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Minimal position and critical martingale convergence in branching random walks, and directed polymers on disordered trees

by

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Summary. We establish an unusual second-order almost sure limit theorem for the minimal position in a one-dimensional super-critical branching random walk, and also prove a martingale convergence theorem which answers a question of Biggins and Kyprianou [7]. Our method applies furthermore to the study of directed polymers on a disordered tree. In particular, we give a rigorous proof of a phase transition phenomenon for the partition function (from the point of view of convergence in probability), already described by Derrida and Spohn [14]. Surprisingly, this phase transition phenomenon disappears in the sense of upper almost sure limits.

Keywords. Branching random walk, minimal position, martingale convergence, spine, marked tree, directed polymer on a tree.

2000 Mathematics Subject Classification. 60J80.

1 Introduction

1.1 Branching random walk and martingale convergence

We consider a branching random walk on the real line \mathbb{R} . Initially, a particle sits at the origin. Its children form the first generation; their displacements from the origin correspond to a point process on the line. These children have children of their own (who form the second generation), and behave – relative to their respective positions – like independent copies of the initial particle. And so on.

We write $|u| = n$ if an individual u is in the n -th generation, and denote its position by $V(u)$. (In particular, for the initial ancestor e , we have $V(e) = 0$.) We assume throughout the paper that there exists a constant $C > 0$ such that $\sup_{|u|=1} |V(u)| \leq C$. For technical reasons, we also assume that

$$(1.1) \quad \mathbf{E} \left\{ \left(\sum_{|u|=1} 1 \right)^{1+\delta} \right\} < \infty, \quad \text{for some } \delta > 0,$$

where \mathbf{E} denotes expectation with respect to \mathbf{P} , the law of the branching random walk.

Let us define the (logarithmic) moment generating function

$$\psi(t) := \log \mathbf{E} \left\{ \sum_{|u|=1} e^{-tV(u)} \right\}, \quad t \geq 0.$$

Following Biggins and Kyprianou [7], we assume

$$(1.2) \quad \psi(0) > 0, \quad \psi(1) = \psi'(1) = 0.$$

Since the number of particles in each generation forms a Galton–Watson tree, the assumption $\psi(0) > 0$ in (1.2) says that this Galton–Watson tree is super-critical.

In the study of the branching random walk, there is a fundamental martingale, defined as follows:

$$(1.3) \quad W_n := \sum_{|u|=n} e^{-V(u)}, \quad n = 0, 1, 2, \dots$$

Since $W_n \geq 0$, it converges almost surely.

When $\psi'(1) < 0$, it is proved by Biggins and Kyprianou [5] that there exists a sequence of constants (a_n) such that $\frac{W_n}{a_n}$ converges in probability to a non-degenerate limit which is (strictly) positive upon the survival of the system. This is called the Seneta–Heyde norming in [5] for branching random walk, referring to Seneta [30] and Heyde [18] on the rate of convergence in the classic Kesten–Stigum theorem for Galton–Watson processes.

The case $\psi'(1) = 0$ is more delicate. In this case, it is known (Lyons [24]) that $W_n \rightarrow 0$ almost surely. The following question is raised in Biggins and Kyprianou [7]: are there deterministic normalizers (a_n) such that $\frac{W_n}{a_n}$ converges?

We aim at answering this question.

Theorem 1.1 *Assume (1.1) and (1.2). There exists a deterministic positive sequence (λ_n) with $0 < \liminf_{n \rightarrow \infty} \frac{\lambda_n}{n^{1/2}} \leq \limsup_{n \rightarrow \infty} \frac{\lambda_n}{n^{1/2}} < \infty$, such that conditionally on the system's survival, $\lambda_n W_n$ converges in distribution to \mathcal{W} , with $\mathcal{W} > 0$ a.s. The distribution of \mathcal{W} is given in (9.6).*

The almost sure behaviour of W_n is described in Theorem 1.3 below. The two theorems together give a clear image of the asymptotics of W_n .

1.2 The minimal position in the branching random walk

A natural question in the study of branching random walks is about $\inf_{|u|=n} V(u)$, the position of the leftmost individual in the n -th generation. In the literature, the concentration (in terms of tightness or even weak convergence) of $\inf_{|u|=n} V(u)$ around its median/quantiles had been studied by many authors. See for example Bachmann [3], Bramson and Zeitouni [11], as well as Section 5 of the survey paper by Aldous and Bandyopadhyay [2]. We also mention the recent paper of Lifshits [21], where an example of branching random walk is constructed such that $\inf_{|u|=n} V(u) - \text{median}(\{\inf_{|u|=n} V(u)\})$ is tight but does not converge weakly.

We are interested in the asymptotic speed of $\inf_{|u|=n} V(u)$. Under assumption (1.2), it is known that, conditionally on the system's survival,

$$(1.4) \quad \frac{1}{n} \inf_{|u|=n} V(u) \rightarrow 0, \quad \text{a.s.}$$

$$(1.5) \quad \inf_{|u|=n} V(u) \rightarrow +\infty, \quad \text{a.s.}$$

The ‘‘law of large numbers’’ in (1.4) is a classic result, and can be found in Hammersley [15], Kingman [19], Biggins [4]. The system's transience to the right, stated in (1.5), follows from the fact that $W_n \rightarrow 0$, a.s.

A refinement of (1.4) is obtained by McDiarmid [26]. By assuming

$$(1.6) \quad \mathbf{E}\left\{\left(\sum_{|u|=1} 1\right)^2\right\} < \infty,$$

it is proved in [26] that for some constant $c_1 < \infty$ and conditionally on the system's survival,

$$\limsup_{n \rightarrow \infty} \frac{1}{\log n} \inf_{|u|=n} V(u) \leq c_1, \quad \text{a.s.}$$

We intend to determine the exact rate at which $\inf_{|u|=n} V(u)$ goes to infinity.

Theorem 1.2 *Assume (1.2) and (1.6). Conditionally on the system's survival, we have*

$$(1.7) \quad \limsup_{n \rightarrow \infty} \frac{1}{\log n} \inf_{|u|=n} V(u) = \frac{3}{2}, \quad \text{a.s.}$$

$$(1.8) \quad \liminf_{n \rightarrow \infty} \frac{1}{\log n} \inf_{|u|=n} V(u) = \frac{1}{2}, \quad \text{a.s.}$$

$$(1.9) \quad \lim_{n \rightarrow \infty} \frac{1}{\log n} \inf_{|u|=n} V(u) = \frac{3}{2}, \quad \text{in probability.}$$

Remark. (i) The most interesting part of Theorem 1.2 is (1.7)–(1.8). It reveals, surprisingly, the presence of fluctuations of $\inf_{|u|=n} V(u)$ on the logarithmic level, which is in contrast with known results of Bramson [10] and Dekking and Host [13] stating that for a class of branching random walks, $\frac{1}{\log \log n} \inf_{|u|=n} V(u)$ converges almost surely to a finite and positive constant.

(ii) Some brief comments on (1.2) are in order. In general (i.e., without assuming $\psi(1) = \psi'(1) = 0$), the law of large numbers (1.4) reads $\frac{1}{n} \inf_{|u|=n} V(u) \rightarrow c$, a.s. (conditionally on the system's survival), where $c := \inf\{a \in \mathbb{R} : g(a) \geq 0\}$, with $g(a) := \inf_{t \geq 0} \{ta + \psi(t)\}$. If

$$(1.10) \quad t^* \psi'(t^*) = \psi(t^*)$$

for some $t_* \in (0, \infty)$, then the branching random walk associated with the point process $\widehat{V}(u) := t^* V(u) + \psi(t^*)|u|$ satisfies (1.2). That is, as long as (1.10) has a solution (which is the case for example if $\psi(1) = 0$ and $\psi'(1) > 0$), the study will boil down to the case (1.2).

It is, however, possible that (1.10) has no solution. In such a situation, Theorem 1.2 does not apply. For example, we have already mentioned a class of branching random walks exhibited in Bramson [10] and Dekking and Host [13], for which $\inf_{|u|=n} V(u)$ has an exotic $\log \log n$ behaviour.

(iii) Under suitable assumptions, Addario-Berry [1] obtains a very precise asymptotic estimate of $\mathbf{E}[\inf_{|u|=n} V(u)]$, which implies (1.9).

(iv) In the case of branching Brownian motion, the analogue of (1.9) was proved by Bramson [9], by means of some powerful explicit analysis.

1.3 Directed polymers on a disordered tree

The following model is borrowed from the well-known paper of Derrida and Spohn [14]: Let \mathbb{T} be a rooted Cayley tree; we study all self-avoiding walks (= directed polymers) of n steps on \mathbb{T} starting from the root. To each edge of the tree, is attached a random variable (= potential). We assume that these random variables are independent and identically distributed. For each walk ω , its energy $E(\omega)$ is the sum of the potentials of the edges visited by the walk. So the partition function is

$$Z_n := \sum_{\omega} e^{-\beta E(\omega)},$$

where the sum is over all self-avoiding walks of n steps on \mathbb{T} , and $\beta > 0$ is the inverse temperature.

More generally, we take \mathbb{T} to be a Galton–Watson tree, and observe that the energy $E(\omega)$ corresponds to (the partial sum of) the branching random walk described in the previous sections. The associated partition function becomes

$$(1.11) \quad W_{n,\beta} := \sum_{|u|=n} e^{-\beta V(u)}, \quad \beta > 0.$$

Clearly, when $\beta = 1$, $W_{n,1}$ is just the W_n defined in (1.3).

If $0 < \beta < 1$, the study of $W_{n,\beta}$ boils down to the case $\psi'(1) < 0$ which was investigated by Biggins and Kyprianou [5]. In particular, conditionally on the system's survival, $\frac{W_{n,\beta}}{\mathbf{E}\{W_{n,\beta}\}}$ converges almost surely to a (strictly) positive random variable.

We study the case $\beta \geq 1$ in the present paper.

Theorem 1.3 *Assume (1.1) and (1.2). Conditionally on the system's survival, we have*

$$(1.12) \quad W_n = n^{-1/2+o(1)}, \quad \text{a.s.}$$

Theorem 1.4 *Assume (1.2) and (1.6), and let $\beta > 1$. Conditionally on the system's survival, we have*

$$(1.13) \quad \limsup_{n \rightarrow \infty} \frac{\log W_{n,\beta}}{\log n} = -\frac{\beta}{2}, \quad \text{a.s.}$$

$$(1.14) \quad \liminf_{n \rightarrow \infty} \frac{\log W_{n,\beta}}{\log n} = -\frac{3\beta}{2}, \quad \text{a.s.}$$

$$(1.15) \quad W_{n,\beta} = n^{-3\beta/2+o(1)}, \quad \text{in probability.}$$

Again, the most interesting part in Theorem 1.4 is (1.13)–(1.14), which describes a new fluctuation phenomenon. Also, there is no phase transition any more for $W_{n,\beta}$ at $\beta = 1$ from the point of view of upper almost sure limits.

The remark on (1.2), stated after Theorem 1.2, applies to Theorems 1.3 and 1.4 as well.

An important step in the proof of Theorems 1.3 and 1.4 is to estimate all small moments of W_n and $W_{n,\beta}$, respectively. This is done in the next theorems.

Theorem 1.5 *Assume (1.1) and (1.2). For any $\gamma \in [0, 1)$, we have*

$$(1.16) \quad 0 < \liminf_{n \rightarrow \infty} \mathbf{E} \left\{ (n^{1/2}W_n)^\gamma \mid \mathcal{S} \right\} \leq \limsup_{n \rightarrow \infty} \mathbf{E} \left\{ (n^{1/2}W_n)^\gamma \mid \mathcal{S} \right\} < \infty,$$

where \mathcal{S} denotes the event that the system survives ultimately.

Theorem 1.6 *Assume (1.2) and (1.6), and let $\beta > 1$. For any $0 < r < \frac{1}{\beta}$, we have*

$$(1.17) \quad \mathbf{E} \{ W_{n,\beta}^r \} = n^{-3r\beta/2+o(1)}, \quad n \rightarrow \infty.$$

The rest of the paper is as follows. In Section 2, we introduce a change-of-measures formula (Proposition 2.1) in terms of spines on marked trees. This formula will be of frequent use throughout the paper. Section 3 contains a few preliminary results of the lower tail probability of the martingale W_n . The proof of the theorems is organized as follows.

- Section 4: Theorem 1.6.
- Section 5: Theorem 1.5.
- Section 6: Theorem 1.3, as well as parts (1.14) and (1.15) of Theorem 1.4.
- Section 7: Theorem 1.2.
- Section 8: part (1.13) of Theorem 1.4.
- Section 9: Theorem 1.1.

Sections 4 and 5 are the technical part of the paper, where a common idea is applied in two different situations.

Throughout the paper, we write

$$q := \mathbf{P}\{\text{the system's survival}\} \in (0, 1).$$

Also, the letter c with a subscript denotes finite and (strictly) positive constants. We also use the notation $\sum_{\emptyset} := 0$ and $\prod_{\emptyset} := 1$.

2 Marked trees and spines

This section is devoted to a change-of-measures result (Proposition 2.1) on marked trees in terms of spines. The material of this section has been presented in the literature in various forms, see for example Chauvin et al. [12], Lyons et al. [25], Biggins and Kyprianou [6], Hardy and Harris [16].

There is a one-to-one correspondence between branching random walks and marked trees. Let us first introduce some notation. We label individuals in the branching random walk by their line of descent, so if $u = i_1 \cdots i_n \in \mathcal{U} := \{\emptyset\} \cup \bigcup_{k=1}^{\infty} (\mathbb{N}^*)^k$ (where $\mathbb{N}^* := \{1, 2, \dots\}$), then u is the i_n -th child of the i_{n-1} -th child of \dots of the i_1 -th child of the initial ancestor e . It is sometimes convenient to consider an element $u \in \mathcal{U}$ as a “word” of length $|u|$, with \emptyset corresponding to e . We identify an individual u with its corresponding word.

If $u, v \in \mathcal{U}$, we denote by uv the concatenated word, with $u\emptyset = \emptyset u = u$.

Let $\overline{\mathcal{U}} := \{(u, V(u)) : u \in \mathcal{U}, V : \mathcal{U} \rightarrow \mathbb{R}\}$. Let Ω be Neveu’s space of marked trees, which consists of all the subsets ω of $\overline{\mathcal{U}}$ such that the first component of ω is a tree. [Recall that a tree t is a subset of \mathcal{U} satisfying: (i) $\emptyset \in t$; (ii) if $uj \in t$ for some $j \in \mathbb{N}^*$, then $u \in t$; (iii) if $u \in t$, then $uj \in t$ if and only if $1 \leq j \leq \nu_u(t)$ for some non-negative integer $\nu_u(t)$.]

Let $\mathbb{T} : \Omega \rightarrow \Omega$ be the identity application. According to Neveu [27], there exists a probability \mathbf{P} on Ω such that the law of \mathbb{T} under \mathbf{P} is the law of the branching random walk described in the introduction.

Let us make a more intuitive presentation. For any $\omega \in \Omega$, let

$$(2.1) \quad \mathbb{T}^{\text{GW}}(\omega) := \text{the set of individuals ever born in } \omega,$$

$$(2.2) \quad \mathbb{T}(\omega) := \{(u, V(u)), u \in \mathbb{T}^{\text{GW}}(\omega), V \text{ such that } (u, V(u)) \in \omega\}.$$

[Of course, $\mathbb{T}(\omega) = \omega$.] In words, \mathbb{T}^{GW} is a Galton–Watson tree, with the population members as the vertices, whereas the *marked tree* \mathbb{T} corresponds to the branching random walk. It is more convenient to write (2.2) in an informal way:

$$\mathbb{T} = \{(u, V(u)), u \in \mathbb{T}^{\text{GW}}\}.$$

For any $u \in \mathbb{T}^{\text{GW}}$, the *shifted* Galton–Watson subtree generated by u is

$$(2.3) \quad \mathbb{T}_u^{\text{GW}} := \{x \in \mathcal{U} : ux \in \mathbb{T}^{\text{GW}}\}.$$

[By shifted, we mean that \mathbb{T}_u^{GW} is also rooted at e .] For any $x \in \mathbb{T}_u^{\text{GW}}$, let

$$(2.4) \quad |x|_u := |ux| - |u|,$$

$$(2.5) \quad V_u(x) := V(ux) - V(u).$$

As such, $|x|_u$ stands for the (relative) generation of x as a vertex of the Galton–Watson tree \mathbb{T}_u^{GW} , and $(V_u(x), x \in \mathbb{T}_u^{\text{GW}})$ the branching random walk which corresponds to the *shifted marked subtree*

$$\mathbb{T}_u := \{(x, V_u(x)), x \in \mathbb{T}_u^{\text{GW}}\}.$$

Let $\mathcal{F}_n := \sigma\{(u, V(u)), u \in \mathbb{T}^{\text{GW}}, |u| \leq n\}$, which is the σ -algebra induced by the first n generations of the branching random walk. Let \mathcal{F}_∞ be the σ -algebra induced by the whole branching random walk.

We assume (1.2). Let \mathbf{Q} be a probability on Ω such that for any $n \geq 1$,

$$(2.6) \quad \mathbf{Q}|_{\mathcal{F}_n} := W_n \bullet \mathbf{P}|_{\mathcal{F}_n}.$$

Fix $n \geq 1$. Let $w_n^{(n)}$ be a random variable taking values in $\{u \in \mathbb{T}^{\text{GW}}, |u| = n\}$ such that for any $|u| = n$,

$$(2.7) \quad \mathbf{Q} \left\{ w_n^{(n)} = u \mid \mathcal{F}_\infty \right\} = \frac{e^{-V(u)}}{W_n}.$$

We write $\llbracket e, w_n^{(n)} \rrbracket = \{e =: w_0^{(n)}, w_1^{(n)}, w_2^{(n)}, \dots, w_n^{(n)}\}$ for the shortest path in \mathbb{T}^{GW} relating the root e to $w_n^{(n)}$, with $|w_k^{(n)}| = k$ for any $1 \leq k \leq n$.

For any individual $u \in \mathbb{T}^{\text{GW}} \setminus \{e\}$, let \overleftarrow{u} be the parent of u in \mathbb{T}^{GW} , and

$$\Delta V(u) := V(u) - V(\overleftarrow{u}).$$

For $1 \leq k \leq n$, we write

$$(2.8) \quad \mathcal{I}_k^{(n)} := \left\{ u \in \mathbb{T}^{\text{GW}} : |u| = k, \overleftarrow{u} = w_{k-1}^{(n)}, u \neq w_k^{(n)} \right\}.$$

In words, $\mathcal{J}_k^{(n)}$ is the set of children of $w_{k-1}^{(n)}$ except $w_k^{(n)}$, or equivalently, the set of the brothers of $w_k^{(n)}$, and is possibly empty. Finally, let us introduce the following sigma-algebra:

$$(2.9) \quad \mathcal{G}_n := \sigma \left\{ \sum_{x \in \mathcal{J}_k^{(n)}} \delta_{\Delta V(x)}, V(w_k^{(n)}), w_k^{(n)}, \mathcal{J}_k^{(n)}, 1 \leq k \leq n \right\}.$$

We present the main result of the section, with δ denoting the Dirac measure. For any marked tree \mathbb{T} , we define its truncation \mathbb{T}^n at level n by $\mathbb{T}^n := \{(x, V(x)), x \in \mathbb{T}^{\text{GW}}, |x| \leq n\}$.

