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SEMPARAMETRIC FREQUENCY ESTIMATION FROM IRREGULARLY SAMPLED OBSERVATIONS

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Abstract: We consider the problem of estimating the period of an unknown periodic function observed in additive noise sampled at irregularly spaced time instants in a semiparametric setting. This problem has been tackled in Hall, Reimann and Rice (2000), but the proposed estimator requires a preliminary estimator of the period whose rate of convergence is assumed to be closed to optimality. In this work, we introduce a novel estimator based on the cumulated Lomb-Scargle periodogram; this estimator does not rely on a preliminary estimator and is able to solve the ambiguities - e.g. pitch halving or doubling - which is inherent to the frequency estimation problem. We establish that this estimator converges at the parametric rate, is an asymptotically Gaussian estimator of the period and provides an explicit expression of the asymptotic variance. A limited Monte-Carlo experiment is presented on both synthetic data and variable stars data to support our claims.

Key words and phrases: Period estimation; Frequency estimation; irregular sampling; semiparametric estimation; cumulated Lomb-Scargle periodogram.

1. INTRODUCTION

The problem of estimating the frequency of a periodic function corrupted by additive noise is ubiquitous and has attracted a lot of research efforts in the last three decades. Up to now, most of these contributions have been devoted to regularly sampled observations; see e.g. Quinn and Hannan (2001) and the references therein. In many applications however, the observations are sampled at irregularly spaced time instants: examples occur in different fields, including among others biological rhythm research from free-living animals, see Ruf (1999), unevenly spaced gene expression time-series analysis, see Glynn, Chen and Mushegian (2006), or the analysis of brightness of periodic stars, see Hall, Reimann and Rice (2000), Thiebaut and Roques (2005). In the latter case, for example, irregular observations come from missing observations due to poor weather conditions (a star can be observed on most nights but not all nights), and because of the variability (sometimes intentional) of the observation times.

In the sequel, we consider the following model:

$$Y_j = s_\star(X_j) + \varepsilon_j, \quad j = 1, 2, \dots, \quad (1)$$

where s_\star is an unknown (real-valued) periodic function on \mathbb{R}_+ with least period T_\star , $\{X_k\}$ are the sampling instants and $\{\varepsilon_k\}$ is an additive noise. Our goal is to construct a consistent, rate optimal and easily computable estimator of the frequency $f_\star = 1/T_\star$ based on the observations $\{(X_i, Y_i)\}_{i=1, \dots, n}$ in a semiparametric setting, where s_\star belongs to some function space. To our best knowledge, the only attempt to rigorously derive such semiparametric estimator is due to Hall, Reimann and Rice (2000), who propose to use the least-squares criterion defined by $S(f) = \sum_{k=1}^n (Y_k - \hat{s}(X_k|f))^2$ where $\hat{s}(X_k|f)$ is a nonparametric kernel estimate of s_\star , given a frequency f . This estimator has been shown to converge at the parametric rate and to attain a semiparametric lower-bound which is the same if the shape of the periodic function is unknown as if it were known. A serious limitation of the proposed estimator stems from the fact that the consistency is obtained under the assumption that a preliminary estimator of the period is known, whose rate of convergence is already closed to be optimal (see Hall, Reimann and Rice (2000)[condition (c), p. 554]).

Our objective in this work is to construct an estimator attaining the optimal parametric rate, without assuming the existence of a preliminary estimator. The estimator considered in our work is based on the cumulated Lomb-Scargle periodogram, which is defined as

$$\Lambda_n(f) = \frac{1}{n^2} \sum_{k=1}^{K_n} \left| \sum_{j=1}^n Y_j e^{-2ik\pi f X_j} \right|^2, \quad (2)$$

where K_n denotes the number of cumulated harmonics, assumed to be slowly increasing with n . Our procedure might be seen as an extension of the algorithm by Quinn and Thomson (1991) where the periodogram is replaced by the Lomb-Scargle periodogram to account for irregular sampling time instants. Another difference stems from the fact that the estimator is not the global maximizer of the function $\Lambda_n(f)$, but rather an "approximate" maximizer in the neighborhood of multiple of the global maximizers, allowing to deal automatically with the inherent ambiguity of the frequency estimation problem (such a procedure has been introduced by Golubev (1988) in a frequency estimation problem from continuous observations). This estimator is computationally easier to implement than the estimator proposed by Hall, Reimann and Rice (2000) because the Lomb-Scargle periodogram can be very efficiently computed using Press, Teukolsky, Vetterling and Flannery (1992), p. 581. In addition, this estimator does not rely on a preliminary estimator of the period. We show that this estimator is rate optimal (i.e. attains the parametric rate); we also establish that the estimator is asymptotically normal and determine an expression of the asymptotic variance.

The paper is organized as follows. After relating $\Lambda_n(f)$ to the log-likelihood in a semi-parametric framework, we propose a frequency estimator based on maximizing $\Lambda_n(f)$ and show that it is consistent, rate optimal and asymptotically Gaussian; see Section 2. We then present some numerical experiments in Section 3. The remainder of the paper is devoted to auxiliary results and technical proofs used for showing the main results. A functional central limit theorem for periodic functions of random walks is presented in Section 4. Technical lemmas are given in Sections 5, 6 and Appendix A.

2. MAIN RESULTS

Define the Fourier coefficients of a locally integrable T_\star -periodic function s by

$$c_k(s) = \frac{1}{T_\star} \int_0^{T_\star} s(t) e^{-2ik\pi t/T_\star} dt, \quad k \in \mathbb{Z} \quad \text{so that} \quad s(t) = \sum_{p \in \mathbb{Z}} c_p(s) e^{2ip\pi t/T_\star}. \quad (3)$$

Recall that the least period T_\star (or equivalently $f_\star = 1/T_\star$, the associated frequency) of s_\star is here the parameter of interest. Consider the least-squares criterion based on observations $\{(X_i, Y_i)\}_{i=1, \dots, n}$,

$$L_n(f, \mathbf{c}) = \sum_{j=1}^n \left(Y_j - \sum_{k=0}^{K_n} c_k e^{2ik\pi f X_j} \right)^2, \quad \mathbf{c} = [c_{-K_n}, \dots, c_{K_n}]^T \quad (4)$$

where $\{K_n\}$ is the number of harmonics. For a given frequency f , the Fourier coefficients $\tilde{\mathbf{c}}_n(f) = [\tilde{c}_{-K_n}, \dots, \tilde{c}_{K_n}]$ which minimize (4) solve the system of equations $G_n(f) \tilde{\mathbf{c}}_n(f) = n \hat{\mathbf{c}}_n(f)$, where $G_n(f) = [G_{n,k,l}(f)]_{-K_n \leq k, l \leq K_n}$ is the Gram matrix with coefficients $G_{n,k,l}(f) = \sum_{j=1}^n e^{-2i(k-l)\pi f X_j}$ and $\hat{\mathbf{c}}_n(f) = [\hat{c}_{-K_n}(f), \dots, \hat{c}_{K_n}(f)]$ with

$$\hat{c}_l(f) = \frac{1}{n} \sum_{j=1}^n Y_j e^{-2il\pi f X_j}. \quad (5)$$

An estimator for the frequency f can then be obtained by minimizing the residual sum of squares

$$f \mapsto L_n(f, \{\tilde{c}_k(f)\}) = \sum_{j=1}^n Y_j^2 - n^2 \tilde{\mathbf{c}}_n^T(f) G_n^{-1}(f) \hat{\mathbf{c}}_n(f). \quad (6)$$

Note that computing $\tilde{\mathbf{c}}_n(f)$ is numerically cumbersome when n is large since it requires to solve a system of $2K_n + 1$ equations for each value of the frequency f where the function $L_n(f, \{\tilde{c}_k(f)\})$ should be evaluated. In Lemma 4, we shall prove that asymptotically in n the Gram matrix $G_n(f) \approx n \text{Id}_{2K_n+1}$, where Id_p denotes the $p \times p$ identity matrix; this suggests to approximate $L_n(f, \{\tilde{c}_k(f)\})$ by $f \mapsto \sum_{j=1}^n Y_j^2 - n \sum_{|k| \leq K_n} |\hat{c}_k(f)|^2$. The minimization of this quantity is equivalent to maximizing the cumulated periodogram Λ_n defined by (2).

Let $0 < f_{\min} < f_{\max}$ and define \tilde{f}_n as a maximizer of Λ_n ,

$$\Lambda_n(\tilde{f}_n) = \tilde{\Lambda}_n, \quad \text{where} \quad \tilde{\Lambda}_n = \sup_{f \in [f_{\min}, f_{\max}]} \Lambda_n(f). \quad (7)$$

Since the function s_\star is also (j/f_\star) -periodic for all $j = 1, 2, \dots$, the global maximum may be associated to a sub-multiple of f_\star . To avoid this problem, we will look in the neighborhood of multiples of the absolute maximizer \tilde{f}_n for the existence of an approximate maximizer. For this purpose, we consider the intervals

$$\tilde{B}_n(j) = \{f \in [f_{\min}, f_{\max}] : |f - j\tilde{f}_n| \leq r\} \quad \text{with} \quad 0 < r < f_{\min}/2 \quad (8)$$

and define an estimator \hat{f}_n of f_\star by

$$\hat{f}_n \in \text{Arg sup}_{f \in \tilde{B}_n(\hat{J}_n)} \Lambda_n(f) \quad \text{with} \quad \hat{J}_n = \max \left\{ j : \sup_{f \in \tilde{B}_n(j)} \Lambda_n(f) \geq (1 - \kappa_n) \tilde{\Lambda}_n \right\}, \quad (9)$$

where $\{\kappa_n\}$ is a sequence taking its values in the interval $(0, 1)$ such that $\lim_{n \rightarrow \infty} \kappa_n = 0$. A similar idea is used in Golubev (1988), but without the local maximization over the interval $\tilde{B}_n(j)$ (in other words with $r = 0$). We introduce the following assumptions on the distribution of the increment $V_k = X_k - X_{k-1}$ of the renewal process $\{X_k\}$.

- (H1) s_\star is an unknown (real-valued) periodic function defined on \mathbb{R}_+ with least period T_\star , defined as the smallest T such that $s_\star(t + T) = s_\star(t)$ for all t ,
- (H2) $\{X_j\}$ are the observation time instants, modeled as a renewal process, that is, $X_j = \sum_{k=1}^j V_k$, where $\{V_k\}$ is an i.i.d sequence of non-negative random variables with finite mean. In addition, for all $\epsilon > 0$, $\sup_{|t| \geq \epsilon} |\Phi(t)| < 1$, where Φ denotes the characteristic function of V_1 ,

$$\Phi(t) = \mathbb{E}[\exp(itV_1)]. \quad (10)$$

- (H3) $\{\varepsilon_j\}$ are i.i.d. zero mean Gaussian random variables with (unknown) variance $\sigma_\star^2 > 0$ and are independent from the random variables $\{X_j\}$.
- (H4) The distribution of V_1 has a non-zero absolutely continuous part with respect to the Lebesgue measure.

Assumption (H2) is a Cramer's type condition, which is weaker than (H4). Assumption (H4) ensures that, for any constant $c > 0$, the process $\{\langle c^{-1}X_j \rangle, j \geq 1\}$, where $\langle x \rangle$ denotes the fractional part of x , is ergodic and converges to the uniform distribution on the interval $[0, c]$. We now prove that our estimator is consistent.

Theorem 1. *Assume (H1) to (H3), and that $f_\star \in (f_{\min}, f_{\max})$, for some $0 < f_{\min} < f_{\max}$. Assume in addition that s_\star satisfies*

$$\sum_{k \in \mathbb{Z}} |c_k(s_\star)| < +\infty. \quad (11)$$

Let $\{K_n\}$ be a sequence of positive integers tending to infinity and $\{\kappa_n\}$ be a $(0, 1)$ -valued sequence tending to zero such that

$$\lim_{n \rightarrow +\infty} K_n n^{-1/2+\beta} = 0 \quad \text{for some } \beta > 0, \quad (12)$$

and

$$\lim_{n \rightarrow +\infty} \kappa_n^{-1} \{R(K_n)^2 + K_n R(n^\delta) + n^{-\epsilon}\} = 0 \quad \text{for some } \delta > 0 \text{ and all } \epsilon > 0, \quad (13)$$

where

$$R(m) = \sum_{|k| > m} |c_k(s_\star)|, \quad m \geq 0. \quad (14)$$

Then \hat{f}_n defined by (7)–(9) is a consistent estimator of f_\star . Moreover, we have, for any $\alpha > 0$,

$$\hat{f}_n = f_\star + o_p(n^{-1+\alpha}) \quad \text{and} \quad \mathbb{P}(\hat{\Lambda}_n(\hat{f}_n) = 0) \rightarrow 1 \quad \text{as } n \rightarrow \infty. \quad (15)$$

If we assume in addition that $\mathbb{E}(V_1^2)$ is finite, then

$$n(\hat{f}_n - f_\star) \rightarrow 0 \quad \text{a.s.} \quad (16)$$

Remark 1. Observe that (11) is a very mild assumption. Once it is satisfied, we have $R(m) \rightarrow 0$ as $m \rightarrow \infty$ and one may always construct sequences $\{K_n\}$ and $\{\kappa_n\}$ satisfying the required conditions (12) and (13). Under the additional assumption that there exists $\nu > 0$ such that

$$\sup_{m \geq 0} (1+m)^\nu \sum_{|k| \geq m} |c_k(s_\star)| < \infty, \quad (17)$$

one may for instance choose any $\{K_n\}$ such that $(\log(n))^{1/(2\nu)} = o(K_n)$ and $K_n = o(n^{1/2-\beta})$ for some $\beta > 0$ and $\kappa_n \asymp 1/\log(n)$.

