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1 Introduction

Let $p$ be a rational polytope in $V = \mathbb{R}^d$ and $h(x)$ a polynomial function on $V$. A classical problem in Integer Programming is to compute the sum of values of $h(x)$ over the set of integral points of $p$,

$$S(p, h) = \sum_{x \in p \cap \mathbb{Z}^d} h(x).$$

When $p$ is dilated by an integer $n \in \mathbb{N}$, we obtain a function of $n$ which is quasi-polynomial, the so-called Ehrhart quasi-polynomial of the pair $(p, h)$

$$S(np, h) = \sum_{m=0}^{d+N} E_m(p, h, n)n^m$$

of degree $d + N$ where $N = \deg h$. The coefficients $E_m(p, h, n)$ are periodic functions of $n \in \mathbb{N}$, with period the smallest integer $q$ such that $qp$ is a lattice polytope.

Replacing $h(x)$ by an exponential, we are led to study the analytic function on $V^*$ defined by

$$S(p)(\xi) = \sum_{x \in p \cap \mathbb{Z}^d} e^{\langle \xi, x \rangle}.$$ 

If $p$ is any rational polyhedron, this sum still makes sense as a meromorphic function defined near 0 and the map $p \mapsto S(p)(\xi)$ is a valuation.
In [4], we proved that the meromorphic function $S(p)(\xi)$ has a local Euler-Maclaurin expansion

$$S(p)(\xi) = \sum f(\mu(t(p,f)))(\xi) \int f(e^{\langle \xi, x \rangle} dx.$$ 

The sum is taken over the set of faces $f$ of the polyhedron $p$. For each face $f$, the function $\mu(t(p,f))(\xi)$ is holomorphic near 0, and it depends only on the transverse cone $t(p,f)$ of $p$ along $f$. More precisely, once a rational scalar product is chosen on $V$, we define canonically a map $a \mapsto \mu(a)$ from the set of rational affine cones $a$ in quotient spaces of $V$, with values in the space of functions on $V$ which are analytic near 0, then we prove that these functions satisfy the above formula. The map $a \mapsto \mu(a)$ is invariant under lattice translations, equivariant with respect to lattice preserving isometries, and it is a valuation on the set of affine cones with a fixed vertex ([4], Theorems 17 and 18).

It is easy to see that the Ehrhart quasi-polynomial can be computed in terms of the Taylor coefficients of the functions $\mu(t(p,f))(\xi)$. For example, if $p$ is a lattice polytope and $h(x) = 1$, we have ([4], Corollary 28)

\begin{equation}
\text{Card}(n p \cap \mathbb{Z}^d) = \sum f(\mu(t(p,f)))(0) \text{vol}(f)n^{\dim f}.
\end{equation}

Using the valuation property of $\mu(a)$ and Barvinok’s decomposition of a cone into unimodular cones, we thus obtained in [4] an algorithm for computing the Ehrhart quasi-polynomial. It has polynomial length with respect to the input $(p,h)$, when the dimension $d$ and the degree $N$ are fixed.

The valuation $S(p,h)$ has a natural generalization used by Barvinok in [2], the mixed valuation $S^L(p,h)$, where $L \subseteq V$ is a rational vector subspace. Denote the projected lattice on $V/L$ by $\Lambda_L$. For a polytope $p \subseteq V$ and a polynomial $h(x)$

$$S^L(p,h) = \sum_{y \in \Lambda_V/L} \int_{p \cap (y + L)} h(x) dx.$$ 

In other words, the polytope $p$ is sliced along lattice affine subspaces parallel to $L$ and the integrals of $h$ over the slices are added up. For $L = V$, there is only one term and $S_V(p,h)$ is just the integral of $h(x)$ over $p$, while, for $L = \{0\}$, we recover $S(p,h)$, the sum of values of $h(x)$ over the set of integral points of $p$. 

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In the case \( h(x) = 1 \), we write \( S(p) \) and \( S^L(p) \) in place of \( S(p, 1) \) and \( S^L(p, 1) \).

Using these mixed valuations, Barvinok gave an algorithm which computes the \( r + 1 \) highest degree Ehrhart coefficients of \( S(np) = \text{Card}(np \cap \mathbb{Z}^d) \), when \( p \) is a simplex in \( \mathbb{R}^d \). Barvinok’s algorithm has polynomial length when \( d \) is an input, provided \( r \) is fixed. The method consists in reducing the problem to summations over lattice points in dimension \( \leq r \).

Barvinok considers particular linear combinations

\[
\sum_{L \in \mathcal{L}} \rho(L) S^L(p),
\]

where \( \mathcal{L} \) is a finite set of rational vector subspaces of \( V \) which is closed under sum, and the coefficients \( \rho(L) \) are integers which satisfy the following relation between characteristic functions:

\[
\chi(\bigcup_{L \in \mathcal{L}} L^\perp) = \sum_{L \in \mathcal{L}} \rho(L) \chi(L^\perp),
\]

where \( L^\perp \subseteq V^* \) is the orthogonal of \( L \). We call a function \( \mathcal{L} \to \mathbb{Z} \) with this property a patchwork function on \( \mathcal{L} \).

When \( p \) is dilated by an integer \( n \), \( S^L(np) \) is again given by a quasi-polynomial in \( n \), as is a linear combination

\[
\sum_{L \in \mathcal{L}} \rho(L) S^L(np) = \sum_{m=0}^{d} \nu_m(p, n)n^m.
\]

The main theoretical result of [2], Theorem (1.3), is the following: if \( \mathcal{L} \) is a family of subspaces which is closed under sum and contains the vector subspace \( \text{lin}(f) \) parallel to \( f \), for every face \( f \) of codimension \( \leq r \) of \( p \), and if \( \rho \) is a patchwork function on \( \mathcal{L} \), then the \( r + 1 \) highest degree coefficients \( \nu_m(p, n) \), for \( m = d, \ldots, d - r \), are equal to the corresponding Ehrhart coefficients of \( S(np) = \text{Card}(np \cap \mathbb{Z}^d) \).

In the present article, we introduce the meromorphic functions which extend \( S^L(p) \). For any polyhedron \( p \),

\[
S^L(p)(\xi) = \sum_{y \in \Lambda_V/L} \int_{p \cap (y + L)} e^{\langle \xi, x \rangle} dx
\]
is defined as a meromorphic function near $\xi = 0$. We show that $S^L(p)(\xi)$ also enjoys a local Euler-Maclaurin expansion (Theorem 3),

$$S^L(p)(\xi) = \sum_{f} \mu^L(t(p,f))(\xi) \int_{f} e^{\langle \xi, x \rangle} dx.$$  

Furthermore, for a linear combination of Barvinok type, if $f$ is a face of $p$ such that $\text{lin}(f) \in L$, we prove that the $f$-term in the Euler-Maclaurin expansions of

$$S^{L,f}(p)(\xi) = \sum_{L \in L} \rho(L) S^L(p)(\xi)$$

and of the usual valuation $S(p)(\xi)$ are equal (Theorem 7):

$$\sum_{L \in L} \rho(L) \mu^L(t(p,f))(\xi) = \mu(t(p,f))(\xi).$$

This is the main result of the present article. From the relation between Ehrhart quasi-polynomials and Euler-Maclaurin expansions, it implies Barvinok’s Theorem (1.3).

Actually, we derive from Theorem 7 another computation of the $r + 1$ highest coefficients of the Ehrhart quasi-polynomial for a pair $(p, h)$, based on Brion’s decomposition of a polytope into cones, in the line of [1] and [3].

