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# Asynchronous Cellular Automata and Brownian Motion

Philippe Chassaing<sup>1</sup> and Lucas Gerin<sup>1</sup>

<sup>1</sup> Institut Élie Cartan Nancy.

Université Henri Poincaré Nancy 1.

B.P. 239, 54506 Vandoeuvre-lès-Nancy Cedex. France.

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This paper deals with some very simple interacting particle systems, *elementary cellular automata*, in the fully asynchronous dynamics: at each time step, a cell is randomly picked, and updated. When the initial configuration is simple, we describe the asymptotic behavior of the random walks performed by the borders of the black/white regions. Following a classification introduced by Fatès *et al.*, we show that four kinds of asymptotic behavior arise, two of them being related to Brownian motion.

**Keywords:** cellular automata, asynchronism, random processes, coalescent random walks

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## 1 Introduction

### 1.1 Elementary Cellular Automata

A *cellular automaton* is a dynamical system widely used the two last decades in order to modelize phenomena arising in game theory, economy, theoretical physics, biology, or theoretical computer science (complexity, computation). It consists of a (finite or countable) set of cells, the state of each cell at time  $k$  being a function of the state of its neighbours at time  $k - 1$ . The set of possible states is finite, and, as we see, time is discrete. Cellular automata were introduced by von Neumann [8] in order to emulate *self-replication* in biology.

This paper deals more specifically with *elementary cellular automata* (ECA), introduced by Wolfram [9], that is two-state automata (0/1 or white/black) with a finite and cyclic set of cells. Let us recall a few definitions.

**Definition 1** A (*deterministic*) elementary cellular automaton (ECA) is a triplet  $(n, x(0), \delta)$ , where  $n$  is a fixed integer,  $x(0) \in \{0, 1\}^n$  is the initial configuration and  $\delta : \{0, 1\}^3 \rightarrow \{0, 1\}$  is the local transition function, or local rule.

The first studies focused on the synchronous dynamic of  $(n, x(0), \delta)$ , i.e. the evolution of the configuration under iterations of the function  $A^\delta$  on  $x(0)$ :

$$A^\delta : \quad \{0, 1\}^n \rightarrow \{0, 1\}^n \\ (x_1, \dots, x_n) \mapsto (x'_1, \dots, x'_n)$$

in which, for  $i \in \{1, 2, \dots, n\}$ ,  $x'_i = \delta(x_{i-1}, x_i, x_{i+1})$ , that is, the  $n$  cells are updated simultaneously. It must be understood with the convention  $x_{n+1} = x_1, x_0 = x_n$ , so that the set of configuration is cyclic. Alternatively, we shall consider configurations as doubly infinite periodic sequences  $(x_i)_{i \in \mathbb{Z}}$ , with period  $n$ . We will focus here only on *double-quiescent* ECA, i.e. ECA for which  $\delta(0, 0, 0) = 0$  and  $\delta(1, 1, 1) = 1$ . This terminology has been introduced in [3].

When the  $n$  cells are not updated simultaneously, but randomly picked and sequentially updated, the cellular automaton is called *asynchronous* (versus synchronous).

**Definition 2** The fully asynchronous dynamic of automaton  $\delta$  is the random process on  $\{0, 1\}^n$  defined by :

$$X_0 = x(0), \\ X_k = A_{i_k}^\delta X_{k-1}, \text{ for each } k \geq 1,$$

where  $(i_k)_{k \geq 1}$  is a sequence of i.i.d. random variables, uniform in  $\{1, \dots, n\}$  and  $A_j^\delta$  is the function defined by

$$A_j^\delta : \quad \{0, 1\}^n \rightarrow \{0, 1\}^n \\ (x_1, \dots, x_n) \mapsto (x'_1, \dots, x'_n)$$

in which  $x'_j = \delta(x_{j-1}, x_j, x_{j+1})$ , while, if  $i \neq j$ ,  $x'_i = x_i$ .

