Proof checking for delayed finite field arithmetic using floating point operators

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Abstract Formal proof checkers such as Coq, PVS and HOL light are capable of validating proofs on finite field arithmetics and their implementations but they require extensive training from potential users. We propose here to use a tool named Gappa that can be learned in a matter of minutes to hide technicalities of such formal proof checkers. We present how we overcame one limitation of Gappa for the delayed solution of a triangular system over a finite field using floating point arithmetic. We did not produce a certificate of validity of the delayed algorithm independent of all the parameters but we contributed to the trust that readers and users may have in this algorithm by spanning a large number of domains for the inputs. As they are easily implemented we hope that such practices may become routinely used during software design and tests.

1 Introduction

Introducing a new algorithm is a difficult task. Authors have to persuade readers that their algorithm is correct and efficient. Such goals are usually attained by providing pen-and-paper proofs of correction more or less interlaced with the description of the algorithm. Authors may also provide results of tests to guarantee correction and efficiency on random cases and on known or new hard cases. Alas, this process is known to fail on mundane as well as notorious occurrences [1,2].

Developing a proof of correction in a formal proof checker using higher order logic such as Coq [3,4], PVS [5] or HOL Light [6] would be a nice alternative but such a task usually represents a large amount of work outside the fields of expertise of most authors.

The delayed solver used here is intended to work on a $n \times n$ unitary triangular matrix on $\mathbb{Z}/p\mathbb{Z}$ finite field. The key improvement compared to state of the art lies in the fact that delayed algorithms use floating point units to perform operations with no rounding error and delay computations of remainders as much as possible. Gappa is able to guarantee that no rounding error occurred whatever the value of the input matrix but it cannot operate independently from n and p as these parameters change the control flow.

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We tested exhaustively small values and randomly large values of n and p. We have set up a C++ class in a matter of a few days and validation with Gappa was also performed in a matter of days. We feel that such external validation is welcome specially to persons that lack extensive training in floating point arithmetic as such training is not necessary to develop innovative algorithms for finite field arithmetic. Similar methodology has proved to be sufficient to meet the highest Common Criteria Evaluated Assurance Level (EAL 7) [7,8] and it may now be applied to delayed finite field arithmetic.

We present in Section 2 the background and one algorithm of delayed solutions of a triangular system over a finite field. We continue in Section 3 by the background and tools of formal proof checking including Gappa and excerpts of our C++ trace class. We conclude this work in Section 4.

2 Finite field arithmetic and application to linear algebra

Finite field arithmetic plays a crucial role in nowadays applications. One of the most extensively studied application of finite fields is cryptography with major concerns on security and efficiency (see [9] for an introduction on cryptography). Another key application of finite field arithmetic arises with exact linear algebra computation where modular techniques (e.g. CRT or P-adic lifting) allow some control on expression swell with high performances [10,11,12]. Arithmetic implementations used in cryptography differ from the ones used in linear algebra. While cryptographic applications need finite fields of large cardinality (e.g. 1024-2048 bits for RSA [13], 300 bits for ECC [14]) for security purpose, exact linear algebra restrains to machine word size prime field (e.g. 32 or 64 bits) in order to benefit from machine arithmetic units.

A classical way to perform one arithmetic operation in a prime field, here we refer to integers modulo a prime number, is to first perform the operation on integers and second reduce the result to the destination field. Let $x, y \in \mathbb{Z}/p\mathbb{Z}$ and $* \in \{+, \times\}$. One may compute $z = x * y \in \mathbb{Z}/p\mathbb{Z}$ by computing $t = x * y \in \mathbb{Z}$ and a modular reduction $z = t \mod p$.

In the rest of this section, we present basic facts on word size prime field arithmetics and how to efficiently integrate them into an exact linear algebra application: solutions of triangular systems.

2.1 Word-size prime field arithmetic

When one deals with fixed precision prime field arithmetic, two majors issues arise: performances and cardinality limitation. The latter issue can have a non-negligible impact on the former one. As was just said, the classical way to perform arithmetic operations over a prime field is to perform operations on integers and reduce intermediate results. Therefore, $(p-1)^2$ must be representable to correctly perform multiplications over $\mathbb{Z}/p\mathbb{Z}$. This limitation slightly increase to perform an `AXPY` operation (a multiplication followed by an addition) with only one reduction step. This implies that $p \times (p-1)$ must be representable.

