[t_0, w_1, 2]} + sq_{[t_0 + 25, w_2, 2]} + sq_{[t_0 + 50, w_3, 2]}$$

Class 321:

$$a(t) = -1 + \varepsilon(t) + sq_{[t_0, w_3, 2]} + sq_{[t_0 + 25, w_2, 2]} + sq_{[t_0 + 50, w_1, 2]}$$

Class 132:

$$a(t) = -1 + \varepsilon(t) + sq_{[t_0, w_1, 2]} + sq_{[t_0 + 25, w_3, 2]} + sq_{[t_0 + 50, w_2, 2]}$$

where: $\varepsilon(t)$ is drawn from a standard normal distribution $N(0, 1/10)$,

t_0 is an integer drawn uniformly from $[2, 32]$,

w_1 is an integer drawn uniformly from $[2, 3]$,

w_2 is an integer drawn uniformly from $[4, 5]$,

w_3 is an integer drawn uniformly from $[6, 7]$

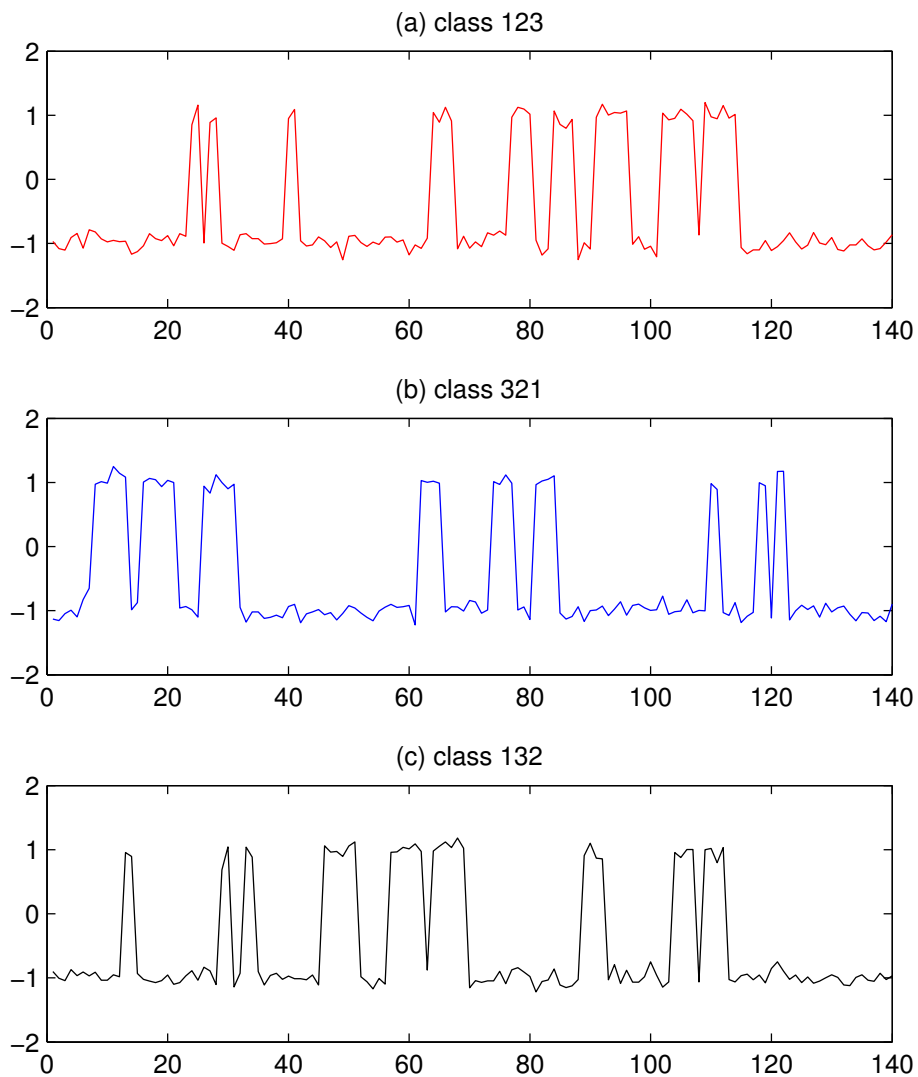


Figure 3: utterances for classes (a) 123 red, (b) 321 blue, (c) 132 black for the PWM2 dataset

3. The files

The 3 classes are equi-likely inside the training and testing datasets for PWM1 and PWM2:

TEST_PWM1 contains 600 time series, 200 per classes

TRAIN_PWM1 contains 60 time series, 20 per classes.

TEST_PWM2 contains 300 time series, 100 per classes

TRAIN_PWM2 contains 30 time series, 10 per classes.

File Format

TEST_PWM1, TEST_PWM2, TRAIN_PWM1 and TRAIN_PWM2 are ASCII files in which each line begins with a label (a number) that characterizes the class (1 for class 123, 2 for class 321 and 3 for class 132; the time series (1D data) samples are then coded in the rest of the line as a sequence of floating point number in ASCII format.

PWM1 and PWM2 datasets can be downloaded at the following URL:

<http://www-valoria.univ-ubs.fr/Pierre-Francois.Marteau/PWM>