Proof of Theorem 1. Using (1), we split Λ_n defined in (2) into three terms: $\Lambda_n(f) = D_n(f) + \xi_n(f) + \zeta_n(f)$ where

$$D_n(f) = \frac{1}{n^2} \sum_{k=1}^{K_n} \left| \sum_{j=1}^n s_\star(X_j) e^{-2ik\pi f X_j} \right|^2, \quad (18)$$

$$\xi_n(f) = \frac{2}{n^2} \sum_{k=1}^{K_n} \sum_{j,j'=1}^n \cos\{2\pi k f (X_j - X_{j'})\} s_\star(X_j) \varepsilon_{j'}, \quad (19)$$

$$\zeta_n(f) = \frac{1}{n^2} \sum_{k=1}^{K_n} \left| \sum_{j=1}^n \varepsilon_j e^{-2ik\pi f X_j} \right|^2. \quad (20)$$

The proof relies on intermediate lemmas establishing uniform bounds of these quantities which are postponed to Section 5 for convenience. In this proof we let $\alpha \in (0, 1)$ (chosen later) and denote by $\{\gamma_n\}$ the sequence

$$\gamma_n = n^{-1+\alpha}. \quad (21)$$

For any integer j and l , we denote by $B_n(j, l)$ and $B_n^c(j, l)$ the following sets

$$B_n(j, l) = \{f \in [f_{\min}, f_{\max}], |f - jf_{\star}/l| \leq \gamma_n\} \quad \text{and} \\ B_n^c(j, l) = \{f \in [f_{\min}, f_{\max}], |f - jf_{\star}/l| > \gamma_n\} \quad (22)$$

and define the set \mathcal{T}_n of near maximizers of Λ_n by $\mathcal{T}_n = \left\{f \in [f_{\min}, f_{\max}], \Lambda_n(f) \geq (1 - \kappa_n)\tilde{\Lambda}_n\right\}$. We first prove that \mathcal{T}_n satisfies

$$\lim_{n \rightarrow +\infty} \mathbb{P} \left(\mathcal{T}_n \subseteq \bigcup_{l=1}^L B_n(1, l) \right) = 1, \quad (23)$$

with

$$L = \sup\{l \geq 1 : B_n(1, l) \neq \emptyset\} \leq f_{\star}/(f_{\min} - \gamma_n) < f_{\star}/(2r), \quad (24)$$

where the second inequality holds for n large enough by (8). The limit in (23) can be equivalently written as $\lim_{n \rightarrow +\infty} \mathbb{P} \left(\sup_{f \in \bigcap_l B_n^c(1, l)} \Lambda_n(f) \geq (1 - \kappa_n)\tilde{\Lambda}_n \right) = 0$ and, for n large enough so that $K_n \geq L$, the probability in this display is less than $P_1 + P_2$ which are defined as

$$P_1 = \mathbb{P} \left(\sup_{f \in \bigcap_{j,l} B_n^c(j, l)} \Lambda_n(f) \geq (1 - \kappa_n)\tilde{\Lambda}_n \right) \quad \text{and} \quad P_2 = \mathbb{P} \left(\sup_{f \in \bigcup'_{j,l} B_n(j, l)} \Lambda_n(f) \geq (1 - \kappa_n)\tilde{\Lambda}_n \right),$$

where $\bigcap_{j,l}$ denotes the intersection over integers $j \geq 1$ and $l = 1, \dots, K_n$ and $\bigcup'_{j,l}$ the union over all $j \geq 2$ and $l = 1, \dots, K_n$ such that j and l are relatively prime. Note that P_1 is bounded above by

$$\mathbb{P} \left(\sup_{f \in \bigcap_{j,l} B_n^c(j, l)} D_n(f) + 2 \sup_{f \in [f_{\min}, f_{\max}]} |\xi_n(f) + \zeta_n(f)| \geq (1 - \kappa_n)D_n(f_{\star}) \right). \quad (25)$$

By (12), applying Lemma 1, we get

$$\sup_{f \in [f_{\min}, f_{\max}]} |\xi_n(f) + \zeta_n(f)| = o_p(n^{-\beta/2}). \quad (26)$$

We now apply Lemma 2. Using (12) again and choosing α small enough in (21), we have $K_n \gamma_n \rightarrow 0$ and, since $n \gamma_n \rightarrow \infty$ and $K_n \rightarrow \infty$, Condition (50) holds. By (13) we have $K_n R(n^\delta)^2 \rightarrow 0$ and, by (12) and (21), taking $m_n = n^\delta$ and ϵ small enough in Lemma 2, we obtain $\sup_{f \in \bigcap_{j,l} B_n^c(j, l)} D_n(f) = o_p(1)$. The last two displays show that the left-hand side of the inequality in (25) converges to zero in probability. Concerning its right-hand side $(1 - \kappa_n)D_n(f_{\star})$, Relation (56) with $j = l = 1$ in Lemma 3 shows that, as $n \rightarrow \infty$,

$$(1 - \kappa_n)D_n(f_{\star}) \xrightarrow{p} \sum_{k \geq 1} |c_k(s_{\star})|^2 > 0. \quad (27)$$

Hence $P_1 \rightarrow 0$. In the same way, we have

$$P_2 \leq \mathbb{P} \left(\sup_{f \in \bigcup_{j,l} B_n(j,l)} D_n(f) + 2 \sup_{f \in [f_{\min}, f_{\max}]} |\xi_n(f) + \zeta_n(f)| \geq (1 - \kappa_n) D_n(f_\star) \right).$$

To prove that P_2 converges to zero, we use the following classical inequality (see Golubev (1988) or Lévy-Leduc (2004, p. 49))

$$\sup_{j \geq 2} \sum_{k=1}^{\infty} |c_{kj}(s_\star)|^2 < \sum_{k=1}^{\infty} |c_k(s_\star)|^2, \quad (28)$$

which directly follows from the fact that f_\star is the maximal frequency of s_\star . Now, we apply Lemma 3. Using (12), Condition (58) holds by choosing α small enough in (21). By (13) we have $K_n R(n^\delta) \rightarrow 0$ and, by (12) and (21), taking $m_n = n^\delta$ and ϵ small enough in (57), we obtain, using (28), $\mathbb{P} \left(\sup_{f \in \bigcup_{j,l} B_n(j,l)} D_n(f) \geq \sum_{k=1}^{\infty} |c_k(s_\star)|^2 \right) \rightarrow 0$. Using (27), we get that $P_2 \rightarrow 0$. This concludes the proof of (23). Let us now prove that

$$\lim_{n \rightarrow +\infty} \mathbb{P}(f_\star \in \mathcal{T}_n) = 1. \quad (29)$$

This amounts to prove that $\lim_{n \rightarrow +\infty} \mathbb{P} \left(\Lambda_n(f_\star) < (1 - \kappa_n) \sup_{f \in [f_{\min}, f_{\max}]} \Lambda_n(f) \right) = 0$. Note that the latter probability is less than

$$\mathbb{P} \left(2 \sup_{f \in [f_{\min}, f_{\max}]} |\xi_n(f) + \zeta_n(f)| > D_n(f_\star) - (1 - \kappa_n) \sup_{f \in [f_{\min}, f_{\max}]} D_n(f) \right). \quad (30)$$

Observe that the right-hand side of the inequality in (30) can be written as

$$D_n(f_\star) - \sum_{k=1}^{\infty} |c_k(s_\star)|^2 - (1 - \kappa_n) \sup_{f \in [f_{\min}, f_{\max}]} \left(D_n(f) - \sum_{k=1}^{\infty} |c_k(s_\star)|^2 \right) + \kappa_n \sum_{k=1}^{\infty} |c_k(s_\star)|^2.$$

Relation (56) with $j = l = 1$ in Lemma 3 gives

$$\left| D_n(f_\star) - \sum_{k=1}^{\infty} |c_k(s_\star)|^2 \right| \leq \left| D_n(f_\star) - \sum_{k=1}^{K_n} |c_k(s_\star)|^2 \right| + R(K_n)^2 = O_p \left(K_n n^{-1/2} \right) + R(K_n)^2,$$

and Lemma 2 and Relation (57) with $m_n = n^\delta$ in Lemma 3 give

$$\begin{aligned} & \sup_{f_{\min} \leq f \leq f_{\max}} \left[D_n(f) - \sum_{k=1}^{\infty} |c_k(s_\star)|^2 \right] \\ & \leq \sup_{f \in \bigcap_{j,l} B_n^\epsilon(j,l)} D_n(f) + \sup_{f \in \bigcup_{j,l} B_n(j,l)} \left(D_n(f) - \sum_{k=1}^{\infty} |c_k(s_\star)|^2 \right) \\ & = O_p \left(K_n R(n^\delta) + (K_n n^{\delta+1})^\epsilon \{ (n\gamma_n)^{-1} + (K_n/n)^{1/2} \} + K_n^2/n \right). \end{aligned}$$

Choosing ϵ small enough so that $(K_n n^{\delta+1})^\epsilon \{ (n\gamma_n)^{-1} + (K_n/n)^{1/2} \} = O(n^{-\epsilon'})$ for some $\epsilon' > 0$, and using the condition (13) on $\{\kappa_n\}$, and (26), we get that the probability in (30) converges

to zero, and (29) is proved. We can now prove the consistency of \hat{f}_n and show (15). By definition of \hat{J}_n in (9), we have $\hat{J}_n \in \{1, \dots, f_{\max}/f_{\min}\}$ since $\tilde{B}_n(j) = \emptyset$ for larger j 's. The proof of (15) is presented in two steps. In the first step, we show that $\check{f}_n = \hat{J}_n \check{f}_n$ is a consistent estimator of f_\star . Then in the second step, the consistency of \hat{f}_n is proved.

Step 1. Since \check{f}_n belongs to \mathcal{T}_n , (23) leads to the existence of $1 \leq l_n \leq L$ defined in (24) such that $\check{f}_n \in B_n(1, l_n)$ with probability tending to 1. Thus, with probability tending to one, $|f_\star - l_n \check{f}_n| \leq r$. Using (29), we get that $\mathbb{P}(\tilde{B}_n(l_n) \cap \mathcal{T}_n \neq \emptyset) \rightarrow 1$ as $n \rightarrow \infty$. Observing that $\tilde{B}_n(l_n) \cap \mathcal{T}_n \neq \emptyset$ implies $\sup_{f \in \tilde{B}_n(l_n)} \Lambda_n(f) \geq (1 - \kappa_n) \tilde{\Lambda}_n$, we get $\hat{J}_n \geq l_n$ with probability tending to 1. On the other hand, by definition of l_n and \check{f}_n , we have, with probability tending to 1,

$$|\check{f}_n - \hat{J}_n f_\star / l_n| \leq \hat{J}_n \gamma_n \leq \frac{f_{\max}}{f_{\min}} \gamma_n. \quad (31)$$

Since, by definition, \hat{f}_n belongs to \mathcal{T}_n , (23) implies $\mathbb{P}(\hat{f}_n \in \bigcup_l B_n(1, l)) \rightarrow 1$ as $n \rightarrow \infty$ and since $\bigcup_l B_n(1, l) \subset [f_{\min}, f_\star + \gamma_n]$, we get $\check{f}_n \leq \hat{f}_n + r \leq f_\star + \gamma_n + r$ with probability tending to 1. This bound, with (31), implies

$$\hat{J}_n \leq \check{f}_n l_n / f_\star + \frac{f_{\max}}{f_{\min} f_\star} \gamma_n l_n \leq l_n \left\{ (f_\star + \gamma_n + r) / f_\star + \frac{f_{\max}}{f_{\min} f_\star} \gamma_n \right\}.$$

Since $l_n \leq L$, (24) implies $l_n r / f_\star < 1/2$, hence, with probability tending to one, $\hat{J}_n \leq l_n$. The converse inequality was shown previously implying, with (31), that $\check{f}_n = f_\star + o_p(1)$.