For each vertex $s$ of $p$, let $c_s$ be the cone of feasible directions of $p$ at $s$. Instead of the full family $\mathcal{L}$ generated by taking sums of the subspaces $\text{lin}(f)$, when $f$ runs over the set of faces of codimension $\leq r$ of the polytope $p$, we consider, for each vertex $s$ of $p$, the family $\mathcal{L}_s$ generated by faces of $c_s$ of codimension $\leq r$. The point in taking a family which depends on $s$ lies in the case where $p$ is simplicial. Then $\mathcal{L}_s$ consists only of the spaces $\text{lin}(f)$ where $f$ is a face of $c_s$ of codimension $\leq r$, as this set is already closed under sum. Moreover the coefficients $\rho(L)$ are just signed binomial coefficients (Lemma 15), and the computation of $S^{L,c_s}(\xi)$ is easier when $L$ is parallel to a face of $c_s$ (Example 3).

Let us describe our method in the simpler case of a lattice polytope $p$ and polynomial $h(x) = 1$. By Brion’s theorem, we have

$$S(p)(\xi) = \sum_{s} e^{\langle \xi, s \rangle} S(c_s)(\xi).$$

For each vertex $s$, let $\rho_s : \mathcal{L}_s \to \mathbb{Z}$ be a patchwork function. We define

$$B(p)(\xi) = \sum_{s} e^{\langle \xi, s \rangle} S^{\mathcal{L}_s, \rho_s}(c_s)(\xi).$$
For the dilated polytope $np$, we have

$$S(np)(\xi) = \sum_s e^{n(\xi,s)} S(c_s)(\xi) = \sum_{m \geq 0} \frac{n^m}{m!} \sum_s \langle \xi, s \rangle^m S(c_s)(\xi).$$

Hence, the meromorphic function $\frac{1}{m!} \sum_s \langle \xi, s \rangle^m S(c_s)(\xi)$ is actually regular at $\xi = 0$ and its value at $\xi = 0$ is the $m$th Ehrhart coefficient of $p$.

We have similarly

$$B(np)(\xi) = \sum_{m \geq 0} \frac{n^m}{m!} \sum_s \langle \xi, s \rangle^m S^{L, \rho_s}(c_s)(\xi).$$

The meromorphic functions $S^{L}(c_s)(\xi)$ and $S^{L, \rho_s}(c_s)(\xi)$ have a special form: they can be written as the quotient of an analytic function by a product of $d' \leq d$ linear forms. Such a function $\phi$ has an expansion into rational functions $\phi = \sum_{j \geq -d} \phi[j]$ where $\phi[j]$ is homogeneous of total degree $j$.

Now it follows from our main theorem that, for $m \geq d - r$, we have

$$S(c_s)[-m](\xi) = S^{L, \rho_s}(c_s)[-m](\xi),$$

hence the zero degree terms of $\sum \langle \xi, s \rangle^m S(c_s)(\xi)$ and $\sum \langle \xi, s \rangle^m S^{L, \rho_s}(c_s)(\xi)$ are equal. Therefore the latter is also analytic at $\xi = 0$ and its value at $\xi = 0$ is the $m$th Ehrhart coefficient of $p$. This is the content of Theorem 20.

Thus, besides taking care of any polynomial $h(x)$, not only $h(x) = 1$, this method to compute the $r + 1$ highest degree Ehrhart coefficients for the pair $(p, h)$ leads to a simpler algorithm than the one proposed in [2]. When $p$ is a rational simplex, the contributions of the terms of the form $S^L(c_s)(\xi)$ when $L \in L_s$ are immediately reduced to the computation of a function $S(a)$ with $a$ a simplicial cone of dimension smaller or equal to $r$.

There is also another possible implementation of an algorithm to compute the $r + 1$ Ehrhart highest degree coefficients for the pair $(p, h)$ based on the results of [4]. As seen from Equation (1), this involves the computation of the analytic function $\mu(t(p, f))$, also associated to simplicial cones in dimension smaller or equal to $r$. We plan to compare the implementation of both methods in the near future.
2 Local Euler-Maclaurin expansion of a mixed valuation $S^L$

We consider a rational vector space $V$ of dimension $d$, that is to say a finite dimensional real vector space with a lattice denoted by $\Lambda_V$ or simply $\Lambda$. We will need to consider subspaces and quotient spaces of $V$, this is why we cannot just let $V = \mathbb{R}^d$ and $\Lambda = \mathbb{Z}^d$. By lattice, we mean a discrete additive subgroup of $V$ which generates $V$ as a vector space. Hence, a lattice is generated by a basis of the vector space $V$. A basis of $V$ which is a $\mathbb{Z}$-basis of $\Lambda_V$ is called an integral basis. The elements of $\Lambda$ are called integral. An element $x \in V$ is called rational if $qx \in \Lambda$ for some integer $q \neq 0$. The space of rational points in $V$ is denoted by $V_\mathbb{Q}$. A subspace $L$ of $V$ is called rational if $L \cap \Lambda$ is a lattice in $L$. If $L$ is a rational subspace, the image of $\Lambda$ in $V/L$ is a lattice in $V/L$, so that $V/L$ is a rational vector space. We will call the image of $\Lambda$ in $V/L$ the projected lattice.

Example 1 Let $V = \mathbb{R}^2$ with standard lattice $\mathbb{Z}^2$. Let $v_1, v_2$ be two primitive integral independent vectors. Using an integral basis with first basis vector $v_1$, a straightforward computation shows that the projected lattice on $\mathbb{R}^2/\mathbb{R}v_1$ is $\mathbb{Z}\overline{v}_2\det(v_1, v_2)$, where $\overline{v}_2$ is the projection of $v_2$ on $\mathbb{R}^2/\mathbb{R}v_1$.

A rational space $V$, with lattice $\Lambda$, has a canonical Lebesgue measure, for which $V/\Lambda$ has measure 1. An affine subspace $L$ of $V$ is called rational if it is a translate of a rational subspace by a rational element. It is similarly provided with a canonical Lebesgue measure. We will denote this measure by $dm_L$.

We will denote elements of $V$ by latin letters $x, y, v, \ldots$ and elements of $V^*$ by greek letters $\xi, \alpha, \ldots$. We denote the duality bracket by $\langle \xi, x \rangle$.

If $S$ is a subset of $V$, we denote by $< S >$ the affine subspace generated by $S$. If $S$ consists of rational points, then $< S >$ is rational. Remark that $< S >$ may contain no integral point. We denote by $\text{lin}(S)$ the vector subspace of $V$ parallel to $< S >$.

If $S$ is a subset of $V$, we denote by $S^\perp$ the subspace of $V^*$ orthogonal to $S$:

$$S^\perp = \{ \xi \in V^* \mid \langle \xi, x \rangle = 0 \text{ for all } x \in S \}. $$

If $L$ is a subspace of $V$, the dual space $(V/L)^*$ is canonically identified with the subspace $L^\perp \subset V^*$. 


A convex rational polyhedron \( p \) in \( V \) (we will simply say polyhedron) is, by definition, the intersection of a finite number of half spaces bounded by rational affine hyperplanes. We say that \( p \) is solid (in \( V \)) if \( \langle p \rangle = V \). A polytope \( p \) is a \textbf{compact} polyhedron.

The set of non negative real numbers is denoted by \( \mathbb{R}_+ \). A convex rational cone \( c \) in \( V \) is a closed convex cone \( \sum_{i=1}^{k} \mathbb{R}_+ v_i \) which is generated by a finite number of elements \( v_i \) of \( V \). In this article, we simply say cone instead of convex rational cone.

