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New entropy estimates for the Oldroyd-B model, and related models

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Abstract

This short note presents the derivation of a new *a priori* estimate for the Oldroyd-B model. Such an estimate may provide useful information when investigating the long-time behaviour of macro-macro models, and the stability of numerical schemes. We show how this estimate can be used as a guideline to derive new estimates for other macroscopic models, like the FENE-P model.

1 Introduction

We consider the Oldroyd-B model:

$$\text{Re} \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = (1 - \varepsilon) \Delta \mathbf{u} - \nabla p + \text{div } \boldsymbol{\tau}, \quad (1)$$

$$\text{div } (\mathbf{u}) = 0, \quad (2)$$

$$\frac{\partial \boldsymbol{\tau}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\tau} = \nabla \mathbf{u} \boldsymbol{\tau} + \boldsymbol{\tau} (\nabla \mathbf{u})^T - \frac{1}{\text{We}} \boldsymbol{\tau} + \frac{\varepsilon}{\text{We}} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \quad (3)$$

where the Reynolds number $\text{Re} > 0$, the Weissenberg number $\text{We} > 0$ and $\varepsilon \in (0, 1)$ are some non-dimensional numbers. We suppose that the space variable \mathbf{x} lives in a bounded domain \mathcal{D} of \mathbb{R}^d . This system is supplied with initial conditions on the velocity \mathbf{u} and on the stress tensor $\boldsymbol{\tau}$. For simplicity, we assume no-slip boundary conditions on the velocity \mathbf{u} :

$$\mathbf{u} = 0 \text{ on } \partial \mathcal{D}. \quad (4)$$

We *suppose* that the initial data and the geometry are such that there exists a unique regular solution to (1)–(3) and our aim is to derive some *a priori* estimates on this solution.

Let us introduce the so-called conformation tensor $\mathbf{A} = \frac{\text{We}}{\varepsilon} \boldsymbol{\tau} + \text{Id}$. The partial differential equation (PDE) on $\boldsymbol{\tau}$ translates into the following PDE on \mathbf{A} :

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{A} = \nabla \mathbf{u} \mathbf{A} + \mathbf{A} (\nabla \mathbf{u})^T - \frac{1}{\text{We}} \mathbf{A} + \frac{1}{\text{We}} \text{Id}. \quad (5)$$

One can check that if

$$\mathbf{A}(t=0) = \frac{\text{We}}{\varepsilon} \boldsymbol{\tau}(t=0) + \text{Id} \text{ is a positive definite symmetric matrix,} \quad (6)$$

then this property is propagated forward in time by (5) (and, in particular, $\boldsymbol{\tau}$ is symmetric). Assuming uniqueness of solution, this can be proven for example by

using the probabilistic interpretation of \mathbf{A} as a covariance matrix, as explained in Section 3. We will assume throughout this note that (6) is satisfied. Concerning the importance of positive-definiteness of \mathbf{A} , we refer for example to [7, Section 9.8.10] and also to the recent work [3, 4].

In Section 2, we recall how the classical *a priori* estimate for the Oldroyd-B model is derived. Next we show how it can be used to derive some bounds on the stress tensor, provided the initial condition satisfies $\det \mathbf{A}(t = 0) > 1$. In Section 3, we establish a new estimate, which comes from an entropy estimate on the micro-macro model associated with the Oldroyd-B model (see [5]). This estimate provides bounds on $(\mathbf{u}, \boldsymbol{\tau})$ without any assumption on $\boldsymbol{\tau}(t = 0)$ (apart from (6)). This new estimate could be useful to study the longtime behaviour of some macro-macro models, or to analyze the stability of some numerical schemes. Current research is directed towards clarifying this.

2 The classical estimate

Let us first introduce the kinetic energy:

$$E(t) = \frac{1}{2} \int_{\mathcal{D}} |\mathbf{u}|^2. \quad (7)$$

We easily obtain from (1)–(2):

$$\operatorname{Re} \frac{dE}{dt} = -(1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 - \int_{\mathcal{D}} \boldsymbol{\tau} : \nabla \mathbf{u}, \quad (8)$$

where for two matrices A and B , we denote $A : B = A_{i,j} B_{i,j} = \operatorname{tr}(A^T B)$. On the other hand, taking the trace of the PDE (3) on $\boldsymbol{\tau}$ and integrating over \mathcal{D} , we get:

$$\frac{d}{dt} \int_{\mathcal{D}} \operatorname{tr} \boldsymbol{\tau} = 2 \int_{\mathcal{D}} \nabla \mathbf{u} : \boldsymbol{\tau} - \frac{1}{\operatorname{We}} \int_{\mathcal{D}} \operatorname{tr} \boldsymbol{\tau}.$$