Proposition 2.1 *Assume (1.2) and fix $n \geq 1$. Under probability \mathbf{Q} ,*

(i) *the random variables $(\sum_{x \in \mathcal{J}_k^{(n)}} \delta_{\Delta V(x)}, \Delta V(w_k^{(n)}))$, $1 \leq k \leq n$, are i.i.d., distributed as $(\sum_{x \in \mathcal{J}_1^{(1)}} \delta_{\Delta V(x)}, \Delta V(w_1^{(1)}))$;*

(ii) *conditionally on \mathcal{G}_n , the truncated shifted marked subtrees $\mathbb{T}_x^{n-|x|}$, for $x \in \cup_{k=1}^n \mathcal{J}_k^{(n)}$, are independent; the conditional distribution of $\mathbb{T}_x^{n-|x|}$ (for any $x \in \cup_{k=1}^n \mathcal{J}_k^{(n)}$) under \mathbf{Q} given \mathcal{G}_n , identical to the distribution of $\mathbb{T}^{n-|x|}$ under \mathbf{P} .*

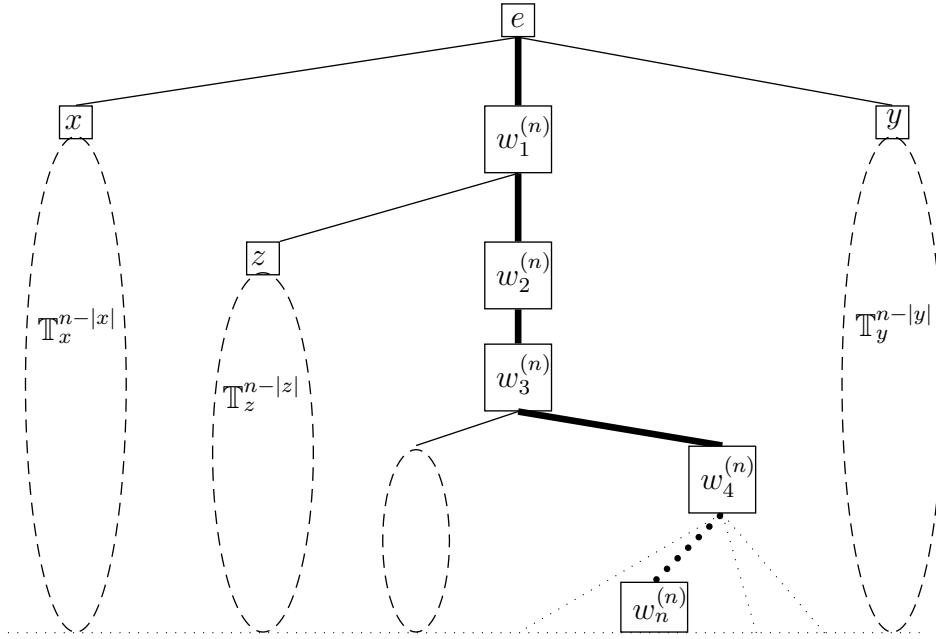


Figure 1: Spine;

The truncated shifted subtrees $\mathbb{T}_x^{n-|x|}, \mathbb{T}_y^{n-|y|}, \mathbb{T}_z^{n-|z|}, \dots$ are actually rooted at e .

For the sake of self-containedness, we present an elementary proof of Proposition 2.1 as an appendix in Section 10.

In general, the (non-truncated) shifted marked subtrees \mathbb{T}_x , for $x \in \cup_{k=1}^n \mathcal{J}_k^{(n)}$, are not conditionally independent under \mathbf{Q} given \mathcal{G}_n .

Throughout the paper, let $((S_i, \sigma_i), i \geq 1)$ be such that $(S_i - S_{i-1}, \sigma_i - \sigma_{i-1})$, for $i \geq 1$ (with $S_0 = \sigma_0 =: 0$), are i.i.d. random vectors under \mathbf{Q} and distributed as $(V(w_1^{(1)}), \#\mathcal{J}_1^{(1)})$.

Corollary 2.2 *Assume (1.2) and fix $n \geq 1$.*

(i) *Under \mathbf{Q} , $((V(w_k^{(n)}), \#\mathcal{J}_k^{(n)}), 1 \leq k \leq n)$ is distributed as $((S_k, \sigma_k), 1 \leq k \leq n)$. In particular, under \mathbf{Q} , $(V(w_k^{(n)}), 1 \leq k \leq n)$ is distributed as $(S_k, 1 \leq k \leq n)$.*

(ii) *We have $\mathbf{E}_{\mathbf{Q}}\{S_1\} = 0$. More generally, for any measurable function $F : \mathbb{R} \rightarrow \mathbb{R}_+$,*

$$(2.10) \quad \mathbf{E}_{\mathbf{Q}}\{F(S_1)\} = \mathbf{E}\left\{\sum_{|u|=1} e^{-V(u)} F(V(u))\right\}.$$

Proof. The first part of the corollary is an immediate consequence of Proposition 2.1 (i). To prove (2.10), we observe that by (2.7),

$$\mathbf{E}_{\mathbf{Q}}\left\{F(V(w_1^{(1)}))\right\} = \mathbf{E}_{\mathbf{Q}}\left\{\sum_{|u|=1} F(V(u)) \mathbf{1}_{\{w_1^{(1)}=u\}}\right\} = \mathbf{E}_{\mathbf{Q}}\left\{\sum_{|u|=1} \frac{e^{-V(u)}}{W_1} F(V(u))\right\},$$

which is $\mathbf{E}\{\sum_{|u|=1} e^{-V(u)} F(V(u))\}$ by the definition of \mathbf{Q} . This implies (2.10). The fact that $\mathbf{E}_{\mathbf{Q}}(S_i) = 0$ follows from (1.2) and (2.10).

We mention that (2.10) can also be found in Biggins and Kyprianou [7]. \square

Corollary 2.3 *Assume (1.1) and (1.2). Let $\delta > 0$ be as in (1.1). Then for any $n \geq k \geq 1$,*

$$\mathbf{E}_{\mathbf{Q}}\left\{(\#\mathcal{J}_k^{(n)})^\delta\right\} = \mathbf{E}_{\mathbf{Q}}\left\{(\#\mathcal{J}_1^{(1)})^\delta\right\} < \infty.$$

Proof. The identity follows from Proposition 2.1 (i). It remains to check $\mathbf{E}_{\mathbf{Q}}\{(\#\mathcal{J}_1^{(1)})^\delta\} < \infty$.

By definition, $\mathbf{E}_{\mathbf{Q}}\{(\sum_{|u|=1} 1)^\delta\} = \mathbf{E}\{(\sum_{|u|=1} 1)^\delta W_1\} = \mathbf{E}\{(\sum_{|u|=1} 1)^\delta \sum_{|v|=1} e^{-V(v)}\} \leq e^C \mathbf{E}\{(\sum_{|u|=1} 1)^{1+\delta}\} < \infty$ (by (1.1)). Since $\#\mathcal{J}_1^{(1)} = (\sum_{|u|=1} 1) - 1$ \mathbf{Q} -a.s., we are done. \square

3 Preliminary: small values of W_n

This preliminary section is devoted to the study of the small values of W_n . We define two important events:

$$(3.1) \quad \mathcal{S} := \{\text{the system's ultimate survival}\},$$

$$(3.2) \quad \mathcal{S}_n := \{\text{the system's survival after } n \text{ generations}\} = \{W_n > 0\}.$$

Clearly, $\mathcal{S} \subset \mathcal{S}_n$. Recall (see for example Harris [17], p. 16) that for some constant c_2 and all $n \geq 1$,

$$(3.3) \quad \mathbf{P} \{ \mathcal{S}_n \setminus \mathcal{S} \} \leq e^{-c_2 n}.$$

Here is the main result of the section.

Proposition 3.1 *For any $\varepsilon > 0$, there exists $\kappa > 0$ such that for all sufficiently large n ,*

$$(3.4) \quad \mathbf{P} \left\{ n^{1/2} W_n < n^{-\varepsilon} \mid \mathcal{S} \right\} \leq \exp(-n^\kappa).$$

The proof of Proposition 3.1 relies on Neveu's multiplicative martingale. Recall that under assumption (1.2), there exists a non-negative random variable ξ^* , with $\mathbf{P} \{ \xi^* > 0 \} > 0$, such that

$$(3.5) \quad \xi^* \stackrel{law}{=} \sum_{|u|=1} \xi_u^* e^{-V(u)},$$

where, given $\{(u, V(u)), |u| = 1\}$, ξ_u^* are independent copies of ξ^* , and " $\stackrel{law}{=}$ " stands for identity in distribution. Moreover, there is uniqueness of the distribution of ξ^* up to a scale change (see Liu [22]); in the rest of the paper, we take the version of ξ^* as the unique one satisfying $\mathbf{E} \{ e^{-\xi^*} \} = \frac{1}{2}$.

Let us introduce the Laplace transform of ξ^* :

$$(3.6) \quad \varphi^*(t) := \mathbf{E} \{ e^{-t\xi^*} \}, \quad t \geq 0.$$

Let

$$(3.7) \quad W_n^* := \prod_{|u|=n} \varphi^*(e^{-V(u)}), \quad n \geq 1.$$

The process $(W_n^*, n \geq 1)$ is also a martingale (Liu [22]). Following Neveu [28], we call W_n^* an associated "*multiplicative martingale*".

The martingale W_n^* being bounded, it converges almost surely (when $n \rightarrow \infty$) to, say, W_∞^* . Let us recall from Liu [22] that

$$(3.8) \quad \log \frac{1}{W_\infty^*} \stackrel{law}{=} \xi^*,$$

$$(3.9) \quad \log \left(\frac{1}{\varphi^*(t)} \right) \sim t \log \left(\frac{1}{t} \right), \quad t \rightarrow 0.$$

We first prove the following estimates:

Lemma 3.2 *There exist $\kappa > 0$ and $a_0 \geq 1$ such that*

$$(3.10) \quad \mathbf{E} \{ (W_\infty^*)^a \mid W_\infty^* < 1 \} \leq \exp(-a^\kappa), \quad \forall a \geq a_0,$$

$$(3.11) \quad \mathbf{E} \{ (W_n^*)^a \mathbf{1}_{\mathcal{S}_n} \} \leq e^{-a^\kappa} + e^{-c_2 n}, \quad \forall n \geq 1, \quad \forall a \geq a_0.$$

Proof of Lemma 3.2. [The best possible value of κ is given by Liu [23], under the additional assumption that $\{u, |u| = 1\}$ contains at least two elements almost surely.]

We first prove (3.10). In view of (3.8), it suffices to show that

$$(3.12) \quad \mathbf{E} \{e^{-a\xi^*} \mid \xi^* > 0\} \leq \exp(-a^\kappa), \quad a \geq a_0.$$

Recall that $q = \mathbf{P}\{\xi^* = 0\}$ is the system's extinction probability. It is well-known for Galton–Watson trees that q satisfies $\mathbf{E}(q^N) = q$, see for example Harris [17], p. 7. Then

$$\varphi^*(t) = \mathbf{E} \{e^{-t\xi^*}\} = q + (1 - q)\mathbf{E} \{e^{-t\xi^*} \mid \xi^* > 0\}.$$

Let $\widehat{\xi}$ be a random variable such that $\mathbf{E}\{e^{-t\widehat{\xi}}\} = \mathbf{E}\{e^{-t\xi^*} \mid \xi^* > 0\}$ for any $t \geq 0$. Let Y be a random variable independent of everything else, such that $\mathbf{P}\{Y = 0\} = q = 1 - \mathbf{P}\{Y = 1\}$. Then ξ^* and $Y\widehat{\xi}$ have the same law, and by (3.5), so do ξ^* and $\sum_{|u|=1} e^{-V(u)} Y_u \widehat{\xi}_u$, where, given $\{u, |u| = 1\}$, $(Y_u, \widehat{\xi}_u)$ are independent copies of $(Y, \widehat{\xi})$, independent of $\{V(u), |u| = 1\}$. Since $\{\sum_{|u|=1} e^{-V(u)} Y_u \widehat{\xi}_u > 0\} = \{\sum_{|u|=1} Y_u > 0\}$, this leads to:

$$\mathbf{E} \{e^{-t\widehat{\xi}}\} = \mathbf{E} \left\{ e^{-t \sum_{|u|=1} e^{-V(u)} Y_u \widehat{\xi}_u} \mid \sum_{|u|=1} Y_u > 0 \right\}, \quad t \geq 0.$$

Let $\widehat{\psi}(t) := -\log \mathbf{E}\{e^{-t\widehat{\xi}}\}$, $t \geq 0$; thus $e^{-\widehat{\psi}(t)} = \mathbf{E}\{e^{-t\widehat{\xi}}\}$. Since $V(u) \leq C$ for $|u| = 1$, we have

$$e^{-\widehat{\psi}(t)} \leq \mathbf{E} \left\{ e^{-te^{-C} \sum_{|u|=1} Y_u \widehat{\xi}_u} \mid \sum_{|u|=1} Y_u > 0 \right\} = \mathbf{E} \left\{ e^{-\sum_{|u|=1} \widehat{\psi}(tY_u e^{-C})} \mid \sum_{|u|=1} Y_u > 0 \right\}.$$

Note that $\widehat{\psi}(0) = 0$, thus $\sum_{|u|=1} \widehat{\psi}(tY_u e^{-C}) = \widehat{\psi}(te^{-C}) \sum_{|u|=1} Y_u$. As a consequence, by writing $h(t) := -\log \mathbf{E}\{e^{-t \sum_{|u|=1} Y_u} \mid \sum_{|u|=1} Y_u > 0\}$, $t \geq 0$, we have $e^{-\widehat{\psi}(t)} \leq e^{-h(\widehat{\psi}(te^{-C}))}$, i.e.,

$$(3.13) \quad \widehat{\psi}(t) \geq h(\widehat{\psi}(te^{-C})), \quad \forall t \geq 0.$$

The function h is smooth and convex on \mathbb{R}_+ , with $h(0) = 0$. Thus $h(t) \geq h'(0)t$ for all $t \geq 0$. Since $\mathbf{P}\{\sum_{|u|=1} Y_u > 0\} = \mathbf{P}\{\sum_{|u|=1} e^{-V(u)} Y_u \widehat{\xi}_u > 0\} = \mathbf{P}\{\xi^* > 0\} = 1 - q$, we have $h'(0) = \mathbf{E}\{\sum_{|u|=1} Y_u \mid \sum_{|u|=1} Y_u > 0\} = \frac{m}{1-q}$, where $m := \mathbf{E}\{\sum_{|u|=1} 1\}$. This yields $h(t) \geq mt$, $t \geq 0$. In view of (3.13), we get $\widehat{\psi}(t) \geq m\widehat{\psi}(te^{-C})$, $\forall t \geq 0$. Iterating the inequality, we see that $\widehat{\psi}(t) \geq m^n \widehat{\psi}(te^{-Cn})$, for $n \geq 1$ and $t \geq 0$. In particular, $\widehat{\psi}(e^{Cn}) \geq m^n \widehat{\psi}(1)$. Since $m > 1$ (assumption (1.2)), this yields (3.12) and thus (3.10).

It remains to check (3.11). Let $a \geq 1$. Since $((W_n^*)^a, n \geq 0)$ is a bounded submartingale, $\mathbf{E}\{(W_n^*)^a \mathbf{1}_{\mathcal{S}_n}\} \leq \mathbf{E}\{(W_\infty^*)^a \mathbf{1}_{\mathcal{S}_n}\}$. Recall that $W_\infty^* \leq 1$; thus

$$\mathbf{E} \{(W_n^*)^a \mathbf{1}_{\mathcal{S}_n}\} \leq \mathbf{E} \{(W_\infty^*)^a \mathbf{1}_{\mathcal{S}}\} + \mathbf{P} \{\mathcal{S}_n \setminus \mathcal{S}\}.$$

By (3.3), $\mathbf{P}\{\mathcal{S}_n \setminus \mathcal{S}\} \leq e^{-c_2 n}$. On the other hand, since $\mathcal{S} = \{W_\infty^* < 1\}$, we have $\mathbf{E}\{(W_\infty^*)^a \mathbf{1}_{\mathcal{S}}\} = \mathbf{E}\{(W_\infty^*)^a \mathbf{1}_{\{W_\infty^* < 1\}}\}$, which, according to (3.10), is bounded by e^{-a^κ} , for $a \geq a_0$. Lemma 3.2 is proved. \square

We are now ready for the proof of Proposition 3.1.

Proof of Proposition 3.1. Let $c_3 > 0$ be such that $\mathbf{P}\{\xi^* \leq c_3\} \geq \frac{1}{2}$. Then $\varphi^*(t) = \mathbf{E}\{e^{-t\xi^*}\} \geq e^{-c_3 t} \mathbf{P}\{\xi^* \leq c_3\} \geq \frac{1}{2} e^{-c_3 t}$, and thus $\log(\frac{1}{\varphi^*(t)}) \leq c_3 t + \log 2$. Together with (3.9), this yields, on the event \mathcal{S}_n ,

$$\begin{aligned} \log\left(\frac{1}{W_n^*}\right) &= \sum_{|u|=n} \log\left(\frac{1}{\varphi^*(e^{-V(u)})}\right) \\ &\leq \sum_{|u|=n} \mathbf{1}_{\{V(u) \geq 1\}} c_4 V(u) e^{-V(u)} + \sum_{|u|=n} \mathbf{1}_{\{V(u) < 1\}} (c_3 e^{-V(u)} + \log 2). \end{aligned}$$

Since $W_n = \sum_{|u|=n} e^{-V(u)}$, we obtain, on \mathcal{S}_n , for any $\lambda \geq 1$,

$$(3.14) \quad \log\left(\frac{1}{W_n^*}\right) \leq c_5 \lambda W_n + c_4 \sum_{|u|=n} \mathbf{1}_{\{V(u) \geq \lambda\}} V(u) e^{-V(u)},$$

where $c_5 := c_3 + c_4 + e \log 2$. Note that c_4 and c_5 do not depend on λ .

Let $0 < y \leq 1$. Since $\mathcal{S} \subset \mathcal{S}_n$, it follows that for $c_6 := c_4 + c_5$,

$$\begin{aligned} &\mathbf{P}\left\{\lambda W_n < y \mid \mathcal{S}_n\right\} \\ &\leq \mathbf{P}\left\{\log\left(\frac{1}{W_n^*}\right) < c_6 y \mid \mathcal{S}_n\right\} + \mathbf{P}\left\{\sum_{|u|=n} \mathbf{1}_{\{V(u) \geq \lambda\}} V(u) e^{-V(u)} \geq y \mid \mathcal{S}_n\right\} \\ (3.15) \quad &=: \text{RHS}_{(3.15)}^1 + \text{RHS}_{(3.15)}^2, \end{aligned}$$

with obvious notation.

Recall that $\mathbf{P}(\mathcal{S}_n) \geq \mathbf{P}(\mathcal{S}) = 1 - q$. By Chebyshev's inequality,

$$\text{RHS}_{(3.15)}^1 \leq e^{c_6} \mathbf{E}\left\{(W_n^*)^{1/y} \mid \mathcal{S}_n\right\} \leq \frac{e^{c_6}}{1 - q} \mathbf{E}\left\{(W_n^*)^{1/y} \mathbf{1}_{\mathcal{S}_n}\right\}.$$

By (3.11), for $n \geq 1$ and $0 < y \leq \frac{1}{a_0}$,

$$(3.16) \quad \text{RHS}_{(3.15)}^1 \leq \frac{e^{c_6}}{1 - q} \left[e^{-1/y^\kappa} + e^{-c_2 n} \right].$$

To estimate $\text{RHS}_{(3.15)}^2$, we use the trivial inequality $V(u) \leq Cn$ for $|u| = n$, to see that

$$\begin{aligned} \text{RHS}_{(3.15)}^2 &\leq \frac{1}{1 - q} \mathbf{P}\left\{Cn \sum_{|u|=n} \mathbf{1}_{\{V(u) \geq \lambda\}} e^{-V(u)} \geq y\right\} \\ &\leq \frac{Cn}{(1 - q)y} \mathbf{E}\left\{\sum_{|u|=n} \mathbf{1}_{\{V(u) \geq \lambda\}} e^{-V(u)}\right\}. \end{aligned}$$

By definition, $\mathbf{E}\{\sum_{|u|=n} \mathbf{1}_{\{V(u) \geq \lambda\}} e^{-V(u)}\} = \mathbf{E}_{\mathbf{Q}}\{\sum_{|u|=n} \mathbf{1}_{\{V(u) \geq \lambda\}} \frac{e^{-V(u)}}{W_n}\} = \mathbf{Q}\{V(w_n^{(n)}) \geq \lambda\}$, which, according to Corollary 2.2 (i), equals $\mathbf{P}\{S_n \geq \lambda\}$. Therefore, if $\lambda \geq n^{1/2}$, then $\mathbf{E}\{\sum_{|u|=n} \mathbf{1}_{\{V(u) \geq \lambda\}} e^{-V(u)}\} \leq e^{-c_7 \lambda^2/n}$ for some constant c_7 . Accordingly, $\text{RHS}_{(3.15)}^2 \leq \frac{Cn e^{-c_7 \lambda^2/n}}{(1-q)y}$. Together with (3.15) and (3.16), it yields that, for $0 < y \leq \frac{1}{a_0}$ and $\lambda \geq n^{1/2} \geq 1$,

$$\mathbf{P}\left\{\lambda W_n < y \mid \mathcal{S}_n\right\} \leq \frac{e^{c_6}}{1-q} \left[e^{-1/y^\kappa} + e^{-c_2 n} \right] + \frac{Cn e^{-c_7 \lambda^2/n}}{(1-q)y}.$$

Let $y := x^{2/(\kappa+2)}$ and $\lambda := n^{1/2} x^{-\kappa/(\kappa+2)}$. The inequality becomes: for $0 < x \leq 1$ and $n \geq 1$,

$$(3.17) \quad \mathbf{P}\left\{n^{1/2} W_n < x \mid \mathcal{S}_n\right\} \leq \frac{e^{c_6}}{1-q} \left[e^{-1/x^{2\kappa/(\kappa+2)}} + e^{-c_2 n} \right] + \frac{Cn e^{-c_7/x^{2\kappa/(\kappa+2)}}}{(1-q)x^{2/(\kappa+2)}}.$$

This readily yields Proposition 3.1. \square

Remark. Recall that $W_n \geq e^{-Cn}$ on \mathcal{S}_n . A consequence of (3.17) (noting that we can replace $\frac{Cn \exp(-c_7/x^{2\kappa/(\kappa+2)})}{(1-q)x^{2/(\kappa+2)}}$ by $\frac{Cn \exp(-c_7/x^{2\kappa/(\kappa+2)})}{(1-q)x^{2/(\kappa+2)}} \wedge 1$ on the right-hand side) is that for any $0 < s < \frac{c_2}{C}$,

$$(3.18) \quad \mathbf{E}\left\{\frac{1}{W_n^s} \mathbf{1}_{\mathcal{S}_n}\right\} \leq n^{(s/2)+o(1)}, \quad n \rightarrow \infty.$$

This estimate will be used in the proof of Theorem 1.5 (Section 5). \square

We complete this section with the following estimate which will be useful in the proof of Theorem 1.5.