Step 2. By definition of \check{f}_n and \hat{f}_n , we have $\hat{f}_n \in \underset{f \in [f_{\min}, f_{\max}]: |f - \check{f}_n| \leq r}{\text{Arg sup}} \Lambda_n(f)$. Since $\check{f}_n = f_\star + o_p(1)$ and $r < f_{\min}/2 \leq f_\star/2$, we have with probability tending to 1, $\hat{f}_n \geq f_\star/2 + \epsilon$ for an arbitrarily small $\epsilon > 0$, which implies $\hat{f}_n \notin \bigcup_{l \geq 2} B_n(1, l)$. Moreover, by (23), since $\hat{f}_n \in \mathcal{T}_n$, $\hat{f}_n \in \bigcup_l B_n(1, l)$ with probability tending to 1 finally leading to $\hat{f}_n = f_\star + o_p(n^{-1+\alpha})$ for any positive α . It remains to show the right-hand side of (15). Since Λ_n is continuously differentiable, this follows from $f_\star \in (f_{\min}, f_{\max})$, $\check{f}_n = f_\star + o_p(1)$ and $\hat{f}_n = f_\star + o_p(1)$. We conclude this proof section by showing (16). We adapt the proof of consistency in Quinn and Thomson (1991) to our random design context by first showing that, for any $\epsilon > 0$,

$$\sup_{|t| \leq n^{-1/2-\epsilon}} \left| |\varphi_{n,X}(t)|^2 - \frac{1}{n} F_n(\mu t) \right| = o_p(1), \quad (32)$$

where $\mu = \mathbb{E}[V_1]$, $F_n(x) = \frac{1}{n} \left| \sum_{k=1}^n e^{ikt} \right|^2$ is the Fejer kernel and $\varphi_{n,X}$ the empirical characteristic function of X_1, \dots, X_n ,

$$\varphi_{n,X}(t) = \frac{1}{n} \sum_{j=1}^n e^{itX_j}. \quad (33)$$

Indeed, using a standard Lipschitz argument and (H2) with the assumption $\mathbb{E}[V_1^2] < \infty$, we have

$$\mathbb{E} \left[\sup_{|t| \leq n^{-1/2-\epsilon}} \left| |\varphi_{n,X}(t)|^2 - \frac{1}{n} F_n(\mu t) \right| \right] \leq 2n^{-1/2-\epsilon} \frac{1}{n} \sum_{k=1}^n \mathbb{E}[|X_k - k\mu|] \leq 2\sqrt{\text{Var}(V_1)} n^{-\epsilon},$$

which gives (32). Now, by definition of \hat{f}_n and \check{f}_n , on the event $\{|f_\star - \check{f}_n| \leq r\}$ whose probability tends to 1, we have $0 \leq \Lambda_n(\hat{f}_n) - \Lambda_n(f_\star)$. Beside, we have, using (26),

$$\Lambda_n(\hat{f}_n) - \Lambda_n(f_\star) \leq D_n(\hat{f}_n) - D_n(f_\star) + 2 \sup_{f \in [f_{\min}, f_{\max}]} |\xi_n(f) + \zeta_n(f)| = D_n(\hat{f}_n) - D_n(f_\star) + o_p(1)$$

and, since $\{\hat{f}_n \in B_n(1, 1)\}$ has probability tending to one, Lemma 3 yields, for α small enough in (21), $D_n(\hat{f}_n) - D_n(f_\star) \leq \sum_{k=1}^{K_n} |c_k(s_\star)|^2 [|\varphi_{n,X}\{2\pi k(f_\star - \hat{f}_n)\}|^2 - 1] + o_p(1)$. Hence, since for α small enough $K_n \gamma_n \leq n^{-1/2-\alpha/2}$, the last three displayed equations and (32) finally yield that, $0 \leq \sum_{k=1}^{K_n} |c_k(s_\star)|^2 \left[\frac{1}{n} F_n\{2\pi \mu k(f_\star - \hat{f}_n)\} - 1 \right] + o_p(1)$. The proof of (16) is concluded as in Quinn and Thomson (1991) by observing that, for any $c > 0$, $\limsup_{n \rightarrow \infty} \sup_{|t| > c/n} \frac{1}{n} F_n(t) < 1$. \square

Theorem 2. Assume (H1) to (H4). Assume in addition that $\mathbb{E}[V_1^4]$ is finite, that s_\star satisfies

$$\sum_{k \in \mathbb{Z}} |k|^3 |c_k(s_\star)| < +\infty \quad (34)$$

and that its maximal frequency f_\star belongs to (f_{\min}, f_{\max}) , where $0 < f_{\min} < f_{\max}$. Let $\{K_n\}$ and $\{\kappa_n\}$ be sequences of positive integers tending to infinity and zero respectively such that

$$\lim_{n \rightarrow +\infty} K_n n^{-\epsilon} = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \kappa_n n^\epsilon = \infty \quad \text{for all } \epsilon > 0. \quad (35)$$

Then the estimator \hat{f}_n defined by (7)–(9) satisfies the following asymptotic linearization:

$$n^{3/2}(\hat{f}_n - f_\star) = \frac{I_\star^{-1} \mu}{n^{3/2} \sigma_\star^2 f_\star} \sum_{j=1}^n \left(j - \frac{n}{2} \right) \dot{s}_\star(X_j) (\varepsilon_j + s_\star(X_j)) + o_p(1), \quad (36)$$

where $\mu = \mathbb{E}(V_1)$ and

$$I_\star = \frac{\mu^2}{12 \sigma_\star^2 f_\star} \int_0^{1/f_\star} \dot{s}_\star^2(t) dt. \quad (37)$$

Moreover \hat{f}_n satisfies the following Central Limit Theorem

$$n^{3/2}(\hat{f}_n - f_\star) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \check{\sigma}^2), \quad (38)$$

where

$$\check{\sigma}^2 = I_\star^{-1} \left\{ 1 + \frac{\sum_{k \neq 0} |c_k(s_\star \dot{s}_\star)|^2 \left(\frac{1 - |\Phi(2k\pi f_\star)|^2}{|1 - \Phi(2k\pi f_\star)|^2} \right)}{\sigma_\star^2 \sum_{k \in \mathbb{Z}} |c_k(\dot{s}_\star)|^2} \right\}, \quad (39)$$

with Φ , defined in (10), denotes the characteristic function of V_1 .

Remark 2. Let $\|\cdot\|_p$ denote the usual L^p norm on $[0, 1/f_\star]$ for $p \in [1, \infty]$. By the Parseval theorem, the ratio inside the curly brackets in (39) is bounded above by

$$\frac{2 \|s_\star \dot{s}_\star\|_2^2}{\sigma_\star^2 \|\dot{s}_\star\|_2^2} \sup_{k \neq 0} \frac{1 - |\Phi(2k\pi f_\star)|^2}{|1 - \Phi(2k\pi f_\star)|^2} \leq \frac{\|s_\star\|_\infty^2}{\sigma_\star^2} \sup_{k \neq 0} \frac{1 - |\Phi(2k\pi f_\star)|^2}{|1 - \Phi(2k\pi f_\star)|^2}. \quad (40)$$

Assumption (H2) implies that $\sup_{k \neq 0} |1 - \Phi(2k\pi f_\star)|^2 > 0$. On the other hand (34) implies that $\|\dot{s}_\star\|_2$ and $\|s_\star\|_\infty$ are both finite. Hence the bound (40) shows that the right-hand side in (39) is well defined. Moreover, when the signal to noise ratio $\|s_\star\|_\infty^2/\sigma_\star^2$ tends to zero, the asymptotic variance formula simplifies to $\check{\sigma}^2 = I_\star^{-1}$. We will see in Section 2.1 that this corresponds to the optimal asymptotic variance.

Remark 3. If the $\{V_k\}$ are exponentially distributed, then

$$\check{\sigma}^2 = I_\star^{-1} \left(1 + \frac{\|s_\star \dot{s}_\star\|_2^2}{\sigma_\star^2 \|\dot{s}_\star\|_2^2} \right).$$

Proof of Theorem 2. We postpone the proof of (36) to Section 6 for convenience. Here we prove that \hat{f}_n satisfies a Central Limit Theorem given by (38). Using (36), this amounts to prove the convergence of

$$S_n = \frac{I_\star^{-1} \mu}{n^{3/2} \sigma_\star^2 f_\star} \sum_{j=1}^n \left(j - \frac{n}{2} \right) \dot{s}_\star(X_j) (\varepsilon_j + s_\star(X_j))$$

to a Gaussian distribution $\mathcal{N}(0, \check{\sigma}^2)$. Using (H3), we have that $S_n \stackrel{d}{=} \frac{I_\star^{-1} \mu}{\sigma_\star^2 f_\star} (A_n Z + U_n)$ where $A_n = n^{-3/2} \left(\sum_{j=1}^n \left(j - \frac{n}{2} \right)^2 \dot{s}_\star^2(X_j) \right)^{1/2}$, $U_n = n^{-3/2} \sum_{j=1}^n \left(j - \frac{n}{2} \right) (s_\star \dot{s}_\star)(X_j)$ and Z has distribution $\mathcal{N}(0, \sigma_\star^2)$ and is independent from the X_j 's. Therefore, using that

$$I_\star = \frac{\mu^2}{12 \sigma_\star^2 f_\star^2} c_0(\dot{s}_\star^2), \quad (41)$$

and since Z and U_n are independent, (38) follows from the two assertions

$$A_n = \left(\frac{1}{12} c_0(\dot{s}_\star^2) \right)^{1/2} (1 + o_p(1)); \quad (42)$$

$$\frac{I_\star^{-1} \mu}{\sigma_\star^2 f_\star} U_n \xrightarrow{\mathcal{L}} \mathcal{N} \left(0, \frac{\sum_{k \neq 0} |c_k(s_\star \dot{s}_\star)|^2 \left(\frac{1 - |\Phi(2k\pi f_\star)|^2}{|1 - \Phi(2k\pi f_\star)|^2} \right)}{I_\star \sigma_\star^2 \sum_{k \in \mathbb{Z}} |c_k(\dot{s}_\star)|^2} \right). \quad (43)$$

We now prove (42) and (43) successively. Let us write

$$A_n^2 = n^{-3} \sum_{j=1}^n \left(j - \frac{n}{2} \right)^2 \{ \dot{s}_\star^2(X_j) - \mathbb{E}(\dot{s}_\star^2(X_j)) \} + n^{-3} \sum_{j=1}^n \left(j - \frac{n}{2} \right)^2 \mathbb{E}(\dot{s}_\star^2(X_j)).$$

Observing that $\mathbb{E}(\dot{s}_\star^2(X_j)) = c_0(\dot{s}_\star^2) + \sum_{p \in \mathbb{Z} \setminus \{0\}} c_p(\dot{s}_\star^2) \Phi^j(2p\pi f_\star)$, we get, using (H2), that the second term in the right-hand side of the previous display has limit $c_0(\dot{s}_\star^2)/12$. Using (45) in Theorem 3, we get that the first term tends to zero in probability. Hence (42) holds. Let us now prove (43). To alleviate the notations, let us write $g = s_\star \dot{s}_\star$ and remark that g thus defined is $(1/f_\star)$ -periodic and is such that $c_0(g) = 0$ and $\sum_k |c_k(g)| < \infty$. Let us write

$$\frac{1}{n^{3/2}} \sum_{j=1}^n \left(j - \frac{n}{2}\right) g(X_j) = \frac{1}{2n^{1/2}} \sum_{j=1}^n g(X_j) - \frac{1}{n^{3/2}} \sum_{k=1}^{n-1} \sum_{j=1}^k g(X_j) .$$

Let $s_n(t)$ denote the following piecewise linear interpolation $s_n(t) = \sum_{j=1}^{[nt]} g(X_j) + (nt - [nt])g(X_{[nt]+1})$, $t \geq 0$. Then, $\int_0^1 s_n(t) dt = s_n(1)/2 + \sum_{k=1}^{n-1} \sum_{j=1}^k g(X_j)$ so that, with the last display,

$$\frac{1}{n^{3/2}} \sum_{j=1}^n \left(j - \frac{n}{2}\right) g(X_j) = \frac{1}{2\sqrt{n}} s_n(1) - \frac{1}{\sqrt{n}} \int_0^1 s_n(t) dt + \frac{1}{2n^{3/2}} s_n(1) .$$

The functional Central Limit Theorem in Theorem 3 yields (43), where the variance is computed by using (41), $c_0(\dot{s}_\star^2) = \sum_{k \in \mathbb{Z}} |c_k(\dot{s}_\star)|^2$, and $\text{Var} \left(B(1)/2 - \int_0^1 B(t) dt \right) = 1/12$, with $\{B(t), t \in [0, 1]\}$ denoting the standard Brownian motion. This concludes the proof of Theorem 2. \square

2.1. Discussion. In Gassiat and Lévy-Leduc (2006), the local asymptotic normality (LAN) of the semiparametric model (1) is established for regular sampling with decreasing sampling periods. Their arguments can be extended to the irregular sampling scheme under Assumptions -(H3). More precisely, any estimator satisfying the asymptotic linearization

$$n^{3/2}(\bar{f}_n - f_\star) = \left(\frac{I_\star^{-1} \mu}{n^{3/2} \sigma_\star^2 f_\star} \sum_{j=1}^n \left(j - \frac{n}{2}\right) \dot{s}_\star(X_j) \varepsilon_j \right) (1 + o_p(1)) , \quad (44)$$

where I_\star is defined in (37), is an efficient semiparametric estimator of f_\star in the sense of McNeney and Wellner (2000). As a byproduct of the proof of Theorem 2, one has that the right hand-side of (44) is asymptotically normal with mean zero and variance I_\star^{-1} . Hence I_\star^{-1} is the optimal asymptotic variance. In view of (36) and (38), we see that the linearization of our estimator \hat{f}_n contains an extra term since ε_j in (44) is replaced by $(\varepsilon_j + s_\star(X_j))$ in (36). This extra term leads to an additional term in the asymptotic variance (39), which highly depends on the distribution of the V_k 's. Hence our estimator enjoys the optimal $n^{-3/2}$ rate but it is not efficient. The estimator proposed in Hall, Reimann and Rice (2000) is efficient and thus, in theory, outperforms the estimator based on the cumulated periodogram. On the other hand, our estimator is only based on the knowledge that the frequency lies in a fixed compact interval on the positive half-line whereas, for the estimator in Hall, Reimann and