We thus obtain the following estimate:

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\operatorname{Re}}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 + \frac{1}{2} \int_{\mathcal{D}} \operatorname{tr} \boldsymbol{\tau} \right) \\ & + (1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 + \frac{1}{2\operatorname{We}} \int_{\mathcal{D}} \operatorname{tr} \boldsymbol{\tau} = 0. \end{aligned} \quad (9)$$

Remark 1 *In terms of \mathbf{A} , the energy estimate (9) writes:*

$$\begin{aligned} & \frac{d}{dt} \left(\frac{\operatorname{Re}}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 + \frac{\varepsilon}{2\operatorname{We}} \int_{\mathcal{D}} \operatorname{tr} \mathbf{A} \right) \\ & + (1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 + \frac{\varepsilon}{2\operatorname{We}^2} \int_{\mathcal{D}} \operatorname{tr} (\mathbf{A} - \operatorname{Id}) = 0. \end{aligned} \quad (10)$$

In Lemma 1 below, we prove that $\operatorname{tr} \boldsymbol{\tau}$ is positive if $\det \mathbf{A}(t = 0) > 1$. This result combined with the estimate (9) thus yields some *a priori* bounds on $(\mathbf{u}, \boldsymbol{\tau})$ provided $\det(\mathbf{A})(t = 0) > 1$. In particular, it shows that \mathbf{u} and $\boldsymbol{\tau}$ go exponentially fast to 0 in the long time limit, using (9) and the Poincaré inequality: $\int_{\mathcal{D}} |\mathbf{u}|^2 \leq C \int_{\mathcal{D}} |\nabla \mathbf{u}|^2$.

Lemma 1 *Let us assume that $\det \mathbf{A}(t = 0) > 1$. Then, we have $\forall t \geq 0$, $\det \mathbf{A}(t) > 1$ and this implies that $\operatorname{tr} \boldsymbol{\tau}(t) > 0$.*

Proof: Using (5) and the Jacobi identity (which states that for any invertible matrix M depending smoothly on a parameter t , $\frac{d}{dt} \ln \det M = \operatorname{tr} (M^{-1} \frac{dM}{dt})$), we have:

$$\frac{\partial \ln(\det \mathbf{A})}{\partial t} + \mathbf{u} \cdot \nabla \ln(\det \mathbf{A}) = \frac{1}{\operatorname{We}} \operatorname{tr} (\mathbf{A}^{-1} - \operatorname{Id}). \quad (11)$$

Since for any symmetric positive matrix M of size $d \times d$,

$$(\det M)^{1/d} \leq (1/d) \operatorname{tr} M, \quad (12)$$

we obtain

$$\frac{\partial \ln(\det \mathbf{A})}{\partial t} + \mathbf{u} \cdot \nabla \ln(\det \mathbf{A}) \geq \frac{d}{\operatorname{We}} \left((\det \mathbf{A})^{-1/d} - 1 \right),$$

which we can rewrite in terms of $y = (\det \mathbf{A})^{1/d}$:

$$\operatorname{We} \left(\frac{\partial y}{\partial t} + \mathbf{u} \cdot \nabla y \right) \geq (1 - y). \quad (13)$$

This shows that $y > 1$ if $y(t=0) > 1$, and thus that $\det \mathbf{A} > 1$ if $\det \mathbf{A}(t=0) > 1$.

Indeed, using the characteristic method (by integrating the vector field $\mathbf{u}(t, \mathbf{x})$), one can rewrite (13) as

$$\operatorname{We} \frac{Dy}{Dt} \geq (1 - y).$$

Now, if y does not remain greater than 1, consider the first time t_0 such that $y(t_0) = 1$. We have on the one hand $\frac{Dy}{Dt}(t_0) < 0$ and, on the other hand $(1 - y(t_0)) = 0$. We reach a contradiction.

We thus have $\det \mathbf{A} > 1$ and therefore, using again (12), $\operatorname{tr} \mathbf{A} > d$. Since $\boldsymbol{\tau} = \frac{\varepsilon}{\operatorname{We}}(\mathbf{A} - \operatorname{Id})$, this is equivalent to $\operatorname{tr} \boldsymbol{\tau} > 0$. \diamond

Remark 2 If $\det \mathbf{A}(t=0) < 1$ (which is the case if $\operatorname{tr} \boldsymbol{\tau}(t=0) < 0$), Equation (13) shows that $\det \mathbf{A}$ grows along the characteristics as long as $\det \mathbf{A} < 1$.