An affine (rational) cone \( a \) is, by definition, the translate of a cone in \( V \) by a rational element \( s \in V \). This cone is uniquely defined by \( a \).

A cone \( c \) is called simplicial if it is generated by independent elements of \( V \). A simplicial cone \( c \) is called unimodular if it is generated by independent integral vectors \( v_1, \ldots, v_k \) such that \( \{v_1, \ldots, v_k\} \) can be completed in an integral basis of \( V \). An affine cone \( a \) is called simplicial (resp. simplicial unimodular) if it is the translate of a simplicial (resp. simplicial unimodular) cone.

An affine cone \( a \) is called pointed if it does not contain any straight line.

The set of faces of an affine cone \( a \) is denoted by \( F(a) \). If \( a \) is pointed, then the vertex of \( a \) is the unique face of dimension 0, while \( a \) is the unique face of maximal dimension \( \dim a \).

Let us recall the definition of the transverse cone \( t(p, f) \) of a polyhedron \( p \) along one of its faces \( f \). Let \( x \) be a point in the relative interior of \( f \). The cone of feasible directions of \( p \) at \( x \) is the set \( c(p, f) := \{ v \in V : x + \epsilon v \in p \text{ for } \epsilon > 0 \text{ small enough} \} \). It does not depend on the choice of \( x \). We denote the projection \( V \to V/\text{lin}(f) \) by \( \pi_f \). Then \( t(p, f) \) is the image \( \pi_f(f + c(p, f)) \) of the affine cone \( f + c(p, f) \) in \( V/\text{lin}(f) \). It is a solid pointed affine cone in the quotient space \( V/\text{lin}(f) \) with vertex \( \pi_f(<f>) \). In particular, if \( v \) is a vertex of \( p \), the transverse cone \( t(p, v) \) coincides with the supporting cone \( v + c(p, v) \subset V \).

If \( a \) is an affine cone and \( f \) is a face of \( a \), then \( c(a, f) = a + \text{lin}(f) \) and the transverse cone \( t(a, f) \) of \( a \) along \( f \) is just the projection \( \pi_f(a) \) of \( a \) on \( V/\text{lin}(f) \).

**Definition 2** Denote by \( \mathcal{H}(V^*) \) the ring of analytic functions around \( 0 \in V^* \). Denote by \( \mathcal{M}(V^*) \) the ring of meromorphic functions defined around \( 0 \in V^* \) and by \( \mathcal{M}_f(V^*) \subset \mathcal{M}(V^*) \) the subring consisting of those meromorphic functions \( \phi(\xi) \) such that there exists a product of linear forms \( D(\xi) \) with \( D(\xi)\phi(\xi) \in \mathcal{H}(V^*) \).
A function $\phi(\xi) \in \mathcal{M}_\ell(V^*)$ has a unique expansion into homogeneous rational functions

$$\phi(\xi) = \sum_{m \gg -\infty} \phi_{|m|}(\xi)$$

where $m$ is the total degree.

If $P$ is a homogeneous polynomial on $V^*$ of degree $p$, and $D$ a product of $r$ linear forms, then $P D$ is an element in $\mathcal{M}_\ell(V^*)$ homogeneous of degree $m = p - r$.

Let us recall the definition of the function $I(p) \in \mathcal{M}_\ell(V^*)$ associated to a polyhedron $p$, (see for instance the survey [3]).

**Proposition 3** There exists a map $I$ which to every polyhedron $p \subset V$ associates a meromorphic function with rational coefficients $I(p) \in \mathcal{M}_\ell(V^*)$, so that the following properties hold:

(a) If $p$ contains a straight line, then $I(p) = 0$.

(b) If $\xi \in V^*$ is such that $|e^{\langle \xi, x \rangle}|$ is integrable over $p$, then

$$I(p)(\xi) = \int_p e^{\langle \xi, x \rangle} d\mu_{<p>}(x).$$

(c) For every point $s \in V_Q$, we have

$$I(s + p)(\xi) = e^{\langle \xi, s \rangle} I(p)(\xi).$$

(d) The map $I$ is a simple valuation: if the characteristic functions $\chi(p_i)$ of a family of polyhedra $p_i$ satisfy a linear relation $\sum_i r_i \chi(p_i) = 0$, then the functions $I(p_i)$ satisfy the relation

$$\sum_{\{i, <p_i>, = V\}} r_i I(p_i) = 0.$$

In the following proposition, we define the *mixed valuation* $p \mapsto S_L^L(p)$ associated to a rational vector subspace $L \subseteq V$. To any polyhedron $p$, we associate a meromorphic function $S_L^L(p)(\xi) \in \mathcal{M}(V^*)$. If $p$ is compact, this function is actually regular at 0, and its value for $\xi = 0$ is the valuation $E_{L^+}(p)$ considered by Barvinok [2].

We denote by $\Lambda_{V/L}$ the projection on $V/L$ of the lattice $\Lambda$. 8
Proposition 4 Let $L \subseteq V$ be a rational subspace. There exists a map $S^L$ which to every rational polyhedron $p \subseteq V$ associates a meromorphic function with rational coefficients $S^L(p) \in \mathcal{M}(V^*)$ so that the following properties hold:

(a) If $p$ contains a line, then $S^L(p) = 0$.

(b) \begin{equation}
S^L(p)(\xi) = \sum_{y \in \Lambda \cap (p + L)} \int_{p \cap (y + L)} e^{\langle \xi, x \rangle} dm_L(x),
\end{equation}

for every $\xi \in V^*$ such that the above sum converges.

(c) For every point $s \in \Lambda$, we have

\begin{equation}
S^L(s + p)(\xi) = e^{\langle \xi, s \rangle} S^L(p)(\xi).
\end{equation}

(d) The map $S^L$ is a valuation: if the characteristic functions $\chi(p_i)$ of a family of polyhedra $p_i$ satisfy a linear relation $\sum_i r_i \chi(p_i) = 0$, then the functions $S^L(p_i)$ satisfy the same relation

$$\sum_i r_i S^L(p_i) = 0.$$ 

For $L = \{0\}$, we recover the valuation $S$ given by

$$S(p)(\xi) = \sum_{x \in p \cap \Lambda} e^{\langle \xi, s \rangle},$$

provided this sum is convergent.

For $L = V$, we have $S^V(p) = I(p)$, if $p$ is solid, and $S^V(p) = 0$ otherwise. The proof is entirely analogous to the case $L = \{0\}$, see Theorem 3.1 in [3], and we omit it.

Remark 5 The function $S^L(p)$ is actually an element of $\mathcal{M}(V^*)$, but we do not prove it at this point. Let $a$ be an affine cone and $\{v_i\}$ the generators of its edges. It will follow from the Euler-Maclaurin expansion of $S^L(a)$ (Theorem 8) that $\prod_i \langle \xi, v_i \rangle S^L(a)(\xi)$ is analytic near zero for any $L$. It would be interesting to prove it a priori. By Brion’s theorem and the valuation property, it follows in particular that $S^L(p) \in \mathcal{M}(V^*)$. 