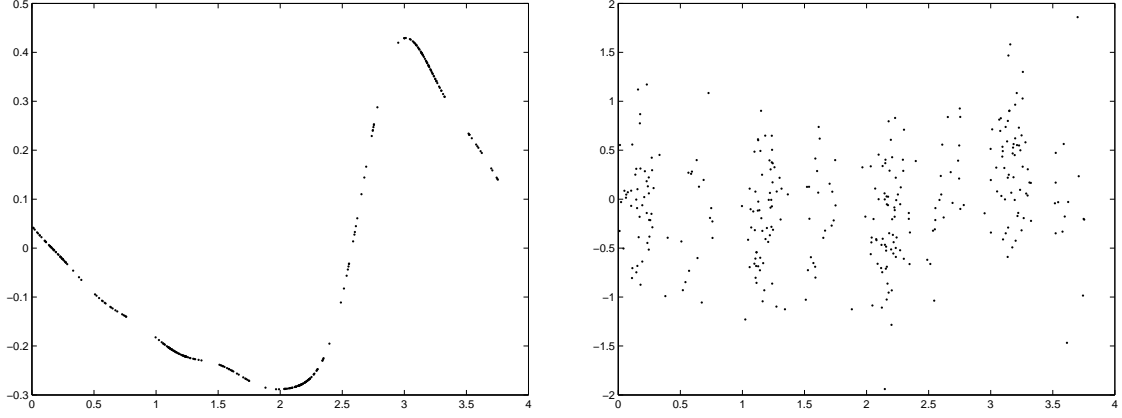
Rice (2000), an interval containing the true frequency with size at $o_p(n^{-(3/2-1/12)})$ is required (see p.554 after conditions (a)–(e)). Since our estimator is rate optimal, it can be used as a preliminary estimator.

3. NUMERICAL EXPERIMENTS

In this section, we apply the proposed estimator to periodic variable stars which are known to emit light whose intensity, or brightness, changes over time in a smooth and periodic manner. The estimation of the period is of direct scientific interest, for instance as an aid to classifying stars into different categories for making inferences about stellar evolution. The irregularity in the observation times comes from the fact that a star might be observed on most nights but not at the same time each night.

We benchmark our estimator against different methods proposed in the literature to estimate frequency of periodic functions from irregularly sampled data. We compare the least-squares method (see (6)) using different number of harmonics $K = 1, 3, 6$, the Hall, Reimann and Rice (2000) estimator based on the Friedman supersmoothing estimator (a crossvalidated variable span local smoother) of the shape of the function; we have also included the Lafler and Kinman (1965) estimator. To analyze the behavior of these algorithms, we follow Hall, Reimann and Rice (2000); we generate synthetic observations corresponding to model (1) where the underlying deterministic function s_* is obtained by smoothing observations from a Cepheid variable star. The time instants are generated using a Pareto distribution whose parameters are fitted to observations times taken from the MACHO database (<http://www.stat.berkeley.edu/users/rice/UBCWorkshop>). This distribution captures the heavy tails in the inter-arrivals $\{V_k\}$. The number of observations is set to 321. The additive noise is i.i.d. Gaussian with standard deviations equal to 0.25 and 0.5, respectively, (the signal to noise ratios are equal to 0 dB and -6 dB). Typical realizations of the light curves with period 3.9936 days are shown in Figure 1 (the times are transformed modulo a period of 3.9936 days so that approximately 80 periods are overlaid).

Estimates of the frequency were computed on a grid ranging from 0.1 to 0.6 cycles per day (the fundamental frequency is equal to 0.2504 cycles per day so the range includes both a multiple and a sub-multiple of the fundamental frequency). For the cumulated periodogram, we used $K = 1, 3, 6$ and $\kappa = 0.75$. The results of 100 Monte-Carlo experiments are summarized in Table 1. We display the percentage of large errors (frequency halving or doubling), the biases, the standardized biases and the mean-squared errors (large errors are eliminated when computing these quantities). The lower bounds for the variances of the frequency estimates are 1.87×10^{-8} and 4.68×10^{-9} for the high and low noise cases respectively. The asymptotic variances of the cumulated periodogram estimator are 2.15×10^{-8} and 4.86×10^{-9} in the high

FIGURE 1. Left: Deterministic signal, Right: noisy observations with $\sigma^* = 0.5$.

and low noise cases respectively. The estimator with the smallest mean-squared error is the least-squares estimator (6) with $K = 3$ harmonics. The mean-squared error of the cumulated periodogram estimator is only slightly larger. Note that the cumulated periodogram estimator displays significantly lower number of large errors and appears to be a reasonably accurate preliminary estimator.

Method	$\sigma^* = 0.25$				$\sigma^* = 0.5$			
	Bias $\times 10^{-5}$	Bias/SE	MSE $\times 10^{-8}$	Large errors in %	Bias $\times 10^{-5}$	Bias/SE	MSE $\times 10^{-8}$	Large errors in %
LS1	1.7	0.15	1.3	0	1.9	0.07	6.8	1
CP1	1.5	0.12	2.1	0	1.5	0.07	7.2	1
LS3	1.5	0.22	0.5	0	0.6	0.04	2.8	5
CP3	1.4	0.16	0.8	0	2.1	0.10	4.2	2
LS6	0.8	0.11	0.6	37	1.6	0.07	4.4	37
CP6	1.6	0.15	1.1	7	2.5	0.11	5.1	8
LK	-0.6	-0.03	4.1	31	261	0.14	3.2×10^4	39
SM	87	-0.05	2.7×10^4	7	1247	0.30	1.6×10^5	15

TABLE 1. Biases, standardized biases, Mean Squared Errors (MSE) of the frequency estimates and percentage of large errors obtained for different methods when the inter-arrival times have a Pareto distribution: Least-squares (LS1,LS3,LS6), Cumulated Periodogram (CP1,CP3,CP6) with $K = 1, 3, 6$, Lafler-Kinman (LK), Supersmoothen (SM)

4. SOME LIMIT RESULTS FOR ADDITIVE FUNCTIONALS OF A RENEWAL PROCESS

Theorem 3. *Assume (H2) and (H4). Let g be a $(1/f_*)$ -periodic real-valued function such that $\int_0^{1/f_*} |g(t)| dt < \infty$. Assume that the Fourier coefficients of g defined by (3) satisfy $c_0(g) = 0$ and $\sum_{k \in \mathbb{Z} \setminus \{0\}} |c_k(g)| < \infty$ then for any non-negative integer k*

$$\frac{1}{n^{k+1}} \sum_{j=1}^n j^k g(X_j) = O_p(n^{-1/2}). \quad (45)$$

Denote by $s_n(t)$ the piecewise linear interpolation

$$s_n(t) = \sum_{k=1}^{[nt]} g(X_k) + (nt - [nt])g(X_{[nt]+1}), \quad t \geq 0,$$

where $[x]$ denotes the integer part of x . Then,

$$\gamma_g^2 = \sum_{k \in \mathbb{Z} \setminus \{0\}} |c_k(g)|^2 \frac{1 - |\Phi(2\pi k f_*)|^2}{|1 - \Phi(2\pi k f_*)|^2} \quad (46)$$

is positive and finite and, as $n \rightarrow \infty$, $(n\gamma_g^2)^{-1/2} s_n(t) \Rightarrow B(t)$, where \Rightarrow denotes the weak convergence in the space of continuous $[0, 1] \rightarrow \mathbb{R}$ functions endowed with the uniform norm and $B(t)$ is the standard Brownian motion on $t \in [0, 1]$.

Proof. Without loss of generality we set $f_* = 1$. Define the Markov chain $\{Y_k\}_{k \geq 0}$, valued in $[0, 1)$ and started at $x \in [0, 1]$ by $Y_0 = x$ and $Y_{k+1} = Y_k + V_{k+1} - [Y_k + V_{k+1}]$, $k \geq 0$. Observe that, with the initial value $x = 0$, we have $g(Y_k) = g(X_k)$ for all $k \geq 1$. Let us show that this Markov Chain is positive Harris and that its invariant probability is the uniform distribution on $[0, 1]$. We first prove that this chain is uniformly Doeblin. By 4, there exists a non-negative and bounded function h such that for all Borel set A , $\mathbb{P}(V \in A) \geq \int_A h(t) dt$. It follows that, for any $k \geq 1$, $\mathbb{P}(X_k \in A) \geq \int_A h^{*k}(t) dt$, where $h^{*k} = h * \dots * h$ (k times) with $*$ denoting the convolution. Observe that, since h is non-negative, bounded and integrable, h^{*2} is non-negative and continuous. It follows that there exists $0 \leq a < b$ and $\delta > 0$ such that $\int_{t \in [a, b]} h^{*2}(t) \geq \delta$. Hence, for k large enough, there exists a non-negative integer l and $\epsilon > 0$ such that $h^{*(2k)}(t) = (h^{*2})^{*k}(t) \geq \epsilon$ for all $t \in [l, l+1]$. Hence, for all $x \in [0, 1)$ and all Borel set $A \subset [0, 1]$,

$$\mathbb{P}_x(Y_{2k} \in A) \geq \mathbb{P}_x(Y_{2k} \in A, X_{2k} \in [l, l+1])$$

which is the uniform Doeblin condition. This implies that Y is a uniformly geometrically ergodic Markov chain; let us compute its invariant probability distribution, denoted by π . For all $x \in [0, 1]$ and $l \in \mathbb{Z}$, $l \neq 0$, we have

$$\mathbb{E}_x[\exp(2i\pi l Y_n)] = \exp(2i\pi x) (\Phi(2\pi l))^n \rightarrow 0,$$

where we used (H2) which is implied by (H4). Hence, for all $l \in \mathbb{Z}, l \neq 0$, $\int_{t=0}^1 \exp(2i\pi lt)\pi(dt) = 0$, which implies that π is the uniform distribution on $[0, 1]$. Define

$$\tilde{g}(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} c_k(g)(1 - \Phi(2\pi k))^{-1} e^{2i\pi kx}.$$

By (H2), $(1 - \Phi(2\pi k))^{-1}$ is bounded uniformly on $k \in \mathbb{Z} \setminus \{0\}$. Hence γ_g is positive and finite. Moreover, $\sum_{k \in \mathbb{Z} \setminus \{0\}} |c_k(g)(1 - \Phi(2\pi k))^{-1}| < \infty$ and we compute

$$\mathbb{E}_x[\tilde{g}(Y_1)] = \sum_{k \in \mathbb{Z} \setminus \{0\}} c_k(g) \frac{\Phi(2\pi k)}{(1 - \Phi(2\pi k))} \exp(2i\pi kx).$$

This yields that \tilde{g} is the solution of the Poisson equation $\tilde{g}(x) - \mathbb{E}_x[\tilde{g}(Y_1)] = g(x) - \int_0^1 g(t)dt$. We now prove (45). Note that, since $\pi(g) = 0$,

$$\begin{aligned} n^{-(k+1)} \sum_{j=1}^n j^k g(X_j) &= n^{-(k+1)} \sum_{j=1}^n j^k (\tilde{g}(X_j) - P\tilde{g}(X_j)) \\ &= n^{-(k+1)} \sum_{j=1}^n j^k (\tilde{g}(X_j) - P\tilde{g}(X_{j-1})) + n^{-(k+1)} \sum_{j=1}^n j^k (P\tilde{g}(X_{j-1}) - P\tilde{g}(X_j)). \end{aligned}$$

Since \tilde{g} is bounded, the variance of the first term is $O(n^{-1})$ as $n \rightarrow \infty$. Integrating by parts yields, using that \tilde{g} is bounded,

$$\begin{aligned} n^{-(k+1)} \sum_{j=1}^n j^k (P\tilde{g}(X_{j-1}) - P\tilde{g}(X_j)) \\ = n^{-(k+1)} P\tilde{g}(X_0) - n^{-1} P\tilde{g}(X_n) + n^{-(k+1)} \sum_{j=1}^n [(j+1)^k - j^k] P\tilde{g}(X_j) = O_p(n^{-1}). \end{aligned}$$

To prove (46) we compute, by the Parseval Theorem,

$$\begin{aligned} \int_0^1 \{\tilde{g}^2(x) - (\mathbb{E}_x[\tilde{g}(Y_1)])^2\} dx \\ = \sum_{k \in \mathbb{Z} \setminus \{0\}} \left\{ |c_k(g)(1 - \Phi(2\pi k))^{-1}|^2 - |c_k(g)\Phi(2\pi k)(1 - \Phi(2\pi k))^{-1}|^2 \right\} = \gamma_g^2, \end{aligned}$$

The end of the proof follows from the functional central limit theorem Meyn and Tweedie (1993, Theorem 17.4.4). \square

5. UNIFORM ESTIMATES OF THE CUMULATED PERIODOGRAM

Lemma 1. *Assume (H2)-(H3) and that s_* is bounded. Define $\xi_n(f)$ and $\zeta_n(f)$ by (19) and (20) where (K_n) is a sequence tending to infinity at most with a polynomial rate. Then,*

for any $0 < f_{\min} < f_{\max}$, $\delta > 0$ and $q = 0, 1, \dots$,

$$\sup_{f \in [f_{\min}, f_{\max}]} \left| \xi_n^{(q)}(f) + \zeta_n^{(q)}(f) \right| = o_p(K_n^{q+1} n^{q-1/2+\delta}), \quad (47)$$

where, for any function h , $h^{(q)}$ denotes the q -th derivative of h .