Asynchronous automata have been introduced in [4, 7], with motivations in physics, and in biology. It turns out that asynchronism actually changes drastically the asymptotic behavior of cellular automata.

Class	$\delta$	#
Identity	$\emptyset$	1
Coupon	E	2
	DE	1
Quadratic	B	4
	FG	2
	BDE	4
	BCDE	2
	BE	4
	EF	4
	BCE	2
	EFG	2
	BCDEF	4
	BEFG	4

Class	$\delta$	#
Cubic	BDEF	2
	BDEG	2
	BCDEFG	1
	BEF	4
	BEG	4
	BCEFG	2
Exponential	BCEF	4
Divergent	BF	2
	BG	2
	BCF	4
	BCFG	1

**Fig. 1:** A classification of the 64 ECA's, according to the asymptotic behavior of  $\text{WECT}_\delta$ .

## 1.2 Worst Expected Convergence Time

In the asynchronous case, for the 64 double-quiescent ECA's, the question of worst expected convergence time has been exhaustively investigated by Fatès *et al.* [3], with surprising results, that we recall below. A local transition function  $\delta$  is given by its eight transitions. A transition is said to be *active* if it changes the cell it is applied to. Of course  $\delta$  is completely determined by its active transitions. Active transitions are labelled with a letter, as follows (a notation that proves to be quite handy when classifying ECA's).

A	B	C	D	E	F	G	H
000	001	100	101	010	011	110	111
1	1	1	1	0	0	0	0

For instance, the only cells possibly changed by automaton D are precisely the white cells surrounded by two black cells. Double-quiescent ECA are those for which neither A nor H appear. Automaton *Identity* is denoted  $\emptyset$ . For an automaton  $\delta$ ,  $\mathfrak{F}_\delta$  denotes the set of fixed points of  $\delta$  (of course, when  $\delta$  is double-quiescent,  $\{0^n, 1^n\} \subset \mathfrak{F}_\delta$ ).

**Definition 3** For a fixed automaton  $(n, x(0), \delta)$  under the fully asynchronous dynamic, let  $T_n = T_n(\delta, x(0))$  be the random variable  $T_n = \inf\{k \geq 0; X_k \in \mathfrak{F}_\delta\}$ , where we use the convention  $\inf\{\emptyset\} = +\infty$ . The Worst Expected Convergence Time  $\text{WECT}_\delta$  is the real number

$$\text{WECT}_\delta = \max_{x(0) \in \{0,1\}^n} \mathbb{E}[T_n(\delta, x(0))].$$

Fatès *et al.* [3] have shown that the 64 double-quiescent ECA's can be classified in five families, according to the asymptotic behavior of  $\text{WECT}_\delta$ , when  $n$  is large. Let  $\Theta(g_n)$  denote the set of sequences  $f_n$  such that  $c_1 \leq f_n/g_n \leq c_2$ , for some constants  $c_1, c_2, 0 < c_1 \leq c_2 < +\infty$ .

**Theorem 1 (Fatès, Morvan, Schabanel & Thierry [3])** For  $\delta \neq \emptyset$ , either  $\text{WECT}_\delta$  is infinite or it belongs to one of these four classes :  $\Theta(n \log n)$ ,  $\Theta(n^2)$ ,  $\Theta(n^3)$ ,  $\Theta(n 2^n)$ . The corresponding families of automata are called respectively Divergent, Coupon Collector, Quadratic, Cubic, and Exponential.

This classification is remarkably similar to that introduced by Wolfram in a completely different context. For reasons of symmetry between black and white, or between left and right, the 64 cases reduce actually to 25. The main results of [3] are summarized in Figure 1 (the third column gives the number of symmetries).

For seek of brevity, we assume that  $n$  is even, and we consider the evolution of the “black region” (i.e. the sequence of consecutive 1's), when the initial configuration is  $x(0) = 0^{n/2}1^{n/2}$ : due to full asynchronism, there exists a unique black region in  $X_k$ , unless the process eventually reaches  $0^n$ .

