



**HAL**  
open science

# Topological Semantics and Decidability

Dmitry Sustretov

► **To cite this version:**

| Dmitry Sustretov. Topological Semantics and Decidability. 2007. hal-00134914

**HAL Id: hal-00134914**

**<https://hal.science/hal-00134914>**

Preprint submitted on 5 Mar 2007

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Topological Semantics and Decidability

Dmitry Sustretov

5th March 2007

## Abstract

It is well-known that the basic modal logic of all topological spaces is  $S4$ . However, the structure of basic modal and hybrid logics of classes of spaces satisfying various separation axioms was until present unclear. We prove that modal logics of  $T_0$ ,  $T_1$  and  $T_2$  topological spaces coincide and are  $S4$ . We also examine basic hybrid logics of these classes and prove their decidability; as part of this, we find out that the hybrid logics of  $T_1$  and  $T_2$  spaces coincide.

## 1 Basic definitions

In this paper we are going to study modal logics that arise as sets of all formulas valid on certain classes of topological spaces. Thus the first definition in this paper is bound to be about how the modal formulas are interpreted on topological spaces (topological semantics was first introduced by Tarski [3]).

**Definition 1 (Topological semantics).** A topological space is a pair  $(T, \tau)$  where  $\tau \subseteq \mathcal{P}(T)$  such that  $\emptyset, T \in \tau$  and  $\tau$  is closed under finite intersections and arbitrary unions. Elements of  $\tau$  are called *opens*, an open containing a point  $x$  is called a *neighborhood* of the point  $x$ .

A *topological model*  $\mathfrak{M}$  is a tuple  $(T, V)$  where  $T = (T, \tau)$  is a topological space and the valuation  $V : \text{PROP} \rightarrow \mathcal{P}(T)$  sends propositional letters to subsets of  $T$ .

Truth of a formula  $\phi$  (of the basic modal language) at a point  $w$  in a topological model  $\mathfrak{M}$  (denoted by  $\mathfrak{M}, w \models \phi$ ) is defined inductively:

$$\begin{aligned} \mathfrak{M}, w \models p & \quad \text{iff} \quad x \in V(p) \\ \mathfrak{M}, w \models \phi \wedge \psi & \quad \text{iff} \quad \mathfrak{M}, w \models \phi \text{ and } \mathfrak{M}, w \models \psi \\ \mathfrak{M}, w \models \neg\phi & \quad \text{iff} \quad \mathfrak{M}, w \not\models \phi \\ \mathfrak{M}, w \models \Box\phi & \quad \text{iff} \quad \exists O \in \tau \text{ such that } w \in O \text{ and } \forall v \in O. (\mathfrak{M}, v \models \phi) \end{aligned}$$

The basic modal language can be extended with nominals and @ operator (in this case we call it  $H(@)$ ) and universal modality  $A$  (we denote the dual modality  $E$  and call the language  $H(E)$ ). Nominals are a special kind of

propositional letters: it is required that their valuation is a singleton set. The semantics of  $@$  and  $E$  is given below:

$$\begin{aligned} \mathfrak{M}, w \models @_i \varphi & \text{ iff } \exists v \mathfrak{M}, v \models i \text{ and } \mathfrak{M}, v \models \varphi \\ & \text{(where } i \text{ is a nominal)} \\ \mathfrak{M}, w \models E\varphi & \text{ iff } \exists v \mathfrak{M}, v \models \varphi \end{aligned}$$

Relational and topological semantics are not completely unrelated; it is possible to transform certain topological spaces into frames and vice versa in a satisfiability-preserving fashion.

**Proposition 1.** *A topological space is called Alexandroff if every point of that space has a minimal neighborhood.*

*For any Alexandroff space  $(T, \tau)$  there exists a binary relation  $R$  such that for any valuation  $V$  and for any formula  $\varphi \in H(E)$ ,  $(T, R, V), w \models \varphi$  iff  $(T, \tau, V), w \models \varphi$ .*

*For any transitive reflexive frame  $(W, R)$  there exists a topology  $\tau$  on  $W$  such that for any valuation  $V$  and for any formula  $\varphi \in H(E)$ ,  $(W, R, V), w \models \varphi$  iff  $(W, \tau, V), w \models \varphi$ .*

*Proof.* For any point  $x$  of an Alexandroff space put  $Rxy$  for all  $y \in O_x$  where  $O_x$  is the minimal neighborhood of  $O_x$ . The frame one obtains is a reflexive transitive frame.

For the second clause, the topology is defined as follows. Call a subset  $O$  of the frame  $\mathfrak{F}$  *downward closed* if it follows from  $x \in O$ ,  $Rxy$  that  $y \in O$ . The topology  $\tau$  consists of all downward closed sets and is an Alexandroff topology.

