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Abstract

We study the qualitative behavior of solutions to the Vlasov equation with strong external magnetic field and oscillating electric field. This model is relevant to the understanding of isotop resonant separation. We show that the effective equation is a kinetic equation with a memory term. This memory term involves a pseudo-differential operator whose kernel is characterized by an integral equation involving Bessel functions. The kernel is explicitly given in some particular cases.

1 Introduction

This paper gives a mathematical analysis of a model related to isotop resonant separation. We undertake this model using homogenization methods applied to the Vlasov equation.

The mathematical framework of this paper is the investigation of the influence of oscillations generated by strong fields in the Vlasov equation. It completes the works led in Frénod [11], Frénod and Hamdache [12], Frénod and Sonnendrücker [14, 15, 16, 17, 18], Frénod, Raviart and Sonnendrücker [13], Golse and Saint Raymond [19, 20], Saint Raymond [37], Brenier [5], Grenier [22, 23], Jabin [25], Schochet [40], Joly, Métivier and Rauch [26]. We also refer to mathematical and physical works where similar methods are used: [24, 27, 28, 41, 32, 30, 10, 6, 33, 21]. The goal here is to exhibit the effect of the interaction of the oscillations induced by a strong magnetic field with the oscillations of the electric field. In the case when both oscillation frequencies are the same, resonant phenomena appear leading to memory effects in the effective equation. Concerning this topic of memory effects, this work is an important step in the understanding of non local homogenization previously analyzed in Sanchez-Palencia [38], Tartar [42, 43], Lions [31], Amirat, Hamdache and Ziani

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[3, 4] and Alexandre [1], in the sense that we are able, here, to exhibit explicit memory terms for a physically relevant problem.

From the physical point of view, this study contributes to the understanding of phenomena appearing during isotop resonant separation experiments. Isotop resonant separation consists, in a plasma made of several kinds of ions, in heating only the ions of one given species. Those ions can be then easily extracted from the plasma by ad hoc devices. In order to reach this goal, the plasma is, inside a cylinder (with length about 1 m and radius about 10 cm), submitted to a strong, static and homogeneous magnetic field B (about 0,4 T). Under its action, each particle moves helicoidally around the magnetic lines with pulsation

$$\omega_c = \frac{q|B|}{m},$$

called cyclotron pulsation, where q and m stand for electric charge and mass of the considered particle. Let $\tilde{m}$ and $\tilde{q}$ be the mass and the charge of the ions to be heated. If an electric field E, oscillating with pulsation $\frac{\tilde{q}|B|}{\tilde{m}}$ is moreover applied to the plasma, the particles of the considered isotop are resonating with it and then acquire energy. For a detailed description of isotop resonant separation, we refer to Omnès [36], Louvet and Omnès [34], Schmitt [39], Dawson *et al.* [9], Compant La Fontaine and Louvet [8, 7].

As a model enabling us to understand some aspects of isotop resonant separation, we introduce a small parameter ε and consider the Vlasov equation

$$\begin{gathered}\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + \big(E(t,\frac{t}{\varepsilon},x) + v\times\big(B(t,\frac{t}{\varepsilon},x)+\frac{\mathcal{M}}{\varepsilon}\big)\big)\cdot\nabla_v f^\varepsilon = 0,\\ f^\varepsilon(0,x,v) = f_0(x,v),\end{gathered}\tag{1.1}$$

where $\frac{\mathcal{M}}{\varepsilon}$ is a strong magnetic field, $\mathcal{M} = e_1$ denoting the first vector of the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$, and where $E(t, \frac{t}{\varepsilon}, x)$ and $B(t, \frac{t}{\varepsilon}, x)$ are fast oscillating electric and magnetic fields. In this equation $f^\varepsilon \equiv f^\varepsilon(t, x, v)$; $t \in [0, T[$, where $T < +\infty$, is the time, $x = (x_1, x_2, x_3)$ stands for the position and $v = (v_1, v_2, v_3)$ is the velocity. This equation models the evolution of the ions to be heated without taking into account their interactions and the interactions with the other ones. This is not completely unreasonable since the ion density met in isotop separation experiment is relatively low (about 10^{10} ions.cm^{-1}). Yet, a model taking into account self induced forces (like Vlasov-Poisson) will be considered in a forthcoming paper.