In this paper, we study the asymptotic behavior of the two borders of the black region. At time  $k$ , the left border  $L_k$  is defined as some integer such that  $x_{L_k}(k) = 0, x_{L_k+1}(k) = 1$ , and the left border  $R_k$  is as some integer such that  $x_{R_k}(k) = 1, x_{R_k+1}(k) = 0$ . We shall assume that  $L_k, R_k \in \{1, 2, \dots, n\}$ , or not, depending on the automaton under study (it can be interesting to track if the black zone shifts, makes several revolutions, for instance). In a longer paper, we shall describe the asymptotic behavior of the borders of black regions for an arbitrary initial state.

Precisely, given some automaton, we exhibit some continuous process with values in  $\mathbb{R}^2$  such that the following weak convergence holds (in a sense to be defined in the next Section):

$$\frac{1}{n} (L_{\lfloor t \mathbb{E}[T_n] \rfloor}, R_{\lfloor t \mathbb{E}[T_n] \rfloor})_{t \geq 0} \Rightarrow (X_t^{(1)}, X_t^{(2)})_{t \geq 0}. \quad (1)$$

### 1.3 Convergence in $\mathcal{D}_p(I)$

If  $I$  is an interval  $[0, T]$ , with  $0 \leq T \leq +\infty$ , let  $\mathcal{D}_p(I)$  be the set of càdlag<sup>(i)</sup> functions:  $I \rightarrow \mathbb{R}^p$ . We address the convergence of random variables in  $\mathcal{D}_p(I)$ , endowed with the Skorohod topology. Recall that, when the limit is a continuous function, convergence in the Skorohod topology is equivalent to uniform convergence on compact sets, that is, convergence for the distance

$$d(f, g) = \sum_{k \geq 1} 2^{-k} \left( 1 \wedge \sup_{t \leq k} \|f(t) - g(t)\|_{\mathbb{R}^p} \right).$$

**Definition 4 (Convergence in  $\mathcal{D}_p(I)$ )** Let  $X$  (resp.  $(X^{(n)})_{n \geq 0}$ ) be a random variable (resp. a sequence of random variables) with values in  $\mathcal{D}_p(I)$ . The sequence  $X^{(n)}$  converges weakly to  $X$ , if for any function  $\mathcal{L} : \mathcal{D}_p(I) \rightarrow \mathbb{R}$ , bounded and continuous,

$$\lim_n \mathbb{E}[\mathcal{L}(X^{(n)})] = \mathbb{E}[\mathcal{L}(X)].$$

We shall use the notation

$$X^{(n)} \Rightarrow X.$$

We use repeatedly the next two results:

**Theorem 2 ([1], Th. 5.1)** Let  $h : \mathcal{D}_p(I) \rightarrow \mathcal{D}_p(I)$ , and  $D_h$  be the set of discontinuity points of  $h$ . Assume that  $X^{(n)} \Rightarrow X$  and that  $\mathbb{P}(X \in D_h) = 0$ . Then

$$h(X^{(n)}) \Rightarrow h(X).$$

Perhaps the most important result of convergence of stochastic processes is the convergence of renormalized random walks to the linear Brownian motion  $(B_t)_{t \geq 0}$  [1, 2, 6]:

**Theorem 3 (Donsker [2])** Let  $X_1, X_2, \dots$  be a sequence of i.i.d. random variables with  $\mathbb{E}[X_1] = 0$  and  $\mathbb{E}[X_1^2] = 1$ . Set  $S_k = \sum_{i \leq k} X_i$ . Then

$$\left( \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}} \right)_{t \geq 0} \Rightarrow (B_t)_{t \geq 0}.$$

### 1.4 The Results

In this paper, we study the case where the initial configuration  $x(0)$  is composed by a single black region:  $x(0) = 0^{n/2} 1^{n/2}$ . The space renormalization must be  $1/n$ , and the time renormalization has to be  $\mathcal{O}(\mathbb{E}[T_n]^{-1})$ , as shown in equation (1). Roughly speaking, renormalization of a discrete process can lead to three different behaviors, ordered by increasing degree of randomness:

- convergence to a non-random process, preferably non null,
- convergence to a random process (e.g. Brownian motion),
- the sequence is tight (relatively compact) but different subsequences converge to different limit processes,
- the sequence is not tight (unbounded).