Using word-size machine integers and classic arithmetic we obtain the following cardinality limitation: $p < 2^{16}$ on 32 bit architectures and $p < 2^{32}$ on 64 bit architectures with `unsigned` types. These bounds are reduced by one bit with signed types such as `long`.

Note that even on 32 bit architectures, one can obtain 64 bit integers by using `long long` data. Full use of `long long` may reduce performances drastically as the reduction phase involves a 64 bit division. Integer multiplications have no impact on performances since most of 32 bit processors provide a multiplier $32\text{bits} \times 32\text{bits} \rightarrow 64\text{bits}$ (see the `imul` instruction on Intel architectures [15]).

Another alternative to increase cardinality of word-size prime fields is to use floating point numbers. According to the IEEE 754 standard [16,17], mantissas of double precision floating point numbers can store 53 bit integers (including the implicit bit). Therefore, we can perform prime field arithmetic with cardinality up to 2^{26} using `double`. Note that, the reduction is easily obtained by the `fmod` function available in standard libraries. This approach is quite interesting in practice since floating point multiplications and divisions are faster than their integer counterparts on most architecture (Compaq Alpha, AMD, Pentium IV, Itanium, Sun Solaris, etc.) with the exception of the Pentium III [18, Table 3.1].

One can get even larger prime fields by mixing integers and floating point numbers. The idea is then to use an approximation of the quotient with floating point and Barret's method to compute the remainder [9, chapter 14] as demonstrated in NTL library [19]. It leads to prime fields with cardinality up to 2^{29} on 32 bit architectures and up to 2^{52} on 64 bit architectures.

Prime field implementations above benefit from the arithmetic units of processors but performances depend on the target architecture (see [20,21] for detailed comparisons). On selected classes of algorithms, delayed prime field arithmetic sustains better performances. The idea is to perform several integer operations before reduction into the field. It has been very fruitful for exact linear algebra [11]. We perform a dot-product or a matrix multiplication with a delayed reduction when $(p-1)^2 \times n < 2^{\beta}$, where n is the dimension of matrices or vectors, p defines the prime field $\mathbb{Z}/p\mathbb{Z}$ and β represents the precision of integers. Delayed exact linear algebra computations also benefit from optimized numerical BLAS (e.g. ATLAS [22], GOTO [23]) libraries for exact computations as shown in [11] and they often reach maximal FPU throughput for operations over a finite field.

Beside basics linear algebra operations such as matrix-vector products and matrix multiplications, delayed arithmetic over a prime field is valuable when expressions swell largely such as solving systems of linear equations. This approach works perfectly for unitary triangular system (only ones along the diagonal) despite the exponential growth of the intermediate variables.

2.2 Triangular system solving with delayed prime field arithmetic

A key application in exact linear algebra is the resolution of triangular systems over finite fields. The resolution of such systems is a classical problem of linear algebra and it is one of the main operations in block Gaussian elimination when right hand side is given as a matrix [24]. Resolution of triangular

systems with matrix as right hand side reduces to matrix multiplication and provides the best known complexity for this problem. The complexity is going from $O(n^3)$ finite field operations with classical algorithm to $O^\sim(n^{2.81})$ using Strassen-Winograd variant and reaches the best asymptotic known value of $O^\sim(n^{2.37})$ with Coppersmith-Winograd [25]. The following algorithm achieves this reduction.

Algorithm `LZ_TRSM`(A, B)
Input: $A \in \mathbb{Z}/p\mathbb{Z}^{N\times N}$, $B \in \mathbb{Z}/p\mathbb{Z}^{N\times K}$.
Output: $X \in \mathbb{Z}/p\mathbb{Z}^{N\times K}$ such that $AX = B$.

---

if N=1 **then**
  $X := A_{1,1}^{-1} \times B$.
else *(splitting matrices into* $\lfloor \frac{N}{2} \rfloor$ *and* $\lceil \frac{N}{2} \rceil$ *blocks)*

$$\overbrace{\begin{bmatrix} A_1 & A_2 \\ & A_3 \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}}^{X} = \overbrace{\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}}^{B}$$

  X_2 :=`LZ_TRSM`(A_3, B_2).
  $B_1 := B_1 - A_2X_2$.
  X_1 :=`LZ_TRSM`(A_1, B_1).
return X.