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Example 6 Let $a$ be a simplicial affine cone in the space $V$, and assume that $L = \text{lin}(f_1)$ for some face $f_1$ of $a$. In this case, $S^L(a)(\xi)$ decomposes as product of an integral and a discrete sum. For simplicity, assume that $a$ is solid. Let $f_2$ be the face of $a$ such $V = \text{lin}(f_1) \oplus \text{lin}(f_2)$. We write $x = x_1 + x_2$ and $\xi = \xi_1 + \xi_2$ for the corresponding decompositions of $x \in V$ and $\xi \in V^*$. Then $a = a_1 + a_2$ where $a_i$ is a simplicial affine cone in $\text{lin}(f_i)$. Let us denote by $\Lambda_2$ the projected lattice in $V / \text{lin}(f_1) \sim \text{lin}(f_2)$. From (2), we obtain immediately

$$S^L(a)(\xi_1 + \xi_2) = I(a_1)(\xi_1) \sum_{x_2 \in a_2 \cap \Lambda_2} e^{\langle \xi_2, x_2 \rangle}.$$  

Notice that the lattice $\Lambda_2$ is usually bigger than $\Lambda \cap \text{lin}(f_2)$.

Example 7 Let $V = \mathbb{R}^2$ with the standard lattice. We compute $S^L(a)$ when $a$ is a cone and $L$ is a line. Let $a = \mathbb{R}_+ v_1 + \mathbb{R}_+ v_2$, where $v_1, v_2$ are two linearly independent primitive integral vectors.

(a) Assume that $L$ is the line supporting an edge of $a$, say $L = \mathbb{R} v_1$. We identify $V / L$ to $\mathbb{R} v_2$. The projected lattice is $\Lambda_2 = \mathbb{Z} \frac{v_2}{\det(v_1, v_2)}$, (Example 6), hence, by (3) in Example 6, we have

$$S^L(a)(\xi) = -\frac{1}{\langle \xi, v_1 \rangle} \frac{1}{1 - e^{\langle \xi, v_2 \rangle / \det(v_1, v_2)}}.$$  

(b) Assume now that $L$ is transverse to both edges of $a$. Assume that $\det(v_1, v_2) > 0$. Let $L = \mathbb{R} u$ where $u$ is a primitive integral vector chosen so that $\det(u, v_2) > 0$. Let $a_i = \mathbb{R}_+ u + \mathbb{R}_+ v_i$ for $i = 1, 2$. We decompose the characteristic function of the cone $a$ as $\chi(a) = \chi(a_2) + \chi(a_1) - \chi(\mathbb{R}_+ u)$ or $\chi(a) = \chi(a_2) - \chi(a_1) + \chi(\mathbb{R}_+ v_1)$, depending on whether $u$ belongs to $a$ or not. Using the valuation property, case (a) and the relation

$$\frac{1}{1 - e^x} + \frac{1}{1 - e^{-x}} = 1,$$

we obtain in both cases

$$S^L(a)(\xi) = -\frac{1}{\langle \xi, u \rangle} \left( \frac{1}{1 - e^{\langle \xi, v_2 \rangle / \det(u, v_2)}} - \frac{1}{1 - e^{\langle \xi, v_1 \rangle / \det(u, v_1)}} \right).$$

In this example, we see that $\langle \xi, v_1 \rangle \langle \xi, v_2 \rangle S^L(a)(\xi)$ is indeed analytic near $\xi = 0$. 

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In the following theorem and its applications, we will consider the functions $S^L(p)$ when the space $V$ is replaced with a quotient space $W$. We denote by $C(W)$ the set of affine cones in $W$. Thus if $a \in C(W)$, and $L$ a rational subspace of $W$, the function $S^L(a)$ is a meromorphic function on $W^*$. We are going to show that the function $S^L(a)$ has a local Euler-Maclaurin expansion, which generalizes the case $L = \{0\}$ of [4].

**Theorem 8** Let $V$ be a rational space and $Q$ a rational scalar product on $V^*$. There exists a unique family of maps $\mu^L_W$, indexed by pairs $(W, L)$ where $W$ is a rational quotient space of $V$ and $L$ is a rational vector subspace of $W$ such that the family enjoys the following properties:

(a) $\mu^L_W$ maps $C(W)$ to $\mathcal{H}(W^*)$, the space of analytic functions on $W^*$.
(b) If $W = \{0\}$, then $\mu^{(0)}_W(\{0\}) = 1$.
(c) For $\dim W > 0$ and $L = W$, then $\mu^L_W(a) = 0$.
(d) If the affine cone $a \in C(W)$ contains a straight line, then $\mu^L_W(a) = 0$.
(e) For any affine cone $a$ in $W$, the following formula holds

$$S^L(a) = \sum_{f \in F(a)} \mu^{L+\dim(f)/\dim(f)}_{W/\dim(f)}(t(a, f))I(f)$$

where the sum is over all faces of the cone $a$.

In this last formula, the function $\mu^{L+\dim(f)/\dim(f)}_{W/\dim(f)}(t(a, f))$ is considered as a function on $W^*$ itself by means of the orthogonal projection $W^* \to (W/\dim(f))^* = (\dim(f))^\perp$ with respect to the scalar product on $W^* \subset V^*$.

**Proof.** The proof is entirely similar to the case $L = \{0\}$ studied in [4]. Note that $\mu^{(0)}_W$ coincides with the map denoted by $\mu^L_W$ in [4]. The only new item is (c). It follows immediately from the relation $S^W(a) = I(a)$.

**Remark 9** Let $a$ be a solid cone in $W$, and let $f$ be a face of $a$ such that $\dim f < \dim W$. If $L$ is transverse to the face $f$, that is, if $L + \dim(f) = W$, then $\mu^{L+\dim(f)/\dim(f)}_{W/\dim(f)}(t(a, f)) = 0$. This follows from (c).

From now on we omit the subscript $W$, thus we write $\mu^L$ in place of $\mu^L_W$. The next theorem and its proof are also entirely similar to the case $L = \{0\}$ in [4].
Theorem 10 The analytic functions defined in Theorem 8 have the following properties:

(a) For any \( x \in \Lambda \), one has \( \mu^L(x + a) = \mu^L(a) \).

(b) The map \((a, L) \mapsto \mu^L(a)\) is equivariant with respect to lattice-preserving isometries. In other words, let \( g \) be an isometry of \( W \) which preserves the lattice \( \Lambda_W \). Then \( \mu^{g(L)}(g(a))(g^{-1}\xi) = \mu^L(a)(\xi) \).

(c) If \( W \) is an orthogonal sum \( W = W_1 \oplus W_2 \) of rational spaces, \( L_i \subseteq W_i \) and \( a_i \) is an affine cone in \( W_i \) for \( i = 1, 2 \), then
\[
\mu^{L_1 \oplus L_2}(a_1 + a_2) = \mu^{L_1}(a_1)\mu^{L_2}(a_2).
\]

(d) For a fixed \( s \in W_Q \), the map \( c \to \mu^L(s + c) \) is a valuation on the set of cones in \( W \).

(e) Let \( p \subseteq W \) be a rational polyhedron, then
\[
S^L(p) = \sum_{f \in F(p)} \mu^{L + \text{lin}(f)/\text{lin}(f)}((t(p, f))I(f)).
\]

Example 11 Let us compute the function \( \mu^L \) for the various transverse cones of Example 8. We define a function \( B(u) \) on \( \mathbb{C} \), holomorphic near 0, by
\[
B(u) = \frac{1}{1 - e^u} + \frac{1}{u}.
\]
We have
\[
I(a)(\xi) = \frac{|\det(v_1, v_2)|}{\langle \xi, v_1 \rangle \langle \xi, v_2 \rangle}.
\]
Consider case (b) where \( L \) is transverse to both edges \( f_i = \mathbb{R}_+v_i \) of \( a \).