We introduce the notations $\Omega = \mathbb{R}^3_x \times \mathbb{R}^3_v$ and $\mathcal{Q} = [0, T[\times\Omega$. The initial data satisfies

$$f_0 \geq 0, \quad 0 < \int_\Omega f_0^2\,dx\,dv < \infty. \tag{1.2}$$

The electric and magnetic fields $E(t, \tau, x)$, $B(t, \tau, x)$, are C^∞ and 2π periodic in τ. Under these assumptions we have the classical a priori estimate that there exists a constant C independent of ε such that the solution f^ε of the Vlasov equation (1.1) satisfies

$$\|f^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq C.$$

We deduce that up to a subsequence still denoted by ε,

$$f^\varepsilon \rightharpoonup f \quad \text{in } L^\infty(0,T;L^2(\Omega)) \quad \text{weak-}\star\,.$$

The aim of this paper is to find out an equation satisfied by the limit f. Let us introduce some more notations: for any vector V in $\mathbb{R}^3$ we denote by $V_\parallel = V_1 e_1$, $V_\perp = V_2 e_2 + V_3 e_3$, respectively, the parallel and perpendicular components of V with respect to e_1. If τ is in $[0,2\pi]$, we denote by $r(V,\tau)$ the image of V under the rotation of angle τ around the axis $\mathcal{M} = e_1$ and we define:

$$\tilde{E}(t,x) = \frac{1}{2\pi}\int_0^{2\pi} r\big(E(t,\tau,x),\tau\big)\,d\tau,$$

$$\tilde{B}(t,x) = \frac{1}{2\pi}\int_0^{2\pi} r\big(B(t,\tau,x),\tau\big)\,d\tau.$$

We shall denote by J_0 (respectively J_1) the Bessel function of order zero (respectively of order one):

$$J_0(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\tau)\,d\tau = \frac{1}{\pi}\int_0^\pi \cos(-z\cos\tau)\,d\tau,$$

$$J_1(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\tau - \tau)\,d\tau = -J_0'(z).$$

About definitions and properties of Bessel functions, see for instance [29] or [45]. To finish with the notations let us precise that we shall use the notation of Taylor ([44]) for pseudodifferential operators:

$$K(z;D)h(z) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} K(z,ik)\mathcal{F}h(k)e^{iz\cdot k}\,dk.$$

Now we are ready to state our main result:

Theorem 1.1 (Main result) *We assume that $f_0(x,v)$ depends only on x, $v_\parallel$, and the modulus of $v_\perp$, and satisfies (1.2). We assume also that $\tilde{E}(t,x)$ does not depend on $x_\parallel$, that $\tilde{E}_\parallel = 0$ and that $\tilde{B} = 0$. Then the sequence (f^ε) of solutions of (1.1) satisfies, for any $T > 0$,*

$$f^\varepsilon \rightharpoonup f \quad \text{in } L^\infty(0,T;L^2(\Omega)) \quad \text{weak-}\star,$$

and f formally satisfies the following partial integro differential equation

$$\begin{gathered}\partial_t f(t,x,v) + v_1\cdot\partial_{x_1} f(t,x,v) = \int_0^t K(s,t,y;D)f(s,x,v)\,ds,\\ f(0,x,v) = f_0(x,v),\end{gathered} \tag{1.3}$$

where $K(s,t,y;D)$ is a pseudo-differential operator parametrized by (s,t) in $[0,T[^2$ and $y = (v_1,x_2,x_3)$ in $\mathbb{R}^3$, acting on functions of $z = (x_1,v_2,v_3)$ variables. The kernel

$$K(s,t,y;ik) = \exp(-ik_1y_1(t-s))\tilde{K}(s,t,y;ik) \tag{1.4}$$

where $\tilde{K}(s,t,y;ik)$ *is the unique solution of the Volterra equation*

$$\int_s^t J_0\big(|k_\perp||L(s,\sigma,y)|\big)\tilde{K}(\sigma,t,y,ik)\,d\sigma = -\frac{L(s,t,y)\cdot\tilde{E}(t,y)}{|L(s,t,y)|}|k_\perp|J_1\big(|k_\perp||L(s,t,y)|\big) \quad (1.5)$$

with

$$L(s,t,y)=\int_s^t \tilde{E}(\sigma,y)d\sigma=\int_s^t \tilde{E}_\perp(\sigma,y)d\sigma. \quad (1.6)$$

Remark 1.1 The assumption on variable-dependancy as the one on $\tilde{E}_\parallel$ we make in the heading of the Theorem is to ensure a transversality property required in order to apply non local homogenization methods. Nevertheless, from physical point of view, as original repartition functions are often maxwellian, considering that f_0 depends only on x, $v_\parallel$ and the modulus of $v_\perp$ is relevant. The two facts $\tilde{E}_\parallel=0$ and $\tilde{B}=0$ can be easily realized experimentally, of course as soon as the self induced fields are neglected.