Lemma 3.3 *For any $0 < s < 1$,*

$$(3.19) \quad \sup_{n \geq 1} \mathbf{E}\left\{\left(\log \frac{1}{W_n^*}\right)^s\right\} < \infty.$$

Proof of Lemma 3.3. Let $x > 1$. By Chebyshev's inequality, $\mathbf{P}\{\log(\frac{1}{W_n^*}) \geq x\} = \mathbf{P}\{e^x W_n^* \leq 1\} \leq e \mathbf{E}\{e^{-e^x W_n^*}\}$. Since W_n^* is a martingale, it follows from Jensen's inequality that $\mathbf{E}\{e^{-e^x W_n^*}\} \leq \mathbf{E}\{e^{-e^x W_\infty^*}\} \leq \mathbf{P}\{W_\infty^* \leq e^{-x/2}\} + \exp(-e^{x/2})$. Therefore,

$$(3.20) \quad \mathbf{P}\left\{\log\left(\frac{1}{W_n^*}\right) \geq x\right\} \leq e \mathbf{P}\{W_\infty^* \leq e^{-x/2}\} + \exp\left(1 - e^{x/2}\right).$$

On the other hand, by integration by parts, $\int_0^\infty e^{-ty} \mathbf{P}(\xi^* \geq y) dy = \frac{1 - \mathbf{E}(e^{-t\xi^*})}{t} = \frac{1 - \phi^*(t)}{t}$, which, according to (3.8), is $\leq c_8 \log(\frac{1}{t})$ for $0 < t \leq \frac{1}{2}$. Therefore, for $a \geq 2$, $c_8 \log a \geq \int_0^\infty e^{-y/a} \mathbf{P}(\xi^* \geq y) dy \geq \int_0^a e^{-y/a} \mathbf{P}(\xi^* \geq a) dy = (1 - e^{-1})a \mathbf{P}(\xi^* \geq a)$. That is, $\mathbf{P}(\xi^* \geq a) \leq c_9 \frac{\log a}{a}$, or equivalently, $\mathbf{P}(W_\infty^* \leq e^{-a}) \leq c_9 \frac{\log a}{a}$, for $a \geq 2$. Substituting this in (3.20) gives that, for any $x \geq 4$,

$$\mathbf{P}\left\{\log\left(\frac{1}{W_n^*}\right) \geq x\right\} \leq c_{10} \frac{\log(x/2)}{x} + \exp\left(1 - e^{x/2}\right),$$

where $c_{10} := 2e c_9$. Lemma 3.3 follows immediately. \square

4 Proof of Theorem 1.6

Before proving Theorem 1.6, we need two preliminary results.

Lemma 4.1 *Let X_1, X_2, \dots, X_N be independent non-negative random variables, and let $T_N := \sum_{i=1}^N X_i$. For any non-increasing function $F : (0, \infty) \rightarrow \mathbb{R}_+$, we have*

$$\mathbf{E} \{F(T_N) \mathbf{1}_{\{T_N > 0\}}\} \leq \max_{1 \leq i \leq N} \mathbf{E} \{F(X_i) \mid X_i > 0\}.$$

Moreover,

$$\mathbf{E} \{F(T_N) \mathbf{1}_{\{T_N > 0\}}\} \leq \sum_{i=1}^N b^{i-1} \mathbf{E} \{F(X_i) \mathbf{1}_{\{X_i > 0\}}\},$$

where $b := \max_{1 \leq i \leq N} \mathbf{P}\{X_i = 0\}$.

Proof. Let $\tau := \min\{i \geq 1 : X_i > 0\}$ (with $\min \emptyset := \infty$). Then $\mathbf{E} \{F(T_N) \mathbf{1}_{\{T_N > 0\}}\} = \sum_{i=1}^N \mathbf{E} \{F(T_N) \mathbf{1}_{\{\tau=i\}}\}$. Since F is non-increasing, we have $F(T_N) \mathbf{1}_{\{\tau=i\}} \leq F(X_i) \mathbf{1}_{\{\tau=i\}} = F(X_i) \mathbf{1}_{\{X_i > 0\}} \mathbf{1}_{\{X_j = 0, \forall j < i\}}$. By independence, this leads to

$$\mathbf{E} \{F(T_N) \mathbf{1}_{\{T_N > 0\}}\} \leq \sum_{i=1}^N \mathbf{E} \{F(X_i) \mathbf{1}_{\{X_i > 0\}}\} \mathbf{P}\{X_j = 0, \forall j < i\}.$$

This yields immediately the second inequality of the lemma, since $\mathbf{P}\{X_j = 0, \forall j < i\} \leq b^{i-1}$.

To prove the first inequality of the lemma, we observe that $\mathbf{E}\{F(X_i) \mathbf{1}_{\{X_i > 0\}}\} \leq \mathbf{P}\{X_i > 0\} \max_{1 \leq k \leq N} \mathbf{E}\{F(X_k) \mid X_k > 0\}$. Therefore,

$$\mathbf{E} \{F(T_N) \mathbf{1}_{\{T_N > 0\}}\} \leq \max_{1 \leq k \leq N} \mathbf{E}\{F(X_k) \mid X_k > 0\} \sum_{i=1}^N \mathbf{P}\{X_i > 0\} \mathbf{P}\{X_j = 0, \forall j < i\}.$$

The $\sum_{i=1}^N \dots$ expression on the right-hand side is $= \sum_{i=1}^N \mathbf{P}\{X_i > 0, X_j = 0, \forall j < i\} = \sum_{i=1}^N \mathbb{P}\{\tau = i\} = \mathbb{P}\{T_N > 0\} \leq 1$. This yields the first inequality of the lemma. \square

To state our second lemma, let $\underline{w}^{(n)} \in \llbracket e, w_n^{(n)} \llbracket$ be a vertex such that

$$(4.1) \quad V(\underline{w}^{(n)}) = \min_{u \in \llbracket e, w_n^{(n)} \llbracket} V(u).$$

[If there are several such vertices, we choose, say, the oldest.] The following estimate gives a (stochastic) lower bound for $\frac{1}{W_{n,\beta}}$ under \mathbf{Q} outside a “small” set. We recall that $W_{n,\beta} > 0$, \mathbf{Q} -almost surely (but not necessarily \mathbf{P} -almost surely).

Lemma 4.2 *Let $\beta > 0$ and $K > 0$. There exist $\theta > 0$ and $n_0 < \infty$ such that for $n \geq n_0$ and $z \in \mathbb{R}$,*

$$\mathbf{Q}\left\{\frac{e^{-\beta V(\underline{w}^{(n)})}}{W_{n,\beta}} > z, E_n\right\} \leq \frac{1}{1-q} \max_{1 \leq k < n} \mathbf{P}\left\{\frac{n^\theta}{W_{k,\beta}} > z, \mathcal{S}_k\right\},$$

where E_n is a measurable event such that

$$\mathbf{Q}\{E_n\} \geq 1 - \frac{1}{n^K}, \quad n \geq n_0.$$

Remark. We insist on the fact that n_0 does not depend on z . As a consequence, for any non-decreasing function $G : (0, \infty) \rightarrow (0, \infty)$ and $n \geq n_0$,

$$(4.2) \quad \mathbf{E}\mathbf{Q}\left\{G\left(\frac{e^{-\beta V(\underline{w}^{(n)})}}{W_{n,\beta}}\right) \mathbf{1}_{E_n}\right\} \leq \frac{1}{1-q} \max_{1 \leq k < n} \mathbf{E}\left\{G\left(\frac{n^\theta}{W_{k,\beta}}\right) \mathbf{1}_{\mathcal{S}_k}\right\}.$$

We will later apply the lemma to a function G depending on n .

Proof of Lemma 4.2. Recall from (2.8) that $\mathcal{S}_k^{(n)}$ is the set of the brothers of $w_k^{(n)}$. For any pair $0 \leq k < n$, we say that *the level k is n -good* if

$$\mathcal{S}_k^{(n)} \neq \emptyset, \quad \text{and } \mathbb{T}_u^{\text{GW}} \text{ survives at least } n - k \text{ generations, } \forall u \in \mathcal{S}_k^{(n)},$$

where \mathbb{T}_u^{GW} is the shifted Galton–Watson subtree generated by u (see (2.3)). By \mathbb{T}_u^{GW} surviving at least $n - k$ generations, we mean that there exists $v \in \mathbb{T}_u^{\text{GW}}$ such that $|v|_u = n - k$ (see (2.4) for notation).

In words, k is n -good means any subtree generated by any of the brothers of $w_k^{(n)}$ has offspring for at least $n - k$ generations.

Let \mathcal{G}_n be the sigma-algebra defined in (2.9). By Proposition 2.1,

$$\mathbf{Q}\{k \text{ is } n\text{-good} \mid \mathcal{G}_n\} = \mathbf{1}_{\{\mathcal{S}_k^{(n)} \neq \emptyset\}} (\mathbf{P}\{\mathcal{S}_{n-k}\})^{\#\mathcal{S}_k^{(n)}},$$

where \mathcal{S}_n denotes the system's survival after n generations (see (3.2)). Since $\mathbf{P}\{\mathcal{S}_{n-k}\} \geq \mathbf{P}\{\mathcal{S}\} = 1 - q$, whereas $\#\mathcal{S}_k^{(n)}$ and $\#\mathcal{S}_1^{(1)}$ have the same distribution under Q (Proposition 2.1), we have

$$\mathbf{Q}\{k \text{ is } n\text{-good}\} \geq \mathbf{E}\mathbf{Q}\left\{\mathbf{1}_{\{\#\mathcal{S}_1^{(1)} \geq 1\}} (1 - q)^{\#\mathcal{S}_1^{(1)}}\right\} = c_{11} \in (0, 1).$$

As a consequence, for all sufficiently large n and all $1 \leq \ell < n$, by Proposition 2.1 again,

$$\mathbf{Q}\left\{\bigcup_{k=1}^{n-\ell} \bigcap_{j=k}^{k+\ell-1} \{j \text{ is not } n\text{-good}\}\right\} \leq \sum_{k=1}^{n-\ell} \prod_{j=k}^{k+\ell-1} \mathbf{Q}\{j \text{ is not } n\text{-good}\}^\ell \leq n(1 - c_{11})^\ell,$$

which is bounded by $ne^{-c_{11}\ell}$ (using the inequality $1 - x \leq e^{-x}$, for $x \geq 0$). Let $K > 0$. We take $\ell = \ell(n) := \lfloor c_{12} \log n \rfloor$ with $c_{12} := \frac{K+2}{c_{11}}$, and write

$$(4.3) \quad E_n := \bigcap_{k=1}^{n - \lfloor c_{12} \log n \rfloor} \bigcup_{j=k}^{k + \lfloor c_{12} \log n \rfloor - 1} \{j \text{ is } n\text{-good}\}.$$

Thus, for large n ,

$$(4.4) \quad \mathbf{Q}\{E_n\} \geq 1 - ne^{-c_{11}(c_{12} \log n - 1)} = 1 - \frac{e^{c_{11}}}{n^{K+1}} \geq 1 - \frac{1}{n^K}.$$

By definition,

$$(4.5) \quad \begin{aligned} W_{n,\beta} &= \sum_{j=1}^n \sum_{u \in \mathcal{I}_j^{(n)}} e^{-\beta V(u)} \sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|_u = n-j} e^{-\beta V_u(x)} + e^{-\beta V(\underline{w}^{(n)})} \\ &\geq \sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} e^{-\beta V(u)} \sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|_u = n-j} e^{-\beta V_u(x)} \end{aligned}$$

for any $\mathcal{L} \subset \{1, 2, \dots, n\}$. We choose $\mathcal{L} := \{1 \leq j \leq n : |j - |\underline{w}^{(n)}|| \leq c_{12} \log n\}$.

For $u \in \mathcal{I}_j^{(n)}$ with some $j \in \mathcal{L}$, we have $V(u) \leq V(\underline{w}^{(n)}) + C(c_{12} \log n + 2) \leq V(\underline{w}^{(n)}) + 2Cc_{12} \log n$ (for large n). Writing $\theta := 2\beta Cc_{12}$, this leads to:

$$W_{n,\beta} \geq \frac{e^{-\beta V(\underline{w}^{(n)})}}{n^\theta} \sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} \sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|_u = n-j} e^{-\beta V_u(x)} =: \frac{e^{-\beta V(\underline{w}^{(n)})}}{n^\theta} \sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} \xi_u.$$

On the event E_n , we have $\sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} \xi_u > 0$. Therefore,

$$\frac{e^{-\beta V(\underline{w}^{(n)})}}{W_{n,\beta}} \mathbf{1}_{E_n} \leq \frac{n^\theta}{\sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} \xi_u} \mathbf{1}_{\{\sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} \xi_u > 0\}}.$$

Let \mathcal{G}_n be the sigma-algebra in (2.9). We observe that \mathcal{L} and $\mathcal{I}_j^{(n)}$ are \mathcal{G}_n -measurable. Moreover, an application of Proposition 2.1 tells us that under \mathbf{Q} , conditionally on \mathcal{G}_n , the random variables ξ_u , $u \in \mathcal{I}_j^{(n)}$, $j \in \mathcal{L}$, are independent, and distributed as $W_{n-j,\beta}$ under \mathbf{P} . We are thus entitled to apply Lemma 4.1, to see that

$$\begin{aligned} \mathbf{Q}\left\{\frac{e^{-\beta V(\underline{w}^{(n)})}}{W_{n,\beta}} > z, E_n \mid \mathcal{G}_n\right\} &\leq \max_{j \in \mathcal{L}} \mathbf{P}\left\{\frac{n^\theta}{W_{n-j,\beta}} > z \mid W_{n-j,\beta} > 0\right\} \\ &\leq \max_{0 \leq k \leq n} \mathbf{P}\left\{\frac{n^\theta}{W_{k,\beta}} > z \mid W_{k,\beta} > 0\right\}. \end{aligned}$$

Since $\mathbf{P}\{W_{k,\beta} > 0\} = \mathbf{P}\{\mathcal{S}_k\} \geq \mathbf{P}\{\mathcal{S}\} = 1 - q$, this yields Lemma 4.2. \square

We are now ready for the proof of Theorem 1.6. For the sake of clarity, the upper and lower bounds are proved in distinct parts. Let us start with the upper bound.

Proof of Theorem 1.6: the upper bound. Fix $\beta > 1$. For any $Z \geq 0$ which is \mathcal{F}_n -measurable, we have $\mathbf{E}\{W_{n,\beta}Z\} = \mathbf{E}_{\mathbf{Q}}\{\sum_{|u|=n} \frac{e^{-\beta V(u)}}{W_n} Z\} = \mathbf{E}_{\mathbf{Q}}\{\sum_{|u|=n} \mathbf{1}_{\{w_n^{(n)}=u\}} e^{-(\beta-1)V(u)} Z\}$, and thus

$$(4.6) \quad \mathbf{E}\{W_{n,\beta}Z\} = \mathbf{E}_{\mathbf{Q}}\{e^{-(\beta-1)V(w_n^{(n)})} Z\}.$$

Let $s \in (\frac{\beta-1}{\beta}, 1)$, and $\lambda > 0$. Then

$$\begin{aligned} \mathbf{E}\{W_{n,\beta}^{1-s}\} &\leq n^{-(1-s)\beta\lambda} + \mathbf{E}\{W_{n,\beta}^{1-s} \mathbf{1}_{\{W_{n,\beta} > n^{-\beta\lambda}\}}\} \\ &= n^{-(1-s)\beta\lambda} + \mathbf{E}_{\mathbf{Q}}\left\{\frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \mathbf{1}_{\{W_{n,\beta} > n^{-\beta\lambda}\}}\right\}. \end{aligned}$$

Since $e^{-\beta V(w_n^{(n)})} \leq W_{n,\beta}$, we have $\frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \leq \frac{1}{W_{n,\beta}^{s-(\beta-1)/\beta}}$; thus on the event $\{W_{n,\beta} > n^{-\beta\lambda}\}$, we have $\frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \leq n^{[\beta s - (\beta-1)]\lambda}$.

Let $K := [\beta s - (\beta - 1)]\lambda + (1 - s)\beta\lambda$, and let E_n be the event in Lemma 4.2. Since $\mathbf{Q}(E_n^c) \leq n^{-K}$ for all sufficiently large n (see Lemma 4.2), we obtain: for large n ,

$$(4.7) \quad \begin{aligned} \mathbf{E}\{W_{n,\beta}^{1-s}\} &\leq n^{-(1-s)\beta\lambda} + n^{[\beta s - (\beta-1)]\lambda - K} + \mathbf{E}_{\mathbf{Q}}\left\{\frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \mathbf{1}_{\{W_{n,\beta} > n^{-\beta\lambda}\} \cap E_n}\right\} \\ &= 2n^{-(1-s)\beta\lambda} + \mathbf{E}_{\mathbf{Q}}\left\{\frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \mathbf{1}_{\{W_{n,\beta} > n^{-\beta\lambda}\} \cap E_n}\right\}. \end{aligned}$$

We now estimate the expectation expression $\mathbf{E}_{\mathbf{Q}}\{\cdot\}$ on the right-hand side of (4.7). Let $a > 0$ and $\varrho > b > 0$ be constants such that $(\beta - 1)a > s\beta\lambda + \frac{3}{2}$ and $[\beta s - (\beta - 1)]b > \frac{3}{2}$. (The choice of ϱ will be precised later on.) We recall that $\underline{w}_n^{(n)} \in \llbracket e, w_n^{(n)} \llbracket$ satisfies $V(\underline{w}_n^{(n)}) = \min_{u \in \llbracket e, w_n^{(n)} \llbracket} V(u)$, and consider the following events:

$$\begin{aligned} E_{1,n} &:= \{V(w_n^{(n)}) > a \log n\} \cup \{V(w_n^{(n)}) \leq -b \log n\}, \\ E_{2,n} &:= \{V(\underline{w}_n^{(n)}) < -\varrho \log n, V(w_n^{(n)}) \geq -b \log n\}, \\ E_{3,n} &:= \{V(\underline{w}_n^{(n)}) \geq -\varrho \log n, -b \log n < V(w_n^{(n)}) \leq a \log n\}. \end{aligned}$$

Clearly, $\cup_{i=1}^3 E_{i,n}$ covers the whole space.