In both cases it can be easily checked that the satisfiability of  $H(E)$  formulas is preserved.  $\square$

It is well-known that the (basic modal) logic of all topological spaces is  $S4$ . In what follows, we are going to deal with three classes of topological spaces, defined by the so-called separation axioms.

**Definition 2 (Separation axioms).**  $T_0$  for any two distinct points  $x, y$  there is either an open neighborhood of  $x$  that does not contain  $y$ , or an open neighborhood of  $y$  that does not contain  $x$ .

$T_1$  any singleton set is closed

$T_2$  any two distinct points  $x, y$  can be separated by two open neighborhoods, i.e. there exist  $O_x \ni x, O_y \ni y$  such that  $O_x \cap O_y = \emptyset$ .

There are necessary and sufficient conditions (given in [2]) of whether a class of spaces is definable in  $H(@)$  (and  $H(E)$ ). Thus, axioms  $T_0$  and  $T_1$  are definable in  $H(@)$ , the formulas are, respectively,  $@_i \neg j \rightarrow (@_i \Box \neg j \vee @_j \Box \neg i)$  and  $\Diamond i \rightarrow i$ . On the other hand, [2] show that  $T_2$  is not definable even in

$H(E)$ . Basic modal language is even less expressible: none of the separation axioms is definable in it. The situation with separation axioms is a bit strange: although we know the boundaries of expressivity of modal and hybrid languages, we know very little about the structure of the logics. Are the logics of separation axioms distinct? Are they decidable? If yes, what is the complexity? In this paper we will address the first two questions, and we hope to use the techniques presented here to get an answer for the third one.

## 2 Basic modal logic

In this section we will denote by  $\text{Log}(K)$  a set of formulas in basic modal language which are valid on all topological spaces in class  $K$ . We will prove that  $\text{Log}(T_0) = \text{Log}(T_1) = \text{Log}(T_2) = S4$ , our technique will be to build a topobisimulation between a finite topological space and a space from each respective class.

**Definition 3 (Topobisimulation).** Let  $(T, \tau, V)$  and  $(S, \sigma, W)$  be two topological models and consider a relation  $R \subseteq T \times S$ . Denote

$$\begin{aligned} R(X) &= \{y \mid \exists x \in X, (x, y) \in R\} \\ R^{-1}(Y) &= \{x \mid \exists y \in Y, (x, y) \in R\} \end{aligned}$$

for any subset  $X \subseteq T, Y \subseteq S$ .

The relation  $R$  is called a *topobisimulation* if

**Prop** if  $Rxy$  then for all  $p \in \text{PROP}$ ,  $(T, \tau), V, x \models p$  iff  $(S, \sigma), W, y \models p$

**Zig** for any  $O \in \tau$ ,  $R(O) \in \sigma$

**Zag** for any  $U \in \sigma$ ,  $R^{-1}(U) \in \tau$

A bisimulation is called *total* iff for any  $x \in T$  there is  $y \in S$  such that  $Rxy$  and for any  $y \in S$  there is  $x \in T$  such that  $Rxy$ .

In topological semantics just like in the relational semantics, two points connected by a topobisimulation satisfy the same formulas (if the topobisimulation is total, this is true for the formulas with universal modality).

**Definition 4 (Finite model property).** A logic  $L$  has *finite model property* with respect to a class of topological models  $K$  if  $K \models L$  and for any  $\varphi \notin L$  there exists a finite  $\mathfrak{M} \in K$  such that  $\varphi$  is satisfiable on  $\mathfrak{M}$ .

**Proposition 2.** *The logic S4 has a finite model property.*

*Proof.* The proof is Proposition 1 combined with the fact that S4 has a finite model property with respect to transitive reflexive frames.  $\square$

**Theorem 3.** *The logic of  $T_0$  spaces is  $S4$ .*

*Proof.* The inclusion  $S4 \subseteq \text{Log}(T_0)$  is obvious, so we only have to prove  $\text{Log}(T_0) \subseteq S4$ . Take an arbitrary topological space  $(T, \tau)$  and define an equivalence relation:  $x \rightsquigarrow y$  iff for all  $O \in \tau, x \in O$  iff  $y \in O$ . The quotient set with the maximal topology that makes the natural projection continuous (this topological space is known as Kolmogorov quotient of  $T$ ) is a  $T_0$  space. The graph of the natural projection map is a topobisimulation. It follows that every formula, that is not an  $S4$  validity is not a  $T_0$  validity either.  $\square$

**Theorem 4.** *The logic of  $T_1$  spaces is  $S4$ .*

*Proof.* By Proposition 2,  $S4$  has finite (topological) model property, i.e. every formula that is not valid on the class of all topological spaces can be refuted on a finite topological model. We are going to build a topobisimulation between a finite topological model and a model based on a topological space with countable domain and  $T_1$  topology. It will follow that any  $S4$  non-theorem can be refuted on a  $T_1$  space based model, hence  $\text{Log}(T_1) \subseteq S4$ , hence  $\text{Log}(T_1) = S4$ .

