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► **To cite this version:**

Dan Hu, Tony Lelièvre. New entropy estimates for the Oldroyd-B model, and related models. 2007.  
hal-00135377

**HAL Id: hal-00135377**

**<https://hal.science/hal-00135377v1>**

Preprint submitted on 7 Mar 2007

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# New entropy estimates for the Oldroyd-B model, and related models

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7th March 2007

## Abstract

This short note presents the derivation of a new *a priori* estimate for the Oldroyd-B model. Such an estimate may provide useful information when investigating the long-time behaviour of macro-macro models, and the stability of numerical schemes. We show how this estimate can be used as a guideline to derive new estimates for other macroscopic models, like the FENE-P model.

## 1 Introduction

We consider the Oldroyd-B model:

$$\text{Re} \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = (1 - \varepsilon) \Delta \mathbf{u} - \nabla p + \text{div } \boldsymbol{\tau}, \quad (1)$$

$$\text{div } (\mathbf{u}) = 0, \quad (2)$$

$$\frac{\partial \boldsymbol{\tau}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\tau} = \nabla \mathbf{u} \boldsymbol{\tau} + \boldsymbol{\tau} (\nabla \mathbf{u})^T - \frac{1}{\text{We}} \boldsymbol{\tau} + \frac{\varepsilon}{\text{We}} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \quad (3)$$

where the Reynolds number  $\text{Re} > 0$ , the Weissenberg number  $\text{We} > 0$  and  $\varepsilon \in (0, 1)$  are some non-dimensional numbers. We suppose that the space variable  $\mathbf{x}$  lives in a bounded domain  $\mathcal{D}$  of  $\mathbb{R}^d$ . This system is supplied with initial conditions on the velocity  $\mathbf{u}$  and on the stress tensor  $\boldsymbol{\tau}$ . For simplicity, we assume no-slip boundary conditions on the velocity  $\mathbf{u}$ :

$$\mathbf{u} = 0 \text{ on } \partial \mathcal{D}. \quad (4)$$

We suppose that the initial data and the geometry are such that there exists a unique regular solution to (1)–(3) and our aim is to derive some *a priori* estimates on this solution.

Let us introduce the so-called conformation tensor  $\mathbf{A} = \frac{\text{We}}{\varepsilon} \boldsymbol{\tau} + \text{Id}$ . The partial differential equation (PDE) on  $\boldsymbol{\tau}$  translates into the following PDE on  $\mathbf{A}$ :

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{A} = \nabla \mathbf{u} \mathbf{A} + \mathbf{A} (\nabla \mathbf{u})^T - \frac{1}{\text{We}} \mathbf{A} + \frac{1}{\text{We}} \text{Id}. \quad (5)$$

One can check that if

$$\mathbf{A}(t=0) = \frac{\text{We}}{\varepsilon} \boldsymbol{\tau}(t=0) + \text{Id} \text{ is a positive definite symmetric matrix,} \quad (6)$$

then this property is propagated forward in time by (5) (and, in particular,  $\boldsymbol{\tau}$  is symmetric). Assuming uniqueness of solution, this can be proven for example by

using the probabilistic interpretation of  $\mathbf{A}$  as a covariance matrix, as explained in Section 3. We will assume throughout this note that (6) is satisfied. Concerning the importance of positive-definiteness of  $\mathbf{A}$ , we refer for example to [7, Section 9.8.10] and also to the recent work [3, 4].

In Section 2, we recall how the classical *a priori* estimate for the Oldroyd-B model is derived. Next we show how it can be used to derive some bounds on the stress tensor, provided the initial condition satisfies  $\det \mathbf{A}(t = 0) > 1$ . In Section 3, we establish a new estimate, which comes from an entropy estimate on the micro-macro model associated with the Oldroyd-B model (see [5]). This estimate provides bounds on  $(\mathbf{u}, \boldsymbol{\tau})$  without any assumption on  $\boldsymbol{\tau}(t = 0)$  (apart from (6)). This new estimate could be useful to study the longtime behaviour of some macro-macro models, or to analyze the stability of some numerical schemes. Current research is directed towards clarifying this.