Using the equation \( \det(v_1, v_2)u = \det(u, v_2)v_1 - \det(u, v_1)v_2 \), we have
\[
I(a)(\xi) = \frac{1}{\langle \xi, u \rangle} \left( \frac{\det(u, v_2)}{\langle \xi, v_2 \rangle} - \frac{\det(u, v_1)}{\langle \xi, v_1 \rangle} \right).
\]
Thus we can rewrite (3) as
\[
S^L(a) = \mu^L(a) + \sum_{i=1,2} \mu^{L + \text{lin}(f_i)/\text{lin}(f_i)}((t(a, f_i))I(f_i) + I(a),
\]
with
\[
\mu^L(a)(\xi) = \frac{1}{\langle \xi, u \rangle} \left[ B \left( \frac{\langle \xi, v_1 \rangle}{\det(u, v_1)} \right) - B \left( \frac{\langle \xi, v_2 \rangle}{\det(u, v_2)} \right) \right],
\]
\[
\mu^{L + \text{lin}(f_i)/\text{lin}(f_i)}((t(a, f_i)) = 0 \text{ for } i = 1, 2.
\]
Observe that (7) is indeed regular at $\xi = 0$.

In case (a) where $L = \mathbb{R}v_1$, we have $L + \text{lin}(f_1)/\text{lin}(f_1) = \{0\}$. Let us assume that $\det(v_1, v_2) > 0$. Then we have, by (3),

$$\mu^{(0)}(t(a, f_1))(\xi) = B \left( \frac{-C_1 \langle \xi, v_1 \rangle + \langle \xi, v_2 \rangle}{\det(v_1, v_2)} \right)$$

with $C_1 = Q(v_1, v_1)/Q(v_1, v_2)$.

As $I(f_1)(\xi) = -1/\langle \xi, v_1 \rangle$, we can rewrite (3) as

$$S^L(a) = \mu^L(a) + \mu^{(0)}(t(a, f_1))I(f_1) + \mu^{L + \text{lin}(f_1)/\text{lin}(f_1)}((t(a, f_2))I(f_2) + I(a),$$

with

$$\mu^L(a)(\xi) = \frac{1}{\langle \xi, v_1 \rangle} \left[ B \left( \frac{-C_1 \langle \xi, v_1 \rangle + \langle \xi, v_2 \rangle}{\det(v_1, v_2)} \right) - B \left( \frac{\langle \xi, v_2 \rangle}{\det(v_1, v_2)} \right) \right],$$

which is indeed regular at $\xi = 0$, and, again,

$$\mu^{L + \text{lin}(f_1)/\text{lin}(f_2)}((t(a, f_2)) = 0.$$

### 3 Barvinok valuations

Let $\mathcal{L}$ be a finite family of rational vector subspaces of $V$, and let $\rho(L), L \in \mathcal{L}$, be a set of rational coefficients. The linear combination $\sum_{L \in \mathcal{L}} \rho(L)S^L(p)$ is again a valuation on the set of polyhedra $p \subset V$, with values in $\mathcal{M}_\ell(V^*)$. By taking linear combinations, we obtain a local Euler-Maclaurin expansion for the function $\sum_{L \in \mathcal{L}} \rho(L)S^L(p)(\xi)$.

**Definition 12** Let $p \subset V$ be a polyhedron.

(a) We denote

$$S^{(\mathcal{L}, \rho)}(p) = \sum_{L \in \mathcal{L}} \rho(L)S^L(p).$$

(b) We define the $f$-term in the local Euler-Maclaurin expansion of $S^{(\mathcal{L}, \rho)}(p)$ to be

$$\mu^{(\mathcal{L}, \rho)}(t(p, f)) = \sum_{L \in \mathcal{L}} \rho(L)\mu^{L + \text{lin}(f)/\text{lin}(f)}(t(p, f)).$$
Thus we have
\[ S^{(\mathcal{L}, \rho)}(p) = \sum_{f \in \mathcal{F}(p)} \mu^{(\mathcal{L}, \rho)}(t(p, f)) I(f). \]

We are going to compute the \( f \)-term in the case of the following particular linear combinations introduced by Barvinok [2].

**Definition 13** The valuation \( S^{(\mathcal{L}, \rho)} \) is called a Barvinok valuation if
(a) the family of subspaces \( \mathcal{L} \) is stable under sum,
(b) \( \rho \) is an integer valued function on the set \( \mathcal{L} \) such that the characteristic function of the union of the subspaces \( L^+ \subseteq V^* \) can be written as the linear combination
\[ \chi(\bigcup_{L \in \mathcal{L}} L^+) = \sum_{L \in \mathcal{L}} \rho(L) \chi(L^+). \]

**Definition 14** We call a function \( \mathcal{L} \rightarrow \mathbb{Z} \) which satisfies (8) a patchwork function on \( \mathcal{L} \).

As the set of orthogonal subspaces \( L^+ \subseteq V^* \) is stable under intersection, a particular function \( \rho \) with this property can be computed in terms of the Moebius function of the partially ordered set \( \mathcal{L} \) ([7], vol I, section 3.7), as explained in [2].

Let us compute a patchwork function \( \rho \) in the following case. \( V = \mathbb{R}^d \) with standard basis \( e_i, i = 1, \ldots, d, \) and \( \mathcal{L}_{d,q} \) is the set of subspaces \( L_I = \bigoplus_{i \in I} \mathbb{R} e_i \) with cardinal \( |I| \geq q \). The function \( \rho_{d,q} \) defined below is actually the one associated to the Moebius function, but we will not need this fact.

We denote the binomial coefficient \( \binom{m!}{k!(m-k)!} \) by \( C_m^k \).

**Lemma 15** The function \( \rho_{d,q} \) on \( \mathcal{L}_{d,q} \) defined by
\[ \rho_{d,q}(L_I) = (-1)^{n-q} C_n^{q-1} \quad \text{if } |I| = n, \]
satisfies Equation (8).

**Proof.** If \( e^i \) is the dual basis, the orthogonal space \( L_I^+ \) is equal to \( \sum_{i \in I} \mathbb{R} e^i \). Let \( \xi = \sum_{i=1}^d \xi_i e^i \in \bigcup_{L \in \mathcal{L}_n} L^+ \). Let \( I_0 \) be the set of indices \( i \in [1, \ldots, d] \) such that \( \xi_i = 0 \). Then \( |I_0| \geq q \), and \( \xi \in L_J^+ \) if and only if \( I \subseteq I_0 \). Let \( |I_0| = N \). The value at \( \xi \) of the right-hand side of (8) is equal to
\[ E(N, q) = \sum_{I \subseteq I_0} \rho_{d,q}(L_I) = \sum_{n=q}^N (-1)^{n-q} C_n^q C_{n-1}^{q-1}. \]
We want to prove that $E(N, q) = 1$. Writing $n = q + i$, we have

$$E(N, q) = \sum_{i=0}^{N-q} (-1)^i \frac{N!}{(q+i)(N-q-i)!i!(q-1)!}.$$ 

Let us compute $(q-1)! \left( E(N+1, q) - E(N, q) \right)$. This is equal to

$$(-1)^{N+1-q} \frac{(N+1)!}{(N+1)(N+1-q)!} \cdot \sum_{i=0}^{N-q} (-1)^i \frac{1}{(q+i)!} \left( \frac{(N+1)!}{(N+1-q-i)!} - \frac{N!}{(N-q-i)!} \right)$$

$$= N! \sum_{i=0}^{N+1-q} (-1)^i \frac{N!}{i!(N+1-q-i)!} = 0.$$ 

We obtain $E(N, q) = E(q, q) = 1$. □

In the case of a Barvinok valuation, it turns out that the $f$-term in the Euler-Maclaurin expansion of $S(L, \rho)(p)$ coincides with that of $S(p)$, if the vector subspace $\text{lin}(f)$ belongs to the set $L$. This is the crucial result of the present article. It is an easy consequence of the following combinatorial lemma.