3 Entropy estimate

We now consider a micro-macro (or multiscale) formulation of the Oldroyd-B model and some estimates based on entropy, inspired from [5].

3.1 General derivation of the entropy estimate for micro-macro models

We consider the following system:

$$\begin{cases} \operatorname{Re} \left(\frac{\partial \mathbf{u}}{\partial t}(t, \mathbf{x}) + \mathbf{u}(t, \mathbf{x}) \cdot \nabla \mathbf{u}(t, \mathbf{x}) \right) = (1 - \varepsilon) \Delta \mathbf{u}(t, \mathbf{x}) - \nabla p(t, \mathbf{x}) + \operatorname{div} \boldsymbol{\tau}(t, \mathbf{x}), \\ \operatorname{div} (\mathbf{u}(t, \mathbf{x})) = 0, \\ \boldsymbol{\tau}(t, \mathbf{x}) = \frac{\varepsilon}{\operatorname{We}} \left(\int_{\mathbb{R}^d} (\mathbf{X} \otimes \nabla \Pi(\mathbf{X})) \psi(t, \mathbf{x}, \mathbf{X}) d\mathbf{X} - \operatorname{Id} \right), \\ \frac{\partial \psi}{\partial t}(t, \mathbf{x}, \mathbf{X}) + \mathbf{u}(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} \psi(t, \mathbf{x}, \mathbf{X}) \\ = -\operatorname{div}_{\mathbf{X}} \left(\left(\nabla_{\mathbf{x}} \mathbf{u}(t, \mathbf{x}) \mathbf{X} - \frac{1}{2\operatorname{We}} \nabla \Pi(\mathbf{X}) \right) \psi(t, \mathbf{x}, \mathbf{X}) \right) + \frac{1}{2\operatorname{We}} \Delta_{\mathbf{X}} \psi(t, \mathbf{x}, \mathbf{X}). \end{cases} \quad (14)$$

This system is supplied with initial conditions on the velocity \mathbf{u} and on the distribution ψ . We recall that we suppose no-slip boundary conditions (4) on the velocity \mathbf{u} . This system corresponds to a micro-macro model of polymeric fluids, the polymer being modelled by two beads linked by a spring with potential energy Π . The configurational variable $\mathbf{X} \in \mathbb{R}^d$ models the end-to-end vector of the polymer. For more details on the modelling, we refer to [1, 8].

Notice that we could rewrite the former system as a system coupling a PDE and a stochastic differential equation (SDE), replacing the last two equations by:

$$\boldsymbol{\tau}(t, \mathbf{x}) = \frac{\varepsilon}{\operatorname{We}} \left(\mathbb{E} (\mathbf{X}_t(\mathbf{x}) \otimes \nabla \Pi(\mathbf{X}_t(\mathbf{x}))) - \operatorname{Id} \right), \quad (15)$$

$$\begin{aligned} d\mathbf{X}_t(\mathbf{x}) + \mathbf{u}(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} \mathbf{X}_t(\mathbf{x}) dt \\ = \left(\nabla_{\mathbf{x}} \mathbf{u}(t, \mathbf{x}) \mathbf{X}_t(\mathbf{x}) - \frac{1}{2\operatorname{We}} \nabla \Pi(\mathbf{X}_t(\mathbf{x})) \right) dt + \frac{1}{\sqrt{\operatorname{We}}} d\mathbf{W}_t. \end{aligned} \quad (16)$$

There, \mathbb{E} denotes the expectation, \mathbf{W}_t denotes a d -dimensional standard Brownian motion independent from the initial condition $(\mathbf{X}_0(\mathbf{x}))_{\mathbf{x} \in \mathcal{D}}$ which is such that, $\forall \mathbf{x} \in \mathcal{D}$, the law of $\mathbf{X}_0(\mathbf{x})$ is $\psi(0, \mathbf{x}, \mathbf{X}) d\mathbf{X}$.