Proof. By (19) and (20),

$$\xi_n^{(q)}(f) = 2 \frac{(2\pi)^q}{n^2} \sum_{k=1}^{K_n} k^q L_q(X, kf)^T \varepsilon, \quad \zeta_n^{(q)}(f) = \frac{(2\pi)^q}{n^2} \sum_{k=1}^{K_n} k^q \varepsilon^T \Gamma_q(X, kf) \varepsilon, \quad (48)$$

where $\varepsilon = [\varepsilon_1, \dots, \varepsilon_n]^T$, $L_q(X, f) = \left[\sum_{j=1}^n (X_j - X_{j'})^q \cos^{(q)} \{2\pi f(X_j - X_{j'})\} s_*(X_j) \right]_{1 \leq j' \leq n}$ and $\Gamma_q(X, f) = [i^q (X_l - X_j)^q e^{2i\pi(X_l - X_j)f}]_{1 \leq l, j \leq n}$. Hence,

$$\sup_{f \in [f_{\min}, f_{\max}]} \left| \xi_n^{(q)}(f) + \zeta_n^{(q)}(f) \right| \leq C n^{-2} K_n^{q+1} \sup_{0 < f \leq K_n f_{\max}} |2L_q^T(X, f) \varepsilon + \varepsilon^T \Gamma_q(X, f) \varepsilon|. \quad (49)$$

Note that $\text{Tr}[\Gamma_q(X, f)] = 0$ for $q \geq 1$, $\text{Tr}[\Gamma_q(X, f)] = n$ for $q = 0$ and that the spectral radius of the matrix $\Gamma_q(X, f)$ is at most $\sup_{j=1, \dots, n} \sum_{l=1}^n |X_l - X_j|^q \leq n X_n^q$. For any hermitian matrix Λ having all its eigenvalues less than $1/4$, $\mathbb{E}[\exp(Z^T \Lambda Z)] \leq \exp(\text{Tr}(\Lambda) + 2\text{Tr}(\Lambda^2))$. Therefore, for any $\lambda > 0$, on the event $\{\lambda \sigma^{*2} n X_n^q \leq 1/8\}$,

$$\begin{aligned} \mathbb{E} \left[e^{\lambda (2L_q(X, f)^T \varepsilon + \varepsilon^T \Gamma_q(X, f) \varepsilon)} \middle| X \right] &\leq C' \exp \left\{ C \lambda^2 (L_q(X, f)^T L_q(X, f) + \text{Tr}(\Gamma_q^2(X, f))) \right\} \\ &\leq C' \exp \left\{ C \lambda^2 n^3 X_n^{2q} \right\}, \end{aligned}$$

where we have used $L_q^T(X, f) L_q(X, f) \leq C n^3 X_n^{2q}$ and $\text{Tr}[\Gamma_q^2(X, f)] \leq C n^2 X_n^{2q}$. Using (H3), we similarly get that $\mathbb{E}[|L_{q+1}(X, f)^T \varepsilon|^2 | X] \leq C L_{q+1}^T(X, f) L_{q+1}(X, f) \leq C n^3 X_n^{2(q+1)}$ and $\mathbb{E}[|\varepsilon^T \Gamma_{q+1}(X, f) \varepsilon|^2 | X] \leq C \text{Tr}[\Gamma_{q+1}^2(X, f)] \leq C n^2 X_n^{2(q+1)}$. Applying Lemma 6, we get that, for all positive numbers λ and R , on the event $\{\lambda \sigma^{*2} n X_n^q \leq 1/8\}$,

$$\begin{aligned} \mathbb{P} \left[\sup_{0 < f \leq K_n f_{\max}} |2L_q(X, f)^T \varepsilon + \varepsilon^T \Gamma_q(X, f) \varepsilon| \geq R \middle| X \right] &\leq \\ &C' e^{-\lambda R + C \lambda^2 n^3 X_n^{2q}} \left(1 + C K_n \lambda n^{3/2} X_n^{q+1} \right). \end{aligned}$$

Let $\delta > 0$. Applying this inequality with $\lambda = n^{-3/2} X_n^{-q}$ and $R = n^{\delta+3/2} X_n^q$, we get

$$\mathbb{P} \left[\sup_{0 < f \leq K_n f_{\max}} |2L_q(X, f)^T \varepsilon + \varepsilon^T \Gamma_q(X, f) \varepsilon| \geq n^{\delta+3/2} X_n^q \middle| X \right] \leq C \exp(-n^\delta) (1 + K_n X_n).$$

Now, using (49),

$$\begin{aligned} & \mathbb{P} \left(\sup_{f \in [f_{\min}, f_{\max}]} \left| \xi_n^{(q)}(f) + \zeta_n^{(q)}(f) \right| \geq n^{q-1/2+2\delta} K_n^{q+1} \right) \\ & \leq \mathbb{P} \left(\sup_{0 < f \leq K_n f_{\max}} |2L_X(f)^T \varepsilon + \varepsilon^T \Gamma_X(f) \varepsilon| \geq n^{\delta+3/2} X_n^q \right) + \mathbb{P} \left(X_n^q \geq n^{q+\delta} \right) \\ & \leq C \exp(-n^\delta) (1 + K_n n) + n^{-\delta/q}, \end{aligned}$$

which concludes the proof. \square

Lemma 2. Assume (H1)–(H2) and that s_\star satisfies (11). Define $D_n(f)$ by (18), with a sequence (K_n) tending to infinity. Then, for any $\epsilon > 0$, as n tends to infinity,

$$\sup_{f \in \bigcap_{j,l} B_n^c(j,l)} D_n(f) = O_p \left(\left\{ K_n R(m_n)^2 + \frac{\{K_n n m_n\}^\epsilon}{n \gamma_n} \right\} \right),$$

where (m_n) is a sequence of positive integers, R is defined by (14), $\bigcap_{j,l}$ is the intersection over integers $j \geq 1$, $l = 1, \dots, K_n$ and $B_n^c(j,l)$ is defined by (22) with (γ_n) satisfying

$$K_n \gamma_n \rightarrow 0 \quad \text{and} \quad n \gamma_n \rightarrow \infty. \quad (50)$$

Proof. Inserting the Fourier expansion (3) of s_\star in (18) and using the definition of $\varphi_{n,X}$ in (33) and of $R(m)$ in (14), we get

$$\begin{aligned} D_n(f) & \leq 2 \sum_{k=1}^{K_n} \left| \sum_{|p| \leq m} c_p(s_\star) \varphi_{n,X} \{2\pi(p f_\star - k f)\} \right|^2 + 2K_n R(m)^2 \\ & \leq 4 \sum_{k=1}^{K_n} \left| \sum_{|p| \leq m} c_p(s_\star) \mathbb{E} [\varphi_{n,X} \{2\pi(p f_\star - k f)\}] \right|^2 + 4\tilde{D}_{n,m}(f) + 2K_n R(m)^2, \quad (51) \end{aligned}$$

where we defined

$$\tilde{D}_{n,m}(f) = \sum_{k=1}^{K_n} \left| \sum_{|p| \leq m} c_p(s_\star) (\varphi_{n,X} \{2\pi(p f_\star - k f)\} - \mathbb{E} [\varphi_{n,X} \{2\pi(p f_\star - k f)\}]) \right|^2. \quad (52)$$

For all positive integers j and $l \leq K_n$, and $f \in A_n = \bigcap_{j',l'} B_n^c(j',l')$, $2\pi|j f_\star - l f| \geq 2\pi l \gamma_n$. Thus, using (77) with $k = 0$ in Lemma 4, (11), and $\lim_{n \rightarrow \infty} n \gamma_n = \infty$, we get, for all $f \in A_n$, and n large enough,

$$\sum_{k=1}^{K_n} \left| \sum_{|p| \leq m} c_p(s_\star) \mathbb{E} [\varphi_{n,X} \{2\pi(p f_\star - k f)\}] \right|^2 \leq \frac{C}{n^2} \sum_{k=1}^{K_n} \frac{1}{k^2 \gamma_n^2} = O(n^{-2} \gamma_n^{-2}). \quad (53)$$

Consider now $\tilde{D}_{n,m}$. For $\rho > 0$ and $q = 1, \dots, Q(\rho) = [\rho^{-1}(f_{\max} - f_{\min})]$, define $I_q = [f_{\min} + (q-1)\rho, f_{\min} + q\rho] \cap A_n$. Observe that $u \mapsto \varphi_{n,X}(u)$ is a Lipschitz function with

Lipshitz norm less than $n^{-1} \sum_{j=1}^n X_j$ and bounded by 1. It follows that $f \mapsto \tilde{D}_{n,m}(f)$ is a Lipschitz function with Lipschitz norm less than

$$8\pi \left(\sum_p |c_p(s_\star)| \right)^2 \sum_{k=1}^{K_n} kn^{-1} \sum_{j=1}^n (X_j + \mathbb{E}[X_j]) \leq C \left\{ \frac{K_n^2}{n} \sum_{j=1}^n X_j + nK_n^2 \right\}.$$

Thus, for any $q = 1, \dots, Q(\rho)$ such that I_q is non-empty, and any $f_q \in I_q$, $\sup_{f \in I_q} \tilde{D}_{n,m}(f) \leq \tilde{D}_{n,m}(f_q) + C \rho \left\{ K_n^2 n^{-1} \sum_{j=1}^n X_j + nK_n^2 \right\}$, which implies

$$\sup_{f \in \mathbf{A}_n} \tilde{D}_{n,m}(f) \leq \sup_{q=1, \dots, Q(\rho)} \tilde{D}_{n,m}(f_q) + O_p(\rho n K_n^2), \quad (54)$$

where, by convention, $\tilde{D}_{n,m}(f_q) = 0$ if I_q is empty. Since by (H2), $\inf_{t \in \mathbb{R}} |1 - \Phi(t)| / (1 \wedge |t|) > 0$, and for n large enough, $K_n \gamma_n \leq 1$, Lemma 5 shows that, for any $f \in \mathbf{A}_n$, $2\pi |pf_\star - kf| \geq 2\pi k \gamma_n$, and $y > 0$,

$$\mathbb{P}(|\varphi_{n,X}\{2\pi(pf_\star - kf)\} - \mathbb{E}[\varphi_{n,X}\{2\pi(pf_\star - kf)\}]| \geq y) \leq 4e^{-Cny^2(1 \wedge k\gamma_n)} \leq 4e^{-Cny^2 k \gamma_n}.$$

Using this bound with the definition of $\tilde{D}_{n,m}(f)$ in (52), we get, for all $x > 0$ and $f \in \mathbf{A}_n$,

$$\mathbb{P}\left(\sup_{q=1, \dots, Q(\rho)} \tilde{D}_{n,m}(f_q) \geq x\right) \leq Q(\rho) \sup_{f \in \mathbf{A}_n} \mathbb{P}(\tilde{D}_{n,m}(f) \geq x) \leq 4Q(\rho) \sum_{k=1}^{K_n} \sum_{|p| \leq m} e^{-Cnx\beta_k \alpha_p^2 k \gamma_n},$$

where $\beta_k, \alpha_p, k = 1, \dots, K_n, |p| \leq m$ are positive weights such that $\sum_k \beta_k = 1$ and $\sum_p \alpha_p |c_p| = 1$. With $\beta_k = k^{-1} / (\sum_{k \leq K_n} k^{-1}) \geq 2k^{-1} / \log(K_n)$ for n large enough, and $\alpha_p = \left(\sum_p |c_p|\right)^{-1} \geq C$, we get

$$\mathbb{P}\left(\sup_{q=1, \dots, Q(\rho)} \tilde{D}_{n,m}(f_q) \geq x\right) \leq 4Q(\rho) K_n (2m+1) e^{-Cnx\gamma_n / \log(K_n)}. \quad (55)$$

Let $\delta > 0$. Defining $\rho_n = (n^{-2} \gamma_n^{-1} K_n^{-2} \log(K_n))$ and $x = (Q(\rho_n) K_n m_n)^\delta \log(K_n) / (n \gamma_n)$, implying $Q(\rho_n) \rightarrow \infty$ and $Q(\rho_n) K_n m_n \rightarrow \infty$, we obtain

$$\sup_{f \in \mathbf{A}_n} \tilde{D}_{n,m_n}(f) = o_p\left(\{n^2 \gamma_n K_n^3 m_n\}^\delta \log(K_n) \{n \gamma_n\}^{-1}\right).$$