Our results are roughly summarized below:

quadratic	→	non-random limit
cubic	→	reflected (and-or) coalescent Brownian motions
exponential	→	no limit (untight)
divergent	→	reflected Brownian motions

so that three of the four previous cases occur when renormalizing ECA's as in (1).

## 2 Quadratic Automata : non-random limit

### 2.1 Automaton FG

In the case of the automaton FG, the stochastic process defined by the borders  $(L_k, R_k)$  of the black domain is a Markov chain. For  $t \geq 0$ , set

$$\psi(t) = (\psi_1(t), \psi_2(t)) = \left( \frac{1}{2} + t, 1 - t \right).$$

Due to Theorem 1, only the time-renormalization  $n^2$  can, eventually, lead to a nontrivial limit process. Actually, a limit process exists, and this limit is non-random. For  $t \geq 0$ , set

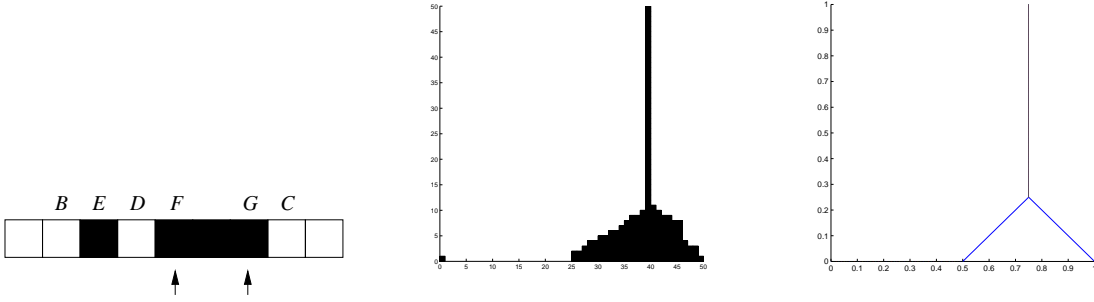
$$\ell_n(t) = L_{\lfloor tn^2 \rfloor \wedge T_n} / n, \quad r_n(t) = R_{\lfloor tn^2 \rfloor \wedge T_n} / n.$$

---

<sup>(i)</sup> càdlag from *continue à droite, limitée à gauche* (french): right-continuous functions that admit a left-limit at each point of  $(0, T]$ .

**Theorem 4** *The following convergence holds in  $\mathcal{D}_2(\mathbb{R}_+)$  :*

$$(\ell_n, r_n) \Rightarrow (\psi(t \wedge \frac{1}{4}))_{t \geq 0}.$$



**Fig. 2:** Automaton FG, and its limit process  $\psi$ .

**Proof:** First, consider the Markov chain  $(\tilde{L}_k, \tilde{R}_k)_{k \geq 0}$  defined by  $(\tilde{L}_0, \tilde{R}_0) = (n/2, 0)$ , and

$$(\tilde{L}_{k+1}, \tilde{R}_{k+1}) = \begin{cases} (\tilde{L}_k, \tilde{R}_k) & \text{with probability } \frac{n-2}{n} \\ (\tilde{L}_k + 1, \tilde{R}_k) & \text{with probability } \frac{1}{n} \\ (\tilde{L}_k, \tilde{R}_k - 1) & \text{with probability } \frac{1}{n} \end{cases}$$

For  $t \geq 0$ , set

$$\tilde{\ell}_n(t) = \tilde{L}_{\lfloor tn^2 \rfloor} / n, \quad \tilde{r}_n(t) = \tilde{R}_{\lfloor tn^2 \rfloor} / n.$$