**Lemma 16** Let $\mathcal{L}$ be a finite family of vector subspaces of $V$, stable under sum and let $\rho$ be a patchwork function on $\mathcal{L}$.

(a) Let $L_0 \in \mathcal{L}$. Then
\[ \sum_{\{L \in \mathcal{L}, L \subseteq L_0\}} \rho(L) = 1. \]

(b) Let $L_0 \nsubseteq L_1$ be two subspaces in the family $\mathcal{L}$. Then
\[ \sum_{\{L \in \mathcal{L}, L + L_0 = L_1\}} \rho(L) = 0. \]

**Proof.** There exists a $\xi_0 \in L_0^\perp$ such that, for $L \in \mathcal{L}$, $\xi_0 \in L^\perp$ if and only if $L \subseteq L_0$. We obtain (a) by evaluating both sides of (8) at this particular element $\xi_0$. Next, we deduce (b) from (a), by induction on $\dim L_1 - \dim L_0$.

For two subspaces $M \subseteq M'$ in the family $\mathcal{L}$, let us denote
\[ f(M, M') = \sum_{\{L \in \mathcal{L}, L+M = M'\}} \rho(L). \]
If $M = M'$, we have $f(M, M) = \sum_{L \in \mathcal{L}, L \subseteq M} \rho(L) = 1$ by (a).

We apply this with $M = L_1$. Thus

$$\sum_{\{L \in \mathcal{L}, L \subseteq L_1\}} \rho(L) = 1.$$ 

In this sum, we group the $L \in \mathcal{L}$ such that $L + L_0$ is equal to a given $M_1 \in \mathcal{L}$ together.

First we consider the case when $\dim L_1 - \dim L_0 = 1$. Then $M_1$ is either $L_0$ or $L_1$, hence we obtain

$$f(L_0, L_0) + f(L_0, L_1) = 1.$$ 

Since $f(L_0, L_0) = 1$ by (a), we obtain $f(L_0, L_1) = 0$ as required.

Next we consider the case when $\dim L_1 - \dim L_0 > 1$. We obtain

$$\sum_{\{M_1 \in \mathcal{L}, L_0 \subseteq M_1 \subseteq L_1\}} f(L_0, M_1) = 1.$$ 

For $M_1 = L_0$, we have $f(L_0, L_0) = 1$ by (a). For $M_1 \not\subseteq L_1$, we have $f(L_0, M_1) = 0$ by induction hypothesis. Hence there remains only the term $f(L_0, L_1)$ which must be equal to 0. □

We study now the Euler-Maclaurin expansion of a Barvinok valuation.

**Theorem 17** Let $p \subset V$ be a rational polyhedron and let $f$ be a face of $p$. Let $\mathcal{L}$ be a finite family of rational vector subspaces of $V$, stable under sum. Let $\rho$ be a patchwork function on $\mathcal{L}$, and let $S(\mathcal{L}, \rho) = \sum_{L \in \mathcal{L}} \rho(L)S^L$.

Assume that $\text{lin}(f)$ belongs to $\mathcal{L}$. Then

$$\mu(\mathcal{L}, \rho)(t(p, f)) = \mu^{[0]}(t(p, f)).$$

In other words, the $f$-term in the local Euler-Maclaurin expansion of $S(\mathcal{L}, \rho)(p)$ coincides with that of $S(p)$.

**Proof.** In the sum $\sum_{L \in \mathcal{L}} \rho(L)\mu^{L+\text{lin}(f)/\text{lin}(f)}(t(p, f))$, we group the terms for which $L + \text{lin}(f)$ is equal to a given $L_1$ together. By Lemma [10] we obtain $\mu^{[0]}(t(p, f))$ for $L_1 = \text{lin}(f)$ and 0 otherwise. □
Corollary 18 Let $p \subseteq V$ be a rational polyhedron. Let $0 \leq k \leq d$. Let $\mathcal{L}$ be a finite family of rational vector subspaces of $V$, stable under sum, such that $\text{lin}(f) \in \mathcal{L}$ for every $k$-dimensional face $f$ of $p$ and let $\rho$ be a patchwork function on $\mathcal{L}$.

- Let $0 < k \leq d$. Then the meromorphic function
  \[ S(p)(\xi) - S^{(\mathcal{L}, \rho)}(p)(\xi) \]
  has lowest degree $\geq -k + 1$.
- Let $k = 0$. Then
  \[ S(p)(\xi) = S^{(\mathcal{L}, \rho)}(p)(\xi). \]

Proof. By Theorem 17, the local Euler-Maclaurin expansion of the difference involves only faces of dimension $< k$.

\[ S(p)(\xi) - S^{(\mathcal{L}, \rho)}(p)(\xi) = \sum_{\{f \in F(p), \dim f < k\}} \left( \mu^{[0]}(t(p, f))(\xi) - \mu^{(\mathcal{L}, \rho)}(t(p, f))(\xi) \right) I(f)(\xi). \]

For a face of dimension $j$, the function $I(f)(\xi)$ is homogeneous of degree $-j$. Multiplied by the holomorphic function $\mu^{[0]}(t(p, f))(\xi) - \mu^{(\mathcal{L}, \rho)}(t(p, f))(\xi)$, its lowest degree can only increase. $\square$

Remark that the statement of Corollary 18 above does not involve any scalar product. In the next section, we will show that this corollary implies our main Theorem (Theorem 20).

4 Ehrhart quasi-polynomial

Let $p$ be a rational polytope and let $h(x)$ be a polynomial function on $V$. Let

\[ S(p, h) = \sum_{x \in p \cap \mathbb{Z}^d} h(x). \]

When $p$ is dilated by a non negative integer $n$, we obtain the Ehrhart quasi-polynomial of the pair $(p, h)$

\[ S(np, h) = \sum_{m=0}^{d+N} E_m(p, h, n)n^m, \]
where $N = \deg h$. The coefficients $E_m(p, h, n)$ are periodic functions of $n \in \mathbb{N}$, with period the smallest integer $q$ such that $q \mathbf{p}$ is a lattice polytope.

If an integer $r \leq d$ is fixed, and $h = 1$, Barvinok proved that the $r + 1$ highest Ehrhart coefficients $E_d(p, 1, n), \ldots, E_{d-r}(p, 1, n)$ of $S(np, 1)$ can be computed in polynomial time with respect to $d$, when $p$ is a rational simplex.

Let $L \subseteq V$ be a rational vector subspace. Denote the projected lattice on $V/L$ by $\Lambda_L$. Consider the mixed valuation

$$S^L(p, h) = \sum_{y \in \Lambda_{V/L}} \int_{p \cap (y + L)} h(x) dx.$$ 

As shown by Barvinok, and as we will see here, we can use linear combination of these mixed valuations to approximate $S(np, h)$ when $n$ is big.

For any polyhedron $a$, we define the meromorphic function $S^a(h)(\xi) \in \mathcal{M}_d(V^*)$ similarly to $S^L(p, h)$. For $\xi \in V^*$ such that the sum converges, we have

$$S^a(h)(\xi) = \sum_{y \in \Lambda_{V/L}} \int_{a \cap (y + L)} h(x) e^{\langle \xi, x \rangle} dm_{L}(x).$$ 

Remark 19 It is clear that $S^a(h)(\xi) = h(\partial_\xi) \cdot S^a(h)(\xi)$.