A delayed prime field arithmetic version of this algorithm can be constructed by simply doing a delayed matrix multiplication on the operation $B_1 := B_1 - A_2X_2$. One can easily see that modular reductions will be performed at each level of the reduction. However, it appears that enough bits are available to combine a few recursion steps without reduction. This has been the purpose of the implementation based on BLAS triangular solver function (i.e. `dtrsm` routine) proposed in a previous work from one of the authors (see [11, §3.2]).

Bounds on the coefficient growth during backward substitution allow to guarantee that numerical results are exact. In [11], an optimal bound on integer coefficients has been proposed. For the sake of completeness we recall it here.

Corollary 21 *[11, corollary 3.3]: Let $A \in \mathbb{Z}^{n\times n}$ be a unit diagonal upper triangular matrix, and $b \in \mathbb{Z}^n$, with $|A|, |b| \leq p-1$. Then $x \in \mathbb{Z}^n$ the solution of the system $Ax = b$ is such that*

$$|x| \leq \frac{p-1}{2}\left[p^{n-1} + (p-2)^{n-1}\right],$$

and this bound is optimal.

We construct an improved version for the delayed prime field arithmetic of algorithm `LZ_TRSM` by replacing the last levels of the recursion by calls to `dtrsm` numerical solver according to the above corollary. Recursion is stopped for the

maximal integer n_{max} such that

$$\frac{p-1}{2}\left[p^{n_{max}-1}+(p-2)^{n_{max}-1}\right] < 2^{53}. \quad (1)$$

3 Formal proof checking and Gappa

Gappa [26,27] has been created to generate formal certificates of correction for programs that use floating point arithmetic [28,29,30,31]. As it was developed we extended it to simple bounds in general calculus. It will in the future be able to interact with Why [32], a tool to certify programs written in a generic language (C and Java can be converted to this language), and Fluctuat [33,34], a tool that proposes loop invariants.

Gappa manipulates arithmetic expressions on real and rational numbers and their evaluations on computers. Exact and rounded expressions are bounded using interval arithmetic [35], forward error analysis and properties of dyadic fractions. For delayed finite field arithmetic, we have to build a list of expressions that were **intended** to be computed with no round-off error and a list of expressions that were **actually** computed possibly with round-off errors. Gappa generates proofs that these two lists are identical.

To the authors' best knowledge, Gappa is the first tool that can convert the simple task performed here into a formal proof validated by an automatic proof checker. Such behavior has previously been quoted as *invisible formal methods* [36] in the sense that Gappa delivers formal certificates to users that are not expected to ever write any piece of proof in any formal proof system.

3.1 A C++ trace class to generate input scripts to Gappa

Gappa produces a Coq file, for example Listing 1.5, for a given input script, in this case Listing 1.2. The Coq file contains properties and proofs. Validity of proofs can automatically be checked by Coq. The C++ trace class of Listing 1.1 is responsible to produce Listing 1.2.

The later file contains three sections each associated with a `static ostringstream` variable in our C++ class. The first section is built in `bout` (beginning), the second in `mout` (middle) and the last one in `eout` (end).

The first section defines aliases. Gappa uses these aliases for its outputs and in the formal proof instead of machine generated names. It starts with the definitions of `rnd` rounding operator. It is a real function yielding rounded values according to the target data format (`ieee_64` - double precision in this case) and a predefined rounding mode (`ne` - nearest with even tie breaking as specified in IEEE 754 standard in this case).

Statements `DGxxx_exact = expr;` build the list of **intended** results and statements on `DGxxx` the list of results **actually** computed. When a rounding operator appears left of the equal symbol (as used in `DGxxx rnd= expr;`), all the arithmetic operations on the right side are rounded individually. We introduced dummy identifiers in aliases `DGxxx = int<ne>(DGxxx_dum);` to express that `DGxxx` are integer numbers.