Let us introduce the kinetic energy:

$$E(t) = \frac{1}{2} \int_{\mathcal{D}} |\mathbf{u}|^2. \quad (17)$$

We easily obtain:

$$\text{Re} \frac{dE}{dt} = -(1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 - \frac{\varepsilon}{\text{We}} \int_{\mathcal{D}} \int_{\mathbf{R}^d} (\mathbf{X} \otimes \nabla \Pi(\mathbf{X})) : \nabla \mathbf{u} \psi. \quad (18)$$

We now introduce the entropy of the system, namely:

$$\begin{aligned} H(t) &= \int_{\mathcal{D}} \int_{\mathbf{R}^d} \psi(t, \mathbf{x}, \mathbf{X}) \ln \left(\frac{\psi(t, \mathbf{x}, \mathbf{X})}{\psi_{\infty}(\mathbf{X})} \right), \\ &= \int_{\mathcal{D}} \int_{\mathbf{R}^d} \Pi(\mathbf{X}) \psi(t, \mathbf{x}, \mathbf{X}) + \int_{\mathcal{D}} \int_{\mathbf{R}^d} \psi(t, \mathbf{x}, \mathbf{X}) \ln(\psi(t, \mathbf{x}, \mathbf{X})) + C, \end{aligned} \quad (19)$$

with

$$\psi_{\infty}(\mathbf{X}) = \frac{\exp(-\Pi(\mathbf{X}))}{\int_{\mathbf{R}^d} \exp(-\Pi(\mathbf{X}))}, \quad (20)$$

and $C = \ln(\int_{\mathbf{R}^d} \exp(-\Pi(\mathbf{X})) d\mathbf{X})$. The function H is actually the relative entropy of ψ with respect to the equilibrium distribution ψ_{∞} .

After some computations (see [5]), we obtain:

$$\frac{dH}{dt} = -\frac{1}{2\text{We}} \int_{\mathcal{D}} \int_{\mathbf{R}^d} \psi \left| \nabla \ln \left(\frac{\psi}{\psi_{\infty}} \right) \right|^2 + \int_{\mathcal{D}} \int_{\mathbf{R}^d} (\mathbf{X} \otimes \nabla \Pi(\mathbf{X})) : \nabla \mathbf{u} \psi. \quad (21)$$

Therefore, introducing the free energy $F(t) = E(t) + \frac{\varepsilon}{\text{We}} H(t)$ of the system, we have:

$$\boxed{\begin{aligned} &\frac{d}{dt} \left(\frac{\text{Re}}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 + \frac{\varepsilon}{\text{We}} \int_{\mathcal{D}} \int_{\mathbf{R}^d} \psi \ln \left(\frac{\psi}{\psi_{\infty}} \right) \right) \\ &+ (1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 + \frac{\varepsilon}{2\text{We}^2} \int_{\mathcal{D}} \int_{\mathbf{R}^d} \psi \left| \nabla \ln \left(\frac{\psi}{\psi_{\infty}} \right) \right|^2 = 0. \end{aligned}} \quad (22)$$

Using a logarithmic Sobolev inequality with respect to ψ_{∞} and a Poincaré inequality for $\mathbf{u} \in H_0^1(\mathcal{D})$, one can then obtain exponential convergence to equilibrium $\lim_{t \rightarrow \infty} (\mathbf{u}, \psi) = (0, \psi_{\infty})$ (see [5]). For some generalizations to the case $\mathbf{u} \neq 0$ on $\partial\mathcal{D}$, we refer to [5].

3.2 The Oldroyd-B case

Let us consider the Hookean dumbbell model, for which the potential Π of the entropic force is:

$$\Pi(\mathbf{X}) = \frac{\|\mathbf{X}\|^2}{2}. \quad (23)$$

By Itô's calculus, it is easy to derive from (16) that $\mathbf{A} = \mathbb{E}(\mathbf{X}_t \otimes \mathbf{X}_t)$ satisfies the following PDE:

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{A} = \nabla \mathbf{u} \mathbf{A} + \mathbf{A} (\nabla \mathbf{u})^T - \frac{1}{\text{We}} \mathbf{A} + \frac{1}{\text{We}} \text{Id}. \quad (24)$$

This translates into the following PDE for $\boldsymbol{\tau} = \frac{\varepsilon}{\text{We}} (\mathbf{A} - \text{Id})$:

$$\frac{\partial \boldsymbol{\tau}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\tau} = \nabla \mathbf{u} \boldsymbol{\tau} + \boldsymbol{\tau} (\nabla \mathbf{u})^T - \frac{1}{\text{We}} \boldsymbol{\tau} + \frac{\varepsilon}{\text{We}} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T). \quad (25)$$

The Hookean dumbbell model is thus equivalent to the Oldroyd-B model (at least for regular enough solutions).