On the event $E_{1,n} \cap \{W_{n,\beta} > n^{-\beta\lambda}\}$, we have either $V(w_n^{(n)}) > a \log n$, in which case $\frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \leq n^{s\beta\lambda - (\beta-1)a}$, or $V(w_n^{(n)}) \leq -b \log n$, in which case we use the trivial inequality

$W_{n,\beta} \geq e^{-\beta V(w_n^{(n)})}$ to see that $\frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \leq e^{[\beta s - (\beta-1)]V(w_n^{(n)})} \leq n^{-[\beta s - (\beta-1)]b}$ (recalling that $\beta s > \beta - 1$). Since $s\beta\lambda - (\beta - 1)a < -\frac{3}{2}$ and $[\beta s - (\beta - 1)]b > \frac{3}{2}$, we obtain:

$$(4.8) \quad \mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \mathbf{1}_{E_{1,n} \cap \{W_{n,\beta} > n^{-\beta\lambda}\}} \right\} \leq n^{-3/2}.$$

At this stage, it is convenient to mention that the tail probability of $\inf_{|u|=n} V(u)$ has been studied by McDiarmid [26] (this is the only place we need (1.6) instead of (1.2)). In view of the inequality $W_{n,\beta} \geq \exp\{-\beta \inf_{|u|=n} V(u)\}$, McDiarmid's result yields the existence of positive constants c_{13} , c_{14} and c_{15} such that

$$(4.9) \quad \mathbf{E} \left\{ \frac{\mathbf{1}_{\mathcal{I}_n}}{W_{n,\beta}^{c_{13}}} \right\} \leq c_{14} n^{c_{15}}, \quad n \geq 1.$$

We now study the integral on $E_{2,n} \cap \{W_{n,\beta} > n^{-\beta\lambda}\} \cap E_n$. Since $s > 0$, we can choose $s_1 > 0$ and $0 < s_2 \leq c_{13}$ such that $s = s_1 + s_2$. We have, on $E_{2,n} \cap \{W_{n,\beta} > n^{-\beta\lambda}\}$,

$$\frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} = \frac{e^{\beta s_2 V(w_n^{(n)}) - (\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^{s_1}} \frac{e^{-\beta s_2 V(w_n^{(n)})}}{W_{n,\beta}^{s_2}} \leq n^{-\beta s_2 \varrho + (\beta-1)b + \beta \lambda s_1} \frac{e^{-\beta s_2 V(w_n^{(n)})}}{W_{n,\beta}^{s_2}}.$$

Therefore, by an application of Lemma 4.2 (or rather, by an application of (4.2) to $G(x) := x^{s_2}$, $x > 0$), we obtain: when $n \rightarrow \infty$,

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \mathbf{1}_{E_{2,n} \cap \{W_{n,\beta} > n^{-\beta\lambda}\} \cap E_n} \right\} &\leq \frac{n^{-\beta s_2 \varrho + (\beta-1)b + \beta \lambda s_1}}{1 - q} \max_{1 \leq k < n} \mathbf{E} \left\{ \frac{n^{s_2 \theta}}{W_{k,\beta}^{s_2}} \mathbf{1}_{\mathcal{I}_k} \right\} \\ &\leq c_{16} n^{-\beta s_2 \varrho + (\beta-1)b + \beta \lambda s_1 + s_2 \theta + c_{15}}, \end{aligned}$$

the last inequality being a consequence of (4.9). We choose (and fix) the constant ϱ so large that $-\beta s_2 \varrho + (\beta - 1)b + \beta \lambda s_1 + s_2 \theta + c_{15} < -\frac{3}{2}$. Therefore, for all large n ,

$$\mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \mathbf{1}_{E_{2,n} \cap \{W_{n,\beta} > n^{-\beta\lambda}\} \cap E_n} \right\} \leq n^{-3/2}.$$

Combining this with (4.8) yields that for all large n ,

$$\sum_{i=1}^2 \mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \mathbf{1}_{E_{i,n} \cap \{W_{n,\beta} > n^{-\beta\lambda}\} \cap E_n} \right\} \leq 2n^{-3/2}.$$

Substituting this in (4.7), we see that

$$\mathbf{E} \left\{ W_{n,\beta}^{1-s} \right\} \leq 2n^{-(1-s)\beta\lambda} + 2n^{-3/2} + \mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \mathbf{1}_{E_{3,n} \cap \{W_{n,\beta} > n^{-\beta\lambda}\}} \right\}.$$

We make a partition of $E_{3,n}$: let $M \geq 2$ be an integer, and let $a_i := -b + \frac{i(a+b)}{M}$, $0 \leq i \leq M$. By definition,

$$E_{3,n} = \bigcup_{i=0}^{M-1} \{V(\underline{w}^{(n)}) \geq -\varrho \log n, a_i \log n < V(w_n^{(n)}) \leq a_{i+1} \log n\} =: \bigcup_{i=0}^{M-1} E_{3,n,i}.$$

Let $0 \leq i \leq M-1$. There are two possible situations. First situation: $a_i \leq \lambda$. In this case, we argue that on the event $E_{3,n,i}$, we have $W_{n,\beta} \geq e^{-\beta V(w_n^{(n)})} \geq n^{-\beta a_{i+1}}$ and $e^{-(\beta-1)V(w_n^{(n)})} \leq n^{-(\beta-1)a_i}$, thus $\frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \leq n^{\beta a_{i+1}s - (\beta-1)a_i} = n^{\beta a_i s - (\beta-1)a_i + \beta s(a+b)/M} \leq n^{[\beta s - (\beta-1)]\lambda + \beta s(a+b)/M}$. Accordingly, in this situation,

$$\mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \mathbf{1}_{E_{3,n,i}} \right\} \leq n^{[\beta s - (\beta-1)]\lambda + \beta s(a+b)/M} \mathbf{Q}(E_{3,n,i}).$$

Second (and last) situation: $a_i > \lambda$. We have, on $E_{3,n,i} \cap \{W_{n,\beta} > n^{-\beta\lambda}\}$, $\frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \leq n^{\beta\lambda s - (\beta-1)a_i} \leq n^{[\beta s - (\beta-1)]\lambda}$; thus, in this situation,

$$\mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \mathbf{1}_{E_{3,n,i} \cap \{W_{n,\beta} > n^{-\beta\lambda}\}} \right\} \leq n^{[\beta s - (\beta-1)]\lambda} \mathbf{Q}(E_{3,n,i}).$$

We have therefore proved that

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \mathbf{1}_{E_{3,n} \cap \{W_{n,\beta} > n^{-\beta\lambda}\}} \right\} &= \sum_{i=1}^M \mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \mathbf{1}_{E_{3,n,i} \cap \{W_{n,\beta} > n^{-\beta\lambda}\}} \right\} \\ &\leq n^{[\beta s - (\beta-1)]\lambda + \beta s(a+b)/M} \mathbf{Q}(E_{3,n}). \end{aligned}$$

By Corollary 2.2, $\mathbf{Q}(E_{3,n}) = \mathbf{P}\{\min_{0 \leq k < n} S_k \geq -\varrho \log n, 0 \leq S_n \leq a \log n\} = n^{-(3/2)+o(1)}$. As a consequence,

$$\mathbf{E} \{W_{n,\beta}^{1-s}\} \leq 2n^{-(1-s)\beta\lambda} + 2n^{-3/2} + n^{[\beta s - (\beta-1)]\lambda + \beta s(a+b)/M - (3/2)+o(1)}.$$

We choose $\lambda := \frac{3}{2}$. Since M can be as large as possible, this yields the upper bound in Theorem 1.6 by posing $r := 1 - s$. \square

Proof of Theorem 1.6: the lower bound. Let $\beta > 1$ and $s \in (1 - \frac{1}{\beta}, 1)$. By means of (4.5) and the elementary inequality $(a+b)^{1-s} \leq a^{1-s} + b^{1-s}$ (for $a \geq 0$ and $b \geq 0$), we have

$$\begin{aligned} W_{n,\beta}^{1-s} &\leq \sum_{j=1}^n \sum_{u \in \mathcal{J}_j^{(n)}} e^{-(1-s)\beta V(u)} \left(\sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|_u = n-j} e^{-\beta V_u(x)} \right)^{1-s} + e^{-\beta(1-s)V(w_n^{(n)})} \\ &\leq c_{17} \sum_{j=1}^n e^{-(1-s)\beta V(w_j^{(n)})} \sum_{u \in \mathcal{J}_j^{(n)}} \left(\sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|_u = n-j} e^{-\beta V_u(x)} \right)^{1-s} + e^{-\beta(1-s)V(w_n^{(n)})}, \end{aligned}$$

where $c_{17} := e^{(1-s)\beta C} \geq 1$. Let \mathcal{G}_n be the sigma-algebra defined in (2.9). Since $V(w_j^{(n)})$ and $\mathcal{I}_j^{(n)}$, for $1 \leq j \leq n$, are \mathcal{G}_n -measurable, it follows from Proposition 2.1 (ii) that

$$\mathbf{E}_{\mathbf{Q}} \{W_{n,\beta}^{1-s} \mid \mathcal{G}_n\} \leq c_{17} \sum_{j=1}^n e^{-(1-s)\beta V(w_j^{(n)})} \mathbf{E}\{W_{n-j,\beta}^{1-s}\} (\#\mathcal{I}_j^{(n)} + 1).$$

Let $\varepsilon > 0$ be small, and let $r := \frac{3}{2}(1-s)\beta - \varepsilon$. By means of the already proved upper bound for $\mathbf{E}(W_{n,\beta}^{1-s})$, this leads to:

$$(4.10) \quad \mathbf{E}_{\mathbf{Q}} \{W_{n,\beta}^{1-s} \mid \mathcal{G}_n\} \leq c_{18} \sum_{j=1}^n e^{-(1-s)\beta V(w_j^{(n)})} (n-j+1)^{-r} (\#\mathcal{I}_j^{(n)} + 1).$$

Since $\mathbf{E}(W_{n,\beta}^{1-s}) = \mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)V(w_n^{(n)})}}{W_{n,\beta}^s} \right\}$ (see (4.6)), we have, by Jensen's inequality (noticing that $V(w_n^{(n)})$ is \mathcal{G}_n -measurable),

$$\mathbf{E} (W_{n,\beta}^{1-s}) \geq \mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)V(w_n^{(n)})}}{\{\mathbf{E}_{\mathbf{Q}}(W_{n,\beta}^{1-s} \mid \mathcal{G}_n)\}^{s/(1-s)}} \right\},$$

which, in view of (4.10), yields (with $c_{19} := c_{18}^{-s/(1-s)}$)

$$\mathbf{E} (W_{n,\beta}^{1-s}) \geq c_{19} \mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)V(w_n^{(n)})}}{\left\{ \sum_{j=1}^n e^{-(1-s)\beta V(w_j^{(n)})} (n-j+1)^{-r} (\#\mathcal{I}_j^{(n)} + 1) \right\}^{s/(1-s)}} \right\}.$$

By Corollary 2.2 and in its notation, this leads to

$$\begin{aligned} \mathbf{E} (W_{n,\beta}^{1-s}) &\geq c_{19} \mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{-(\beta-1)S_n}}{\left\{ \sum_{j=1}^n (n-j+1)^{-r} e^{-(1-s)\beta S_j} (\sigma_j + 1) \right\}^{s/(1-s)}} \right\} \\ &= c_{19} \mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{[\beta s - (\beta-1)]\tilde{S}_n}}{\left\{ \sum_{k=0}^{n-1} (k+1)^{-r} e^{(1-s)\beta \tilde{S}_k} (\tilde{\sigma}_k + 1) \right\}^{s/(1-s)}} \right\}, \end{aligned}$$

where

$$\tilde{S}_k := S_n - S_{n-k}, \quad \tilde{\sigma}_k := \sigma_{n-k}, \quad 0 \leq k \leq n-1,$$

and $\tilde{S}_n := S_n$. Let $c_{20} > 0$ be a small constant, and define

$$\begin{aligned} E_{n,1}^{\tilde{S}} &:= \bigcap_{k=1}^{\lfloor n^\varepsilon \rfloor - 1} \left\{ \tilde{S}_k \leq -c_{20} k^{1/3} \right\} \cap \left\{ -2n^{\varepsilon/2} \leq \tilde{S}_{\lfloor n^\varepsilon \rfloor} \leq -n^{\varepsilon/2} \right\}, \\ E_{n,2}^{\tilde{S}} &:= \bigcap_{k=\lfloor n^\varepsilon \rfloor + 1}^{n - \lfloor n^\varepsilon \rfloor - 1} \left\{ \tilde{S}_k \leq -[k^{1/3} \wedge (n-k)^{1/3}] \right\} \cap \left\{ -2n^{\varepsilon/2} \leq \tilde{S}_{n - \lfloor n^\varepsilon \rfloor} \leq -n^{\varepsilon/2} \right\}, \\ E_{n,3}^{\tilde{S}} &:= \bigcap_{k=n - \lfloor n^\varepsilon \rfloor + 1}^{n-1} \left\{ \tilde{S}_k \leq \frac{3}{2} \log n \right\} \cap \left\{ \frac{3-\varepsilon}{2} \log n \leq \tilde{S}_n \leq \frac{3}{2} \log n \right\}. \end{aligned}$$

Let

$$\begin{aligned}
E_{n,1}^{\tilde{\sigma}} &:= \bigcap_{k=0}^{\lfloor n^\varepsilon \rfloor} \left\{ \tilde{\sigma}_k \leq n^{3\varepsilon/\delta} \right\}, \\
E_{n,2}^{\tilde{\sigma}} &:= \bigcap_{k=\lfloor n^\varepsilon \rfloor+1}^{n-\lfloor n^\varepsilon \rfloor} \left\{ \tilde{\sigma}_k \leq e^{n^{\varepsilon/4}} \right\}, \\
E_{n,3}^{\tilde{\sigma}} &:= \bigcap_{k=n-\lfloor n^\varepsilon \rfloor+1}^{n-1} \left\{ \tilde{\sigma}_k \leq n^{3\varepsilon/\delta} \right\}.
\end{aligned}$$

On $\bigcap_{i=1}^3 (E_{n,i}^{\tilde{S}} \cap E_{n,i}^{\tilde{\sigma}})$, we have $\sum_{k=0}^{n-1} (k+1)^{-r} e^{(1-s)\beta\tilde{S}_k} (\tilde{\sigma}_k + 1) \leq c_{21} n^{2\varepsilon+(3\varepsilon/\delta)}$, whereas $e^{\lfloor \beta s - (\beta-1) \rfloor \tilde{S}_n} \geq n^{(3-\varepsilon)\lfloor \beta s - (\beta-1) \rfloor/2}$ (recalling that $\beta s > \beta - 1$). Therefore, with $c_{22} := c_{19}/c_{21}^{s/(1-s)}$ and $c_{23} := (2 + \frac{3}{\delta}) \frac{s}{1-s}$,

$$(4.11) \quad \mathbf{E} (W_{n,\beta}^{1-s}) \geq c_{22} n^{-c_{23} \varepsilon} n^{(3-\varepsilon)\lfloor \beta s - (\beta-1) \rfloor/2} \mathbf{Q} \{ \bigcap_{i=1}^3 (E_{n,i}^{\tilde{S}} \cap E_{n,i}^{\tilde{\sigma}}) \}.$$

We need to bound $\mathbf{Q}(\bigcap_{i=1}^3 (E_{n,i}^{\tilde{S}} \cap E_{n,i}^{\tilde{\sigma}}))$ from below. For $j \leq n$, let $\tilde{\mathcal{G}}_j$ be the sigma-algebra generated by $(\tilde{S}_k, \tilde{\sigma}_k)$, $1 \leq k \leq j$. Then $E_{n,1}^{\tilde{S}}$, $E_{n,2}^{\tilde{S}}$, $E_{n,1}^{\tilde{\sigma}}$ and $E_{n,2}^{\tilde{\sigma}}$ are $\tilde{\mathcal{G}}_{n-\lfloor n^\varepsilon \rfloor}$ -measurable, whereas $E_{n,3}^{\tilde{\sigma}}$ is independent of $\tilde{\mathcal{G}}_{n-\lfloor n^\varepsilon \rfloor}$. Therefore,

$$\mathbf{Q}(\bigcap_{i=1}^3 (E_{n,i}^{\tilde{S}} \cap E_{n,i}^{\tilde{\sigma}}) \mid \tilde{\mathcal{G}}_{n-\lfloor n^\varepsilon \rfloor}) \geq [\mathbf{Q}(E_{n,3}^{\tilde{S}} \mid \tilde{\mathcal{G}}_{n-\lfloor n^\varepsilon \rfloor}) + \mathbf{Q}(E_{n,3}^{\tilde{\sigma}}) - 1] \mathbf{1}_{E_{n,1}^{\tilde{S}} \cap E_{n,2}^{\tilde{S}} \cap E_{n,1}^{\tilde{\sigma}} \cap E_{n,2}^{\tilde{\sigma}}}.$$

It is easy to estimate $\mathbf{Q}(E_{n,3}^{\tilde{\sigma}})$: since $c_{24} := \mathbf{E}_{\mathbf{Q}}(\sigma_1^\delta) < \infty$ (Corollary 2.3), we have $\mathbf{Q}\{\sigma_1 > n^{3\varepsilon/\delta}\} \leq c_{24} n^{-3\varepsilon}$, and thus $\mathbf{Q}(E_{n,3}^{\tilde{\sigma}}) = (\mathbf{Q}\{\sigma_1 \leq n^{3\varepsilon/\delta}\})^{\lfloor n^\varepsilon \rfloor - 1} \geq (1 - c_{24} n^{-3\varepsilon})^{\lfloor n^\varepsilon \rfloor - 1} \geq 1 - c_{25} n^{-2\varepsilon}$. To estimate $\mathbf{Q}(E_{n,3}^{\tilde{S}} \mid \tilde{\mathcal{G}}_{n-\lfloor n^\varepsilon \rfloor})$, we use the Markov property to see that, if $\tilde{S}_{n-\lfloor n^\varepsilon \rfloor} \in I_n := [-2n^{\varepsilon/2}, -n^{\varepsilon/2}]$, the conditional probability is (writing $N := \lfloor n^\varepsilon \rfloor$)

$$\geq \inf_{z \in I_n} \mathbf{Q} \left\{ S_i \leq \frac{3}{2} \log n - z, \forall 1 \leq i \leq N-1, \frac{3-\varepsilon}{2} \log n - z \leq S_N \leq \frac{3}{2} \log n - z \right\},$$

which is greater than $N^{-(3/2)+o(1)}$. Therefore,

$$\mathbf{Q}(E_{n,3}^{\tilde{S}} \mid \tilde{\mathcal{G}}_{n-\lfloor n^\varepsilon \rfloor}) + \mathbf{Q}(E_{n,3}^{\tilde{\sigma}}) - 1 \geq n^{-(3\varepsilon/2)+o(1)} - c_{25} n^{-2\varepsilon} = n^{-(3\varepsilon/2)+o(1)}.$$

As a consequence,

$$(4.12) \quad \mathbf{Q} \{ \bigcap_{i=1}^3 (E_{n,i}^{\tilde{S}} \cap E_{n,i}^{\tilde{\sigma}}) \} \geq n^{-(3\varepsilon/2)+o(1)} \mathbf{Q}(E_{n,1}^{\tilde{S}} \cap E_{n,2}^{\tilde{S}} \cap E_{n,1}^{\tilde{\sigma}} \cap E_{n,2}^{\tilde{\sigma}}).$$

To estimate $\mathbf{Q}(E_{n,1}^{\tilde{S}} \cap E_{n,2}^{\tilde{S}} \cap E_{n,1}^{\tilde{\sigma}} \cap E_{n,2}^{\tilde{\sigma}})$, we condition on $\tilde{\mathcal{G}}_{\lfloor n^\varepsilon \rfloor}$, and note that $E_{n,1}^{\tilde{S}}$ and $E_{n,1}^{\tilde{\sigma}}$ are $\tilde{\mathcal{G}}_{\lfloor n^\varepsilon \rfloor}$ -measurable, whereas $E_{n,2}^{\tilde{\sigma}}$ is independent of $\tilde{\mathcal{G}}_{\lfloor n^\varepsilon \rfloor}$. Since $\mathbf{Q}(E_{n,2}^{\tilde{S}} \mid \tilde{\mathcal{G}}_{\lfloor n^\varepsilon \rfloor}) \geq$