If $p$ is a polytope, then $S^L(p, h)(\xi)$ is regular at $\xi = 0$ and $S^L(p, h)(0) = S^L(p, h)$.

For a family $\mathcal{L}$ and a function $L \mapsto \rho(L), L \in \mathcal{L}$, we define

$$S^{(\mathcal{L}, \rho)}(a, h)(\xi) = \sum_{L \in \mathcal{L}} \rho(L) S^L(a, h)(\xi).$$

If $p$ is a polytope, and we dilate by $n \in \mathbb{N}$, we have again a quasi-polynomial

$$S^{(\mathcal{L}, \rho)}(np, h)(0) = \sum_{m=0}^{d+N} E_m(\mathcal{L}, \rho, p, h, n)n^m.$$

We can replace the quasi-polynomial $S^{(\mathcal{L}, \rho)}(np, h)(0)$ by $q$ legal polynomials in the variable $u$, by splitting $\mathbb{N}$ into classes modulo $q$. Writing $n = qu + k$, for $k = 0, \ldots, q - 1$, we obtain the polynomial function of $u$:

$$S^{(\mathcal{L}, \rho)}((qu + k)p, h)(0) = \sum_{m=0}^{d+N} E_m^{(k)}(\mathcal{L}, \rho, p, h) u^m.$$
We briefly recall how the usual Ehrhart quasi-polynomial of a polytope can be computed using Brion’s theorem. We will then use a similar method in order to compute efficiently the \( r + 1 \) highest coefficients only, using Barvinok valuations.

Let \( \mathcal{V}(p) \) be the set of vertices of \( p \). For each vertex \( s \), let \( c_s \) be the cone of feasible directions of \( p \) at \( s \), so that the supporting cone at \( s \) is \( s + c_s \). By Brion’s theorem \[5\], we have

\[
S(p, h)(\xi) = \sum_{s \in \mathcal{V}(p)} S(s + c_s, h)(\xi).
\]

Let \( n \in \mathbb{N} \) and consider the dilated polytope \( np \). The supporting cone at the vertex \( ns \) is \( ns + c_s \). Let \( q \in \mathbb{N} \) such that \( qp \) is a lattice polytope and fix \( k \in \mathbb{N}, 0 \leq k \leq q - 1 \). Let \( n = qu + k \). As \( qus \) is an integral point, we have

\[
S((qu + k)s + c_s, h)(\xi) = e^{nu(\xi,s)} S^{c_s, h}(ks + c_s, h)(\xi).
\]

Expanding in powers of \( u \), we obtain

\[
S((qu + k)p, h)(\xi) = \sum_{m \geq 0} u^m q^m \sum_{s \in \mathcal{V}(p)} \langle \xi, s \rangle^m S(ks + c_s, h)(\xi).
\]

It follows that for each \( m \), the sum of meromorphic functions

\[
\frac{q^m}{m!} \sum_{s \in \mathcal{V}(p)} \langle \xi, s \rangle^m S(ks + c_s, h)(\xi)
\]

is actually analytic. Its value at \( \xi = 0 \) is obtained by taking the zero degree term. We obtain

\[
S((qu + k)p, h)(0) = \sum_{m \geq 0} E_m^{(k)}(p, h) u^m,
\]

with

\[
E_m^{(k)}(p, h) = \frac{q^m}{m!} \sum_{s \in \mathcal{V}(p)} \langle \xi, s \rangle^m S(ks + c_s, h)_{[-m]}(\xi).
\]

The right-hand side of this relation, a priori a meromorphic function of \( \xi \), is actually constant. Moreover, we have \( S(ks + c_s, h)_{[-m]}(\xi) = 0 \) if \( m > d + N \), hence the Ehrhart quasi-polynomial has degree \( \leq d + N \).
We apply now Brion’s theorem to $S_{\mathcal{L},\rho}(p,h)(\xi)$. We obtain

$$S_{\mathcal{L},\rho}(p,h)(\xi) = \sum_{s \in V(p)} S_{\mathcal{L},\rho}(s + c_s,h)(\xi).$$

For reasons to be explained later on, instead of one family $\mathcal{L}$, we take a family of subspaces $\mathcal{L}_s$ for each vertex $s$. Let $\rho_s : \mathcal{L}_s \to \mathbb{Z}$ be a function on $\mathcal{L}_s$. We denote now by $(\mathcal{L},\rho)$ the map $s \mapsto (\mathcal{L}_s,\rho_s)$.

We define:

$$B_{\mathcal{L},\rho}(p,h)(\xi) = \sum_{s \in V(p)} S_{\mathcal{L},\rho}(s + c_s,h)(\xi).$$

If the family does not depend on $s$, $(\mathcal{L}_s,\rho_s) = (\mathcal{L}_0,\rho_0)$ for every vertex $s$, then, by Brion’s theorem, we have

$$B_{\mathcal{L},\rho}(p,h)(\xi) = S_{\mathcal{L}_0,\rho_0}(p,h)(\xi).$$

We dilate (9). Let $n = qu + k$. We obtain

$$B_{\mathcal{L},\rho}((qu+k)p,h)(\xi) = \sum_{s \in V(p)} e^{qu(\xi,s)} S_{\mathcal{L},\rho}(ks + c_s,h)(\xi).$$

Expanding in powers of $u$, we obtain

$$B_{\mathcal{L},\rho}((qu+k)p,h)(\xi) = \sum_{m \geq 0} u^m E_m^{(k)}(\mathcal{L},\rho,p,h)(\xi)$$

with

$$E_m^{(k)}(\mathcal{L},\rho,p,h)(\xi) = \frac{q^m}{m!} \sum_{s \in V(p)} \langle \xi, s \rangle^m S_{\mathcal{L},\rho}(ks + c_s,h)(\xi).$$

If the family does not depend on $s$, $(\mathcal{L}_s,\rho_s) = (\mathcal{L}_0,\rho_0)$ for all vertices, then $B_{\mathcal{L},\rho}(p,h)(\xi) = S_{\mathcal{L}_0,\rho_0}(p,h)(\xi)$ is analytic near $\xi = 0$, and so are the coefficients (11).

On the contrary, if we take a different family $\mathcal{L}_s$ for each vertex $s$, the coefficient $E_m^{(k)}(\mathcal{L},\rho,p,h)(\xi)$ of $u^m$ in (11) is no longer analytic near $\xi = 0$, in general. However, the meromorphic function $\xi \mapsto E_m^{(k)}(\mathcal{L},\rho,p,h)(\xi)$ belongs to $\mathcal{M}_\ell(V^*)$, thus it has a term of degree 0 with respect to $\xi$, given by

$$E_m^{(k)}(\mathcal{L},\rho,p,h)|_0(\xi) = \frac{q^m}{m!} \sum_{s \in V(p)} \langle \xi, s \rangle^m S_{\mathcal{L},\rho}(ks + c_s,h)|_{-m}(\xi).$$
For a family \((\mathcal{L}, \rho)\) as described in the next theorem, it turns out that, for large \(m\), this zero-degree part \(E_m^{(k)}(\mathcal{L}, \rho, p, h)|_0(\xi)\) is actually analytic, hence constant, and its value is equal to the \(m\)-th Ehrhart coefficient \(E_m^{(k)}(p, h)\) of \(S((k + qu)p, h)(0)\).