For any $\epsilon > 0$, we set $\delta > 0$ small enough such that $\{n^2 K_n^2 m_n\}^\delta \log(K_n) = O(\{K_n n m_n\}^\epsilon)$. The previous bound, with (50), (51), (53) and (54) yields the result. \square

Lemma 3. Assume (H1)–(H2) and that s_\star satisfies (11). Define $D_n(f)$ by (18), with a sequence (K_n) tending to infinity. Then, as n tends to infinity, for all relatively prime integers j and l ,

$$D_n(jf_\star/l) = \sum_{k=1}^{[K_n/l]} |c_{kj}(s_\star)|^2 + O_p\left(K_n l^{1/2} n^{-1/2}\right). \quad (56)$$

Moreover, for any $\epsilon > 0$,

$$\sup_{f \in \bigcup_{j,l} B_n(j,l)} \left| D_n(f) - \sum_{k=1}^{\lfloor K_n/l \rfloor} |c_{kj}(s_*) \varphi_{n,X} \{2\pi(kjf_* - klf)\}|^2 \right| = O_p \left(K_n R(m_n) + K_n^2 n^{-1} + (K_n n m_n)^\epsilon K_n^{1/2} n^{-1/2} \right), \quad (57)$$

where (m_n) is a sequence of positive integers, R is defined by (14), $\bigcup_{j,l}$ is the union over $j \geq 1, 1 \leq l \leq K_n$, $B_n(j,l)$ is defined by (22) with (γ_n) satisfying

$$\gamma_n K_n^2 \rightarrow 0. \quad (58)$$

Proof. Let j and $l \leq K_n$ be two relatively prime integers. In the following, C denotes a positive constant independent of j, l and f that may change upon each appearance. Inserting the Fourier expansion (3) of s_* in (18), the leading term in $D_n(f)$ for f close to jf_*/l will be given by the indices k and p such that k/l and p/j are equal to the same integer, say q . Thus we split $D_n(f)$ into

$$D_n(f) = \sum_{q=1}^{\lfloor K_n/l \rfloor} |c_{qj}(s_*) \varphi_{n,X} \{2\pi(qjf_* - qlf)\}|^2 + A_n(f), \quad (59)$$

where

$$A_n(f) = \sum'_{k,p,p'} c_p(s_*) \overline{c_{p'}(s_*)} \varphi_{n,X} \{2\pi(pf_* - kf)\} \overline{\varphi_{n,X} \{2\pi(p'f_* - kf)\}}$$

with $\sum'_{k,p,p'}$ denoting the sum over indices $k = 1, \dots, K_n$ and $p, p' \in \mathbb{Z}$ such that, for any integer q , we have $k \neq ql$, $p \neq jq$ or $p' \neq jq$. It follows from this definition and from (11), since $|\varphi_{n,X}| \leq 1$, that

$$|A_n(f)| \leq C \sum'_{k,p} |c_p(s_*) \varphi_{n,X} \{2\pi(pf_* - kf)\}|, \quad (60)$$

where $\sum'_{k,p}$ denotes the sum over indices $k = 1, \dots, K_n$ and $p \in \mathbb{Z}$ such that, for any integer q , we have $k \neq ql$ or $p \neq jq$. Using that j and l are relatively prime, if for any integer q , $k \neq ql$ or $p \neq jq$, then $|pl - kj| \geq 1$, which implies, by (78) with $k = 0$ in Lemma 4,

$$\mathbb{E}[|\varphi_{n,X} \{2\pi(pf_* - kf)\}|] \leq \mathbb{E} \left[|\varphi_{n,X} \{2\pi(pf_* - kf)\}|^2 \right]^{1/2} \leq C (n^{-1}l)^{1/2}.$$

Hence, using (11) and this bound in (60), Relation (59) yields (56). We now proceed in bounding $A_n(f)$ uniformly for $f \in \bigcup_{j,l} B_n(j,l)$. We use the same line of reasoning as for bounding $D_n(f)$ in Lemma 2. First we split the sum in p appearing in (60) and introduce the centering term $\mathbb{E}[\varphi_{n,X} \{2\pi(pf_* - kf)\}]$ so that

$$|A_n(f)| \leq C \left(A_{n,m}(f) + \sum'_{k,p} |c_p(s_*) \mathbb{E}[\varphi_{n,X} \{2\pi(pf_* - kf)\}]| + K_n R(m) \right), \quad (61)$$

where

$$A_{n,m}(f) = \sum_{k,p}'' |c_p(s_*)(\varphi_{n,X}\{2\pi(pf_\star - kf)\}) - \mathbb{E}[\varphi_{n,X}\{2\pi(pf_\star - kf)\})]| ,$$

with $\sum_{k,p}''$ denoting the sum over indices $k = 1, \dots, K_n$ and $p = 0, \pm 1, \dots, \pm m$ such that $|pl - kj| \geq 1$. Using (77) with $k = 0$ in Lemma 4 and (11), we have

$$\sum_{k,p}' |c_p(s_*) \mathbb{E}[\varphi_{n,X}\{2\pi(pf_\star - kf)\})]| \leq C l K_n n^{-1} . \quad (62)$$

As for obtaining (54), we cover $[f_{\min}, f_{\max}]$ with Q intervals of size $\rho = (f_{\max} - f_{\min})/Q$, and obtain

$$\sup_{f \in \bigcup_{j,l} B_n(j,l)} A_{n,m}(f) \leq \sup_{q=1,\dots,Q} A_{n,m}(f_q) + O_p(\rho n K_n^2) ,$$

where either $A_{n,m}(f_q) = 0$ by convention, or $f_q \in \bigcup_{j,l} B_n(j,l)$, implying for all indices k and p in the summation term $\sum_{k,p}''$, that there exists integers j and $l \leq K_n$ such that

$$|pf_\star - kf_q| \geq |pf_\star - kjf_\star/l| - \gamma_n k \geq f_\star/l - \gamma_n K_n \geq f_\star/K_n - \gamma_n K_n \geq C K_n^{-1} ,$$

for n large enough, by (58). Now, we apply the deviation estimate in Lemma 5, so that, as in (55), we have

$$\mathbb{P} \left(\sup_{q=1,\dots,Q} A_{n,m}(f_q) > x \right) \leq 4Q K_n (2m+1) e^{-C n x^2 K_n^{-1}} .$$

Let $\delta > 0$. Setting $Q = [K_n^{3/2} n^{3/2}]$ and $x = (Q K_n m_n)^\delta K_n^{1/2} n^{-1/2}$ so that $Q \rightarrow \infty$ and $Q K_n m_n \rightarrow \infty$ as $n \rightarrow \infty$, we finally obtain

$$\sup_{f \in \bigcup_{j,l} B_n(j,l)} A_{n,m}(f) = O_p \left((Q K_n m_n)^\delta K_n^{1/2} n^{-1/2} \right) .$$

For any $\epsilon > 0$, we set $\delta > 0$ such that $(Q K_n m_n)^\delta = O((K_n n m_n)^\epsilon)$. Applying this bound in (61) and using (62), Relation (59) yields (57). \square

6. LINEARIZATION OF THE ESTIMATOR \hat{f}_n

In this section we suppose that the conditions of Theorem 2 hold and we show (36). A Taylor expansion of $\dot{\Lambda}_n(f)$ gives

$$\dot{\Lambda}_n(\hat{f}_n) = \dot{\Lambda}_n(f_\star) + (\hat{f}_n - f_\star) \ddot{\Lambda}_n(f'_n) ,$$

where f'_n is random and lies between \hat{f}_n and f_\star . We prove hereafter that

$$\dot{\Lambda}_n(f_\star) = \sum_{j=1}^n \left(\frac{X_j}{n} - \frac{\mu}{2} \right) \frac{\dot{s}_\star(X_j)}{f_\star} (\varepsilon_j + s_\star(X_j)) + o_p(\sqrt{n}) , \quad (63)$$

$$\ddot{\Lambda}_n(f'_n) = -n^2 \sigma_\star^2 I_\star + o_p(n^2) . \quad (64)$$

By (35) the conditions of Theorem 1 are met so that (15) and (16) hold. The last three displayed equations thus yield

$$n^{3/2}(\hat{f}_n - f_\star) = \frac{I_\star^{-1}}{n^{3/2}\sigma_\star^2 f_\star} \sum_{j=1}^n \left(X_j - \frac{n\mu}{2}\right) \dot{s}_\star(X_j) (\varepsilon_j + s_\star(X_j)) + o_p(1) ,$$

and Relation (36) then follows from

$$n^{-3/2} \sum_{j=1}^n (X_j - j\mu) \dot{s}_\star(X_j) \varepsilon_j = o_p(1) \text{ and } n^{-3/2} \sum_{j=1}^n (X_j - j\mu) \dot{s}_\star(X_j) s_\star(X_j) = o_p(1) . \quad (65)$$

The rest of this section provides the proofs of (63), (64) and (65).

Proof of equation (63). We use that $\dot{\Lambda}_n(f_\star) = \dot{\xi}_n(f_\star) + \dot{\zeta}_n(f_\star) + \dot{D}_n(f_\star)$ so that (63) follows from

$$\dot{\xi}_n(f_\star) = \frac{1}{f_\star} \sum_{j=1}^n \dot{s}_\star(X_j) \left(\frac{X_j}{n} - \frac{\mu}{2}\right) \varepsilon_j + o_p(\sqrt{n}) , \quad (66)$$

$$\dot{\zeta}_n(f_\star) = o_p(\sqrt{n}) , \quad (67)$$

$$\dot{D}_n(f_\star) = \frac{1}{f_\star} \sum_{j=1}^n \left(\frac{X_j}{n} - \frac{\mu}{2}\right) \dot{s}_\star(X_j) s_\star(X_j) + o_p(\sqrt{n}) . \quad (68)$$

Consider $\dot{\xi}_n(f_\star)$. Differentiating (19), we obtain $\dot{\xi}_n(f_\star) = n^{-1} \sum_{j=1}^n A_n(j) \varepsilon_j$ where $A_n(j) = n^{-1} \sum_{j'=1}^n \sum_{|k| \leq K_n} 2i\pi k (X_j - X_{j'}) e^{2i\pi k (X_j - X_{j'}) f_\star} s_\star(X_{j'})$. Inserting the Fourier expansion (3) of $s_\star(X_{j'})$ and using the definition of $\varphi_{n,X}$ in (33), we obtain for any $j = 1, \dots, n$, $A_n(j) = \sum_{|k| \leq K_n} e^{2i\pi k X_j f_\star} \sum_{p \in \mathbb{Z}} c_p(s_\star) (2i\pi k) \{X_j \varphi_{n,X}[2\pi(p-k)f_\star] + i\dot{\varphi}_{n,X}[2\pi(p-k)f_\star]\}$. In the sequel, $\overline{X_n} = n^{-1} \sum_{j=1}^n X_j$ and $\|Y\|_2 = \mathbb{E}(|Y|^2)^{1/2}$ is the classical L^2 -norm. By Minkowski inequality, $\dot{\xi}_n(f_\star) - (nf_\star)^{-1} \sum_{j=1}^n \dot{s}_\star(X_j) (X_j - n\mu/2) \varepsilon_j = O_p \left(n^{-1} \sum_{k=1}^3 \left\{ \sum_{j=1}^n \|A_{n,k}(j)\|_2^2 \right\}^{1/2} \right)$, where

$$A_{n,1}(j) = -f_\star^{-1} \dot{s}_\star(X_j) (\overline{X_n} - n\mu/2) ,$$

$$A_{n,2}(j) = - \sum_{|k| > K_n} (2i\pi k) c_k(s_\star) e^{2i\pi k f_\star X_j} (X_j - \overline{X_n}) ,$$

$$A_{n,3}(j) = \sum_{|k| \leq K_n} \sum_{p \neq k} (2i\pi k) c_p(s_\star) e^{2i\pi k X_j f_\star} (X_j \varphi_{n,X}[2\pi(p-k)f_\star] + i\dot{\varphi}_{n,X}[2\pi(p-k)f_\star]) .$$

Note that for all $j = 1, \dots, n$, $\|A_{n,1}(j)\|_2^2 \leq (\sum_{p \in \mathbb{Z}} |k| |c_k(s_\star)|)^2 \mathbb{E}\{(\overline{X_n} - n\mu/2)^2\} = O(n)$ and $n^{-1} (\sum_{j=1}^n \|A_{n,2}(j)\|_2^2)^{1/2} \leq C n^{1/2} (\sum_{|k| \geq K_n} |k| |c_k(s_\star)|) = o(\sqrt{n})$, using (34). Using Minkowski inequality, we obtain, for all $j = 1, \dots, n$,

$$\|A_{n,3}(j)\|_2 \leq 2\pi \sum_{|k| \leq K_n} \sum_{p \neq k} |k| |c_p(s_\star)| (\|X_j \varphi_{n,X}\{2\pi(p-k)f_\star\}\|_2 + \|\dot{\varphi}_{n,X}\{2\pi(p-k)f_\star\}\|_2) .$$

Using that $|\varphi_{n,X}| \leq 1$, Lemma 4 which gives $\mathbb{E}[|\varphi_{n,X}\{2\pi(p-k)f_\star\}|^2] = O(n^{-1})$ uniformly in $p \neq k$, we obtain $\|X_j \varphi_{n,X}\{2\pi(p-k)f_\star\}\|_2 \leq \|X_j - j\mu\|_2 + j\mu n^{-1/2} = O(jn^{-1/2} + j^{1/2})$. By Lemma 4, $\|\dot{\varphi}_{n,X}\{2\pi(p-k)f_\star\}\|_2 = O(n^{1/2})$ uniformly in $p \neq k$ leading thus to $n^{-1}(\sum_{j=1}^n \|A_{n,3}(j)\|_2^2)^{1/2} = O(K_n^2) = o(\sqrt{n})$ by (35). This concludes the proof of (66).