Listing 1.1. A C++ trace class to implement single-assignement behavior and ask Gappa to prove that no operation introduced any round-off error

```
class trace;
ostream &operator<<(ostream &os, trace const &a);

class trace {
public :
  static int lastid;
  static ostringstream bout, mout, eout;
  int id;
  trace() {}
  void init(int p) {
    id = lastid++;
    bout << *this
         << " = int<ne>(" << *this << "_dum);" << endl;
    mout << *this
         << " in [0, " << p - 1 << "] ->"      << endl;
  }
  trace(char * oper, trace const &a, trace const &b) {
    id = lastid++;
    bout << *this << "    rnd= "
         << a     << oper  << b << ";"         << endl;
    bout << *this << "_exact = "
         << a     << oper  << b << ";"         << endl;
    eout << *this << " - "
         << *this << "_exact in [0, 0] /\\"    << endl;
    eout << *this << " in ? /\\"               << endl;
  }
  void dump() {
    cout << bout.str() << "{"                  << endl
         << mout.str()
         << eout.str()
         << *this      << " in ?"              << endl
                       << "}"                  << endl;
  }
  void bmes (char * mes, int n = 0)
  {bout << mes ; if (n) bout << n; bout << endl;}
};

ostream &operator<<(ostream &os,trace const &a)
  {return os << "DG" << a.id;}

trace operator+(trace const &a, trace const &b)
  {return trace(" + ", a, b);}
trace operator-(trace const &a, trace const &b)
  {return trace(" - ", a, b);}
trace operator*(trace const &a, trace const &b)
  {return trace(" * ", a, b);}

int trace::lastid = 0;
ostringstream trace::bout, trace::mout, trace::eout;
```

Listing 1.2. Reordered Gappa input script for a small example (4×4 matrices on $\mathbb{Z}/101\mathbb{Z}$)

```
@rnd = float< ieee_64, ne>;
# Matrix
DG0 = int<ne>(DG0_dum);
...
DG15 = int<ne>(DG15_dum);
# Solution
DG16 = int<ne>(DG16_dum);
...
DG19 = int<ne>(DG19_dum);
# DTRSM 4
DG20     rnd= DG5 * DG19;
DG21     rnd= DG18 - DG20;
DG22     rnd= DG3 * DG21;
DG23     rnd= DG17 - DG22;
DG24     rnd= DG4 * DG19;
DG25     rnd= DG23 - DG24;
DG26     rnd= DG1 * DG25;
DG27     rnd= DG16 - DG26;
DG28     rnd= DG2 * DG21;
DG29     rnd= DG27 - DG28;
DG30     rnd= DG3 * DG19;
DG31     rnd= DG29 - DG30;
DG20_exact = DG5 * DG19;
...
DG31_exact = DG29 - DG30;
# DREMM 3
DG32 = int<ne>(DG32_dum);
DG33 = int<ne>(DG33_dum);
DG34 = int<ne>(DG34_dum);
{
DG0 in [0, 100] ->
...
DG19 in [0, 100] ->
DG32 in [0, 100] ->
...
DG34 in [0, 100] ->
DG20 - DG20_exact in [0, 0] /\
...
DG31 - DG31_exact in [0, 0] /\
DG20 in ? /\
...
DG31 in ? /\
DG0 in ?
}
```

The second and third sections are written together between brackets ({ }) and in this work, it is a large implication (->) of a large conjunction (/\) of interval enclosures of mathematical expressions.

The second section contains a set of hypotheses each stating that a variable or an expression is within an interval. Note that $p1 \rightarrow p2 \rightarrow p3$ is logically equivalent to $p1 \wedge p2 \rightarrow p3$ and this section is build from the lines finished by an implication sign (->). Identifiers without definition, such as the `DGxxx_dum` variables used in the first section are assumed to be universally quantified over the set of real numbers the first time Gappa encounters them.

The third section contains goals. It is a large conjunction (/\). Statements `DGxxx - DGxxx_exact in [0, 0]` mean that we ask Gappa to prove that `DGxxx` and `DGxxx_exact` are identical. Statements `DGxxx in ?` are inserted for diagnostics as Gappa proposes enclosing intervals.

Listing 1.3 is the direct transcription of the algorithm presented in Section 2 to solve unitary triangular systems. We used naming conventions of BLAS and LAPack for the function and the parameter names. For the sake of simplicity some parameters have be omitted and function names were slightly modified.