## 2 The classical estimate

Let us first introduce the kinetic energy:

$$E(t) = \frac{1}{2} \int_{\mathcal{D}} |\mathbf{u}|^2. \quad (7)$$

We easily obtain from (1)–(2):

$$\operatorname{Re} \frac{dE}{dt} = -(1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 - \int_{\mathcal{D}} \boldsymbol{\tau} : \nabla \mathbf{u}, \quad (8)$$

where for two matrices  $A$  and  $B$ , we denote  $A : B = A_{i,j} B_{i,j} = \operatorname{tr}(A^T B)$ . On the other hand, taking the trace of the PDE (3) on  $\boldsymbol{\tau}$  and integrating over  $\mathcal{D}$ , we get:

$$\frac{d}{dt} \int_{\mathcal{D}} \operatorname{tr} \boldsymbol{\tau} = 2 \int_{\mathcal{D}} \nabla \mathbf{u} : \boldsymbol{\tau} - \frac{1}{\operatorname{We}} \int_{\mathcal{D}} \operatorname{tr} \boldsymbol{\tau}.$$

We thus obtain the following estimate:

$$\begin{aligned} & \frac{d}{dt} \left( \frac{\operatorname{Re}}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 + \frac{1}{2} \int_{\mathcal{D}} \operatorname{tr} \boldsymbol{\tau} \right) \\ & + (1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 + \frac{1}{2\operatorname{We}} \int_{\mathcal{D}} \operatorname{tr} \boldsymbol{\tau} = 0. \end{aligned} \quad (9)$$

**Remark 1** *In terms of  $\mathbf{A}$ , the energy estimate (9) writes:*

$$\begin{aligned} & \frac{d}{dt} \left( \frac{\operatorname{Re}}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 + \frac{\varepsilon}{2\operatorname{We}} \int_{\mathcal{D}} \operatorname{tr} \mathbf{A} \right) \\ & + (1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 + \frac{\varepsilon}{2\operatorname{We}^2} \int_{\mathcal{D}} \operatorname{tr} (\mathbf{A} - \operatorname{Id}) = 0. \end{aligned} \quad (10)$$

In Lemma 1 below, we prove that  $\operatorname{tr} \boldsymbol{\tau}$  is positive if  $\det \mathbf{A}(t = 0) > 1$ . This result combined with the estimate (9) thus yields some *a priori* bounds on  $(\mathbf{u}, \boldsymbol{\tau})$  provided  $\det(\mathbf{A})(t = 0) > 1$ . In particular, it shows that  $\mathbf{u}$  and  $\boldsymbol{\tau}$  go exponentially fast to 0 in the long time limit, using (9) and the Poincaré inequality:  $\int_{\mathcal{D}} |\mathbf{u}|^2 \leq C \int_{\mathcal{D}} |\nabla \mathbf{u}|^2$ .

**Lemma 1** *Let us assume that  $\det \mathbf{A}(t = 0) > 1$ . Then, we have  $\forall t \geq 0$ ,  $\det \mathbf{A}(t) > 1$  and this implies that  $\operatorname{tr} \boldsymbol{\tau}(t) > 0$ .*

*Proof:* Using (5) and the Jacobi identity (which states that for any invertible matrix  $M$  depending smoothly on a parameter  $t$ ,  $\frac{d}{dt} \ln \det M = \operatorname{tr} (M^{-1} \frac{dM}{dt})$ ), we have:

$$\frac{\partial \ln(\det \mathbf{A})}{\partial t} + \mathbf{u} \cdot \nabla \ln(\det \mathbf{A}) = \frac{1}{\operatorname{We}} \operatorname{tr} (\mathbf{A}^{-1} - \operatorname{Id}). \quad (11)$$