Let  $(T, \tau, V)$  be a finite topological model, let us construct a topobisimilar model  $(S, \sigma, W)$  where  $(S, \sigma)$  is a  $T_1$  topological space.

We will identify  $T$  with the initial segment of natural numbers, so  $T = \{1, \dots, n\}$ . First, let us introduce some notation:

$$X_k = \{nx + k \mid 0 \leq x < \infty\}, 1 \leq k \leq n$$

Let  $\sigma_0$  be a cofinite topology on  $\mathbb{N}$ , that is

$$\sigma_0 = \{O \mid \mathbb{N} - O \text{ is finite}\}$$

and for any subset  $O \subseteq T$  denote

$$\bar{O} = \{X_i \mid i \in O\}$$

Then define the topology  $\sigma$  on  $S = \mathbb{N}$  to be generated by the set

$$\beta = \sigma_0 \cup \{\bar{O} \mid O \in \tau\}$$

$\beta$  satisfies the finite intersection property, hence it indeed can serve as a base for topology. The topology generated by  $\beta$  consists of sets from  $\sigma_0$  and sets of the form  $\bar{O} - F$ , where  $F$  is finite.

Define valuation to be

$$W(p) = \overline{V(p)} \text{ for all } p \in \text{PROP}$$

$(S, \sigma)$  is a  $T_1$  space, because  $\sigma$  contains  $\sigma_0$ , hence all complements of singleton sets are open.

Define a relation  $R \subset T \times S$  as follows:

$$R = \{(k, l) \mid l \in X_k\}$$

Let us check that  $R$  is a topobisimulation. First, notice that for any  $O \subseteq T$ ,  $R(O) = \bar{O}$  and therefore is open by the definition of topology on  $R$ .

Second, notice that sets  $X_k$  are dense in  $(S, \sigma_0)$ , as well as sets of the form  $X_k - F$ , where  $F$  is finite. It follows that any open set  $U \in \sigma$  either has a non-empty intersection with every  $X_k$ , or is of the form  $U = \bar{O} - F$  where  $F$  is finite. In the first case,  $R^{-1}(U) = T$ , in the second case  $R^{-1}(U) = O$  and in both cases we get an open set.

It follows from the construction of valuations, that points connected by  $R$  agree on propositional letters.  $\square$

The main idea of the proof is that we could construct a space which is a disjoint union of dense subsets and then add necessary opens to the topology to get a space topobisimilar to  $T$ . We can exploit this idea in a more general setting, leading us to the following

**Theorem 5.** *The logic of  $T_2$  spaces is  $S4$ .*

*Proof.* We will use essentially the same idea as in the proof of the Theorem 4: we will construct a bisimulation between a finite model and a  $T_2$  model. In order to do this we will use a construction by L. Feng and O. Masaveu. In the paper [1] they prove that for any cardinal  $\alpha$  there exists a  $T_2$  space which is a disjoint union of  $\alpha$  dense subsets (such a space is called  $\alpha$ -resolvable). We will apply this statement for a finite  $\alpha = n$ , so let  $(S, \sigma_0)$  be such a space and  $S = \bigcup_{k=1}^n X_k$  where  $X_k$  are disjoint dense subsets from the theorem of Feng and Masaveu.

Now define  $\bar{O}$ ,  $\beta$ ,  $\sigma$ , the valuation  $W$  and the relation  $R \subset T \times S$  the same way as in the proof of the previous theorem.

$(S, \sigma)$  stays a  $T_2$  space (because we have only added more opens to it). For any  $O \subseteq T$ ,  $R(O) = \bar{O}$  and is open by the definition of topology on  $R$ .

In order to prove **Zag** condition for  $R$ , we are going to prove that for any  $U$  which is a finite intersection of elements of  $\beta$ ,  $R^{-1}(U)$  is open. Since any open of  $\sigma$  is a union of sets of this form, **Zag** will follow.