The `DREMM` function computes the remainder modulo p of all the components of a matrix. As visible from the proof of Corollary 21, we do not use the exact value of these remainders but only the fact that they are between 0 and $p-1$.

The `TEST` function in Listing 1.4 presents memory allocation and parts of the Coq script related to initial aliases and some comments.

3.2 Insight into the Coq generated script

The generated Coq script visible in Listing 1.5 contains `Variables`, `Definitions`, `Notations` and `Lemmas`. Users do not need to be able to write such a file but they can check the work of Gappa by reading it or parsing it. Comments are between `(*` and `*)` signs. They were generated by Gappa.

Lemma `l134` is one of the tasks assigned to Gappa. It proves that property `p134` is valid under a large number of assumptions (`p3`, `p18`, `p43`, `p92`...) that is a subset of the hypotheses given in the second section of Gappa input script. The lines between the statement of a lemma and the `Qed.` sign are proof scripts intended for Coq only. All the properties are defined in the script. The `BND` predicate of `p134` holds, when its first argument, an expression on real numbers, is an element of its second argument, an interval defined by dyadic fraction bounds.

Proof of Lemma `l134` requires a bound on `DG31 - DG31_exact` provided by Lemma `t696`. Although enclosure (`BND`) is the only predicate available to users, Gappa internally relies on more predicates to describe properties on expressions.

Predicate	Definition
$\texttt{BND}(x,[a,b])$	$a \leq x \leq b$
$\texttt{ABS}(x,[a,b])$	$0 \leq a \leq \lvert x\rvert \leq b$
$\texttt{FIX}(x,e)$	$\exists m \in \mathbb{Z}, x = m \cdot 2^e$
$\texttt{FLT}(x,p)$	$\exists m,e \in \mathbb{Z}, x = m \cdot 2^e \wedge \lvert m\rvert < 2^p$

Listing 1.3. Delayed solution of a unitary triangular system over a finit field

```
// Solutions to a small unitary triangular system
void DTRSM (int N, int K,
            trace *A, int LDA,
            trace *X, int LDX) {
  A->bmes("# DTRSM ", N);
  for (int i = N - 2; i >= 0; i--)
    for (int j = i+1 ; j < N; j++)
      for (int k = 0; k < K; k++)
        X[i*LDX+k] = X[i*LDX+k] - A[i*LDA+j] * X[j*LDX+k];
}

// Remainder of a matrix modulo p
void DREMM (int N, int K, int p,
            trace *X, int LDX) {
  X->bmes("# DREMM ", N);
  for (int i = 0; i < N; i++)
    for (int k = 0; k < K; k++)
      X[i*LDX+k].init(p);
}

// Matrix-matrix multiplication Y <- Y - AX
void DGEMM_NEG (int N, int M, int K,
                trace *A, int LDA,
                trace *X, int LDX,
                trace *Y, int LDY) {
  A->bmes ("# DGEMM ", N);
  for (int i = 0; i < N; i++)
    for (int j = 0 ; j < M; j++)
      for (int k = 0; k < K; k++)
        Y[i*LDY+k] = Y[i*LDY+k] - A[i*LDA+j] * X[j*LDX+k];
}

// Inductive solutions to a unitary triangular system
void LZ_TRSM (int N, int K, int nmax, int p,
              trace *A, int LDA,
              trace *B, int LDB) {
  if (N <= nmax) {
    DTRSM (N, K, A, LDA, B, LDB);
    DREMM (N-1, K, p, B, LDB);
  } else {
    int P = N/2, G = N - P;
    LZ_TRSM (G, K, nmax, p, A+P*(LDA+1), LDA, B+P*LDB, LDB);
    DGEMM_NEG (P, G, K, A+P, LDA, B+P*LDB, LDB, B, LDB);
    DREMM (P, K, p, B, LDB);
    LZ_TRSM (P, K, nmax, p, A, LDA, B, LDB);
  }
}
```

Listing 1.4. Test

```
void TEST (int n, int nmax, int p, int k) {
  int i;
  trace *A = new trace[n*n], *B = new trace[k*n];
  A->bmes("@rnd = float< ieee_64, ne>;");
  A->bmes("# Matrix");
  for (i = 0; i < n*n; i++) {A[i].init(p);}
  A->bmes("# Solution");
  for (i = 0; i < k*n; i++) {B[i].init(p);}
  LZ_TRSM (n, k, nmax, p, A, k, B, k);
  A->dump();
  delete []A; delete []B;
}
```

The proof of Lemma `t696` uses `bnd_of_bnd_fix` of Gappa support library with the `apply` tactic. Automatic validation of the hypotheses is triggered by the `finalize` tactic that checks that the current goal can be reduced to $true = true$. More insights to Gappa are presented in [27].