Since for any symmetric positive matrix  $M$  of size  $d \times d$ ,

$$(\det M)^{1/d} \leq (1/d)\text{tr } M, \quad (12)$$

we obtain

$$\frac{\partial \ln(\det \mathbf{A})}{\partial t} + \mathbf{u} \cdot \nabla \ln(\det \mathbf{A}) \geq \frac{d}{\text{We}} \left( (\det \mathbf{A})^{-1/d} - 1 \right),$$

which we can rewrite in terms of  $y = (\det \mathbf{A})^{1/d}$ :

$$\text{We} \left( \frac{\partial y}{\partial t} + \mathbf{u} \cdot \nabla y \right) \geq (1 - y). \quad (13)$$

This shows that  $y > 1$  if  $y(t=0) > 1$ , and thus that  $\det \mathbf{A} > 1$  if  $\det \mathbf{A}(t=0) > 1$ .

Indeed, using the characteristic method (by integrating the vector field  $\mathbf{u}(t, \mathbf{x})$ ), one can rewrite (13) as

$$\text{We} \frac{Dy}{Dt} \geq (1 - y).$$

Now, if  $y$  does not remain greater than 1, consider the first time  $t_0$  such that  $y(t_0) = 1$ . We have on the one hand  $\frac{Dy}{Dt}(t_0) < 0$  and, on the other hand  $(1 - y(t_0)) = 0$ . We reach a contradiction.

We thus have  $\det \mathbf{A} > 1$  and therefore, using again (12),  $\text{tr } \mathbf{A} > d$ . Since  $\boldsymbol{\tau} = \frac{\varepsilon}{\text{We}}(\mathbf{A} - \text{Id})$ , this is equivalent to  $\text{tr } \boldsymbol{\tau} > 0$ .  $\diamond$

**Remark 2** If  $\det \mathbf{A}(t=0) < 1$  (which is the case if  $\text{tr } \boldsymbol{\tau}(t=0) < 0$ ), Equation (13) shows that  $\det \mathbf{A}$  grows along the characteristics as long as  $\det \mathbf{A} < 1$ .

### 3 Entropy estimate

We now consider a micro-macro (or multiscale) formulation of the Oldroyd-B model and some estimates based on entropy, inspired from [5].

#### 3.1 General derivation of the entropy estimate for micro-macro models

We consider the following system:

$$\left\{ \begin{array}{l} \text{Re} \left( \frac{\partial \mathbf{u}}{\partial t}(t, \mathbf{x}) + \mathbf{u}(t, \mathbf{x}) \cdot \nabla \mathbf{u}(t, \mathbf{x}) \right) = (1 - \varepsilon) \Delta \mathbf{u}(t, \mathbf{x}) - \nabla p(t, \mathbf{x}) + \text{div } \boldsymbol{\tau}(t, \mathbf{x}), \\ \text{div}(\mathbf{u}(t, \mathbf{x})) = 0, \\ \boldsymbol{\tau}(t, \mathbf{x}) = \frac{\varepsilon}{\text{We}} \left( \int_{\mathbb{R}^d} (\mathbf{X} \otimes \nabla \Pi(\mathbf{X})) \psi(t, \mathbf{x}, \mathbf{X}) d\mathbf{X} - \text{Id} \right), \\ \frac{\partial \psi}{\partial t}(t, \mathbf{x}, \mathbf{X}) + \mathbf{u}(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} \psi(t, \mathbf{x}, \mathbf{X}) \\ = -\text{div}_{\mathbf{X}} \left( \left( \nabla_{\mathbf{x}} \mathbf{u}(t, \mathbf{x}) \mathbf{X} - \frac{1}{2\text{We}} \nabla \Pi(\mathbf{X}) \right) \psi(t, \mathbf{x}, \mathbf{X}) \right) + \frac{1}{2\text{We}} \Delta_{\mathbf{X}} \psi(t, \mathbf{x}, \mathbf{X}). \end{array} \right. \quad (14)$$

This system is supplied with initial conditions on the velocity  $\mathbf{u}$  and on the distribution  $\psi$ . We recall that we suppose no-slip boundary conditions (4) on the velocity  $\mathbf{u}$ . This system corresponds to a micro-macro model of polymeric fluids, the polymer being modelled by two beads linked by a spring with potential energy  $\Pi$ . The configurational variable  $\mathbf{X} \in \mathbb{R}^d$  models the end-to-end vector of the polymer. For more details on the modelling, we refer to [1, 8].