Without loss of generality we can suppose that  $U = I \cap \bar{O}$  for  $I \in \sigma_0$  and some open  $O \in \tau$ . Since  $I \in \sigma_0$  and  $\bar{O}$  has a non-empty intersection with any set from  $\sigma_0$ ,  $R^{-1}(U) = R^{-1}(\bar{O}) = O$ , which is open. **Prop** is immediate by construction of valuation.  $\square$

In fact, nothing in the proof depends on the  $T_2$  condition, except the existence of  $n$ -resolvable sets. That leads us to the following more general result.

**Theorem 6.** *Let  $K$  be a class of topological spaces that contains an  $n$ -resolvable space for all finite  $n$  and for any space  $(T, \tau) \in K$ , it is true that  $(T, \tau') \in K$  for all  $\tau' \supset \tau$  (i.e.  $K$  is closed under refinement of topology). Then  $\text{Log}(K) = S4$ .*

*Proof.* In this theorem we extract the key properties of the class  $T_2$  used in the proof of the Theorem 5. Indeed, we need an  $n$ -resolvable space to start our construction, then we add new opens to this space in order to obtain a space bisimilar to the given finite topological space. If the class of topological spaces in question is closed under refinement of topology, we can do it the same way as we have done in the Theorem 5.  $\square$

**Remark 1.** *Note that the proofs of Theorems 3, 4 and 5 still work for the basic modal language enriched with universal modality, because the bisimulations we construct are all total.*

### 3 Basic hybrid logic

In this section we will denote by  $\text{Log}(K)$  a set of formulas in the hybrid language  $H(@)$  (with nominals and  $@$ ) which are valid on all topological spaces in the class  $K$ .

In the subsequent subsections we will prove decidability of logics of different separation axioms. Our main tool will be the notion of topological filtration, which allows to present the information relevant for the satisfiability of a formula in a finite structure.

**Definition 5 (Topological filtration).** Let  $\Sigma$  be a subformula-closed set of formulas and  $\mathfrak{M} = (T, \tau, V)$  be a topological model. Define an equivalence relation  $\leftrightarrow_{\Sigma}$  on  $T$  as follows:

$$w \leftrightarrow_{\Sigma} v \text{ iff } \forall \varphi \in \Sigma \ \mathfrak{M}, w \models \varphi \text{ iff } \mathfrak{M}, v \models \varphi$$

A *filtration* of  $\mathfrak{M}$  through  $\Sigma$  is a model  $\mathfrak{N} = (S, \sigma, W)$ , defined as follows. Let  $S = T / \leftrightarrow_{\Sigma}$  and let us denote by  $[s]$  an equivalence class of  $\leftrightarrow_{\Sigma}$  with a representative  $s$ . For a formula  $\varphi \in \Sigma$  define

$$\llbracket \varphi \rrbracket^{\mathfrak{N}} = \{[x] \mid M, x \models \varphi\}$$

and  $W(p) = \llbracket p \rrbracket^{\mathfrak{N}}$ . This is well-defined, because points from the same equivalence class satisfy the same formulas from  $\Sigma$ .

Let  $\pi$  be a natural projection map  $t \mapsto [t]$ . Define  $\sigma$  to be the finest topology that makes  $\pi$  continuous.

Note that if  $\Sigma = Cl(\varphi)$  (all subformulas of a single formula  $\varphi$ ), then any filtration by  $\Sigma$  is finite (there is only finite number of subsets of  $Cl(\varphi)$ ).

### 3.1 $T_1$ spaces

The class  $T_1$  does not have a finite model property: for example, the formula  $i \rightarrow \Box i$  can only be falsified on an infinite model with  $T_1$  topology. In order to prove decidability of  $\text{Log}(T_1)$  we will introduce a special kind of finite structures that represent infinite models — quasi-models. Quasi-models definition is designed in such a way that you can get a quasi-model out of any model by taking a filtration of it. At the same time any quasi-model can be obtained like this. That means that by exhausting the class of all possible quasi-models for a formula we exhaust the class of all possible models a formula can be satisfied on. Since there is a bound on a number of possible quasi-models for a formula, we get decidability.