For  $x, T$  two positive constants,

$$\mathbb{P} \left( \sup_{t \leq T} |\tilde{\ell}_n(t) - \psi_1(t)| \geq x \right) \leq \mathbb{P} \left( \sup_{t \leq T} |\tilde{\ell}_n(t) - \mathbb{E}[\tilde{\ell}_n(t)]| \geq \frac{x}{2} \right) + \mathbb{P} \left( \sup_{t \leq T} |\mathbb{E}[\tilde{\ell}_n(t)] - \psi_1(t)| \geq \frac{x}{2} \right).$$

For  $n$  large enough, the last probability on the right-hand side cancels. For the first one, we need the following bound:

**Lemma 4.1 ([5], Chap.3, Th. 18)** *Let  $(Y_{k,n})_{k \geq 0}$  denote sequences of i.i.d. random variables such that  $\mathbb{E}[Y_{1,n}] = 0$ ,  $\mathbb{E}[Y_{1,n}^2] = c_n < \infty$ . One notes  $S_{k,n} = Y_{1,n} + \dots + Y_{k,n}$ . For any  $k$  and  $x > 0$ ,*

$$\mathbb{P} \left( \max_{1 \leq l \leq k} |S_{l,n}| \geq x \right) \leq c_n k / x^2. \quad (2)$$

Let us write  $\tilde{L}_k - (n/2) = B_1 + \dots + B_k$ , in which the  $B_i$ 's are i.i.d. random variables with  $\mathbb{P}(B_i = 1) = 1 - \mathbb{P}(B_i = 0) = 1/n$ . Applying Lemma 4.1 with  $S_{k,n} = \tilde{L}_k - (n/2) - (k/n)$ ,  $Y_{i,n} = B_i - (1/n)$ ,  $c_n = \frac{n-1}{n^2}$  and  $k = \lfloor Tn^2 \rfloor$ , one obtains

$$\begin{aligned} \mathbb{P} \left( \sup_{t \leq T} |\tilde{\ell}_n(t) - \mathbb{E}[\tilde{\ell}_n(t)]| \geq \frac{x}{2} \right) &= \mathbb{P} \left( \max_{1 \leq \ell \leq \lfloor Tn^2 \rfloor} |S_{\ell,n}| \geq \frac{nx}{2} \right) \\ &\leq 4 \lfloor Tn^2 \rfloor n^{-3} x^{-2}. \end{aligned}$$

With  $x = n^{-1/2+\delta}$ , for some  $\delta \in (0, 1/2)$ , it leads to

$$\mathbb{P} \left( \sup_{t \leq T} |\tilde{\ell}_n(t) - \psi_1(t)| \geq x \right) \leq T n^{-2\delta}.$$

The same argument holds for the right border  $\tilde{R}_k$ . It follows that

$$(\tilde{\ell}_n, \tilde{r}_n) \Rightarrow \psi. \quad (3)$$

Now, the process  $(\tilde{L}_k, \tilde{R}_k)_{k \geq 0}$  is designed to have the same distribution as  $(L_k, R_k)_{k \geq 0}$ , as long as  $L_k \leq R_k - 1$ . More precisely, if  $\tau$  and  $\mathcal{L}$  denote the operators defined on  $\mathcal{D}_2(0, +\infty)$  by

$$\tau(f) = \inf\{t \geq 0; f_1(t) \geq f_2(t) - 1\},$$

and

$$\mathcal{L}(f) = (f(t \wedge \tau(f)))_{t \geq 0},$$

then we have:

$$(\ell_n, r_n) \stackrel{\text{law}}{=} \mathcal{L}(\tilde{\ell}_n, \tilde{r}_n). \quad (4)$$