Notice that we could rewrite the former system as a system coupling a PDE and a stochastic differential equation (SDE), replacing the last two equations by:

$$\boldsymbol{\tau}(t, \mathbf{x}) = \frac{\varepsilon}{\text{We}} \left( \mathbb{E}(\mathbf{X}_t(\mathbf{x}) \otimes \nabla \Pi(\mathbf{X}_t(\mathbf{x}))) - \text{Id} \right), \quad (15)$$

$$\begin{aligned} d\mathbf{X}_t(\mathbf{x}) + \mathbf{u}(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} \mathbf{X}_t(\mathbf{x}) dt \\ = \left( \nabla_{\mathbf{x}} \mathbf{u}(t, \mathbf{x}) \mathbf{X}_t(\mathbf{x}) - \frac{1}{2\text{We}} \nabla \Pi(\mathbf{X}_t(\mathbf{x})) \right) dt + \frac{1}{\sqrt{\text{We}}} d\mathbf{W}_t. \end{aligned} \quad (16)$$

There,  $\mathbf{E}$  denotes the expectation,  $\mathbf{W}_t$  denotes a  $d$ -dimensional standard Brownian motion independent from the initial condition  $(\mathbf{X}_0(\mathbf{x}))_{\mathbf{x} \in \mathcal{D}}$  which is such that,  $\forall \mathbf{x} \in \mathcal{D}$ , the law of  $\mathbf{X}_0(\mathbf{x})$  is  $\psi(0, \mathbf{x}, \mathbf{X}) d\mathbf{X}$ .

Let us introduce the kinetic energy:

$$E(t) = \frac{1}{2} \int_{\mathcal{D}} |\mathbf{u}|^2. \quad (17)$$

We easily obtain:

$$\operatorname{Re} \frac{dE}{dt} = -(1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 - \frac{\varepsilon}{\operatorname{We}} \int_{\mathcal{D}} \int_{\mathbf{R}^d} (\mathbf{X} \otimes \nabla \Pi(\mathbf{X})) : \nabla \mathbf{u} \psi. \quad (18)$$

We now introduce the entropy of the system, namely:

$$\begin{aligned} H(t) &= \int_{\mathcal{D}} \int_{\mathbf{R}^d} \psi(t, \mathbf{x}, \mathbf{X}) \ln \left( \frac{\psi(t, \mathbf{x}, \mathbf{X})}{\psi_{\infty}(\mathbf{X})} \right), \\ &= \int_{\mathcal{D}} \int_{\mathbf{R}^d} \Pi(\mathbf{X}) \psi(t, \mathbf{x}, \mathbf{X}) + \int_{\mathcal{D}} \int_{\mathbf{R}^d} \psi(t, \mathbf{x}, \mathbf{X}) \ln(\psi(t, \mathbf{x}, \mathbf{X})) + C, \end{aligned} \quad (19)$$

with

$$\psi_{\infty}(\mathbf{X}) = \frac{\exp(-\Pi(\mathbf{X}))}{\int_{\mathbf{R}^d} \exp(-\Pi(\mathbf{X}))}, \quad (20)$$

and  $C = \ln(\int_{\mathbf{R}^d} \exp(-\Pi(\mathbf{X})) |D|)$ . The function  $H$  is actually the relative entropy of  $\psi$  with respect to the equilibrium distribution  $\psi_{\infty}$ .