**Definition 6 (Hintikka set).** Let  $\Sigma$  be a set of formulas closed under subformulas and single negations. A set  $A \subseteq \Sigma$  is called a *Hintikka set* if it is maximal subset satisfying the following conditions:

1.  $\perp \notin A$
2. if  $\neg\varphi \in \Sigma$  then  $\varphi \in A$  iff  $\neg\varphi \notin A$
3. if  $\varphi \wedge \psi \in \Sigma$  then  $\varphi \wedge \psi \in A$  iff  $\varphi \in A$  and  $\psi \in A$

**Definition 7 (Quasi-model).** Let  $\varphi$  be a formula and  $Cl(\varphi)$  be its subformula closure. A tuple  $(T, \tau, \lambda)$ , where  $(T, \tau)$  is a finite topological space and  $\lambda$  is a function from  $T$  to  $Cl(\varphi)$  is called a *quasi-model* for  $\varphi$  if the following holds:

1.  $\lambda(t)$  is a Hintikka set for any  $t \in T$
2. at least for one  $t \in T$ ,  $\varphi \in \lambda(t)$
3. if  $O \in \tau$  and there is a formula  $\Box\psi \in \Sigma$  such that  $\forall t \in O \psi \in \lambda(t)$  then  $\forall t \in O \Box\psi \in \lambda(t)$
4. if  $i \in \lambda(t)$  where  $i$  is a nominal, then  $T - \{t\} \in \tau$  ( $T_1$  condition for quasi-models)

Here is a property of filtration that we will need for the main theorem

**Lemma 7.** For any formula  $\varphi \in \Sigma$ ,  $\llbracket \Box\varphi \rrbracket^{\mathfrak{N}}$  is open.

*Proof.* Suppose  $\llbracket \Box\varphi \rrbracket^{\mathfrak{N}}$  is not open. Then let  $\sigma'$  be generated by  $\sigma \cup \llbracket \Box\varphi \rrbracket^{\mathfrak{N}}$ . Then we have for  $O = \pi^{-1}(\llbracket \Box\varphi \rrbracket^{\mathfrak{N}})$  that  $\forall x \in O \mathfrak{M}, x \models \Box\varphi$  hence  $O$  is open which means that  $\sigma'$  makes  $\pi$  continuous. That contradicts the definition of  $\sigma$ .  $\square$

**Theorem 8.** A formula  $\varphi$  is satisfiable on a  $T_1$  space iff there exists a topological quasi-model for it.



*Proof.* The left-to-right direction can be proved using topological filtrations. Suppose we have a topological model  $\mathfrak{M} = (T, \tau, V)$ , where  $(T, \tau)$  is a  $T_1$ -space and  $\mathfrak{M}, v \models \varphi$ , let  $\Sigma = Cl(\varphi)$  and let  $\mathfrak{N} = (S, \sigma, W)$  be the filtration of  $\mathfrak{M}$  through  $\Sigma$ . Then define a quasi-model  $Q = (S, \sigma, \lambda)$ , where  $\lambda([s]) = \{\psi \in Cl(\varphi) \mid \mathfrak{M}, s \models \psi\}$ . It can be easily checked that  $\lambda([x])$  is a Hintikka set for any  $[x] \in T$ . Next,  $\varphi \in \lambda([v])$ . The  $T_1$  condition for quasi-models follows from Lemma 7. Let us finally show that the condition 3 holds. Suppose  $O \in \sigma$ ,  $\Box\psi \in \Sigma$  and  $\forall s \in O \psi \in \lambda(s)$ . Then  $U = \pi^{-1}(O)$  is open and  $\forall t \in U \mathfrak{M}, t \models \psi$ , hence  $\forall t \in U \mathfrak{M}, t \models \Box\psi$  and by construction of quasi-model  $\forall s \in O \Box\psi \in \lambda(s)$ .

To prove right-to-left direction we will construct a  $T_1$  model  $\mathfrak{M} = (S, \sigma, W)$  which satisfies  $\varphi$  from a given quasi-model  $(T, \tau, \lambda)$ . The model will have  $\mathbb{N}$  as support. This is a reasonable choice, because any satisfiable hybrid formula is satisfiable on a countable or finite model, as we are going to prove. We can always allocate a finite open subspace to satisfy  $\varphi$  on by imposing a certain topology on  $\mathbb{N}$ , and in case  $\varphi$  can only be satisfied on an infinite model, we just have to add enough opens to cofinite topology on  $\mathbb{N}$ .

Just like in the previous section we identify  $T$  with the set of natural numbers  $\{1, \dots, n\}$ . Suppose there are  $m$  points  $t_1, \dots, t_m \in T$  such that there is a nominal in  $\lambda(t_k)$  for  $1 \leq k \leq m$ . If  $m = n$ , then every point is “named” by a nominal and should be represented by a singleton. In this case the model construction process described below will produce a model with a finite discrete submodel.