If $\psi(0, \mathbf{x}, \cdot)$ is Gaussian (with zero mean), so is $\psi(t, \mathbf{x}, \cdot)$:

$$\psi(t, \mathbf{x}, \mathbf{X}) = \frac{1}{(2\pi)^{d/2} \sqrt{\det(\mathbf{A})}} \exp\left(-\frac{\mathbf{X}^T \mathbf{A}^{-1} \mathbf{X}}{2}\right)$$

where $\mathbf{A} = \mathbb{E}(\mathbf{X}_t \otimes \mathbf{X}_t) = \int_{\mathbb{R}^d} \mathbf{X} \otimes \mathbf{X} \psi(t, \mathbf{x}, \mathbf{X}) d\mathbf{X}$ denotes as above the covariance matrix of \mathbf{X}_t , which depends on time and also on the space variable \mathbf{x} . The covariance matrix \mathbf{A} is symmetric and nonnegative. Moreover, since for almost all $t \geq 0$, $\int_{\mathcal{D}} \int_{\mathbb{R}^d} \psi(t, \mathbf{x}, \mathbf{X}) \ln\left(\frac{\psi(t, \mathbf{x}, \mathbf{X})}{\psi_\infty(\mathbf{X})}\right) < \infty$, then for almost all $t \geq 0$ and for almost all $\mathbf{x} \in \mathcal{D}$, \mathbf{A} is positive.

The following explicit expression of the relative entropy can then be derived:

$$\int_{\mathcal{D}} \int_{\mathbb{R}^d} \psi(t, \mathbf{x}, \mathbf{X}) \ln\left(\frac{\psi(t, \mathbf{x}, \mathbf{X})}{\psi_\infty(\mathbf{X})}\right) d\mathbf{X} = \int_{\mathcal{D}} \frac{1}{2} (-\ln(\det \mathbf{A}) - d + \text{tr} \mathbf{A}) .$$

On the other hand,

$$\int_{\mathcal{D}} \int_{\mathbb{R}^d} \psi(t, \mathbf{x}, \mathbf{X}) \left| \nabla_{\mathbf{X}} \ln\left(\frac{\psi(t, \mathbf{x}, \mathbf{X})}{\psi_\infty(\mathbf{X})}\right) \right|^2 d\mathbf{X} = \int_{\mathcal{D}} \text{tr}((\text{Id} - \mathbf{A}^{-1})^2 \mathbf{A}) .$$

Rewriting (22), we thus obtain the following estimate, in terms of \mathbf{A} :

$$\boxed{\begin{aligned} & \frac{d}{dt} \left(\frac{\text{Re}}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 + \frac{\varepsilon}{2\text{We}} \int_{\mathcal{D}} (-\ln(\det \mathbf{A}) - d + \text{tr} \mathbf{A}) \right) \\ & + (1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 + \frac{\varepsilon}{2\text{We}^2} \int_{\mathcal{D}} \text{tr}((\text{Id} - \mathbf{A}^{-1})^2 \mathbf{A}) = 0. \end{aligned}} \quad (26)$$

This is, in the specific case of Hookean dumbbells (that is Oldroyd-B model) the macroscopic version of (22).

Since $-\ln(\det \mathbf{A}) - d + \text{tr}(\mathbf{A}) \geq 0$, this energy estimate yields some *a priori* bounds on (\mathbf{u}, \mathbf{A}) , and thus on $(\mathbf{u}, \boldsymbol{\tau})$. In sharp contrast to the classical estimate (9), it provides bounds on $(\mathbf{u}, \boldsymbol{\tau})$ without any assumption on $\boldsymbol{\tau}(t=0)$ (apart from (6)). Using a Poincaré inequality and the fact¹ that, for any symmetric positive matrix M of size $d \times d$,

$$-\ln(\det M) - d + \text{tr} M \leq \text{tr}((\text{Id} - M^{-1})^2 M)$$

exponential convergence to equilibrium ($\lim_{t \rightarrow \infty} (\mathbf{u}, \mathbf{A}) = (0, \text{Id})$) can be obtained from (26).

Remark 3 Notice that (26) can be schematically obtained as (10) $- \frac{\varepsilon}{2\text{We}} \int_{\mathcal{D}} (11)$.

Remark 4 If $\psi(0, \mathbf{x}, \cdot)$ is not Gaussian, it is always possible to replace it by a Gaussian initial condition with the same mean and variance, so that the macroscopic quantities $(\mathbf{u}, p, \mathbf{A})$ would be the same for the two initial conditions.