After some computations (see [5]), we obtain:

$$\frac{dH}{dt} = -\frac{1}{2\operatorname{We}} \int_{\mathcal{D}} \int_{\mathbf{R}^d} \psi \left| \nabla \ln \left( \frac{\psi}{\psi_{\infty}} \right) \right|^2 + \int_{\mathcal{D}} \int_{\mathbf{R}^d} (\mathbf{X} \otimes \nabla \Pi(\mathbf{X})) : \nabla \mathbf{u} \psi. \quad (21)$$

Therefore, introducing the free energy  $F(t) = E(t) + \frac{\varepsilon}{\operatorname{We}} H(t)$  of the system, we have:

$$\boxed{\begin{aligned} \frac{d}{dt} \left( \frac{\operatorname{Re}}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 + \frac{\varepsilon}{\operatorname{We}} \int_{\mathcal{D}} \int_{\mathbf{R}^d} \psi \ln \left( \frac{\psi}{\psi_{\infty}} \right) \right) \\ + (1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 + \frac{\varepsilon}{2\operatorname{We}^2} \int_{\mathcal{D}} \int_{\mathbf{R}^d} \psi \left| \nabla \ln \left( \frac{\psi}{\psi_{\infty}} \right) \right|^2 = 0. \end{aligned}} \quad (22)$$

Using a logarithmic Sobolev inequality with respect to  $\psi_{\infty}$  and a Poincaré inequality for  $\mathbf{u} \in H_0^1(\mathcal{D})$ , one can then obtain exponential convergence to equilibrium  $\lim_{t \rightarrow \infty} (\mathbf{u}, \psi) = (0, \psi_{\infty})$  (see [5]). For some generalizations to the case  $\mathbf{u} \neq 0$  on  $\partial \mathcal{D}$ , we refer to [5].

### 3.2 The Oldroyd-B case

Let us consider the Hookean dumbbell model, for which the potential  $\Pi$  of the entropic force is:

$$\Pi(\mathbf{X}) = \frac{\|\mathbf{X}\|^2}{2}. \quad (23)$$

By Itô's calculus, it is easy to derive from (16) that  $\mathbf{A} = \mathbf{E}(\mathbf{X}_t \otimes \mathbf{X}_t)$  satisfies the following PDE:

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{A} = \nabla \mathbf{u} \mathbf{A} + \mathbf{A} (\nabla \mathbf{u})^T - \frac{1}{\operatorname{We}} \mathbf{A} + \frac{1}{\operatorname{We}} \operatorname{Id}. \quad (24)$$

This translates into the following PDE for  $\boldsymbol{\tau} = \frac{\varepsilon}{\operatorname{We}} (\mathbf{A} - \operatorname{Id})$ :

$$\frac{\partial \boldsymbol{\tau}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\tau} = \nabla \mathbf{u} \boldsymbol{\tau} + \boldsymbol{\tau} (\nabla \mathbf{u})^T - \frac{1}{\operatorname{We}} \boldsymbol{\tau} + \frac{\varepsilon}{\operatorname{We}} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T). \quad (25)$$

The Hookean dumbbell model is thus equivalent to the Oldroyd-B model (at least for regular enough solutions).

If  $\psi(0, \mathbf{x}, \cdot)$  is Gaussian (with zero mean), so is  $\psi(t, \mathbf{x}, \cdot)$ :

$$\psi(t, \mathbf{x}, \mathbf{X}) = \frac{1}{(2\pi)^{d/2} \sqrt{\det(\mathbf{A})}} \exp\left(-\frac{\mathbf{X}^T \mathbf{A}^{-1} \mathbf{X}}{2}\right)$$

where  $\mathbf{A} = \mathbb{E}(\mathbf{X}_t \otimes \mathbf{X}_t) = \int_{\mathbb{R}^d} \mathbf{X} \otimes \mathbf{X} \psi(t, \mathbf{x}, \mathbf{X}) d\mathbf{X}$  denotes as above the covariance matrix of  $\mathbf{X}_t$ , which depends on time and also on the space variable  $\mathbf{x}$ . The covariance matrix  $\mathbf{A}$  is symmetric and nonnegative. Moreover, since for almost all  $t \geq 0$ ,  $\int_{\mathcal{D}} \int_{\mathbb{R}^d} \psi(t, \mathbf{x}, \mathbf{X}) \ln\left(\frac{\psi(t, \mathbf{x}, \mathbf{X})}{\psi_\infty(\mathbf{X})}\right) < \infty$ , then for almost all  $t \geq 0$  and for almost all  $\mathbf{x} \in \mathcal{D}$ ,  $\mathbf{A}$  is positive.