We now prove (67). Using (48) with $q = 1$, since, for any hermitian matrix Γ , $\mathbb{E}[(\varepsilon^T \Gamma \varepsilon)^2] \leq 2\sigma^4 \text{Tr}(\Gamma^2)$, we get

$$\mathbb{E}[(\dot{\zeta}_n(f_\star))^2] = \text{Var}(\dot{\zeta}_n(f_\star)) \leq 8\pi^2 \frac{\sigma^4}{n^4} K_n \sum_{k=1}^{K_n} k^2 \mathbb{E}[\text{Tr}(\Gamma_X^2(kf_\star))] = O(K_n^4) = o(n)$$

by (35). Hence (67). Let us now prove (68). Using (18), we get

$$\begin{aligned} \dot{D}_n(f_\star) = \sum_{k=1}^{K_n} \sum_{p,q \in \mathbb{Z}} c_p(s_\star) \overline{c_q(s_\star)} (-2\pi k) \left\{ \dot{\varphi}_{n,X}[2\pi(p-k)f_\star] \overline{\varphi_{n,X}[2\pi(q-k)f_\star]} \right. \\ \left. + \varphi_{n,X}[2\pi(p-k)f_\star] \overline{\dot{\varphi}_{n,X}[2\pi(q-k)f_\star]} \right\}. \end{aligned} \quad (69)$$

If $p \neq k$ and $q \neq k$, by using Lemma 4, there exists a constant $C > 0$, such that $\mathbb{E}(|\dot{\varphi}_{n,X}[2\pi(p-k)f_\star] \overline{\varphi_{n,X}[2\pi(q-k)f_\star]}|) \leq \|\dot{\varphi}_{n,X}[2\pi(p-k)f_\star]\|_2 \|\varphi_{n,X}[2\pi(q-k)f_\star]\|_2 \leq C$. Using (35) and $\sum_p |c_p(s_\star)| < \infty$, we get that the term $\sum_k \sum_{p \neq k, q \neq k}$ in the right-hand side of (69) is $o_p(\sqrt{n})$. Now, if $p = q = k$, the term in the curly brackets is equal to zero. Hence (69) can be rewritten as $\dot{D}_n(f_\star) = \sum_{k=1}^{K_n} D_{n,k}(f_\star) + o_p(\sqrt{n})$ where

$$\begin{aligned} D_{n,k}(f_\star) = \sum_{q \in \mathbb{Z}} c_k(s_\star) \overline{c_q(s_\star)} (-2\pi k) \left\{ \dot{\varphi}_{n,X}(0) \overline{\varphi_{n,X}[2\pi(q-k)f_\star]} + \overline{\dot{\varphi}_{n,X}[2\pi(q-k)f_\star]} \right\} \\ + \sum_{p \in \mathbb{Z}} c_p(s_\star) \overline{c_k(s_\star)} (-2\pi k) \left\{ \dot{\varphi}_{n,X}[2\pi(p-k)f_\star] + \varphi_{n,X}[2\pi(p-k)f_\star] \overline{\dot{\varphi}_{n,X}(0)} \right\}. \end{aligned}$$

We will check that $\sum_{k > K_n} D_{n,k}(f_\star) = o_p(\sqrt{n})$. Using the Fourier expansion of s_\star and \dot{s}_\star , we obtain after some algebra, $\dot{D}_n(f_\star) = (nf_\star)^{-1} \sum_{j=1}^n (X_j - \overline{X_n}) \dot{s}_\star(X_j) s_\star(X_j) + o_p(\sqrt{n})$. This yields (68) by Slutsky's Lemma. Indeed, $\mu/2 - \sum_{l=1}^n X_l/n^2 = o_p(1)$ and, by Theorem 3, $n^{-1/2} \sum_{j=1}^n (s_\star \dot{s}_\star)(X_j) = O_p(1)$, thus we have $(\mu/2 - n^{-2} \sum_{l=1}^n X_l) \sum_{j=1}^n \dot{s}_\star(X_j) s_\star(X_j) = o_p(\sqrt{n})$. To conclude the proof of (63), we have to prove that $\sum_{k > K_n} D_{n,k}(f_\star) = o_p(\sqrt{n})$. By Minkowski inequality, $\|\sum_{k > K_n} D_{n,k}(f_\star)\|_2 \leq 2\pi \sum_{k > K_n} \sum_{q \neq k} |k| |c_k(s_\star)| |c_q(s_\star)| (\|\dot{\varphi}_{n,X}(0) \overline{\varphi_{n,X}[2\pi(q-k)f_\star]}\|_2 + \|\dot{\varphi}_{n,X}[2\pi(q-k)f_\star]\|_2)$. Using that $|\varphi_{n,X}| \leq 1$ and $\|\varphi_{n,X}[2\pi(q-k)f_\star]\|_2 = O(n^{-1/2})$ by Lemma 4 uniformly in $q \neq k$, we get $\|\dot{\varphi}_{n,X}(0) \overline{\varphi_{n,X}[2\pi(q-k)f_\star]}\|_2 \leq \|\overline{X_n} - (n+1)\mu/2\|_2 + (n+1)\mu/2 \|\varphi_{n,X}[2\pi(q-k)f_\star]\|_2 = O(\sqrt{n})$. By (34) and Lemma 4, we obtain $\|\sum_{k > K_n} D_{n,k}(f_\star)\|_2 = o(\sqrt{n})$. \square

Proof of Equation (64). Using that $\ddot{\Lambda}_n = \ddot{D}_n + \ddot{\xi}_n + \ddot{\zeta}_n$, applying Lemma 1 with $q = 2$ and using (16), the bound (64) is a consequence of the two following estimates, proved below,

$$\ddot{D}_n(f_\star) = -n^2 \sigma_\star^2 I_\star (1 + o_p(1)) , \quad (70)$$

$$\sup_{f: |f-f_\star| \leq \rho_n/n} |\ddot{D}_n(f_\star) - \ddot{D}_n(f)| = o_p(n^2) , \quad (71)$$

for any decreasing sequence (ρ_n) tending to zero. We now prove (70). Using (18), we obtain

$$\begin{aligned} \ddot{D}_n(f) &= 4\pi^2 \sum_{k=1}^{K_n} \sum_{p,q \in \mathbb{Z}} c_p(s_\star) \overline{c_q(s_\star)} k^2 \left\{ \ddot{\varphi}_{n,X}[2\pi(pf_\star - kf)] \overline{\varphi_{n,X}[2\pi(qf_\star - kf)]} \right. \\ &\quad \left. + 2\dot{\varphi}_{n,X}[2\pi(pf_\star - kf)] \overline{\dot{\varphi}_{n,X}[2\pi(qf_\star - kf)]} + \varphi_{n,X}[2\pi(pf_\star - kf)] \overline{\varphi_{n,X}[2\pi(qf_\star - kf)]} \right\} . \end{aligned} \quad (72)$$

For $f = f_\star$, since $\varphi_{n,X}(0) = 1$, $\dot{\varphi}_{n,X}(0) = i/n \sum_{j=1}^n X_j$ and $\ddot{\varphi}_{n,X}(0) = -1/n \sum_{j=1}^n X_j^2$, we get

$$\frac{1}{n^2} \ddot{D}_n(f_\star) = \left(4\pi^2 \sum_{k=1}^{K_n} |c_k(s_\star)|^2 k^2 \right) \left\{ -\frac{2}{n^3} \sum_{j=1}^n X_j^2 + \frac{2}{n^4} \left(\sum_{j=1}^n X_j \right)^2 \right\} + G_n(f_\star) , \quad (73)$$

$$\begin{aligned} \text{where } G_n(f_\star) &= \frac{4\pi^2}{n^2} \sum_{k=1}^{K_n} \sum_{(p,q) \neq (k,k)} c_p(s_\star) \overline{c_q(s_\star)} k^2 \left\{ \ddot{\varphi}_{n,X}[2\pi(p-k)f_\star] \overline{\varphi_{n,X}[2\pi(q-k)f_\star]} \right. \\ &\quad \left. + 2\dot{\varphi}_{n,X}[2\pi(p-k)f_\star] \overline{\dot{\varphi}_{n,X}[2\pi(q-k)f_\star]} + \varphi_{n,X}[2\pi(p-k)f_\star] \overline{\varphi_{n,X}[2\pi(q-k)f_\star]} \right\} . \end{aligned}$$

As n tends to infinity, the term between parentheses in (73) tends to $1/(2f_\star) \int_0^{1/f_\star} \dot{s}_\star^2(t) dt$ and the term between curly brackets converges to $-2\mu^2/3 + \mu^2/2$ in probability, and hence, using (37), their product converges to $-\sigma_\star^2 I_\star$ in probability. We conclude the proof of (70) by showing that $G_n(f_\star) = o_p(1)$. We split the summation $\sum_{p,q}$ in the definition of $G_n(f_\star)$ into three terms $\sum_{p \neq k, q \neq k} + \sum_{p=k, q \neq k} + \sum_{p \neq k, q=k}$ leading to $G_n(f_\star) = \sum_{i=1}^3 G_{n,i}(f_\star)$. Observe that, setting $C = 2\pi \sum_p |c_p(s_\star)|$,

$$\mathbb{E}[|G_{n,1}|] \leq C^2 K_n^3 n^{-2} \inf_{|t| > 2\pi f_\star} \{ \mathbb{E}[|\ddot{\varphi}_{n,X}(t) \varphi_{n,X}(t)|] + \mathbb{E}[|\dot{\varphi}_{n,X}(t)|^2] \} .$$

Using that $\mathbb{E}[|\ddot{\varphi}_{n,X}(t) \varphi_{n,X}(t)|]^2 \leq \mathbb{E}[|\ddot{\varphi}_{n,X}(t)|^2] \mathbb{E}[|\varphi_{n,X}(t)|^2]$, Lemma 4 yields $G_{n,1} = o_p(1)$. Note that

$$\begin{aligned} \mathbb{E}[|G_{n,2} + G_{n,3}|] &\leq \frac{8\pi^2}{n^2} \sum_{k=1}^{K_n} k^2 |c_k(s_\star)| \sum_{q \neq k} |c_q| \left\{ \mathbb{E}[|\ddot{\varphi}_{n,X}(0)|^2]^{1/2} \mathbb{E}[|\varphi_{n,X}[2\pi(q-k)f_\star]|^2]^{1/2} \right. \\ &\quad \left. + 2\mathbb{E}[|\dot{\varphi}_{n,X}(0)|^2]^{1/2} \mathbb{E}[|\dot{\varphi}_{n,X}[2\pi(q-k)f_\star]|^2]^{1/2} + \mathbb{E}[|\varphi_{n,X}[2\pi(q-k)f_\star]|^2]^{1/2} \right\} = O(n^{-1/2}), \end{aligned}$$

by using that $\mathbb{E} \left[n^{-2} \left(\sum_{j=1}^n X_j \right)^2 \right] = O(n^2)$, $\mathbb{E} \left[n^{-2} \left(\sum_{j=1}^n X_j^2 \right)^2 \right] = O(n^4)$ and Lemma 4. We now prove (71). In the expression of $\ddot{D}_n(f)$ given by the right-hand side of (72), we separate the summation $\sum_{p,q}$ into three terms $\sum_{p=k,q} + \sum_{p \neq k, q=k} + \sum_{p \neq k, q \neq k}$ resulting into

$$\ddot{D}_n(f) = \ddot{D}_{n,1}(f) + \ddot{D}_{n,2}(f) + \ddot{D}_{n,3}(f). \quad (74)$$

Using that $\sum_k |c_k(s_\star)| |k|^3$ and $\sum_p |c_p(s_\star)|$ are finite, and that $\ddot{\varphi}_{n,X} \overline{\varphi_{n,X}} + \dot{\varphi}_{n,X} \overline{\dot{\varphi}_{n,X}} + \varphi_{n,X} \overline{\ddot{\varphi}_{n,X}}$ is Lipschitz with Lipschitz constant at most $n^{-1} \sum_{j=1}^n X_j^3 + n^{-2} \sum_{j=1}^n X_j \sum_{j=1}^n X_j^2 = O_p(n^3)$, one easily gets that

$$\sup_{f: |f-f_\star| \leq \rho_n/n} \left| \ddot{D}_{n,1}(f) + \ddot{D}_{n,2}(f) - \ddot{D}_{n,1}(f_\star) - \ddot{D}_{n,2}(f_\star) \right| = O_p(\rho_n n^2) = o_p(n^2). \quad (75)$$