Denote  $X_i = \{k\}$  for  $i = t_k, 1 \leq k \leq m$  and let  $X_i$  for  $i \in T - t_1, \dots, t_m$  form a partition of  $\mathbb{N} - \{1, \dots, m\}$  such that every  $X_i$  is an infinite coinfinite set. Let  $\sigma_0$  be a collection of cofinite subsets of  $\mathbb{N}$  and for any subset  $O \subseteq T$  denote

$$\bar{O} = \bigcup_{i \in O} X_i$$

Then define the topology on  $\mathbb{N}$  to be generated by the following set:

$$\sigma = \sigma_0 \cup \{\bar{O} \mid O \in \tau\}$$

The valuation is defined the following way

$$W(p) = \bigcup_{p \in \lambda(k)} X_k \text{ for all } p \in \text{PROP} \cup \text{NOM}$$

The definition looks similar to the definition in Section 2 (indeed, the only real difference is in the definition of  $X_k$ ); however, the proof works differently because of nominals.

For any formula  $\psi$  let us write  $W(\psi)$  for  $\{v \in S \mid \mathfrak{M}, v \models \psi\}$  and  $V(\psi)$  for  $\{v \in T \mid \psi \in \lambda(v)\}$ . We will now prove the following

**Claim 8.1.** For any  $\psi \in Cl(\varphi)$ ,  $W(\psi) = \overline{V(\psi)}$ .

*Proof.* The proof proceeds by induction on formula structure.

The statement for propositional letters and nominals follows from the construction of  $\mathfrak{M}$ . The Boolean connectives case as well as universal modality are easy to verify too.

Now, suppose that  $\psi = \Box\chi \in Cl(\varphi)$  and the statement is true for  $\chi$ .  $W(\Box\chi) = \mathbb{I}W(\chi)$  is the largest open contained in  $W(\chi)$ . There are two possibilities here.

If  $W(\chi)$  is cofinite, then it is open. Then  $W(\Box\chi) = W(\chi) = \overline{V(\psi)}$ . Note that by construction of topology  $\sigma$ , if  $\bar{O}$  is open then  $O$  is open. And since  $\overline{V(\psi)}$  is open (by condition 3 in the definition of a quasi-model),  $\overline{V(\psi)} = \overline{V(\Box\psi)}$ , which gives us the necessary statement.

If  $W(\chi)$  is coinfinite (either finite, or infinite), then the only open sets that are contained in this set (if any) are of the form  $\bar{O} - F$ , where  $O \in \tau$  and  $F$  is finite. So the largest open contained in  $W(\chi)$  is of the form  $\bar{O}$ , where  $O$  is the largest open contained in  $V(\chi)$ , i.e.  $O = \mathbb{I}V(\chi)$ . In other words  $W(\Box\chi) = \mathbb{I}W(\chi) = \overline{\mathbb{I}V(\chi)}$ . By condition 3 in the definition of quasi-model  $\mathbb{I}V(\chi) = V(\Box\chi)$ .  $\square$

A direct consequence of this claim is that if  $\varphi \in \lambda(k)$  then  $\mathfrak{M}, v \models \varphi$  for all  $v \in X_k$ , which finishes the proof of the right-to-left direction.  $\square$

Note that the size of a quasi-model for  $\varphi$  is bounded by  $2^{|Cl(\varphi)|}$  which allows us to deduce

**Theorem 9.**  $Log(T_1)$  is decidable.

### 3.2 $Log(T_1) = Log(T_2)$

In the Theorem 5 we used the construction of the Theorem 4 and replaced the naturals with cofinite topology with an  $n$ -resolvable  $T_2$  space whose existence is guaranteed by the theorem of Feng and Masaveu. In a similar fashion we are going to reuse the notion of the  $(T_1)$  quasi-model and modify the construction of the Theorem 8 in order to construct  $T_2$  models out of the  $(T_1)$  quasi-models.

**Theorem 10.** A formula  $\varphi$  is satisfiable on a  $T_2$  space iff there exists a topological quasi-model for it.

*Proof.* Since all  $T_2$  spaces are  $T_1$ , the same filtration argument as in Theorem 8 applies here.

Now suppose we are given a quasi-model  $\mathfrak{M} = (T, \tau, \lambda)$  for the formula  $\varphi$ . Let  $(S, \sigma_0)$  be an  $n - m$ -resolvable  $T_2$  space, where  $n = |T|$  and  $m$  is the number of points in the quasi-model, named by a nominal. Let  $X'_1, \dots, X'_{n-m}$  be the dense subsets of  $S$  which form the partition of  $S$ . Note that if  $n > 1$

then these sets have empty interiors, because if one of them doesn't then no other can be dense. Let  $X_{n-m+1}, \dots, X_n$  be arbitrary singleton subsets of  $S$ . Finally, denote

$$X_i = X'_i - \bigcup_{j=n-m+1}^n X_j, \text{ for } 1 \leq i \leq 1, n-m$$

Since  $S$  is a  $T_1$  space,  $X_1, \dots, X_{n-m}$  are still dense in  $S$ . As usual, denote

$$\bar{O} = \bigcup_{i \in O} X_i$$

and consider a new topology  $\sigma$  on  $S$  generated by

$$\sigma_0 \cup \{\bar{O} \mid O \in \tau\}$$

and the valuation

$$W(p) = \bigcup_{p \in \lambda(k)} X_k \text{ for all } p \in \text{PROP} \cup \text{NOM}$$