4 Perspectives and concluding remarks

The authors had two goal in writing this report. Our first goal was to show that Gappa can be used to partially validate theorems for finite field arithmetic. Our second goal was to present Gappa on an example in sufficient details so that this report is a first tutorial on how to use Gappa, how to read files generated by Gappa and how Gappa works internally.

`http://lipforge.ens-lyon.fr/www/gappa/`

This paper addresses the problem of formally certificate expression swell within delayed finite field arithmetic. In particular, proofs have been validated for the first prime numbers and for random prime numbers on the problem of solving unitary triangular system. Our approach can be easily reproduced to other exact linear applications over finite fields. More precisely, the FFLAS-FFPACK project has been successful on using delayed prime field arithmetic for linear algebra applications.

`http://ljk.imag.fr/membres/Jean-Guillaume.Dumas/FFLAS/`

More validation work will continue to contribute to the trust of such libraries. We will also perform validation of symmetric representations of prime field in the future.

Gappa is also capable of validating bounds used for Winograd-Strassen matrix multiplication [37] by rewriting expressions. This feature not demonstrated here is useful in proving that Winograd-Strassen algorithm is correct. Gappa transfers the validation of additional formulas to a strategy of Coq that is able to validate it component by component in this case.

Listing 1.5. Excerpts of the 3548 line Coq script generated for a small example (4×4 matrices on $\mathbb{Z}/101\mathbb{Z}$)

```
Require Import Gappa_library.
Section Generated_by_Gappa.
Definition f1 := Float2 (25) (2).
Definition f2 := Float2 (0) (0).
...
Variable _DG19_dum : R.
Notation _DG19 := ((rounding_fixed roundNE (0)) (_DG19_dum)).
Notation p3 := (BND _DG19 i1). (* BND(DG19, [0, 100]) *)
...
Definition i3 := makepairF f2 f2.
...
Definition f15 := Float2 (1) (-27).
Definition f16 := Float2 (-1) (-27).
Definition i13 := makepairF f16 f15.
...
Variable _DG16_dum : R.
Notation _DG16 := ((rounding_fixed roundNE (0)) (_DG16_dum)).
Notation p92 := (BND _DG16 i1). (* BND(DG16, [0, 100]) *)
...
Notation _DG31_exact := ((_DG29 - _DG30)%R).
Notation _DG31 :=
  ((rounding_float roundNE (53) (1074)) (_DG31_exact)).
...
Notation r70 := ((_DG31 - _DG31_exact)%R).
Notation p134 := (BND r70 i3).
  (* BND(DG31 - DG31_exact, [0, 0]) *)
Notation p135 := (BND r70 i13).
  (* BND(DG31 - DG31_exact, [-7.45058e-09, 7.45058e-09]) *)
...
Notation p141 := (FIX r70 (0)). (* FIX(DG31 - DG31_exact, 0) *)
...
Lemma l133 : p141. (* FIX(DG31 - DG31_exact, 0) *)
 assert (h0 : p142). apply l132.
 assert (h1 : p143). apply l131.
 apply t695. exact h0. exact h1.
Qed.
Lemma t696 : p135 -> p141 -> p134.
 intros h0 h1.
 apply bnd_of_bnd_fix with (1 := h0) (2 := h1) ; finalize.
Qed.
Lemma l134 : ... -> p92 -> p43 -> p18 -> p3 -> p134.
  (* BND(DG31 - DG31_exact, [-0, 0]) *)
 intros h0 h1 h2 h3 h4 h5 h6 h7 h8.
 assert (h9 : p135). apply l130. exact h0. exact h1...
 assert (h10 : p141). apply l133.
 apply t696. exact h9. exact h10.
Qed.
End Generated_by_Gappa.
```

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