The following explicit expression of the relative entropy can then be derived:

$$\int_{\mathcal{D}} \int_{\mathbb{R}^d} \psi(t, \mathbf{x}, \mathbf{X}) \ln\left(\frac{\psi(t, \mathbf{x}, \mathbf{X})}{\psi_\infty(\mathbf{X})}\right) d\mathbf{X} = \int_{\mathcal{D}} \frac{1}{2} (-\ln(\det \mathbf{A}) - d + \text{tr} \mathbf{A}).$$

On the other hand,

$$\int_{\mathcal{D}} \int_{\mathbb{R}^d} \psi(t, \mathbf{x}, \mathbf{X}) \left| \nabla_{\mathbf{X}} \ln\left(\frac{\psi(t, \mathbf{x}, \mathbf{X})}{\psi_\infty(\mathbf{X})}\right) \right|^2 d\mathbf{X} = \int_{\mathcal{D}} \text{tr}((\text{Id} - \mathbf{A}^{-1})^2 \mathbf{A}).$$

Rewriting (22), we thus obtain the following estimate, in terms of  $\mathbf{A}$ :

$$\boxed{\begin{aligned} & \frac{d}{dt} \left( \frac{\text{Re}}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 + \frac{\varepsilon}{2\text{We}} \int_{\mathcal{D}} (-\ln(\det \mathbf{A}) - d + \text{tr} \mathbf{A}) \right) \\ & + (1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 + \frac{\varepsilon}{2\text{We}^2} \int_{\mathcal{D}} \text{tr}((\text{Id} - \mathbf{A}^{-1})^2 \mathbf{A}) = 0. \end{aligned}} \quad (26)$$

This is, in the specific case of Hookean dumbbells (that is Oldroyd-B model) the macroscopic version of (22).

Since  $-\ln(\det \mathbf{A}) - d + \text{tr} \mathbf{A} \geq 0$ , this energy estimate yields some *a priori* bounds on  $(\mathbf{u}, \mathbf{A})$ , and thus on  $(\mathbf{u}, \boldsymbol{\tau})$ . In sharp contrast to the classical estimate (9), it provides bounds on  $(\mathbf{u}, \boldsymbol{\tau})$  without any assumption on  $\boldsymbol{\tau}(t=0)$  (apart from (6)). Using a Poincaré inequality and the fact<sup>1</sup> that, for any symmetric positive matrix  $M$  of size  $d \times d$ ,

$$-\ln(\det M) - d + \text{tr} M \leq \text{tr}((\text{Id} - M^{-1})^2 M)$$

exponential convergence to equilibrium ( $\lim_{t \rightarrow \infty} (\mathbf{u}, \mathbf{A}) = (0, \text{Id})$ ) can be obtained from (26).

**Remark 3** Notice that (26) can be schematically obtained as (10)– $\frac{\varepsilon}{2\text{We}} \int_{\mathcal{D}}$  (11).

**Remark 4** If  $\psi(0, \mathbf{x}, \cdot)$  is not Gaussian, it is always possible to replace it by a Gaussian initial condition with the same mean and variance, so that the macroscopic quantities  $(\mathbf{u}, p, \mathbf{A})$  would be the same for the two initial conditions.

### 3.3 Application to related macroscopic models

The energy estimate (26) can be used as a guideline to derive energy estimates for other macroscopic models, even though they cannot be recast as a microscopic model of the form (14).