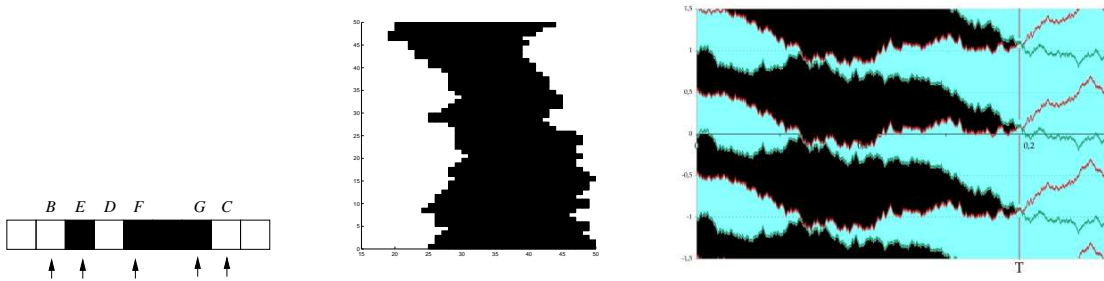
Theorem 2 allows us to conclude, since, in the relation (3), the limit point  $\psi$  is a point of continuity of  $\mathcal{L}$ , and since  $(\psi(t \wedge \frac{1}{4}))_{t \geq 0} = \mathcal{L}\psi$ .  $\square$

## 2.2 Other Quadratic Automata

Quadratic automata are roughly divided into two sub-families. FG belongs to the first one, with automata B, EF, EFG, BDE, BE, BCDE and BCE. The proof adapts easily to all of them, and they converge to non-random limits. The second family contains BCDEF and BEFG. Their behavior is slightly different. A border (say, the left-border) essentially drifts to the right (with small random perturbations), whereas the right-border performs a symmetric random walk. However, these random perturbations are of order  $\mathcal{O}(n^{1/2})$  and are erased by the space renormalization factor  $1/n$ , so that the limit is also deterministic. For quadratic automata, our results extend easily to the case of several black regions, for the regions essentially do not interact.

## 3 Cubic Automata : Interactions between Brownian Motions

### 3.1 Automaton BCEFG



**Fig. 3:** Automaton BCEFG: a simulation for  $n = 50$  and the limit process (here  $T \sim 0.195 \dots$ ).

The class of cubic automata provides a variety of interesting limit processes, related with the standard linear Brownian motion [6]. In this section, we choose the unique determination of  $(L_k, R_k)_{k \geq 0}$  such that  $(L_0, R_0) = (n/2, 0)$  and  $|L_{k+1} - L_k| \leq 1$ ,  $|R_{k+1} - R_k| \leq 1$ , for all  $k < T_n$ . For sake of brevity, we focus on the automaton BCEFG: its limit process can be described by reflection and coalescence of two independent standard linear Brownian motions  $W_1$  and  $W_2$ : set  $B_t^{(1)} = 0.5 + \sqrt{2}W_1(t)$  (resp.  $B_t^{(2)} = \sqrt{2}W_2(t)$ ). For  $tn^3 \leq T_n$ , set

$$\ell_n(t) = \frac{L_{\lfloor tn^3 \rfloor}}{n}, \quad r_n(t) = \frac{R_{\lfloor tn^3 \rfloor}}{n}.$$

We have

**Theorem 5** Set

$$(B_t^+, B_t^-) = (B_t^{(1)} \vee B_t^{(2)}, B_t^{(1)} \wedge B_t^{(2)}),$$

and

$$T = \inf\{t \geq 0; |B_t^{(1)} - B_t^{(2)}| \geq 1\} = \inf\{t \geq 0; B_t^+ - B_t^- \geq 1\}.$$

Then

$$(\ell_n(t), r_n(t))_{0 \leq tn^3 \leq T_n} \Rightarrow (B_t^+, B_t^-)_{0 \leq t \leq T}.$$

**Proof:** First, we study a simpler Markov chain,  $(\tilde{L}_k^{(n)}, \tilde{R}_k^{(n)})_{k \geq 0} = (\tilde{L}_k, \tilde{R}_k)_{k \geq 0}$ , with values in  $\mathbb{Z}^2$ , starting at  $(\frac{n}{2}, 0)$ . Its transition probabilities  $p_{(x,y),(z,t)}$  are defined as follows:

- if  $y = x - 1$ ,

$$p_{(x,y),(x+1,y)} = p_{(x,y),(x,y-1)} = p_{(x,y),(y,x)} = \frac{1}{n}, \quad p_{(x,y),(x,y)} = \frac{n-3}{n},$$

- if  $y = x - n + 1$ ,

$$p_{(x,y),(x-1,y)} = p_{(x,y),(x,y+1)} = p_{(x,y),(x+1,y-1)} = \frac{1}{n}, \quad p_{(x,y),(x,y)} = \frac{n-3}{n},$$

- else,

$$P_{(x,y),(x-1,y)} = P_{(x,y),(x,y-1)} = P_{(x,y),(x+1,y)} = P_{(x,y),(x,y+1)} = \frac{1}{n}, \quad P_{(x,y),(x,y)} = \frac{n-4}{n}.$$

We take  $p$  symmetric, that is:  $p_{(y,x),(t,z)} = p_{(x,y),(z,t)}$ . The transitions of  $(\tilde{L}_k, \tilde{R}_k)_{k \geq 0}$  are designed with the purpose that the Markov chain

$$\left( \tilde{L}_k^+, \tilde{R}_k^- \right) = \left( \tilde{L}_k \vee \tilde{R}_k, \tilde{L}_k \wedge \tilde{R}_k \right)$$

has the same distribution as  $(L_k, R_k)_{k \geq 0}$ , as long as  $L_k - n \leq R_k - 1$ . These processes, when suitably renormalized, converges to Brownian-like stochastic processes. More precisely, for  $t \geq 0$ , set

$$\left( \tilde{\ell}_n, \tilde{r}_n, \tilde{\ell}_n^+, \tilde{r}_n^- \right) (t) = \frac{1}{n} \left( \tilde{L}_{\lfloor tn^3 \rfloor}, \tilde{R}_{\lfloor tn^3 \rfloor}, \tilde{L}_{\lfloor tn^3 \rfloor}^+, \tilde{R}_{\lfloor tn^3 \rfloor}^- \right).$$

Then

$$\left( \tilde{\ell}_n, \tilde{r}_n \right) \Rightarrow \left( B_t^{(1)}, B_t^{(2)} \right)_{t \geq 0}. \quad (5)$$

The proof, though not really diffi cult, would exceed the 10 pages limit. Since the operator  $\Lambda$  defined on  $\mathcal{D}_2(0, +\infty)$  by

$$\Lambda(f) = (f_1(t) \vee f_2(t), f_1(t) \wedge f_2(t))_{t \geq 0}$$

is continuous, it follows that

$$\left( \tilde{\ell}_n^+, \tilde{r}_n^- \right) \Rightarrow \left( B_t^+, B_t^- \right)_{t \geq 0}.$$

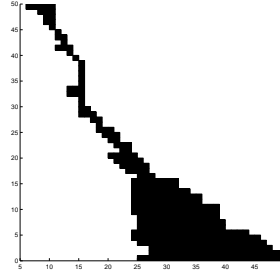
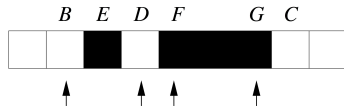
The stochastic process  $(B_t^+, B_t^-)_{t \geq 0}$  is often called a planar Brownian motion reflected at a line (here the first bisectrix). Finally, using the operators  $\tau$  and  $\tilde{\mathcal{L}}$  defined at Section 2.1, we have again:

$$(\ell_n, r_n) \stackrel{\text{law}}{=} \mathcal{L} \left( \tilde{\ell}_n^+, \tilde{r}_n^- \right). \quad (6)$$

Again, Theorem 2 allows us to conclude, since, due to properties of sample paths of the standard Brownian motion (cf. [6], Chap.2, Th.2.2), the limit point  $(B_t^+, B_t^-)_{t \geq 0}$  is almost surely a point of continuity of  $\mathcal{L}$ .  $\square$

The other cubic automata have basically the same limit, differing only through the scaling factor, and the black/white symmetry. There is little doubt the results can be extended to general initial configurations, with  $k$  black regions, the limit involving  $2k$  Brownian motions.