Let $(f_l)_{1 \leq l \leq L_n}$ be a regular grid with mesh δ_n covering $[f_\star - \rho_n/n, f_\star + \rho_n/n]$. Then,

$$\begin{aligned} & \sup_{f: |f-f_\star| \leq \rho_n/n} \left| \ddot{D}_{n,3}(f) - \ddot{D}_{n,3}(f_\star) \right| \\ & \leq \sup_{l=1, \dots, L_n} \left| \ddot{D}_{n,3}(f_l) - \ddot{D}_{n,3}(f_\star) \right| + \sup_{l=1, \dots, L_n} \sup_{f \in [f_l, f_{l+1}]} \left| \ddot{D}_{n,3}(f) - \ddot{D}_{n,3}(f_l) \right|. \end{aligned} \quad (76)$$

Using the same argument as above with $\sum_p |c_p(s_\star)| < \infty$ and $\sum_{k=1}^{K_n} k^3 = O(K_n^4)$, we get that $\sup_{l=1, \dots, L_n} \sup_{f \in [f_l, f_{l+1}]} \left| \ddot{D}_{n,3}(f) - \ddot{D}_{n,3}(f_l) \right| = O_p(K_n^4 \delta_n n^3)$. Since $K_n = o(n^{-1})$, there exists N such that, for any $n \geq N$, any f such that $|f - f_\star| \leq 1/n$ and any $p \in \mathbb{Z}$ and $k = 1, \dots, K_n$ such that $p \neq k$, we have $|pf_\star - kf| \geq f_\star/2$. Then proceeding as for bounding $G_n(f_\star)$ above, we have, for any $n \geq N$ and any f such that $|f - f_\star| \leq 1/n$, $\mathbb{E} \left[\left| \ddot{D}_{n,3}(f) \right| \right] \leq C K_n^3 n$, where C is some positive constant. From this, we obtain $\sup_{l=1, \dots, L_n} \left| \ddot{D}_{n,3}(f_l) - \ddot{D}_{n,3}(f_\star) \right| = O_p(L_n K_n^3 n)$, so that, for $\delta_n = n^{-3/2}$, implying $L_n = [\rho_n/(n\delta_n)] = o(n^{1/2})$, (76) finally yields $\sup_{f: |f-f_\star| \leq \rho_n/n} \left| \ddot{D}_{n,3}(f) - \ddot{D}_{n,3}(f_\star) \right| = O_p(K_n^4 n^{3/2})$, which, with (75) and (74), gives (71). \square

Proof of equation (65). The left-hand side of (65) follows from (H2) and (H3) by computing the variance and observing that \dot{s}_\star is uniformly bounded. Let us now prove the right-hand side of (65). Define

$$H_n = \frac{1}{n^{3/2}} \sum_{j=1}^n (X_j - j\mu)(s_\star \dot{s}_\star)(X_j) = \sum_{p \neq 0} c_p(s_\star \dot{s}_\star) \left(\frac{1}{n^{3/2}} \sum_{j=1}^n (X_j - j\mu) e^{2ip\pi f_\star X_j} \right).$$

By Cauchy-Schwarz inequality, setting $\gamma = \sum_{p \neq 0} |c_p(s_\star \dot{s}_\star)|$, we get

$$\begin{aligned} \|H_n\|_2^2 &\leq \gamma \sum_{p \neq 0} |c_p(s_\star \dot{s}_\star)| \left\| n^{-3/2} \sum_{j=1}^n (X_j - j\mu) e^{2ip\pi f_\star X_j} \right\|_2^2 \leq \frac{\gamma^2}{n^3} \sum_{j=1}^n \|X_j - j\mu\|_2^2 \\ &\quad + \frac{2\gamma}{n^3} \mathcal{R}e \left\{ \sum_{p \neq 0} \sum_{1 \leq j < k \leq n} |c_p(s_\star \dot{s}_\star)| \mathbb{E} \left[(X_j - j\mu)(X_k - k\mu) e^{2ip\pi f_\star (X_k - X_j)} \right] \right\}. \end{aligned}$$

The first term in the last right-hand side is $O(n^{-1})$. For the second term, we write $X_k - k\mu = X_j - j\mu + \sum_{\ell=j+1}^k (V_\ell - \mu)$ and, using (H2), we obtain

$$\|H_n\|_2^2 \leq O(n^{-1}) + \frac{2C}{n^3} \mathcal{R}e \left\{ \sum_{p \neq 0} \sum_{1 \leq j < k \leq n} |c_p(s_\star \dot{s}_\star)| \mathbb{E} \left[(X_j - j\mu)^2 e^{2ip\pi f_\star \sum_{\ell=j+1}^k V_\ell} \right] \right\}.$$

Since, by (H2) and (H3),

$$\sup_{p \neq 0} \left| \mathbb{E} \left[(X_j - j\mu)^2 e^{2ip\pi f_\star \sum_{\ell=j+1}^k V_\ell} \right] \right| = \mathbb{E} \left[(X_j - j\mu)^2 \right] \sup_{p \neq 0} |\Phi(2p\pi f_\star)|^{k-j} = O(j\delta^{k-j}),$$

where $\delta \in (0, 1)$, and since $\sum_p |c_p(s_\star \dot{s}_\star)| < \infty$, we finally get

$$\|H_n\|_2^2 = O(n^{-1}) + O \left(n^{-3} \sum_{1 \leq j < k \leq n} j\delta^{k-j} \right) = O(n^{-1}),$$

which achieves the proof of (65). \square

APPENDIX A. TECHNICAL LEMMAS

The following Lemma provides upper bounds for the moments of $\varphi_{n,X}$ defined in (33).

Lemma 4. *Let (H2) hold. Then, for any non-negative integer k , there exists a positive constant C such that for all $t \in \mathbb{R}$,*

$$\left| \mathbb{E}[\varphi_{n,X}^{(k)}(t)] \right| \leq C \max_{1 \leq l \leq k} \mathbb{E}[V_1^l] n^{k-1} (n \wedge |t|^{-1}) \leq C (1 + \mathbb{E}(V_1^k)) n^{k-1} (n \wedge |t|^{-1}), \quad (77)$$

$$\mathbb{E} \left[\left| \varphi_{n,X}^{(k)}(t) \right|^2 \right] \leq \mathbb{E}(V_1^{2k}) n^{2k-1} + C(1 + \mathbb{E}(V_1^k)) n^{2k-1} (n \wedge |t|^{-1}). \quad (78)$$

Proof. In order to prove (77), we suppose that $\mathbb{E}[V_1^k]$ is finite. Then by (33) and (H2), we may write, for all $t \in \mathbb{R}$,

$$\mathbb{E}[\varphi_{n,X}^{(k)}(t)] = n^{-1} \frac{d^k}{dt^k} \sum_{j=1}^n \Phi^j(t) = n^{-1} \frac{d^k}{dt^k} \left\{ \Phi(t) \frac{1 - \Phi^n(t)}{1 - \Phi(t)} \right\}.$$

By (H2), there exists some constant $C > 0$ such that the following inequality holds:

$$|1 - \Phi(t)|^{-1} \leq C (1 + |t|^{-1}), \quad t \in \mathbb{R}.$$

Standard computations yield, for all $j = 0, \dots, k$,

$$|\Phi^{(j)}(t)| \leq \mathbb{E}[V_1^j] \quad \text{and} \quad \left| \frac{d^j}{dt^j} \Phi^n(t) \right| \leq C n^j \max_{0 \leq l \leq j} \mathbb{E}[V_1^l], \quad t \in \mathbb{R}$$

for some constant $C > 0$ only depending on k . The three last displays yield

$$\left| \mathbb{E}[\varphi_{n,X}^{(k)}(t)] \right| \leq C \max_{1 \leq l \leq k} \mathbb{E}[V_1^l] \sum_{j=0}^k n^{j-1} (1 + |t|^{j-k-1}).$$

If $|t| \geq n^{-1}$, then $\sum_{j=0}^k n^{j-1} (1 + |t|^{j-k-1}) \leq C n^{k-1} |t|^{-1}$. If $|t| < n^{-1}$, then $\sum_{j=0}^k n^{j-1} (1 + |t|^{j-k-1}) > C n^k$. Using that, $\left| \mathbb{E}[\varphi_{n,X}^{(k)}(t)] \right| \leq n^{-1} \sum_{j=1}^n \mathbb{E}(X_j^k) \leq n^k$, for all t in \mathbb{R} , we get (77).

Now, we prove (78) when $\mathbb{E}[V_1^{2k}]$ is finite. We have, for all $t \in \mathbb{R}$,

$$\mathbb{E} \left[\left| \varphi_{n,X}^{(k)}(t) \right|^2 \right] = n^{-2} \left[\sum_{j=1}^n \mathbb{E}(X_j^{2k}) + 2 \operatorname{Re} \left\{ \mathbb{E} \left(\sum_{1 \leq j' < j \leq n} X_j^k X_{j'}^k e^{it(X_j - X_{j'})} \right) \right\} \right].$$

Writing, $X_j = X_{j'} + (V_{j'+1} + \dots + V_j)$, we obtain using (H2) and (77) that

$$n^{-2} \mathbb{E} \left(\sum_{1 \leq j' < j \leq n} X_j^k X_{j'}^k e^{it(X_j - X_{j'})} \right) \leq C(1 + \mathbb{E}(V_1^k)) n^{2k-1} (n \wedge |t|^{-1}),$$

where C is a positive constant depending only on k . Note also that $n^{-2} \sum_{j=1}^n \mathbb{E}(X_j^{2k}) \leq n^{2k-1} \mathbb{E}(V_1^{2k})$. The last two inequalities provide (78). \square

Lemma 5. *Under Assumption (H2), we have, for all $x > 0$ and $t \in \mathbb{R}$,*

$$\mathbb{P}(|\varphi_{n,X}(t) - \mathbb{E}(\varphi_{n,X}(t))| \geq x) \leq 4 \exp \left(- \frac{nx^2 |1 - \Phi(t)|}{16(2 + \sqrt{2})} \right), \quad (79)$$

where Φ is the characteristic function of V defined in (10).

Proof. Note that $\prod_{k=1}^j e^{itV_k} - \Phi^j(t) = \sum_{q=1}^j \Phi^{j-q}(t) \Pi_q(t)$ where $\Pi_q(t) = \prod_{k=1}^q e^{itV_k} - \Phi(t) \prod_{k=1}^{q-1} e^{itV_k}$. Thus,

$$n(\varphi_{n,X}(t) - \mathbb{E}[\varphi_{n,X}(t)]) = \sum_{j=1}^n \left[\left(\prod_{k=1}^j e^{itV_k} \right) - \Phi^j(t) \right] = \sum_{j=1}^n \sum_{q=1}^j \Phi^{j-q}(t) \Pi_q(t) = \sum_{q=1}^n \alpha_{n,q}(t) \Pi_q(t)$$

where $\alpha_{n,q}(t) = \sum_{j=q}^n \Phi^{j-q}(t) = (1 - \Phi(t))^{-1} (1 - \Phi^{n-q+1}(t))$, the last equality being valid as soon as $\Phi(t) \neq 1$. Let \mathcal{F}_q denotes the σ -field generated by V_1, \dots, V_q . Note that $\{\alpha_{n,q}(t) \Pi_q(t), q \geq 1\}$ is a martingale difference adapted to the filtration $(\mathcal{F}_q)_{q \geq 1}$ and

$$|\alpha_{n,q}(t) \Pi_q(t)| \leq 4|1 - \Phi(t)|^{-1} \quad \text{and} \quad \mathbb{E}^{\mathcal{F}_{q-1}} [|\alpha_{n,q}(t) \Pi_q(t)|^2] \leq \frac{4(1 - |\Phi(t)|^2)}{|1 - \Phi(t)|^2} \leq 8|1 - \Phi(t)|^{-1},$$

where $\mathbb{E}^{\mathcal{F}_q}$ denotes the conditional expectation given \mathcal{F}_q . The proof then follows from Bernstein inequality for martingale (see Steiger (1969) and Freedman (1975)). \square

For completeness, we state the following result, due to Golubev (1988).

Lemma 6. *Let \mathcal{L} be a stochastic process defined on an interval $I \subseteq \mathbb{R}$. Then, for all $\lambda, R > 0$,*

$$\mathbb{P} \left(\sup_{\tau \in I} \mathcal{L}(\tau) > R \right) \leq e^{-\lambda R} \sup_{\tau \in I} \left(\sqrt{\mathbb{E} \left(e^{2\lambda \mathcal{L}(\tau)} \right)} \right) \left(1 + \lambda \int_{\tau \in I} \sqrt{\mathbb{E} \left(\left| \dot{\mathcal{L}}(\tau) \right|^2 \right)} d\tau \right).$$

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