$n^{-(3/2)+o(1)}$, whereas $\mathbf{Q}(E_{n,2}^{\tilde{\sigma}}) = [\mathbf{Q}\{\sigma_1 \leq e^{n^{\varepsilon/4}}\}]^{n-2\lfloor n^\varepsilon \rfloor} \geq [1 - c_{24} e^{-\delta n^{\varepsilon/4}}]^{n-2\lfloor n^\varepsilon \rfloor} \geq 1 - e^{-n^{\varepsilon/5}}$ (for large n), we have

$$\begin{aligned} \mathbf{Q}(E_{n,1}^{\tilde{S}} \cap E_{n,2}^{\tilde{S}} \cap E_{n,1}^{\tilde{\sigma}} \cap E_{n,2}^{\tilde{\sigma}} | \tilde{\mathcal{G}}_{\lfloor n^\varepsilon \rfloor}) &\geq [\mathbf{Q}(E_{n,2}^{\tilde{S}} | \tilde{\mathcal{G}}_{\lfloor n^\varepsilon \rfloor}) + \mathbf{Q}(E_{n,2}^{\tilde{\sigma}}) - 1] \mathbf{1}_{E_{n,1}^{\tilde{S}} \cap E_{n,1}^{\tilde{\sigma}}} \\ &\geq n^{-(3/2)+o(1)} \mathbf{1}_{E_{n,1}^{\tilde{S}} \cap E_{n,1}^{\tilde{\sigma}}}. \end{aligned}$$

Thus, $\mathbf{Q}(E_{n,1}^{\tilde{S}} \cap E_{n,2}^{\tilde{S}} \cap E_{n,1}^{\tilde{\sigma}} \cap E_{n,2}^{\tilde{\sigma}}) \geq n^{-(3/2)+o(1)} \mathbf{Q}(E_{n,1}^{\tilde{S}} \cap E_{n,1}^{\tilde{\sigma}})$. Going back to (4.12), we have

$$\begin{aligned} \mathbf{Q}\{\cap_{i=1}^3 (E_{n,i}^{\tilde{S}} \cap E_{n,i}^{\tilde{\sigma}})\} &\geq n^{-(3\varepsilon/2)-(3/2)+o(1)} \mathbf{Q}(E_{n,1}^{\tilde{S}} \cap E_{n,1}^{\tilde{\sigma}}) \\ &\geq n^{-(3\varepsilon/2)-(3/2)+o(1)} [\mathbf{Q}(E_{n,1}^{\tilde{S}}) + \mathbf{Q}(E_{n,1}^{\tilde{\sigma}}) - 1]. \end{aligned}$$

We choose the constant $c_{20} > 0$ sufficiently small so that $\mathbf{Q}(E_{n,1}^{\tilde{S}}) \geq n^{-(3\varepsilon/2)+o(1)}$, whereas $\mathbf{Q}(E_{n,1}^{\tilde{\sigma}}) = \mathbf{Q}(E_{n,3}^{\tilde{\sigma}}) \geq 1 - c_{25} n^{-2\varepsilon}$. Accordingly,

$$\mathbf{Q}\{\cap_{i=1}^3 (E_{n,i}^{\tilde{S}} \cap E_{n,i}^{\tilde{\sigma}})\} \geq n^{-3\varepsilon-(3/2)+o(1)}, \quad n \rightarrow \infty.$$

Substituting this into (4.11) yields

$$\mathbf{E}(W_{n,\beta}^{1-s}) \geq c_{22} n^{-c_{23}\varepsilon} n^{(3-\varepsilon)[\beta s - (\beta-1)]/2} n^{-3\varepsilon-(3/2)+o(1)}.$$

Since ε can be as small as possible, this implies the lower bound in Theorem 1.6. \square

5 Proof of Theorem 1.5

The basic idea in the proof of Theorem 1.5 is the same as in the proof of Theorem 1.6. Again, we prove the upper and lower bounds in distinct parts, for the sake of clarity.

Proof of Theorem 1.5: the upper bound. Clearly, $n^{1/2}W_n \leq \bar{Y}_n$, where

$$\bar{Y}_n := \sum_{|u|=n} (n^{1/2} \vee V(u)^+) e^{-V(u)}.$$

Recall W_n^* from (3.7). Applying (3.14) to $\lambda = 1$, we see that $\bar{Y}_n \geq c_{26} \log(\frac{1}{W_n^*})$, with $c_{26} := \frac{1}{c_4+c_5}$. Thus $\mathbf{P}\{\bar{Y}_n < x, \mathcal{S}_n\} \leq \mathbf{P}\{\log(\frac{1}{W_n^*}) < c_{26} x, \mathcal{S}_n\} \leq e^{c_{26}} \mathbf{E}\{(W_n^*)^{1/x} \mathbf{1}_{\mathcal{S}_n}\}$, which, according to (3.11), is bounded by $e^{c_{26}} (e^{-1/x^\kappa} + e^{-c_2 n})$ for $0 < x \leq \frac{1}{a_0}$. Since $\bar{Y}_n \geq e^{-Cn}$ on \mathcal{S}_n , this yields

$$(5.1) \quad \sup_{n \geq 1} \mathbf{E}\left\{\frac{1}{\bar{Y}_n^s} \mathbf{1}_{\mathcal{S}_n}\right\} < \infty, \quad \forall s \in (0, \frac{c_2}{C}).$$

We now fix $0 < s < \min\{\frac{1}{2}, \frac{\infty}{C}\}$. Let $K \geq 1$ and let E_n be the event in (4.3), satisfying $\mathbf{Q}\{E_n\} \geq 1 - n^{-K}$ for large n (say $n \geq n_0$). We write

$$\mathbf{E} \left\{ (n^{1/2}W_n)^{1-s} \mathbf{1}_{\mathcal{I}_n} \right\} \leq \mathbf{E} \left\{ (n^{1/2}W_n)^{1-s} \mathbf{1}_{\mathcal{I}_n \cap E_n} \right\} + \mathbf{E} \left\{ (n^{1/2}W_n)^{1-s} \mathbf{1}_{E_n^c} \right\}.$$

For $n \geq n_0$, $\mathbf{E}\{W_n^{1-s} \mathbf{1}_{E_n^c}\} \leq [\mathbf{E}\{W_n^{1-2s}\}]^{1/2} [\mathbf{E}\{W_n \mathbf{1}_{E_n^c}\}]^{1/2} = [\mathbf{E}\{W_n^{1-2s}\}]^{1/2} [\mathbf{Q}\{E_n^c\}]^{1/2} \leq [\mathbf{E}\{W_n\}]^{(1/2)-s} n^{-K/2}$, which equals $n^{-K/2}$ (since $\mathbf{E}\{W_n\} = 1$). Therefore, for $n \rightarrow \infty$,

$$\mathbf{E} \left\{ (n^{1/2}W_n)^{1-s} \mathbf{1}_{\mathcal{I}_n} \right\} \leq \mathbf{E} \left\{ \bar{Y}_n^{1-s} \mathbf{1}_{\mathcal{I}_n \cap E_n} \right\} + o(1).$$

Exactly as in (4.6), we have $\mathbf{E}\{\bar{Y}_n^{1-s} \mathbf{1}_{\mathcal{I}_n \cap E_n}\} = \mathbf{E}_{\mathbf{Q}}\{(n^{1/2} \vee V(w_n^{(n)+}) \bar{Y}_n^{-s} \mathbf{1}_{E_n})\}$. Thus, for $n \rightarrow \infty$,

$$(5.2) \quad \mathbf{E} \left\{ (n^{1/2}W_n)^{1-s} \mathbf{1}_{\mathcal{I}_n} \right\} \leq \mathbf{E}_{\mathbf{Q}} \left\{ (n^{1/2} + V(w_n^{(n)+}) \bar{Y}_n^{-s} \mathbf{1}_{E_n}) \right\} + o(1).$$

Recall that $\underline{w}^{(n)} \in \llbracket e, w_n^{(n)} \rrbracket$ is such that $V(\underline{w}^{(n)}) = \min_{u \in \llbracket e, w_n^{(n)} \rrbracket} V(u)$. We have, for any subset $\mathcal{L} \subset \{1, 2, \dots, n\}$,

$$\begin{aligned} \bar{Y}_n &\geq \sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} \sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|_u = n-j} \max \{n^{1/2}, V(x)^+\} e^{-V(u)} \\ &= \sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} e^{-V(u)} \sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|_u = n-j} \max \{n^{1/2}, [V(u) + V_u(x)]^+\} e^{-V_u(x)}. \end{aligned}$$

We choose $\mathcal{L} := \{1 \leq j \leq n : |j - \lfloor \underline{w}^{(n)} \rfloor| \leq c_{12} \log n\}$ (c_{12} being the constant in E_n , defined in (4.3)), and observe that for any $u \in \mathcal{I}_j^{(n)}$ (with $j \in \mathcal{L}$), we have $|V(u) - V(\underline{w}^{(n)})| \leq C(c_{12} \log n + 2) \leq 2Cc_{12} \log n$ (for $n \geq 2$).

We now distinguish two possible situations, depending on whether $V(\underline{w}^{(n)}) \geq -c_{27} \log n$, where $c_{27} := \frac{1}{s} + 2Cc_{12}$. In both situations, we consider a sufficiently large n and an arbitrary $u \in \mathcal{I}_j^{(n)}$ (with $j \in \mathcal{L}$).

On $\{V(\underline{w}^{(n)}) \geq -c_{27} \log n\}$, we have $\max\{n^{1/2}, [V(u) + V_u(x)]^+\} \geq \frac{1}{2}(n^{1/2} \vee V_u(x)^+)$ (this holds trivially in case $V_u(x) \leq n^{1/2}$; otherwise $[V(u) + V_u(x)]^+ \geq V_u(x) - (c_{27} + 2Cc_{12}) \log n \geq \frac{1}{2}V_u(x)^+$), and thus

$$\begin{aligned} \bar{Y}_n &\geq \frac{1}{2} \sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} e^{-V(u)} \sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|_u = n-j} \max \{(n-j)^{1/2}, V_u(x)^+\} e^{-V_u(x)} \\ &\geq \frac{e^{-2C}}{2} \sum_{j \in \mathcal{L}} e^{-V(w_j^{(n)})} \sum_{u \in \mathcal{I}_j^{(n)}} \sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|_u = n-j} \max \{(n-j)^{1/2}, V_u(x)^+\} e^{-V_u(x)} \\ &=: \frac{e^{-2C}}{2} \sum_{j \in \mathcal{L}} e^{-V(w_j^{(n)})} \sum_{u \in \mathcal{I}_j^{(n)}} \xi_u. \end{aligned}$$

If, however, $V(\underline{w}^{(n)}) < -c_{27} \log n$, then $V(u) \leq -\frac{1}{s} \log n$, and since $\max\{n^{1/2}, [V(u) + V_u(x)]^+\} \geq n^{1/2}$, we have, in this case,

$$\bar{Y}_n \geq n^{(1/s)+(1/2)} \sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} \sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|_u = n-j} e^{-V_u(x)} =: n^{(1/s)+(1/2)} \sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} \eta_u.$$

Therefore, in both situations, we have ($c_{28} := 2^s e^{2sC}$)

$$(5.3) \quad \bar{Y}_n^{-s} \mathbf{1}_{E_n} \leq c_{28} \left(\sum_{j \in \mathcal{L}} e^{-V(w_j^{(n)})} \sum_{u \in \mathcal{I}_j^{(n)}} \xi_u \right)^{-s} \mathbf{1}_{E_n} + n^{-(s/2)-1} \left(\sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} \eta_u \right)^{-s} \mathbf{1}_{E_n}.$$

[By definition, on the event E_n , we have $\sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} \sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|_u = n-j} 1 > 0$, so the $(\dots)^{-s}$ expressions on the right-hand side are well-defined.]

Recall \mathcal{G}_n from (2.9). We observe that by Proposition 2.1, under \mathbf{Q} and conditionally on \mathcal{G}_n , the random variables ξ_u , for $u \in \mathcal{I}_j^{(n)}$ and $j \in \mathcal{L}$, are i.i.d. and are distributed as \bar{Y}_{n-j} under \mathbf{P} . By applying the second part of Lemma 4.1, we obtain:

$$\begin{aligned} & \mathbf{E}_{\mathbf{Q}} \left\{ \left(\sum_{j \in \mathcal{L}} e^{-V(w_j^{(n)})} \sum_{u \in \mathcal{I}_j^{(n)}} \xi_u \right)^{-s} \mathbf{1}_{E_n} \mid \mathcal{G}_n \right\} \\ & \leq \sum_{j \in \mathcal{L}} b^{|j-j_0|} \mathbf{E}_{\mathbf{Q}} \left\{ \frac{e^{sV(w_j^{(n)})}}{[\sum_{u \in \mathcal{I}_j^{(n)}} \xi_u]^s} \mathbf{1}_{\{\sum_{u \in \mathcal{I}_j^{(n)}} \xi_u > 0\}} \mid \mathcal{G}_n \right\}, \end{aligned}$$

for any $j_0 \in \mathcal{L}$, with $b := \max_{j \in \mathcal{L}} \max_{u \in \mathcal{I}_j^{(n)}} \mathbf{Q}\{\xi_u = 0\} \leq \max_{1 \leq j \leq n} \mathbf{P}\{\mathcal{S}_{n-j}^c\} \leq \mathbf{P}\{\mathcal{S}^c\} = q$. [We mention that Lemma 4.1 applies to any (deterministic) permutation of the random variables X_1, \dots, X_N , which allows us to choose an arbitrary j_0 here.]

The first part of Lemma 4.1 tells us that for any $j \in \mathcal{L}$,

$$\mathbf{E}_{\mathbf{Q}} \left\{ \frac{1}{[\sum_{u \in \mathcal{I}_j^{(n)}} \xi_u]^s} \mathbf{1}_{\{\sum_{u \in \mathcal{I}_j^{(n)}} \xi_u > 0\}} \mid \mathcal{G}_n \right\} \leq \frac{1}{1-q} \max_{u \in \mathcal{I}_j^{(n)}} \mathbf{E}_{\mathbf{Q}} \left\{ \frac{1}{\xi_u^s} \mathbf{1}_{\{\xi_u > 0\}} \mid \mathcal{G}_n \right\},$$

which equals $\frac{1}{1-q} \mathbf{E}\{\frac{1}{\bar{Y}_{n-j}^s} \mathbf{1}_{\mathcal{S}_{n-j}}\}$ (Proposition 2.1), and is hence bounded in n and j (by (5.1)). Summarizing, we have proved that

$$\mathbf{E}_{\mathbf{Q}} \left\{ \left(\sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{I}_j^{(n)}} e^{-V(w_j^{(n)})} \xi_u \right)^{-s} \mathbf{1}_{E_n} \mid \mathcal{G}_n \right\} \leq c_{29} \sum_{j \in \mathcal{L}} q^{|j-j_0|} e^{sV(w_j^{(n)})},$$

where the constant c_{29} does not depend on the choice of j_0 . We choose $j_0 := |\underline{w}_n| \in \mathcal{L}$. Then $V(w_j^{(n)}) \leq V(\underline{w}_n) + C|j - j_0|$. For $s < \frac{1}{C} \log(\frac{1}{q})$, we have $\sum_{j \in \mathcal{L}} q^{|j-j_0|} e^{sV(w_j^{(n)})} \leq c_{30} e^{sV(\underline{w}_n)}$.

As a consequence,

$$\begin{aligned}
& \mathbf{E}_{\mathbf{Q}} \left\{ (n^{1/2} + V(w_n^{(n)}))^+ \left(\sum_{j \in \mathcal{L}} e^{-V(w_j^{(n)})} \sum_{u \in \mathcal{J}_j^{(n)}} \xi_u \right)^{-s} \mathbf{1}_{E_n} \right\} \\
& \leq c_{31} \mathbf{E}_{\mathbf{Q}} \left\{ (n^{1/2} + V(w_n^{(n)}))^+ e^{sV(w_n)} \right\} \\
& = c_{31} \mathbf{E}_{\mathbf{Q}} \left\{ (n^{1/2} + S_n^+) \exp \left(s \min_{1 \leq j < n} S_j \right) \right\} \leq c_{32}.
\end{aligned}$$

Similarly, using (3.18) in place of (5.1), we have

$$\mathbf{E}_{\mathbf{Q}} \left\{ (n^{1/2} + V(w_n^{(n)}))^+ \left(\sum_{j \in \mathcal{L}} \sum_{u \in \mathcal{J}_j^{(n)}} \eta_u \right)^{-s} \mathbf{1}_{E_n} \right\} \leq n^{(1/2)+(s/2)+o(1)}.$$

In view of (5.3), we obtain: for $0 < s < s_0 := \min\{\frac{1}{2}, \frac{c_2}{C}, \frac{1}{C} \log(\frac{1}{q})\}$,

$$\mathbf{E}_{\mathbf{Q}} \left\{ (n^{1/2} + V(w_n^{(n)}))^+ \bar{Y}_n^{-s} \mathbf{1}_{E_n} \right\} \leq c_{28} c_{32} + n^{-(1/2)+o(1)}.$$

Substituting this in (5.2), we see that $\sup_{n \geq 1} \mathbf{E}\{(n^{1/2} W_n)^{1-s} \mathbf{1}_{\mathcal{J}_n}\} < \infty$ for any $s \in (0, s_0)$. This yields the last inequality in (1.16) when γ is close to 1. By Hölder's inequality, it holds for all $\gamma \in [0, 1)$. \square

Proof of Theorem 1.5: the lower bound. We start with

$$n^{1/2} W_n \geq \underline{Y}_n := \sum_{|u|=n} (n^{1/2} \wedge V(u)^+) e^{-V(u)}.$$

Let $s \in (0, 1)$. Exactly as in (4.6), we have

$$(5.4) \quad \mathbf{E} \{ \underline{Y}_n^{1-s} \mathbf{1}_{\mathcal{J}_n} \} = \mathbf{E}_{\mathbf{Q}} \{ (n^{1/2} \wedge V(w_n^{(n)}))^+ \underline{Y}_n^{-s} \}.$$

By definition,

$$\begin{aligned}
\underline{Y}_n &= \sum_{j=1}^n \sum_{u \in \mathcal{J}_j^{(n)}} e^{-V(u)} \sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|=n-j} \min \{ n^{1/2}, [V(u) + V_u(x)]^+ \} e^{-V_u(x)} \\
&\quad + \min \{ n^{1/2}, V(w_n^{(n)})^+ \} e^{-V(w_n^{(n)})} \\
&\leq e^C \sum_{j=1}^n e^{-V(w_j^{(n)})} \sum_{u \in \mathcal{J}_j^{(n)}} \sum_{x \in \mathbb{T}_u^{\text{GW}}, |x|=n-j} [V(w_j^{(n)})^+ + C + V_u(x)^+] e^{-V_u(x)} \\
&\quad + V(w_n^{(n)})^+ e^{-V(w_n^{(n)})}.
\end{aligned}$$

By means of the elementary inequality $(\sum_i a_i)^{-s} \geq (\sum_i a_i^s)^{-1}$ and $(\sum_i b_i)^s \leq \sum_i b_i^s$ for non-negative a_i and b_i , this leads to, on \mathcal{S}_n (writing $\Theta_n := (V(w_n^{(n)}))^s e^{-sV(w_n^{(n)})}$ for brevity):

$$\underline{Y}_n^{-s} \geq \frac{e^{-sC}}{\sum_j e^{-sV(w_j^{(n)})} \sum_u \{(V(w_j^{(n)}))^s + C^s\} (\sum_x e^{-V_u(x)} + (\sum_x V_u(x) + e^{-V_u(x)})^s) + \Theta_n},$$

where, as before, $\sum_j := \sum_{j=1}^n$, $\sum_u := \sum_{u \in \mathcal{S}_j^{(n)}}$, and $\sum_x := \sum_{x \in \mathbb{T}_u^{\text{CW}}, |x|_u = n-j}$. We now condition upon \mathcal{G}_n , and note that $V(w_j^{(n)})$ and $\mathcal{S}_j^{(n)}$ are \mathcal{G}_n -measurable. We apply Jensen's inequality $\mathbf{E}\{\frac{1}{Z} | \mathcal{G}_n\} \geq \frac{1}{\mathbf{E}\{Z | \mathcal{G}_n\}}$ (for positive random variables Z). By Proposition 2.1, we have

$$\mathbf{E}\{\underline{Y}_n^{-s} | \mathcal{G}_n\} \geq \frac{e^{-sC}}{\sum_j e^{-sV(w_j^{(n)})} \sum_u \{(V(w_j^{(n)}))^s + C^s\} \mathbf{E}(W_{n-j}^s) + \mathbf{E}(U_{n-j}^s) + \Theta_n},$$

where, for any $k \geq 0$, $U_k := \sum_{|y|=k} V(y) + e^{-V(y)}$. By Jensen's inequality, $\mathbf{E}(W_{n-j}^s) \leq [\mathbf{E}(W_{n-j})]^s = 1$. On the other hand, by (3.9), $U_k \leq c_{33} \log \frac{1}{W_k^*}$, and thus by Lemma 3.3, $\mathbf{E}(U_k^s) \leq c_{33}^s \mathbf{E}\{[\log \frac{1}{W_k^*}]^s\} \leq c_{34}$. Therefore,

$$\mathbf{E}\{\underline{Y}_n^{-s} | \mathcal{G}_n\} \geq \frac{c_{35}}{\sum_{j=1}^n e^{-sV(w_j^{(n)})} \{(V(w_j^{(n)}))^s + 1\} (\#\mathcal{S}_j^{(n)} + 1)}.$$