It is left to prove that this construction preserves satisfiability of subformulas of  $\varphi$ . We use induction on formula structure just like in the proof of Claim 8.1. Boolean connectives and universal modality do not pose any problem, so we will only consider in detail the case when the formula is of the form  $\Box\chi$ .

We will use notation  $V(\psi)$  and  $W(\psi)$  in the same way as in the proof of Theorem 8. Suppose that  $\psi = \Box\chi$  and  $W(\chi) = \overline{V(\chi)}$ .

First note that  $X_1, \dots, X_n$  have an empty interior in  $\sigma_0$ . This is obvious for  $X_{n-m+1}, \dots, X_n$  which are singleton sets,  $X_1, \dots, X_{n-m}$  have empty interiors since  $X'_1, \dots, X'_{n-m}$  have empty interiors. This fact together with the induction hypothesis  $W(\chi) = \bigcup_{i \in A} X_i$  implies that the only opens  $W(\chi)$  contains are of the form  $\bar{O} \cap F$  where  $O \in \tau, O \subseteq A, F \in \sigma_0$ . Thus information given by the topology of the quasi-model is enough to find  $W(\Box\chi)$ .

If  $W(\chi)$  is cofinite then  $W(\chi) = W(\Box\chi) = \overline{V(\Box\chi)}$ , since  $W(\chi)$  is open and condition 3 in the definition of a quasi-model holds. This is the only place in the present proof where we used the " $T_1$  condition" from the definition of the quasi-model; we have to require it because cofinite opens were already present in  $\sigma_0$  and we inherited them in  $\sigma$  which is constructed after the quasi-model.

In case  $W(\chi)$  is not cofinite,

$$\begin{aligned}
W(\Box\chi) &= \mathbb{I}(\bigcup_{i \in A} X_i) = \\
&= \bigcup_{O \in \tau, O \subseteq A, F \in \sigma_0} (\bar{O} \cap F) = \\
&= \bigcup_{O \in \tau, O \subseteq A} \bar{O} \\
&= \frac{O \in \tau, O \subseteq A}{\mathbb{I}(V(\Box\chi))}
\end{aligned}$$

which finishes the proof.  $\square$

Since every  $T_2$  space is a  $T_1$  space, we get the following corollary

**Theorem 11.** *The logic of  $T_2$  spaces coincides with the logic of  $T_1$  spaces (and hence, is decidable).*

### 3.3 $T_0$ spaces

We will next deal with the simplest and the most natural class of topological spaces,  $T_0$  spaces.

**Proposition 12.** *An Alexandroff space corresponding to a partial order by the Proposition 1 is  $T_0$  and the frame that corresponds to a  $T_0$  Alexandroff space is a partial order.*

*Proof.* Let  $T$  be a  $T_0$  Alexandroff space. Take two points  $x, y \in T$  and suppose  $Rxy$  and  $Ryx$ . That means that  $x$  is in the minimal neighbourhood of  $x$  and that  $y$  is in the minimal neighborhood of  $x$  which contradicts the fact that  $T$  is a  $T_0$  space. It follows that  $\mathfrak{F}$  is antisymmetric, hence a partial order.

Now let  $\mathfrak{F}$  be a partial order frame and let  $\tau$  be the corresponding Alexandroff topology. Take two distinct points  $x, y \in \mathfrak{F}$  and consider their corresponding pointwise generated subframes (that is, minimal neighborhoods)  $O_x$  and  $O_y$ . Since  $\mathfrak{F}$  is antisymmetric, either  $x \notin O_y$  or  $y \notin O_x$ , so  $\tau$  is a  $T_0$  topology.  $\square$

By the Proposition above, every  $T_0$  validity is a partial order validity. The converse is not true.

Consider the countable topological space  $(\mathbb{N}, \sigma)$  with cofinite topology. Construct a topological space  $(T, \tau)$  as follows: let  $T = \{*\} \cup \mathbb{N}$  and  $\tau = \{U = \{*\} \cup O \mid O \in \sigma\}$ . This is a  $T_0$  space. Now, introduce a valuation that names  $*$  with a nominal  $i$  and consider a formula  $\varphi = \diamond(\neg i \wedge \diamond i)$ . This formula is satisfied at  $*$ , but it is not satisfiable on any partial order. Hence  $\text{Log}(T_0)$  is a strict subset of the logic of partial order.