## 4 Exponential Automaton



**Fig. 4:** Automaton BDFG.

BDFG is, up to symmetries, the only *exponential automaton*. Simulations suggest that its behavior is quite different from those already encountered. The right border essentially drifts to the left (with small random perturbations), while the left border, that would be a symmetric random walk, is pushed to the left by the right border. Actually, the size of the black region  $Z_k^{(n)} = |R_k - L_k|$  performs a *biased* random walk on  $\{1, \dots, n\}$ , reflected at 1, absorbed at  $n$ . According to [3],

$$\mathbb{E}[T_n] = \frac{1}{9} n 2^n + \mathcal{O}(n^2).$$

However, it turns out that the process

$$z_n = \left( \frac{1}{n} Z_{\lfloor tn 2^n \rfloor}^{(n)} \right)_{t \geq 0}$$

is not weakly convergent, as opposed to the previous cases. Actually the sequence  $z_n$  is not *tight* <sup>(ii)</sup>.

<sup>(ii)</sup> Meaning that its closure is not even compact, cf. [1] for definitions.

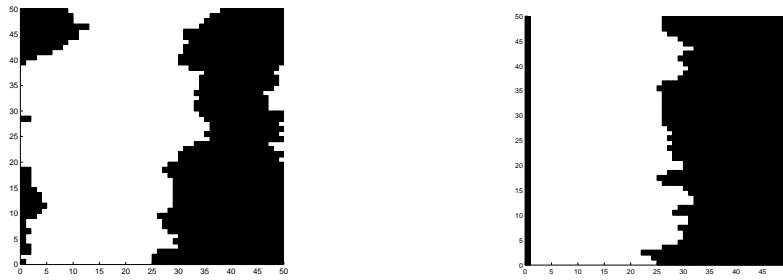


Fig. 5: Simulations for divergent automata BCFG and BF.

## 5 Divergent Automata

### 5.1 Automata BCFG, BF and CF: reflected Brownian Motions

The limit processes of these three divergent automata are related to reflected Brownian motions. The main difference with Section 3 is that coalescence does not occur. In order to state our results for automaton BCFG, we shall use the same tools and notations as in Section 3. Set

$$(L_t, R_t) = \left( B_t^{(2)}, B_t^{(1)} \right) + (-1, 1) \frac{\lfloor B_t^{(2)} - B_t^{(1)} \rfloor}{2} + (0, 1),$$

if  $\lfloor B_t^{(2)} - B_t^{(1)} \rfloor$  is even,

$$(L_t, R_t) = \left( B_t^{(1)}, B_t^{(2)} \right) + (1, -1) \frac{\lfloor B_t^{(2)} - B_t^{(1)} \rfloor}{2} + (0.5, -0.5),$$

if  $\lfloor B_t^{(2)} - B_t^{(1)} \rfloor$  is odd. One can see  $(L, R)$  as two self-reflected Brownian motions on the circle.

**Theorem 6** For automaton BCFG,

$$(\ell_n, r_n) \Rightarrow (L, R).$$

For automaton BF,  $(\ell_n, r_n) \Rightarrow (W, 1)$ , in which  $W$  denotes a standard linear Brownian motion starting at 0.5, reflected at 0 and 1, while for automaton CF, only the renormalized width  $z_n = r_n - \ell_n$  of the black region converges to  $W$ , while  $(\ell_n, r_n)$  is untight: more precisely, one can see that

$$(\ell_n(t)/n, r_n(t)/n)_{t \geq 0} \Rightarrow (t, 0.5 + t)_{t \geq 0}.$$

### 5.2 Automaton BCF

This automaton behaves a lot like the exponential automaton BDFG of Section 4, with the difference that its width is reflected at 0 but also at  $n - 1$ . The hitting time of the barrier  $n - 1$  has again an expectation  $n 2^n$ , but then the whole process starts again. For the same reasons as in Section 4, the sequence of processes  $z_n$  is not tight.

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