As a consequence (in the notation of Corollary 2.2),

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}} \{(n^{1/2} \wedge V(w_n^{(n)}))^s \underline{Y}_n^{-s}\} &\geq c_{35} \mathbf{E}_{\mathbf{Q}} \left\{ \frac{n^{1/2} \wedge S_n^+}{\sum_{j=1}^n e^{-sS_j} [(S_j^+)^s + 1] (\sigma_j + 1)} \right\} \\ &\geq c_{35} \mathbf{E}_{\mathbf{Q}} \left\{ \frac{(n^{1/2} \wedge S_n) \mathbf{1}_{\{\min_{1 \leq j \leq n} S_j > 0\}}}{\sum_{j=1}^n e^{-sS_j} (S_j^s + 1) (\sigma_j + 1)} \right\}. \end{aligned}$$

Note that if $S_j > 0$, then $e^{-sS_j} [S_j^s + 1] \leq c_{36} e^{-tS_j}$ with $t := \frac{s}{2}$. Therefore, by writing

$$\mathbf{Q}^{(n)} \{ \cdot \} := \mathbf{Q} \left\{ \cdot \mid \min_{1 \leq j \leq n} S_j > 0 \right\},$$

and $\mathbf{E}_{\mathbf{Q}}^{(n)}$ the expectation with respect to $\mathbf{Q}^{(n)}$, and $\hat{\sigma}_j := \sigma_j + 1$ for brevity, we get that

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}} \{(n^{1/2} \wedge V(w_n^{(n)}))^s \underline{Y}_n^{-s}\} &\geq c_{37} \mathbf{Q} \left\{ \min_{1 \leq j \leq n} S_j > 0 \right\} \mathbf{E}_{\mathbf{Q}}^{(n)} \left\{ \frac{n^{1/2} \wedge S_n}{\sum_{j=1}^n e^{-tS_j} \hat{\sigma}_j} \right\} \\ &\geq c_{37} \mathbf{Q} \left\{ \min_{1 \leq j \leq n} S_j > 0 \right\} \mathbf{E}_{\mathbf{Q}}^{(n)} \left\{ \frac{\varepsilon n^{1/2} \mathbf{1}_{\{S_n > \varepsilon n^{1/2}\}}}{\sum_{j=1}^n e^{-tS_j} \hat{\sigma}_j} \right\}. \end{aligned}$$

Since $\mathbf{Q}\{\min_{1 \leq j \leq n} S_j > 0\} \geq c_{38} n^{-1/2}$, this leads to

$$\begin{aligned} \mathbf{E}_{\mathbf{Q}} \{(n^{1/2} \wedge V(w_n^{(n)+}) \underline{Y}_n^{-s})\} &\geq c_{39} \varepsilon \mathbf{E}_{\mathbf{Q}}^{(n)} \left\{ \frac{\mathbf{1}_{\{S_n > \varepsilon n^{1/2}\}}}{\sum_{j=1}^n e^{-tS_j} \widehat{\sigma}_j} \right\} \\ &\geq c_{39} \varepsilon \left[\mathbf{E}_{\mathbf{Q}}^{(n)} \left\{ \frac{1}{\sum_{j=1}^n e^{-tS_j} \widehat{\sigma}_j} \right\} - \mathbf{Q}^{(n)}\{S_n \leq \varepsilon n^{1/2}\} \right]. \end{aligned}$$

Of course, $\frac{1}{\sum_{j=1}^n e^{-tS_j} \widehat{\sigma}_j} \geq \frac{1}{1 + \sum_{j=1}^n e^{-tS_j} \widehat{\sigma}_j}$. By Corollary 2.3, $\mathbf{E}_{\mathbf{Q}}\{\widehat{\sigma}_1^\delta\} < \infty$; we are thus entitled to apply Lemma 9.1 (proved in the forthcoming Section 9) to see that $\mathbf{E}_{\mathbf{Q}}^{(n)}\left\{\frac{1}{1 + \sum_{j=1}^n e^{-tS_j} \widehat{\sigma}_j}\right\} \geq c_{40}$ for some $c_{40} \in (0, \infty)$ and all $n \geq n_0$. On the other hand, it is known that $S_n/n^{1/2}$ under $\mathbf{Q}^{(n)}$ converges weakly to the terminal value of a Brownian meander; in particular, $\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \mathbf{Q}^{(n)}\{S_n \leq \varepsilon n^{1/2}\} = 0$. We can thus choose (and fix) a small $\varepsilon > 0$ such that $\mathbf{Q}^{(n)}\{S_n \leq \varepsilon n^{1/2}\} \leq \frac{c_{40}}{2}$ for all $n \geq n_1$. Therefore, for $n \geq n_0 + n_1$,

$$\mathbf{E}_{\mathbf{Q}} \{(n^{1/2} \wedge V(w_n^{(n)+}) \underline{Y}_n^{-s})\} \geq c_{39} \varepsilon \left[c_{40} - \frac{c_{40}}{2} \right].$$

As a consequence, we have proved that for $0 < s < 1$,

$$\liminf_{n \rightarrow \infty} \mathbf{E}_{\mathbf{Q}} \{(n^{1/2} \wedge V(w_n^{(n)+}) \underline{Y}_n^{-s})\} > 0,$$

which, in view of (5.4), yields the first inequality in (1.16), and thus completes the proof of the lower bound in Theorem 1.5. \square

6 Proof of Theorem 1.3 and (1.14)–(1.15) of Theorem 1.4

In this section, we prove Theorem 1.3, as well as parts (1.14)–(1.15) of Theorem 1.4. We assume (1.2) and (1.6).

Proof of Theorem 1.3. We start with the proof of the upper bound. Fix $\gamma \in (0, 1)$. Since W_n^γ is a non-negative supermartingale, the maximal inequality tells that for any $n \leq m$ and any $\lambda > 0$,

$$\mathbf{P} \left\{ \max_{n \leq j \leq m} W_j^\gamma \geq \lambda \right\} \leq \frac{\mathbf{E}(W_n^\gamma)}{\lambda} \leq \frac{1}{\lambda n^{(\gamma/2)+o(1)}},$$

the last inequality being a consequence of Theorem 1.5, and $o(1)$ being uniform in (n, m, λ) . Let $\varepsilon > 0$ and let $n_k := \lfloor k^{2/\varepsilon} \rfloor$. Then $\sum_k \mathbf{P}\{\max_{n_k \leq j \leq n_{k+1}} W_j^\gamma \geq n_k^{-(\gamma/2)+\varepsilon}\} < \infty$. By the Borel–Cantelli lemma, almost surely for all large k , we have $\max_{n_k \leq j \leq n_{k+1}} W_j < n_k^{-(1/2)+(\varepsilon/\gamma)}$. Since $\frac{\varepsilon}{\gamma}$ can be arbitrarily small, this yields the desired upper bound: $W_n \leq n^{-(1/2)+o(1)}$ a.s.

To prove the lower bound in Theorem 1.3, we recall the Paley–Zygmund inequality: for any non-negative random variable ξ , $\mathbf{P}\{\xi > \frac{1}{2}\mathbf{E}(\xi)\} \geq \frac{1}{4} \frac{(\mathbf{E}\xi)^2}{\mathbf{E}(\xi^2)}$. Therefore, by Theorem 1.5, for any $0 < \gamma < \frac{1}{2}$,

$$\mathbf{P}\left\{W_n^\gamma > \frac{1}{2}\mathbf{E}(W_n^\gamma)\right\} \geq \frac{1}{4} \frac{\{\mathbf{E}(W_n^\gamma)\}^2}{\mathbf{E}\{W_n^{2\gamma}\}} = n^{o(1)};$$

thus $\mathbf{P}\{W_n > n^{-(1/2)+o(1)}\} \geq n^{o(1)}$. [Clearly, $n^{-(1/2)+o(1)}$ can be replaced by a constant multiple of $n^{-1/2}$, and $n^{o(1)}$ by a constant; but since we are going to apply the same argument in the proof of Theorems 1.4 and 1.2, we stay with the weaker version of the inequality.]

By the elementary inequality $1 - x \leq e^{-x}$ (for $x \geq 0$), we get

$$(6.1) \quad \mathbf{P}\{W_n \leq n^{-(1/2)+o(1)}\} \leq e^{-n^{o(1)}}.$$

Let $\varepsilon > 0$ and let $\tau_n := \inf\{k \geq 1 : \#\{u : |u| = k\} \geq n^{2\varepsilon}\}$. We have

$$\begin{aligned} & \mathbf{P}\left\{\tau_n < \infty, \min_{k \in [\frac{n}{2}, n]} W_{k+\tau_n} \leq e^{-C\tau_n} n^{-(1/2)-\varepsilon}\right\} \\ & \leq \sum_{k \in [\frac{n}{2}, n]} \mathbf{P}\left\{\tau_n < \infty, W_{k+\tau_n} \leq e^{-C\tau_n} n^{-(1/2)-\varepsilon}\right\} \\ & \leq \sum_{k \in [\frac{n}{2}, n]} (\mathbf{P}\{W_k \leq n^{-(1/2)-\varepsilon}\})^{\lfloor n^{2\varepsilon} \rfloor}. \end{aligned}$$

By (6.1), $\max_{k \in [\frac{n}{2}, n]} \mathbf{P}\{W_k \leq n^{-(1/2)-\varepsilon}\} \leq e^{-n^{-\varepsilon}}$ for all sufficiently large n , say $n \geq n_0$. Therefore, for $n \geq n_0$,

$$\mathbf{P}\left\{\tau_n < \infty, \min_{k \in [\frac{n}{2}, n]} W_{k+\tau_n} \leq e^{-C\tau_n} n^{-(1/2)-\varepsilon}\right\} \leq n \exp(-n^{-\varepsilon} \lfloor n^{2\varepsilon} \rfloor),$$

which is summable in n . By the Borel–Cantelli lemma, almost surely for all large n , we have either $\tau_n = \infty$, or $\min_{k \in [\frac{n}{2}, n]} W_{k+\tau_n} > e^{-C\tau_n} n^{-(1/2)-\varepsilon}$.

Recall that the number of particles in each generation forms a Galton–Watson tree, which is super-critical under assumption (1.2) (because $m := \mathbf{E}\{\sum_{u:|u|=1} 1\} > 1$). In particular, conditionally on the system’s ultimate survival, $\frac{\#\{u:|u|=j\}}{m^j}$ converges a.s. to a (strictly) positive random variable when $j \rightarrow \infty$, which implies $\tau_n \sim \frac{2\varepsilon \log n}{\log m}$ a.s., $n \rightarrow \infty$. As a consequence, upon the system’s survival, we have, almost surely for all large n ,

$$W_n \geq \min_{k \in [\frac{n}{2}, n]} W_{k+\tau_n} > e^{-C\tau_n} n^{-(1/2)-\varepsilon} \geq n^{-(1/2)-\varepsilon-3C\varepsilon/\log m}.$$

This yields the desired lower bound in Theorem 1.3. □

Proof of Theorem 1.4: parts (1.14) and (1.15). Let $\beta > 1$. Along exactly the same lines of the proof of the lower bound in Theorem 1.3 — but using Theorem 1.6 instead of Theorem

1.5 —, we have, conditionally on the system's survival, $W_{n,\beta} \geq n^{-(3\beta/2)+o(1)}$ almost surely (and a fortiori, in probability). This is the claimed lower bound in (1.14) and (1.15).

Let $\varepsilon > 0$. By Theorem 1.6 and Chebyshev's inequality, $\mathbf{P}\{W_{n,\beta} > n^{-(3\beta/2)+\varepsilon}\} \rightarrow 0$. Therefore, $W_{n,\beta} \leq n^{-(3\beta/2)+o(1)}$ in probability, yielding the upper bound in (1.15).

The upper bound in (1.14) follows trivially from the upper bound in (1.15). \square

7 Proof of Theorem 1.2

Assume (1.2) and (1.6). Let $\beta > 1$. We trivially have $W_{n,\beta} \leq W_n \exp\{-(\beta-1) \inf_{|u|=n} V(u)\}$ and $W_{n,\beta} \geq \exp\{-\beta \inf_{|u|=n} V(u)\}$. Therefore, $\frac{1}{\beta} \log \frac{1}{W_{n,\beta}} \leq \inf_{|u|=n} V(u) \leq \frac{1}{\beta-1} \log \frac{W_n}{W_{n,\beta}}$ on \mathcal{S}_n . Since β can be as large as possible, by means of Theorem 1.3 and of parts (1.14)–(1.15) of Theorem 1.4, we immediately get (1.7) and (1.9).

Since $W_n \geq \exp\{-\inf_{|u|=n} V(u)\}$, the lower bound in (1.8) follows immediately from Theorem 1.3. The rest of this section is devoted to proving the upper bound in (1.8): conditionally on the system's survival,

$$(7.1) \quad \liminf_{n \rightarrow \infty} \frac{1}{\log n} \inf_{|u|=n} V(u) \leq \frac{1}{2}, \quad \text{a.s.}$$

The proof borrows some ideas from Bramson [9]. We fix $b > a > \frac{1}{2}$ and $\varepsilon > 0$, and let $n_j := 2^j$, $j \geq j_0$ (with a sufficiently large integer j_0). Consider $j \geq j_0$ and $n \in [n_j, n_{j+1}] \cap \mathbb{Z}$. Let

$$g_n(k) := \min \left\{ c_{20} k^{1/3}, c_{20}(n-k)^{1/3} + a \log n_j, n^\varepsilon \right\}, \quad 0 \leq k \leq n.$$

Let \mathbb{L}_n be the set of individuals $x \in \mathbb{T}^{\text{GW}}$ with $|x| = n$ such that

$$V(x_k) \geq g_n(k), \quad \forall 0 \leq k \leq n, \quad \text{and} \quad a \log n_j \leq V(x) \leq b \log n_j,$$

where $x_0 := e$, $x_1, \dots, x_n := x$ are the vertices on the shortest path in \mathbb{T}^{GW} relating the root e and the vertex x , with $|x_k| = k$ for any $0 \leq k \leq n$. We consider the measurable event

$$F_j := \bigcup_{n=n_j}^{n_{j+1}} \bigcup_{|x|=n} \mathbf{1}_{\{x \in \mathbb{L}_n\}}.$$

We start by estimating the first moment of $\#F_j$: $\mathbf{E}(\#F_j) = \sum_{n=n_j}^{n_{j+1}} \mathbf{E}\{\sum_{|x|=n} \mathbf{1}_{\{x \in \mathbb{L}_n\}}\}$. Since $\mathbf{E}\{\sum_{|x|=n} \mathbf{1}_{\{x \in \mathbb{L}_n\}}\} = \mathbf{E}_{\mathbf{Q}}\{\sum_{|x|=n} \frac{e^{-V(x)}}{W_n} e^{V(x)} \mathbf{1}_{\{x \in \mathbb{L}_n\}}\} = \mathbf{E}_{\mathbf{Q}}\{e^{V(w_n^{(n)})} \mathbf{1}_{\{w_n^{(n)} \in \mathbb{L}_n\}}\}$, we can apply Corollary 2.2 to see that

$$\mathbf{E}(\#F_j) = \sum_{n=n_j}^{n_{j+1}} \mathbf{E}_{\mathbf{Q}} \left\{ e^{S_n} \mathbf{1}_{\{S_k \geq g_n(k), \forall 0 \leq k \leq n, a \log n_j \leq S_n \leq b \log n_j\}} \right\}$$

$$\begin{aligned}
&\geq (n_{j+1} - n_j)n_j^a \mathbf{Q}\left\{S_k \geq g_n(k), \forall 0 \leq k \leq n, a \log n_j \leq S_n \leq b \log n_j\right\} \\
&\geq (n_{j+1} - n_j)n_j^a n_j^{-(3/2)+o(1)} \\
&= n_j^{a-(1/2)+o(1)}.
\end{aligned}$$

We now proceed to estimate the second moment of $\#F_j$. By definition,

$$\begin{aligned}
\mathbf{E}[(\#F_j)^2] &= \sum_{n=n_j}^{n_{j+1}} \sum_{m=n_j}^{n_{j+1}} \mathbf{E}\left\{\sum_{|x|=n} \sum_{|y|=m} \mathbf{1}_{\{x \in \mathbb{L}_n, y \in \mathbb{L}_n\}}\right\} \\
&\leq 2 \sum_{n=n_j}^{n_{j+1}} \sum_{m=n}^{n_{j+1}} \mathbf{E}\left\{\sum_{|x|=n} \sum_{|y|=m} \mathbf{1}_{\{x \in \mathbb{L}_n, y \in \mathbb{L}_n\}}\right\}.
\end{aligned}$$

We look at the double sum $\sum_{|x|=n} \sum_{|y|=m}$ on the right-hand side. By considering z , the youngest common ancestor of x and y , and writing $k := |z|$, we arrive at:

$$\sum_{|x|=n} \sum_{|y|=m} \mathbf{1}_{\{x \in \mathbb{L}_n, y \in \mathbb{L}_n\}} = \sum_{k=0}^n \sum_{|z|=k} \sum_{(u,v)} \mathbf{1}_{\{zu \in \mathbb{L}_n, zv \in \mathbb{L}_n\}}$$

where the double sum $\sum_{(u,v)}$ is over $u, v \in \mathbb{T}_z^{\text{GW}}$ such that $|u|_z = n - k$ and $|v|_z = m - k$ and that the unique common ancestor of u and v in \mathbb{T}_z^{GW} is the root. Therefore,

$$\mathbf{E}[(\#F_j)^2] \leq 2 \sum_{n=n_j}^{n_{j+1}} \sum_{m=n}^{n_{j+1}} \sum_{k=0}^n \mathbf{E}\left\{\sum_{|z|=k} \sum_{(u,v)} \mathbf{1}_{\{zu \in \mathbb{L}_n, zv \in \mathbb{L}_n\}}\right\} =: 2 \sum_{n=n_j}^{n_{j+1}} \sum_{m=n}^{n_{j+1}} \sum_{k=0}^n \Lambda_{k,n,m}.$$

We estimate $\Lambda_{k,n,m}$ according to three different situations.