3.3 Application to related macroscopic models

The energy estimate (26) can be used as a guideline to derive energy estimates for other macroscopic models, even though they cannot be recast as a microscopic model of the form (14).

¹which can be seen as the logarithmic Sobolev inequality for Gaussian random variables translated on their covariance matrices

Let us consider the example of the FENE-P model [9, 2], for which

$$\boldsymbol{\tau} = \frac{\varepsilon}{\text{We}} \left(\frac{\mathbf{A}}{1 - \text{tr}(\mathbf{A})/b} - \text{Id} \right), \quad (27)$$

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{A} = \nabla \mathbf{u} \mathbf{A} + \mathbf{A} (\nabla \mathbf{u})^T - \frac{1}{\text{We}} \frac{\mathbf{A}}{1 - \text{tr}(\mathbf{A})/b} + \frac{1}{\text{We}} \text{Id}. \quad (28)$$

For this model, we assume (6), and also that $\text{tr}(\mathbf{A})(t=0) < b$, and this property is propagated forward in time by (28) (see [6]).

Using the same ideas as for the Oldroyd-B model, we consider the “entropy” $H(t) = -\ln(\det \mathbf{A}) - b \ln(1 - \text{tr}(\mathbf{A})/b)$, and we compute its time-derivative:

$$\frac{d}{dt} \int_{\mathcal{D}} -b \ln(1 - \text{tr}(\mathbf{A})/b) = 2 \int_{\mathcal{D}} \frac{\nabla \mathbf{u} : \mathbf{A}}{1 - \text{tr}(\mathbf{A})/b} + \frac{1}{\text{We}} \int_{\mathcal{D}} \left(-\frac{\text{tr}(\mathbf{A})}{(1 - \text{tr}(\mathbf{A})/b)^2} + \frac{d}{1 - \text{tr}(\mathbf{A})/b} \right), \quad (29)$$

$$\frac{d}{dt} \int_{\mathcal{D}} \ln(\det(\mathbf{A})) = \frac{1}{\text{We}} \int_{\mathcal{D}} \left(-\frac{d}{1 - \text{tr}(\mathbf{A})/b} + \text{tr}(\mathbf{A}^{-1}) \right). \quad (30)$$

Combining these expressions with (8), we obtain

$$\boxed{\begin{aligned} & \frac{d}{dt} \left(\frac{\text{Re}}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 + \frac{\varepsilon}{2\text{We}} \int_{\mathcal{D}} (-\ln(\det \mathbf{A}) - b \ln(1 - \text{tr}(\mathbf{A})/b)) \right) \\ & + (1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 + \frac{\varepsilon}{2\text{We}^2} \int_{\mathcal{D}} \left(\frac{\text{tr}(\mathbf{A})}{(1 - \text{tr}(\mathbf{A})/b)^2} - \frac{2d}{1 - \text{tr}(\mathbf{A})/b} + \text{tr}(\mathbf{A}^{-1}) \right) = 0. \end{aligned}} \quad (31)$$

One can check that for any symmetric positive matrix M of size $d \times d$:

$$-\ln(\det(M)) - b \ln(1 - \text{tr}(M)/b) \geq -(b+d) \ln\left(\frac{b}{b+d}\right) \geq d \quad (32)$$

and that

$$-\ln(\det(M)) - b \ln(1 - \text{tr}(M)/b) + (b+d) \ln\left(\frac{b}{b+d}\right) \quad (33)$$

$$\leq \left(\frac{\text{tr}(M)}{(1 - \text{tr}(M)/b)^2} - \frac{2d}{1 - \text{tr}(M)/b} + \text{tr}(M^{-1}) \right). \quad (34)$$

The proof of these inequalities is tedious and can be done by diagonalizing the matrix M .

Equation (32) shows that

$$\frac{\text{Re}}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 + \frac{\varepsilon}{2\text{We}} \int_{\mathcal{D}} \left(-\ln(\det \mathbf{A}) - b \ln(1 - \text{tr}(\mathbf{A})/b) + (b+d) \ln\left(\frac{b}{b+d}\right) \right)$$

is a non-negative quantity, and thus that (31) indeed yields some *a priori* bounds on (\mathbf{u}, \mathbf{A}) .

Equation (34) (which plays the role of the log-Sobolev inequality in the micro-macro models) shows that the estimate (31) can be used to prove exponential convergence to equilibrium.

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