<sup>1</sup>which can be seen as the logarithmic Sobolev inequality for Gaussian random variables translated on their covariance matrices

Let us consider the example of the FENE-P model [9, 2], for which

$$\boldsymbol{\tau} = \frac{\varepsilon}{\text{We}} \left( \frac{\mathbf{A}}{1 - \text{tr}(\mathbf{A})/b} - \text{Id} \right), \quad (27)$$

$$\frac{\partial \mathbf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{A} = \nabla \mathbf{u} \mathbf{A} + \mathbf{A} (\nabla \mathbf{u})^T - \frac{1}{\text{We}} \frac{\mathbf{A}}{1 - \text{tr}(\mathbf{A})/b} + \frac{1}{\text{We}} \text{Id}. \quad (28)$$

For this model, we assume (6), and also that  $\text{tr}(\mathbf{A})(t=0) < b$ , and this property is propagated forward in time by (28) (see [6]).

Using the same ideas as for the Oldroyd-B model, we consider the “entropy”  $H(t) = -\ln(\det \mathbf{A}) - b \ln(1 - \text{tr}(\mathbf{A})/b)$ , and we compute its time-derivative:

$$\frac{d}{dt} \int_{\mathcal{D}} -b \ln(1 - \text{tr}(\mathbf{A})/b) = 2 \int_{\mathcal{D}} \frac{\nabla \mathbf{u} : \mathbf{A}}{1 - \text{tr}(\mathbf{A})/b} + \frac{1}{\text{We}} \int_{\mathcal{D}} \left( -\frac{\text{tr}(\mathbf{A})}{(1 - \text{tr}(\mathbf{A})/b)^2} + \frac{d}{1 - \text{tr}(\mathbf{A})/b} \right), \quad (29)$$

$$\frac{d}{dt} \int_{\mathcal{D}} \ln(\det(\mathbf{A})) = \frac{1}{\text{We}} \int_{\mathcal{D}} \left( -\frac{d}{1 - \text{tr}(\mathbf{A})/b} + \text{tr}(\mathbf{A}^{-1}) \right). \quad (30)$$

Combining these expressions with (8), we obtain

$$\boxed{\begin{aligned} & \frac{d}{dt} \left( \frac{\text{Re}}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 + \frac{\varepsilon}{2\text{We}} \int_{\mathcal{D}} (-\ln(\det \mathbf{A}) - b \ln(1 - \text{tr}(\mathbf{A})/b)) \right) \\ & + (1 - \varepsilon) \int_{\mathcal{D}} |\nabla \mathbf{u}|^2 + \frac{\varepsilon}{2\text{We}^2} \int_{\mathcal{D}} \left( \frac{\text{tr}(\mathbf{A})}{(1 - \text{tr}(\mathbf{A})/b)^2} - \frac{2d}{1 - \text{tr}(\mathbf{A})/b} + \text{tr}(\mathbf{A}^{-1}) \right) = 0. \end{aligned}} \quad (31)$$

One can check that for any symmetric positive matrix  $M$  of size  $d \times d$ :

$$-\ln(\det(M)) - b \ln(1 - \text{tr}(M)/b) \geq -(b+d) \ln\left(\frac{b}{b+d}\right) \geq d \quad (32)$$

and that

$$-\ln(\det(M)) - b \ln(1 - \text{tr}(M)/b) + (b+d) \ln\left(\frac{b}{b+d}\right) \quad (33)$$

$$\leq \left( \frac{\text{tr}(M)}{(1 - \text{tr}(M)/b)^2} - \frac{2d}{1 - \text{tr}(M)/b} + \text{tr}(M^{-1}) \right). \quad (34)$$

The proof of these inequalities is tedious and can be done by diagonalizing the matrix  $M$ .

Equation (32) shows that

$$\frac{\text{Re}}{2} \int_{\mathcal{D}} |\mathbf{u}|^2 + \frac{\varepsilon}{2\text{We}} \int_{\mathcal{D}} \left( -\ln(\det \mathbf{A}) - b \ln(1 - \text{tr}(\mathbf{A})/b) + (b+d) \ln\left(\frac{b}{b+d}\right) \right)$$

is a non-negative quantity, and thus that (31) indeed yields some *a priori* bounds on  $(\mathbf{u}, \mathbf{A})$ .

Equation (34) (which plays the role of the log-Sobolev inequality in the micro-macro models) shows that the estimate (31) can be used to prove exponential convergence to equilibrium.

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