First situation: $0 \leq k \leq n^\varepsilon$. Let $V_z(u) := V(zu) - V(z)$ as in Section 2. We have $0 \leq g_n(k) \leq V(z) \leq Cn^\varepsilon$, and $V(zu_i) \geq a \log n_j$ for $0 \leq i \leq n - k$ and $V(zu_{n-k}) \leq b \log n_j$, where $u_0 := e, u_1, \dots, u_{n-k}$ are the vertices on the shortest path in \mathbb{T}_z^{GW} relating the root e and the vertex u , with $|u_i|_z = i$ for any $0 \leq i \leq n - k$. Therefore, $V_z(u_i) \geq -Cn^\varepsilon$ for $0 \leq i \leq n - k$, and $V_z(u) \leq b \log n_j$. Accordingly,

$$\Lambda_{k,n,m} \leq \mathbf{E}\left\{\sum_{|z|=k} \sum_{v \in \mathbb{T}_z^{\text{GW}}, |v|_z=m-k} \mathbf{1}_{\{zv \in \mathbb{L}_n\}} B_{n-k}\right\},$$

where

$$\begin{aligned}
B_{n-k} &:= \mathbf{E}\left\{\sum_{|x|=n-k} \mathbf{1}_{\{V(x_i) \geq -Cn^\varepsilon, \forall 0 \leq i \leq n-k, V(x) \leq b \log n_j\}}\right\} \\
&= \mathbf{E}_{\mathbf{Q}}\left\{e^{V(w_{n-k}^{(n-k)})} \mathbf{1}_{\{V(w_i^{(n-k)}) \geq -Cn^\varepsilon, \forall 0 \leq i \leq n-k, V(w_{n-k}^{(n-k)}) \leq b \log n_j\}}\right\}
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{E}_{\mathbf{Q}} \left\{ e^{S_{n-k}} \mathbf{1}_{\{S_i \geq -C n^\varepsilon, \forall 0 \leq i \leq n-k, S_{n-k} \leq b \log n_j\}} \right\} \\
&\leq n_j^b \mathbf{Q} \left\{ S_i \geq -C n^\varepsilon, \forall 0 \leq i \leq n-k, S_{n-k} \leq b \log n_j \right\} \\
&\leq n_j^b n_j^{-(3/2)+\varepsilon+o(1)} \leq n_j^{b-(3/2)+2\varepsilon}.
\end{aligned}$$

Therefore,

$$\Lambda_{k,n,m} \leq n_j^{b-(3/2)+2\varepsilon} \mathbf{E} \left\{ \sum_{|z|=k} \sum_{v \in \mathbb{T}_z^{\text{GW}}, |v|_z = m-k} \mathbf{1}_{\{zv \in \mathbb{L}_n\}} \right\} = n_j^{b-(3/2)+2\varepsilon} \mathbf{E} \left\{ \sum_{|x|=n} \mathbf{1}_{\{x \in \mathbb{L}_n\}} \right\},$$

and thus

$$(7.2) \quad \sum_{n=n_j}^{n_{j+1}} \sum_{m=n}^{n_{j+1}} \sum_{k=0}^{\lfloor n^\varepsilon \rfloor} \Lambda_{k,n,m} \leq n_j^{b-(3/2)+2\varepsilon} (n_{j+1} - n_j) n^\varepsilon \mathbf{E}(\#F_j) = n_j^{b-(1/2)+3\varepsilon} \mathbf{E}(\#F_j).$$

Second situation: $n^\varepsilon + 1 \leq k \leq n - n^\varepsilon$. In this situation, since $V(z) \geq g_n(k) = c_{20} n^\varepsilon$, we simply have $V_z(u) \leq b \log n_j - c_{20} n^\varepsilon$. Exactly as in the first situation, we get

$$\Lambda_{k,n,m} \leq \mathbf{E} \left\{ \sum_{|x|=n} \mathbf{1}_{\{x \in \mathbb{L}_n\}} \right\} \mathbf{E} \left\{ \sum_{|x|=n-k} \mathbf{1}_{\{V(x) \leq b \log n_j - c_{20} n^\varepsilon\}} \right\}.$$

The second $\mathbf{E}\{\dots\}$ on the right-hand side is

$$= \mathbf{E}_{\mathbf{Q}} \left\{ e^{S_{n-k}} \mathbf{1}_{\{S_{n-k} \leq (a+\varepsilon) \log n_j - c_{20} n^\varepsilon\}} \right\} \leq n_j^b e^{-c_{20} n^\varepsilon},$$

and thus

$$(7.3) \quad \sum_{n=n_j}^{n_{j+1}} \sum_{m=n}^{n_{j+1}} \sum_{k=\lfloor n^\varepsilon \rfloor + 1}^{n - \lfloor n^\varepsilon \rfloor} \Lambda_{k,n,m} \leq n_j^b e^{-c_{20} n^\varepsilon} (n_{j+1} - n_j) n_{j+1} \mathbf{E}(\#F_j) = o(1) \mathbf{E}(\#F_j).$$

Third and last situation: $n - n^\varepsilon + 1 \leq k \leq n$. This time, we use $V_z(u) \leq (b-a) \log n_j - c_{20} (m-k)^{1/3}$, to arrive at

$$\begin{aligned}
\Lambda_{k,n,m} &\leq \mathbf{E} \left\{ \sum_{|x|=n} \mathbf{1}_{\{x \in \mathbb{L}_n\}} \right\} \mathbf{E} \left\{ \sum_{|x|=n-k} \mathbf{1}_{\{V(x) \leq (b-a) \log n_j - c_{20} (m-k)^{1/3}\}} \right\} \\
&\leq n_j^{b-a} e^{-c_{20} (m-k)^{1/3}} \mathbf{E} \left\{ \sum_{|x|=n} \mathbf{1}_{\{x \in \mathbb{L}_n\}} \right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{n=n_j}^{n_{j+1}} \sum_{m=n}^{n_{j+1}} \sum_{k=n - \lfloor n^\varepsilon \rfloor + 1}^n \Lambda_{k,n,m} &= \sum_{n=n_j}^{n_{j+1}} \left(\sum_{m=n}^{n + \lfloor n^\varepsilon \rfloor} + \sum_{m=n + \lfloor n^\varepsilon \rfloor + 1}^{n_{j+1}} \right) \sum_{k=n - \lfloor n^\varepsilon \rfloor + 1}^n \Lambda_{k,n,m} \\
&\leq n_j^{b-a} n_{j+1}^\varepsilon n_{j+1} \mathbf{E}(\#F_j) + n_j^{b-a} e^{-c_{20} n_j^{\varepsilon/3}} n_{j+1} n_{j+1}^\varepsilon \mathbf{E}(\#F_j) \\
&\leq n_j^{(b-a)+3\varepsilon} \mathbf{E}(\#F_j).
\end{aligned}$$

Combining this with (7.2) and (7.3), and since $\mathbf{E}[(\#F_j)^2] \leq 2 \sum_{n=n_j}^{n_{j+1}} \sum_{m=n}^{n_{j+1}} \sum_{k=0}^n \Lambda_{k,n,m}$, we obtain: for $j \geq j_0$,

$$\frac{\mathbf{E}[(\#F_j)^2]}{[\mathbf{E}(\#F_j)]^2} \leq \frac{3n_j^{b-(1/2)+3\varepsilon}}{\mathbf{E}(\#F_j)} \leq n_j^{(b-a)+4\varepsilon},$$

by recalling that $\mathbf{E}(\#F_j) \geq n_j^{a-(1/2)+o(1)}$. By the Paley–Zygmund inequality,

$$\mathbf{P}\{F_j \neq \emptyset\} \geq \frac{1}{4} \frac{[\mathbf{E}(\#F_j)]^2}{\mathbf{E}[(\#F_j)^2]} \geq n_j^{-(b-a)-5\varepsilon}, \quad j \geq j_0.$$

A fortiori, $\mathbf{P}\{\min_{n_j \leq n \leq n_{j+1}} \min_{|x|=n} V(x) \leq b \log n_j\} \geq n_j^{-(b-a)-5\varepsilon}$. Of course, we can take $\varepsilon > 0$ even smaller to see that for any $b > \frac{1}{2}$ and any $\varepsilon > 0$, there exists j_0 sufficiently large such that

$$(7.4) \quad \mathbf{P}\left\{ \min_{n_j \leq n \leq n_{j+1}} \min_{|x|=n} V(x) \leq b \log n_j \right\} \geq n_j^{-\varepsilon}, \quad j \geq j_0.$$

This is the analogue of (6.1) for $\min_{|x|=n} V(x)$. From here, the argument follows the lines in the proof of the upper bound in Theorem 1.3 in Section 6, and goes as follows: let $\tau_j := \inf\{k : \#\{u : |u| = k\} \geq n_j^{2\varepsilon}\}$. Then

$$\mathbf{P}\left\{ \min_{\tau_j+n_j \leq |x| \leq \tau_j+n_{j+1}} V(x) \geq C \tau_j + b \log n_j \right\} \leq \left(\mathbf{P}\left\{ \min_{n_j \leq |x| \leq n_{j+1}} V(x) \geq b \log n_j \right\} \right)^{\lfloor n_j^{2\varepsilon} \rfloor},$$

which is, according to (7.4), bounded by $(1 - n_j^{-\varepsilon})^{\lfloor n_j^{2\varepsilon} \rfloor}$ (for $j \geq j_0$), and thus summable in j . By the Borel–Cantelli lemma, almost surely for all sufficiently large j , we have $\min_{\tau_j+n_j \leq |x| \leq \tau_j+n_{j+1}} V(x) < C \tau_j + b \log n_j$. Since $\tau_j \sim 2\varepsilon \frac{\log n_j}{\log m}$ a.s. ($j \rightarrow \infty$), and b can be as close to $\frac{1}{2}$ as possible, this readily yields (7.1), and completes the proof of Theorem 1.2. \square

8 Proof of part (1.13) of Theorem 1.4

The upper bound follows from Theorem 1.3 and the elementary inequality $W_{n,\beta} \leq W_n^\beta$, the lower bound from (1.8) and the relation $W_{n,\beta} \geq \exp\{-\beta \inf_{|u|=n} V(u)\}$. \square

9 Proof of Theorem 1.1

The proof of Theorem 1.1 relies on Theorem 1.5 and a preliminary result, stated below as Proposition 9.2. Theorem 1.5 ensures the tightness of $(n^{1/2}W_n, n \geq 1)$, whereas Proposition 9.2 implies that $\frac{W_{n+1}}{W_n}$ converges to 1 in probability (conditionally on the system's survival).

We start with a lemma, which is a very simple variant of a result of Kozlov [20].

Lemma 9.1 Assume that $\{(X_k, \eta_k), k \geq 1\}$ is a sequence of i.i.d. random vectors defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{P}\{\eta_1 \geq 0\} = 1$ and $\mathbb{E}\{\eta_1^\theta\} < \infty$ for some $\theta > 0$. We assume that X_1 is non-degenerate and bounded with $\mathbb{E}(X_1) = 0$, and set $S_n := X_1 + \dots + X_n$ (for $n \geq 1$). Then

$$(9.1) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left\{ \frac{1}{1 + \sum_{k=1}^n \eta_k e^{-S_k}} \mid \min_{1 \leq k \leq n} S_k > 0 \right\} = c_{41} \in (0, \infty).$$

Proof. The lemma is an analogue of the identity (26) of Kozlov [20], except that the distribution of our η_1 is slightly different from that of Kozlov's, which explains the moment condition $\mathbb{E}\{\eta_1^\theta\} < \infty$: this condition will be seen to guarantee

$$(9.2) \quad \lim_{j \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{E} \left\{ \frac{1}{1 + \sum_{k=1}^j \eta_k e^{-S_k}} - \frac{1}{1 + \sum_{k=1}^n \eta_k e^{-S_k}} \mid \min_{1 \leq k \leq n} S_k > 0 \right\} = 0.$$

The identity (9.2), which plays the role of Kozlov's Lemma 1 in [20], is the key ingredient in the proof of (9.1). Since the rest of the proof goes along the lines of [20] with obvious modifications, we only prove (9.2) here.

Without loss of generality, we assume $\theta \leq 1$ (otherwise, we can replace θ by 1). We observe that for $n > j$, the integrand in (9.2) is non-negative, and is

$$\leq \frac{\sum_{k=j+1}^n \eta_k e^{-S_k}}{1 + \sum_{k=1}^n \eta_k e^{-S_k}} \leq \left(\frac{\sum_{k=j+1}^n \eta_k e^{-S_k}}{1 + \sum_{k=1}^n \eta_k e^{-S_k}} \right)^\theta \leq \left(\sum_{k=j+1}^n \eta_k e^{-S_k} \right)^\theta,$$

which is bounded by $\sum_{k=j+1}^n \eta_k^\theta e^{-\theta S_k}$. Since $\mathbb{P}\{\min_{1 \leq k \leq n} S_k > 0\} \sim c_{42}/n^{1/2}$ (Bingham [8]), we only need to check that

$$(9.3) \quad \lim_{j \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{1/2} \sum_{k=j+1}^n \mathbb{E} \left\{ \eta_k^\theta e^{-\theta S_k} \mathbf{1}_{\{\min_{1 \leq i \leq n} S_i > 0\}} \right\} = 0.$$

Let $j < k < n$. Then $\{\min_{1 \leq i \leq n} S_i > 0\} \subset \{\min_{k \leq i \leq n} (S_i - S_k) > -S_k\}$. By boundedness of X_1 , we have $S_{k-1} - C \leq S_k \leq S_{k-1} + C$ for some constant $C > 0$. Recall that $S_k = \sum_{i=1}^k X_i$ and that (X_i, η_i) are i.i.d. Therefore, the expectation term in (9.3) is bounded by

$$e^{\theta C} \mathbb{E}\{\eta_k^\theta\} \mathbb{E} \left\{ e^{-\theta S_{k-1}} \mathbf{1}_{\{\min_{1 \leq i < k} S_i > 0, \min_{k \leq i \leq n} (S_i - S_k) > -S_{k-1} - C\}} \right\}.$$

By assumption, $\mathbb{E}\{\eta_k^\theta\} = \mathbb{E}\{\eta_1^\theta\} < \infty$. Let $(Y_i, i \geq 1)$ be i.i.d. random variables distributed as X_1 , independent of $(X_i, i \geq 1)$. Since Y_1 is non-degenerate with $\mathbb{E}(Y_1) = 0$, there exists $\varepsilon > 0$ such that $\mathbb{P}\{Y_1 > \varepsilon\} > 0$. Let $N := \lfloor \frac{1}{\varepsilon} \rfloor + 1$. Then $c_{43} := \mathbb{P}\{\min_{1 \leq i \leq N} Y_i > 0, Y_1 + \dots + Y_N > C\} \geq \mathbb{P}\{\min_{1 \leq i \leq N} Y_i > \varepsilon\} > 0$. By independence,

$$\begin{aligned} & \mathbb{E} \left\{ e^{-\theta S_{k-1}} \mathbf{1}_{\{\min_{1 \leq i < k} S_i > 0, \min_{k \leq i \leq n} (S_i - S_k) > -S_{k-1} - C\}} \right\} \\ &= \frac{1}{c_{43}} \mathbb{E} \left\{ e^{-\theta S_{k-1}} \mathbf{1}_{\{\min_{1 \leq i < k} S_i > 0, \min_{k \leq i \leq n} (S_i - S_k) > -S_{k-1} - C\}} \mathbf{1}_{\{\min_{1 \leq i \leq N} Y_i > 0, Y_1 + \dots + Y_N > C\}} \right\} \\ &\leq \frac{1}{c_{43}} \mathbb{E} \left\{ e^{-\theta S_{k-1}} \mathbf{1}_{\{\min_{1 \leq i \leq n+N-1} S_i > 0\}} \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \limsup_{j \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{1/2} \sum_{k=j+1}^n \mathbb{E} \left\{ \eta_k^\theta e^{-\theta S_k} \mathbf{1}_{\{\min_{1 \leq i \leq n} S_i > 0\}} \right\} \\ & \leq c_{44} \limsup_{j \rightarrow \infty} \limsup_{n \rightarrow \infty} n^{1/2} \sum_{k=j}^{n-1} \mathbb{E} \left\{ e^{-\theta S_k} \mathbf{1}_{\{\min_{1 \leq i \leq n+N-1} S_i > 0\}} \right\}, \end{aligned}$$

which equals 0 by Lemma 1 of Kozlov [20] (applied to $\eta_i = 1, \forall i$). This yields (9.3). \square

Proposition 9.2 *For any $\gamma \in (0, 1)$, we have $n \rightarrow \infty$,*

$$(9.4) \quad \mathbf{P} \left\{ \left| \frac{W_{n+1}}{W_n} - 1 \right| \geq n^{-\gamma} \mid \mathcal{S} \right\} \leq n^{-(1-\gamma)/2+o(1)}, \quad n \rightarrow \infty.$$

Proof. Let $1 < \beta \leq 2$. We use a probability estimate of Petrov [29], p. 82: for centered random variables ξ_1, \dots, ξ_ℓ with $\mathbf{E}(|\xi_i|^\beta) < \infty$ (for $1 \leq i \leq \ell$), we have $\mathbf{E}\{|\sum_{i=1}^\ell \xi_i|^\beta\} \leq 2 \sum_{i=1}^\ell \mathbf{E}\{|\xi_i|^\beta\}$.

By definition, on the set \mathcal{S}_n , we have

$$\frac{W_{n+1}}{W_n} - 1 = \sum_{|u|=n} \frac{e^{-V(u)}}{W_n} \left(\sum_{x \in \mathbb{T}_u^{\text{GW}}: |x|=1} e^{-V_u(x)} - 1 \right),$$

where \mathbb{T}^{GW} and $|x|_u$ are as in (2.1) and (2.4), respectively. Conditioning on \mathcal{F}_n , and applying Petrov's probability inequality recalled above, we see that on \mathcal{S}_n ,

$$(9.5) \quad \mathbf{E} \left\{ \left| \frac{W_{n+1}}{W_n} - 1 \right|^\beta \mid \mathcal{F}_n \right\} \leq 2 \sum_{|u|=n} \frac{e^{-\beta V(u)}}{W_n^\beta} \mathbf{E} \left\{ \left| \sum_{|y|=1} e^{-V(y)} - 1 \right|^\beta \right\} = c_{45} \frac{W_{n,\beta}}{W_n^\beta},$$

where $c_{45} := 2\mathbf{E}\{|\sum_{|v|=1} e^{-V(v)} - 1|^\beta\}$, and $W_{n,\beta}$ is as in (1.11).

Fix $0 < \gamma < 1$. Let $\varepsilon > 0$ and $b > 0$. Let $s \in (\frac{\beta-1}{\beta}, 1)$ and $s < \frac{\varepsilon}{C}$ (which is possible if β is close to 1). Define $D_n := \{W_n \geq n^{-(1/2)-\varepsilon}\} \cap \{W_{n,\beta} \leq n^{-(3\beta/2)+b}\}$. By Proposition 3.1, there exists $\kappa = \kappa(\varepsilon)$ such that $\mathbf{P}\{W_n < n^{-(1/2)-\varepsilon}, \mathcal{S}\} \leq \exp(-n^\kappa)$ for all large n , whereas by Theorem 1.6, $\mathbf{P}\{W_{n,\beta} > n^{-(3\beta/2)+b}\} \leq n^{3\beta(1-s)/2-(1-s)b} \mathbf{E}\{W_{n,\beta}^{1-s}\} = n^{-(1-s)b+o(1)}$. Therefore,

$$\mathbf{P}\{\mathcal{S} \setminus D_n\} \leq e^{-n^\kappa} + n^{-(1-s)b+o(1)} = n^{-(1-s)b+o(1)}, \quad n \rightarrow \infty.$$

On the other hand, since $\mathcal{S} \subset \mathcal{S}_n$, it follows from (9.5) and Chebyshev's inequality that for $n \rightarrow \infty$,

$$\begin{aligned} \mathbf{P} \left\{ \left| \frac{W_{n+1}}{W_n} - 1 \right| \geq n^{-\gamma}, D_n, \mathcal{S} \right\} & \leq n^{\gamma\beta} \mathbf{E} \left\{ c_{45} \frac{W_{n,\beta}}{W_n^\beta} \mathbf{1}_{D_n \cap \mathcal{S}_n} \right\} \\ & \leq n^{\gamma\beta - (3\beta/2) + b + [(1/2) + \varepsilon]\beta + o(1)}. \end{aligned}$$

As a consequence, when $n \rightarrow \infty$,

$$\mathbf{P}\left\{\left|\frac{W_{n+1}}{W_n} - 1\right| \geq n^{-\gamma}, \mathcal{S}\right\} \leq n^{-(1-s)b+o(1)} + n^{\gamma\beta-\beta+b+\varepsilon\beta+o(1)}.$$

We choose $b := \frac{1-\gamma}{2}$. By sending $\varepsilon \rightarrow 0$, $\beta \rightarrow 1$ and then $s \rightarrow 0$, we obtain (9.4). \square

We now have all of the ingredients needed for the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\lambda_n > 0$ be such that $\mathbf{E}\{(\lambda_n W_n)^{1/2}\} = 1$. That is,

$$\lambda_n := \{\mathbf{E}(W_n^{1/2})\}^{-2}.$$

By Theorem 1.5, we have $0 < \liminf_{n \rightarrow \infty} \frac{\lambda_n}{n^{1/2}} \leq \limsup_{n \rightarrow \infty} \frac{\lambda_n}{n^{1/2}} < \infty$, and $(\lambda_n W_n)$ is tight. Let \mathscr{W} be any possible (weak) limit of $(\lambda_n W_n)$ along a subsequence. By Theorem 1.5 and dominated convergence, $\mathbf{E}(\mathscr{W}^{1/2}) = 1$. We now prove the uniqueness of \mathscr{W} .