In order to understand the logic of  $T_0$  spaces we will use again the technique of quasi-models. Although the counterexample we have just mentioned

tells us that  $\text{Log}(T_0)$  is more complicated than the logic of partial orders, it will serve us as the source of ideas on how one might build a  $T_0$  model out of a quasi-model. We will need a different notion of a quasi-model than one for  $T_1$  and  $T_2$  spaces (otherwise  $\text{Log}(T_0)$  would coincide with  $\text{Log}(T_1)$  which is impossible).

The change to the definition will only affect one clause, namely the “ $T_1$  condition for quasi-models”. It is replaced with the following one:

**( $T_0$  condition for quasi-models)** for every pair of points  $x, y$  named by nominals, there exists an open neighborhood  $O_x$  of  $x$  such that  $y \notin O_x$  or there exists an open neighborhood  $O_y$  of  $y$  such that  $x \notin O_y$ .

Once again we will describe a way to construct a topological space (this time a  $T_1$  space) that satisfies a given formula given a quasi-model for that formula. We will have as a consequence a

**Theorem 13.** *The  $\text{Log}(T_0)$  is decidable.*

*Proof.* What we really prove here is that  $T_0$  satisfiability is equivalent to satisfiability on a quasi-model.

A filtration of a  $T_0$  space through  $Cl(\varphi)$  gives a  $T_0$  quasi-model of  $\varphi$ , this can be verified directly, so we will not go into the details here.

The other direction of the proof goes as follows. Consider a  $T_0$  quasi-model  $\mathfrak{M} = (T, \tau, \lambda)$ . We will identify  $T$  with natural numbers  $1, \dots, n$  and we will use such a numbering that  $1, \dots, m$  are named by a nominal, i.e. for  $1 \leq km, i \in \lambda(k)$  for some nominal  $i$ . We will construct a topological model  $(S, \sigma, W)$  with a support  $\{1, \dots, m\} \cup \mathbb{N}$  and topology and valuation defined below. We will suppose further that  $n \neq m$  since otherwise the quasi-model is a real model for  $\varphi$  already.

Partition  $S$  into sets  $X_1, \dots, X_n$ : let  $X_k = \{k\}$  for  $1 \leq k \leq m$  and let  $X_m, \dots, X_n$  be the sets of the form  $\{k + j(n - m) \mid 0 \leq j < \infty\}$  for  $m \leq k \leq n$ .

As usual, denote

$$\bar{O} = \bigcup_{i \in O} X_i$$

for  $O \subseteq T$ . Define the topology  $\sigma$  to be generated by the following collection of sets

$$\{\bar{O} - F \mid O \in \tau, F \subseteq \mathbb{N} \text{ finite} \}$$

Valuation is also defined in a usual way

$$W(p) = \bigcup_{p \in \lambda(k)} X_k \text{ for all } p \in \text{PROP} \cup \text{NOM}$$

Note that similarly to the Theorems 8 and 10, the only opens  $\bar{A}$  contains (for  $A \subset T$ ) are of the form  $\bar{O} - F$  where  $O \in \tau, O \subseteq A, F$  is finite. This is what makes the same inductive proof as in the aforementioned theorems go through.

We will only mention that the model thus constructed is  $T_0$ . Any point  $x$  from  $\mathbb{N}$  can be separated from any other point by a set  $S - \{x\}$ . Two points named by nominal can be separated by an open due to the  $T_0$  condition for quasi-models.

□

**Remark 2.** *What happens if we add the universal modality to the language? The only part of proofs of Theorems 13, 8 and 10 that depends on the language is when we prove that a formula is satisfied at a point of a quasi-model iff it is satisfied at a corresponding point of the model we have constructed. This is proved by induction on formula structure, and it is easy to see that if we add a clause that handles universal modality, the statement will still hold. Thus, the aforementioned theorems also apply to  $H(E)$ .*

## References

- [1] L. Feng and O. Masaveu, *Exactly  $n$ -resolvable spaces and  $\omega$ -resolvability*, Math. Japonica **50** (1999), no. 3, 333–339.
- [2] D. Gabelaia, B. ten Cate, and D. Sustretov, *Modal languages for topology: Expressivity and definability*, submitted to Annals of Pure and Applied Logic, preprint available at <http://www.arxiv.org/abs/math.LO/0610357>.
- [3] A. Tarski, *Der Aussagenkalkül und die Topologie*, Fundamenta Mathematicae **31** (1938), 103–134.