**Theorem 20** Let \(p\) be a rational polytope in a rational vector space of dimension \(d\). For each vertex \(s\) of the polytope \(p\), let \(c_s\) be the cone of feasible directions of \(p\) at \(s\), so that the supporting cone at \(s\) is \(s + c_s\). For each vertex \(s\), let \(\mathcal{L}_s\) be a finite family of rational vector subspaces of \(V\), stable under sum, such that \(\text{lin}(f)\) belongs to \(\mathcal{L}_s\) for every face \(f\) of codimension \(r\) of the cone \(c_s\), and let \(\rho_s\) be a patchwork function on \(\mathcal{L}_s\). Let \(q \in \mathbb{N}\) such that \(q\rho\) is a lattice polytope and fix \(k \in \mathbb{N}\), \(0 \leq k \leq q - 1\). Let \(h(x)\) be a homogeneous polynomial of total degree \(N\).

Then, for \(m \geq d + N - r\), the zero-degree term \(E_m^{(k)}(\mathcal{L}, \rho, p, h)|_0(\xi)\) defined by \((12)\) is regular near \(\xi = 0\), hence constant. Its value is the coefficient \(E_m^{(k)}(p, h)\) of \(u^m\) in the Ehrhart quasi-polynomial

\[
S((k + qu)p, h)(0) = \sum_{x \in ((k + qu)p) \cap \Lambda} h(x) = \sum_{m=0}^{d+N} u^m E_m^{(k)}(p, h).
\]

**Proof.** We first consider the case \(h(x) = 1\). We have, for every \(m \geq 0\),

\[
E_m^{(k)}(p, 1) = \frac{q^m}{m!} \sum_{s \in V(p)} \langle \xi, s \rangle^m S(ks + c_s)|_{-m}(\xi)
\]

where the right-hand side is actually a constant function of \(\xi\). For \(m > d - r - 1\), we have, by Corollary 18,

\[
S(ks + c_s)(\xi)|_{-m} = S^{\mathcal{L}, \rho_s}(ks + c_s)|_{-m}(\xi).
\]

This proves the theorem when \(h(x) = 1\). The case of a non constant polynomial \(h(x)\) is quite similar. If \(h(x) = x_1^{N_1} \ldots x_d^{N_d}\), we just have to replace the meromorphic functions \(S(ks + c_s)(\xi)\) and \(S^{\mathcal{L}, \rho_s}(ks + c_s)(\xi)\) by their derivatives under \(\partial_{\xi_1}^{N_1} \ldots \partial_{\xi_d}^{N_d}\).

If for each vertex \(s\), we take \(\mathcal{L}_s = \mathcal{L}\), the full collection generated by all \(r\) codimensional faces of \(p\), we obtain Corollary 21 below, that is Barvinok’s theorem \([2]\), with an extension to the sum of values of any polynomial \(h(x)\) over the set of integral points of a rational polytope (Barvinok considers only the case \(h(x) = 1\)).
Corollary 21 Let $p \subset V$ be a rational polytope and let $h(x)$ be a polynomial function on $V$. Let $\mathcal{L}$ be a finite family of rational vector subspaces of $V$, stable under sum. Assume that $\text{lin}(f)$ belongs to $\mathcal{L}$ for every face $f$ of codimension $r$ of $p$. Let $p$ be a patchwork function on $\mathcal{L}$ and let $S^{L,p} = \sum_{L \in \mathcal{L}} \rho(L) S^L$. Then the $r + 1$ highest Ehrhart coefficients of $S(tp,h)(0)$ and $S^{L,p}(tp,h)(0)$ are equal.

The point in taking a family $\mathcal{L}_s$ which depends on the vertex $s$ lies in the case where $p$ is simplicial. In this case, we can take $\mathcal{L}_s$ to be just the set of subspaces $\text{lin}(f)$, for all faces $f$ of codimension $\leq r$ of the supporting cone $c_s$ at vertex $s$. This family is stable under sum. Moreover the patchwork function on $\mathcal{L}_s$ is simple, (Lemma 15) and the computation of the function $S^L(ks + c_s)(\xi)$, when $L \in \mathcal{L}_s$, is immediately reduced (Example 6) to the computation of a function $S(a)(\xi)$ for a simplicial cone $a$ in a rational vector space of dimension smaller or equal than $r$. When $p$ is a simplex, we obtain in this way a method for computing the $r + 1$ highest Ehrhart coefficients for the pair $(p,h)$.

5 Local Euler-Maclaurin formula for mixed sums

Finally in this last section, we discuss an application of the existence of the coefficients $\mu^L$ (Theorem 8) in the line of [4].

Let $p$ be a rational polytope in a rational vector space $V$ of dimension $d$ and let $h(x)$ be a polynomial function on $V$. Let $L$ be a rational subspace of $V$. Consider the mixed sum

$$S^L(p,h) = \sum_{y \in A_{V/L}} \int_{p(y+L)} h(x) dm_L(x).$$

As in [4], we associate to the analytic function $\mu^L(t(p,f))$ a constant coefficients differential operator (of infinite order) on $V$.

Definition 22 Let $f$ be a face of $p$. We denote by $D^L(p,f)$ the differential operator on $V$ associated to analytic function $\mu^L(t(p,f))$:

$$D^L(p,f)(\partial_{\xi} \cdot e^{\langle \xi,x \rangle}) = \mu^L(t(p,f))(\xi)e^{\langle \xi,x \rangle}.$$
The operators $D^L(p,f)$ are local, that is they depend only of the transverse cone $t(p,f)$ of $p$ along $f$, and they involve only derivatives in directions orthogonal to the face $f$. We can state the following theorem with the same proof as in [4].

**Theorem 23 (Local Euler-Maclaurin formula)** Let $p$ be a polytope in $V$. For any polynomial function $h(x)$ on $V$, we have

\[(13) \quad S^L(p,h) = \sum_{f \in F(p)} \int_{f} D^L(p,f) \cdot h\]

where the integral on the face $f$ is taken with respect to the Lebesgue measure on $<f>$ defined by the lattice $\Lambda \cap \text{lin}(f)$.

In particular, for $h = 1$, we obtain

\[(14) \quad S^L(p,1) = \sum_{f \in F(p)} \mu^L(t(p,f))(0) \text{ vol}(f).\]

Let us dilate the polytope $p$ by a non negative integer $n$. If $f$ is a face of $p$, let $q_f$ be the smallest positive integer such that $q_f < f >$ contains integral points. Define $D(p,f,n) = D(np,nf)$, if $n > 0$, and $D(p,f,0) = D(q_fp,q_ff)$. The function $n \mapsto D(p,f,n)$ is periodic of period $q_f$.

**Proposition 24** Let $p$ be a rational polytope and $h$ a polynomial function of degree $N$ on $V$. Then, for any integer $n \geq 0$, we have

\[(15) \quad S^L(np,h) = \sum_{f \in F(p)} \int_{nf} D^L(p,f,n) \cdot h.\]

Furthermore, if $f \in F(p)$, we have

\[
\int_{nf} D^L(p,f,n) \cdot h = \sum_{i=\text{dim } f}^{\text{dim } f + N} E_i(p,h,f,n) n^i
\]

where the coefficients $E_i(p,h,f,n)$ are periodic with period $q_f$.

Hence the Ehrhart coefficients are given by

\[
E^L_m(p,h,n) = \sum_{f, \text{dim } f \leq m} E_m(p,h,f,n).
\]
When we apply the last proposition to the function $h(x) = 1$, we obtain

$$S^L(np, 1) = \sum_{f \in F(p)} \mu^L(nt(p, f))(0) \text{vol}(f) n^\dim f.$$  \hfill (16)

As $\mu^L(nt(p, f))$ is invariant by integral translations, the function $\mu^L(nt(p, f))(0)$ is of period $q_f$.

References


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