By definition,

$$W_{n+1} = \sum_{|u|=n+1} e^{-V(u)} = \sum_{|v|=1} e^{-V(v)} \sum_{x \in \mathbb{T}_v^{\text{GW}}, |x|_v=n} e^{-V_v(x)}.$$

By assumption, $\lambda_n W_n \rightarrow \mathscr{W}$ in distribution when n goes to infinity along a certain subsequence. Thus $\lambda_n W_{n+1}$ converges weakly (when n goes along the same subsequence) to $\sum_{|v|=1} e^{-V(v)} \mathscr{W}_v$, where, conditionally on $(v, |v|=1)$, \mathscr{W}_v are independent copies of \mathscr{W} .

On the other hand, by Proposition 9.2, $\lambda_n W_{n+1}$ also converges weakly (along the same subsequence) to \mathscr{W} . Therefore,

$$\mathscr{W} \stackrel{\text{law}}{=} \sum_{|v|=1} e^{-V(v)} \mathscr{W}_v.$$

This is the same equation for ξ^* in (3.5). Since (3.5) has a unique solution up to a scale change, and since $\mathbf{E}(\mathscr{W}^{1/2}) = 1$, we have

$$(9.6) \quad \mathscr{W} \stackrel{\text{law}}{=} c_{46} \xi^*,$$

where $c_{46} := [\mathbf{E}\{(\xi^*)^{1/2}\}]^{-2}$. The uniqueness (in law) of \mathscr{W} shows that $\lambda_n W_n$ converges weakly to \mathscr{W} when $n \rightarrow \infty$. Since by (3.3), $\mathbf{P}\{W_n > 0\} = \mathbf{P}\{\mathcal{S}_n\} \rightarrow \mathbf{P}\{\mathcal{S}\} = \mathbf{P}\{\xi^* > 0\}$, it follows that $\mathscr{W} > 0$ a.s., conditionally upon the system's survival. \square

10 Appendix: proof of Proposition 2.1

We start with a simple preliminary result. For any individual $u \in \mathbb{T}^{\text{GW}}$, we write $\llbracket e, u \rrbracket = \{e =: u_0, u_1, u_2, \dots, u_{|u|} := u\}$, with $|u_k| = k$, $\forall 0 \leq k \leq |u|$. Recall \mathcal{U} , Ω and \mathcal{F}_n from Section 2. Let $\mathcal{U}_k := \{u \in \mathcal{U} : |u| = k\}$, $k \geq 1$.

Lemma 10.1 *Fix $n \geq 1$. For $1 \leq k \leq n$, let $f_k : \mathcal{U}_k \times \Omega \rightarrow \mathbb{R}_+$ be such that $\forall u \in \mathcal{U}_k$, $f_k(u, \bullet)$ is \mathcal{F}_n -measurable. Then*

$$(10.1) \quad \mathbf{E}_{\mathbf{Q}} \left\{ \prod_{k=1}^n f_k(w_k^{(n)}) \right\} = \mathbf{E} \left\{ \sum_{|u|=n} e^{-V(u)} \prod_{k=1}^n f_k(u_k) \right\}.$$

Proof. The expectation expression on the left-hand side is

$$= \mathbf{E}_{\mathbf{Q}} \left\{ \sum_{|u|=n} \mathbf{1}_{\{w_n^{(n)}=u\}} \prod_{k=1}^n f_k(w_k^{(n)}) \right\} = \mathbf{E}_{\mathbf{Q}} \left\{ \sum_{|u|=n} \mathbf{1}_{\{w_n^{(n)}=u\}} \prod_{k=1}^n f_k(u_k) \right\},$$

which, by the definition of $w_n^{(n)}$, equals $\mathbf{E}_{\mathbf{Q}} \left\{ \sum_{|u|=n} \frac{e^{-V(u)}}{W_n} \prod_{k=1}^n f_k(u_k) \right\}$. This implies the lemma using the definition of \mathbf{Q} . \square

We now proceed to the proof of Proposition 2.1.

Proof of Proposition 2.1. We start with the proof of (i). It suffice to check that

$$(10.2) \quad \begin{aligned} & \mathbf{E}_{\mathbf{Q}} \left\{ \prod_{k=1}^n \exp \left(- \sum_{u \in \mathcal{J}_k^{(n)}} \varphi_k(\Delta V(u)) - \lambda_k \Delta V(w_k^{(n)}) \right) \right\} \\ &= \prod_{k=1}^n \mathbf{E}_{\mathbf{Q}} \left\{ \exp \left(- \sum_{x \in \mathcal{J}_1^{(1)}} \varphi_k(\Delta V(x)) - \lambda_k \Delta V(w_1^{(1)}) \right) \right\}, \end{aligned}$$

for any positive Borel functions $\varphi_1, \varphi_2, \dots, \varphi_n$, and any real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$.

For any individual $|v| \geq 1$, let

$$(10.3) \quad \mathcal{J}(v) := \{x : \overleftarrow{x} = \overleftarrow{v}, x \neq v\},$$

$$(10.4) \quad \Phi_k(v) := \exp \left(- \sum_{x \in \mathcal{J}(v_k)} \varphi_k(\Delta V(x)) - \lambda_k \Delta V(v_k) \right), \quad 1 \leq k \leq |v|.$$

[In words, $\mathcal{J}(v)$ is the set of the brothers of v ; in particular, $\mathcal{J}(w_k^{(n)}) = \mathcal{J}_k^{(n)}$.] Thus (10.2) can be rewritten as

$$(10.5) \quad \mathbf{E}_{\mathbf{Q}} \left\{ \prod_{k=1}^n \Phi_k(w_k^{(n)}) \right\} = \prod_{k=1}^n \mathbf{E}_{\mathbf{Q}} \left\{ \exp \left(- \sum_{x \in \mathcal{J}_1^{(1)}} \varphi_k(\Delta V(x)) - \lambda_k \Delta V(w_1^{(1)}) \right) \right\}.$$

To prove (10.5), we observe that by (10.1),

$$\mathbf{E}_{\mathbf{Q}}\left\{\prod_{k=1}^n \Phi_k(w_k^{(n)})\right\} = \mathbf{E}\left\{\sum_{|u|=n} e^{-V(u)} \prod_{k=1}^n \Phi_k(u_k)\right\}.$$

Of course,

$$\sum_{|u|=n} e^{-V(u)} \prod_{k=1}^n \Phi_k(u_k) = \sum_{|v|=n-1} e^{-V(v)} \prod_{k=1}^{n-1} \Phi_k(v_k) \sum_{\substack{u: \\ \bar{u}=v}} e^{-[V(u)-V(v)]} \Phi_n(u).$$

By the branching property (see Neveu [27]),

$$\mathbf{E}\left\{\sum_{|u|=n} e^{-V(u)} \prod_{k=1}^n \Phi_k(u_k) \middle| \mathcal{F}_{n-1}\right\} = b_n \sum_{|v|=n-1} e^{-V(v)} \prod_{k=1}^{n-1} \Phi_k(v_k),$$

where

$$b_n := \mathbf{E}\left\{\sum_{|x|=1} e^{-V(x)} \exp\left(-\sum_{|y|=1, y \neq x} \varphi_n(\Delta V(y)) - \lambda_n \Delta V(x)\right)\right\}.$$

Therefore,

$$\mathbf{E}_{\mathbf{Q}}\left\{\prod_{k=1}^n \Phi_k(w_k^{(n)})\right\} = b_n \mathbf{E}\left\{\sum_{|v|=n-1} e^{-V(v)} \prod_{k=1}^{n-1} \Phi_k(v_k)\right\} = b_n \mathbf{E}_{\mathbf{Q}}\left\{\prod_{k=1}^{n-1} \Phi_k(w_k^{(n-1)})\right\},$$

the last identity being a consequence of (10.1). By induction, we obtain: $\mathbf{E}_{\mathbf{Q}}\{\prod_{k=1}^n \Phi_k(w_k^{(n)})\} = \prod_{k=1}^n b_k$, where

$$\begin{aligned} b_k &:= \mathbf{E}\left\{\sum_{|x|=1} e^{-V(x)} \exp\left(-\sum_{|y|=1, y \neq x} \varphi_k(\Delta V(y)) - \lambda_k \Delta V(x)\right)\right\} \\ &= \mathbf{E}_{\mathbf{Q}}\left\{\exp\left(-\sum_{y \in \mathcal{J}_1^{(1)}} \varphi_k(\Delta V(y)) - \lambda_k \Delta V(w_1^{(1)})\right)\right\}. \end{aligned}$$

(We used (10.1) again in the last equality.) This yields (10.5), thus part (i) of Proposition 2.1.

We now turn to the proof of part (ii) of the proposition. Recall that the vertices of the Galton–Watson tree \mathbb{T}^{GW} are elements of $\mathcal{U} := \{\emptyset\} \cup \bigcup_{n=1}^{\infty} (\mathbb{N}^*)^n$. We need to check that

$$\begin{aligned} &\mathbf{E}_{\mathbf{Q}}\left\{\prod_{k=1}^n \prod_{x \in \mathcal{J}_k^{(n)}} h_x(\mathbb{T}_x^{n-k}) \prod_{k=1}^n G_k\left(\sum_{x \in \mathcal{J}_k^{(n)}} \varphi_k(\Delta V(x)), \Delta V(w_k^{(n)}), w_k^{(n)}, \mathcal{J}_k^{(n)}\right)\right\} \\ (10.6) \quad &= \mathbf{E}_{\mathbf{Q}}\left\{\prod_{k=1}^n \prod_{x \in \mathcal{J}_k^{(n)}} H_{n-k}(x) \prod_{k=1}^n G_k\left(\sum_{x \in \mathcal{J}_k^{(n)}} \varphi_k(\Delta V(x)), \Delta V(w_k^{(n)}), w_k^{(n)}, \mathcal{J}_k^{(n)}\right)\right\}, \end{aligned}$$

where $\varphi_1, \varphi_2, \dots, \varphi_n$ are positive Borel functions, $(h_x, x \in \mathcal{U})$ measurable functions $\Omega \rightarrow \mathbb{R}_+$, G_1, \dots, G_n positive Borel functionals on $\mathbb{R}_+ \times \mathbb{R} \times \mathcal{U} \times \cup_{j=0}^{\infty} \mathcal{U}^{\otimes j}$, and $H_j(x) := \int_{\Omega} h_x(T^j(\omega)) d\mathbf{P}(\omega)$ (for $j \geq 0$ and $x \in \mathcal{U}$).

To check (10.6), we see by Lemma 10.1 that the expression on the left-hand side is

$$\mathbf{E} \left\{ \sum_{|u|=n} e^{-V(u)} \prod_{k=1}^n \prod_{x \in \mathcal{I}(u_k)} h_x(\mathbb{T}_x^{n-k}) \prod_{k=1}^n G_k \left(\sum_{x \in \mathcal{I}(u_k)} \varphi_k(\Delta V(x)), \Delta V(u_k), u_k, \mathcal{I}(u_k) \right) \right\},$$

where $\mathcal{I}(\cdot)$ is defined in (10.3). The integrand within $\mathbf{E}\{\dots\}$ on the left-hand side is (vz being the concatenated word of $v \in \mathcal{U}$ and $z \in \mathcal{U}$)

$$\begin{aligned} & \sum_{|v|=1} e^{-V(v)} \prod_{x \in \mathcal{I}(v)} h_x(\mathbb{T}_x^{n-1}) G_1 \left(\sum_{x \in \mathcal{I}(v)} \varphi_1(\Delta V(x)), \Delta V(v), v, \mathcal{I}(v) \right) \times \\ & \times \sum_{y \in \mathbb{T}_v^{\text{GW}}, |y|_v = n-1} e^{-V(y)} \prod_{\ell=1}^{n-1} \prod_{z \in \mathcal{I}_v(y_\ell)} h_{vz}(\mathbb{T}_{vz}^{n-\ell-1}) \times \\ & \times \prod_{\ell=1}^{n-1} G_{\ell+1} \left(\sum_{z \in \mathcal{I}_v(y_\ell)} \varphi_{\ell+1}(\Delta V_v(z)), \Delta V_v(y_\ell), v y_\ell, v \mathcal{I}_v(y_\ell) \right), \end{aligned}$$

where, for $y \in \mathbb{T}_v^{\text{GW}}$ with $|y|_v = n-1$, $\{e =: y_0, y_1, \dots, y_{n-1} := y\}$ denotes the shortest path on \mathbb{T}_v^{GW} relating e and y , with $|y_\ell|_v = \ell$ for $1 \leq \ell \leq n-1$, $\mathcal{I}_v(y_\ell) := \{z \in \mathbb{T}_v^{\text{GW}} : \overleftarrow{z} = y_{\ell-1}, z \neq y_\ell\}$, and $v \mathcal{I}_v(y_\ell) := \{vx : x \in \mathcal{I}_v(y_\ell)\} = \mathcal{I}(v y_\ell)$.

The conditional expectation (under \mathbf{P}) of the above expression given \mathcal{F}_1 is, by the branching property,

$$\sum_{|v|=1} e^{-V(v)} \prod_{x \in \mathcal{I}(v)} H_{n-1}(x) G_1 \left(\sum_{x \in \mathcal{I}(v)} \varphi_1(\Delta V(x)), \Delta V(v), v, \mathcal{I}(v) \right) \Psi_{n-1}(v),$$

where, for any $x \in \mathcal{U}$,

$$\begin{aligned} \Psi_{n-1}(x) & := \mathbf{E} \left\{ \sum_{|y|=n-1} e^{-V(y)} \prod_{\ell=1}^{n-1} \prod_{z \in \mathcal{I}(y_\ell)} h_{xz}(\mathbb{T}_{xz}^{n-\ell-1}) \times \right. \\ & \left. \times \prod_{\ell=1}^{n-1} G_{\ell+1} \left(\sum_{z \in \mathcal{I}(y_\ell)} \varphi_{\ell+1}(\Delta V(z)), \Delta V(y_\ell), x y_\ell, x \mathcal{I}(y_\ell) \right) \right\}. \end{aligned}$$

Using the argument backwards, we obtain:

$$\mathbf{E}_{\mathbf{Q}} \left\{ \prod_{k=1}^n \prod_{x \in \mathcal{I}_k^{(n)}} h_x(\mathbb{T}_x^{n-k}) \prod_{k=1}^n G_k \left(\sum_{x \in \mathcal{I}_k^{(n)}} \varphi_k(\Delta V(x)), \Delta V(w_k^{(n)}), w_k^{(n)}, \mathcal{I}_k^{(n)} \right) \right\}$$

$$\begin{aligned}
&= \mathbf{E}_{\mathbf{Q}} \left\{ \prod_{x \in \mathcal{F}_1^{(n)}} H_{n-1}(x) G_1 \left(\sum_{x \in \mathcal{F}_1^{(n)}} \varphi_1(\Delta V(x)), \Delta V(w_1^{(n)}), w_1^{(n)}, \mathcal{F}_1^{(n)} \right) \times \right. \\
&\quad \left. \times \prod_{k=2}^n \prod_{x \in \mathcal{F}_k^{(n)}} H_{n-k}(x) \prod_{k=2}^n G_k \left(\sum_{x \in \mathcal{F}_k^{(n)}} \varphi_k(\Delta V(x)), \Delta V(w_k^{(n)}), w_k^{(n)}, \mathcal{F}_k^{(n)} \right) \right\}.
\end{aligned}$$

Iterating the procedure by gradually conditioning on $\mathcal{F}_2, \mathcal{F}_3, \dots$, and \mathcal{F}_n , this yields (10.6), and thus part (ii) of Proposition 2.1. \square

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References

- [1] Addario-Berry, L. (2006). Ballot theorems and the heights of trees. PhD thesis, McGill University.
- [2] Aldous, D.J. and Bandyopadhyay, A. (2005). A survey of max-type recursive distributional equations. *Ann. Appl. Probab.* **15**, 1047–1110.
- [3] Bachmann, M. (2000). Limit theorems for the minimal position in a branching random walk with independent logconcave displacements. *Adv. Appl. Probab.* **32**, 159–176.
- [4] Biggins, J.D. (1976). The first- and last-birth problems for a multitype age-dependent branching process. *Adv. Appl. Probab.* **8**, 446–459.
- [5] Biggins, J.D. and Kyprianou, A.E. (1997). Seneta-Heyde norming in the branching random walk. *Ann. Probab.* **25**, 337–360.
- [6] Biggins, J.D. and Kyprianou, A.E. (2004). Measure change in multitype branching. *Adv. Appl. Probab.* **36**, 544–581.
- [7] Biggins, J.D. and Kyprianou, A.E. (2005). Fixed points of the smoothing transform: the boundary case. *Electron. J. Probab.* **10**, Paper no. 17, 609–631.
- [8] Bingham, N.H. (1973). Limit theorems in fluctuation theory. *Adv. Appl. Probab.* **5**, 554–569.
- [9] Bramson, M.D. (1978). Maximal displacement of branching Brownian motion. *Comm. Pure Appl. Math.* **31**, 531–581.
- [10] Bramson, M.D. (1978). Maximal displacement of branching random walk. *Z. Wahrsch. Gebiete* **45**, 89–108.

- [11] Bramson, M.D. and Zeitouni, O. (2006+). Tightness for a family of recursion equations. (preprint available at [ArXiv math.PR/0612382](https://arxiv.org/abs/math.PR/0612382))
- [12] Chauvin, B., Rouault, A. and Wakolbinger, A. (1991). Growing conditioned trees. *Stoch. Proc. Appl.* **39**, 117–130.
- [13] Dekking, F.M. and Host, B. (1991). Limit distributions for minimal displacement of branching random walks. *Probab. Theory Related Fields* **90**, 403–426.
- [14] Derrida, B. and Spohn, H. (1988). Polymers on disordered trees, spin glasses, and traveling waves. *J. Statist. Phys.* **51**, 817–840.
- [15] Hammersley, J.M. (1974). Postulates for subadditive processes. *Ann. Probab.* **2**, 652–680.
- [16] Hardy, R. and Harris, S.C. (2004+) A new formulation of the spine approach to branching diffusions. Mathematics Preprint, University of Bath.
- [17] Harris, T.E. (1963). *The Theory of Branching Processes*. Springer, Berlin.
- [18] Heyde, C.C. (1970). Extension of a result of Seneta for the super-critical Galton–Watson process. *Ann. Math. Statist.* **41**, 739–742.
- [19] Kingman, J.F.C. (1975). The first birth problem for an age-dependent branching process. *Ann. Probab.* **3**, 790–801.
- [20] Kozlov, M.V. (1976). The asymptotic behavior of the probability of non-extinction of critical branching processes in a random environment. *Theory Probab. Appl.* **21**, 791–804.
- [21] Lifshits, M.A. (2007+). Some limit theorems on binary trees. (in preparation)
- [22] Liu, Q.S. (2000). On generalized multiplicative cascades. *Stoch. Proc. Appl.* **86**, 263–286.
- [23] Liu, Q.S. (2001). Asymptotic properties and absolute continuity of laws stable by random weighted mean. *Stoch. Proc. Appl.* **95**, 83–107.
- [24] Lyons, R. (1997). A simple path to Biggins’ martingale convergence for branching random walk. In: *Classical and Modern Branching Processes* (Eds.: K.B. Athreya and P. Jagers). *IMA Volumes in Mathematics and its Applications* **84**, 217–221. Springer, New York.
- [25] Lyons, R., Pemantle, R. and Peres, Y. (1995). Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.* **23**, 1125–1138.
- [26] McDiarmid, C. (1995). Minimal positions in a branching random walk. *Ann. Appl. Probab.* **5**, 128–139.

- [27] Neveu, J. (1986). Arbres et processus de Galton-Watson. *Ann. Inst. H. Poincaré Probab. Statist.* **22**, 199–207.
- [28] Neveu, J. (1988). Multiplicative martingales for spatial branching processes. In: *Seminar on Stochastic Processes 1987* (Eds.: E. Çinlar et al.). *Progr. Probab. Statist.* **15**, 223–242. Birkhäuser, Boston.
- [29] Petrov, V.V. (1995). *Limit Theorems of Probability Theory*. Clarendon Press, Oxford.
- [30] Seneta, E. (1968). On recent theorems concerning the supercritical Galton–Watson process. *Ann. Math. Statist.* **39**, 2098–2102.

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