Remark 1.2 Equation (1.3) does not contain oscillations. It only contains the mean effect, via the memory term, of the oscillations contained in equation (1.1). According to our knowledge, it is the first time that such a mean model is exhibited to describe the dynamic of ions during isotop resonant separation.

In some cases, we may obtain explicit expressions of the kernel K:

Theorem 1.2 *In the particular case where*

$$E_\perp(t,\tau,x_\perp)=\mathcal{E}(t,x_\perp)g(\tau,x_\perp),$$

with $\mathcal{E}$ $\mathbb{R}$*-valued and under the assumptions of Theorem 1.1, the kernel is exactly equal to*

$$K(s,t,y,ik) = -\exp(-ik_1y_1(t-s))\,\mathcal{E}(t,y)\mathcal{E}(s,y)\ |k_\perp||\tilde{g}(y)|\frac{J_1\big(|k_\perp||\tilde{g}(y)|\int_s^t\mathcal{E}(\sigma,y)\,d\sigma\big)}{\int_s^t\mathcal{E}(\sigma,y)\,d\sigma}, \quad (1.7)$$

where

$$\tilde{g}(y)=\frac{1}{2\pi}\int_0^{2\pi} r\big(g(\tau,x),\tau\big)\,d\tau.$$

Remark 1.3 As it is stated in Alexandre [1], it is immediate to check that the kernel K appearing in (1.7) satisfies:

$$|K(s,t,y,ik)|\le\|\mathcal{E}\|_\infty|\tilde{g}(y)|^2(1+|k|^2).$$

Moreover, we notice that for $s\neq t$, the kernel $K(s,t,y,ik)$ decreases like $\sqrt{|k|}$ when k goes to infinity.

The following Theorem is a simplified version of the previous one, but as its proof is much simpler; we state it separately.

Theorem 1.3 *In the very particular case of an electric field independent of time:*

$$E_\perp = E_\perp(\tau, x_\perp) \tag{1.8}$$

and under the assumptions of Theorem 1.1, we have:

$$K(s,t,y,ik) = -\exp(-ik_1 y_1(t-s))\,|k_\perp||\tilde{E}_\perp(y)|\frac{J_1\big(|k_\perp||\tilde{E}_\perp(y)|(t-s)\big)}{t-s}. \tag{1.9}$$

The proof uses the notion of two scale convergence introduced by N'Guetseng [35] and Allaire [2]. Their result is the following:

Theorem 1.4 (N'Guetseng and Allaire) *If a sequence (f^ε) is bounded in $L^\infty(0,T;L^2(\Omega))$, then there exists a 2π-periodic in τ profile $F(t,\tau,x,v)$ in $L^\infty(0,T;L^\infty(\mathbb{R}_\tau;L^2(\Omega)))$ such that, for every $\psi(t,\tau,x,v)$ regular, compactly supported with respect to (t,x,v), and 2π-periodic with respect to τ, we have, up to a subsequence,*

$$\int_{\mathcal{Q}} f^\varepsilon \psi^\varepsilon \; dt\; dx\; dv \to \int_{\mathcal{Q}} \int_0^{2\pi} F\psi \, d\tau\; dt\; dx\; dv,$$

where $\psi^\varepsilon(t,x,v) = \psi(t, \frac{t}{\varepsilon}, x, v)$. The profile F is called the 2π-periodic two scale limit of f^ε and the link between F and the weak-$\star$ limit f is given by

$$\int_0^{2\pi} F(t,\tau,x,v)\, d\tau = f(t,x,v).$$

It has been used by Frénod and Sonnendrücker [15, 17, 18] in the context of homogenization of the Vlasov equation but under an assumption of strong convergence of the electric field. Using the same ideas we obtain first the following result concerning the two-scale limit of f^ε:

Theorem 1.5 *The sequence (f^ε) of solutions of (1.1) two scale converges towards the 2π-periodic in τ profile F which is the unique solution of*

$$\begin{gathered} \partial_\tau F + (v \times \mathcal{M}) \cdot \nabla_v F = 0, \\ \partial_t F + v_1 \cdot \partial_{x_1} F + \big(r(\tilde{E}, -\tau) + v \times r(\tilde{B}, -\tau)\big) \cdot \nabla_v F = 0 \\ F(0,\tau,x,v) = \frac{1}{2\pi} f_0\big(x, r(v,\tau)\big). \end{gathered} \tag{1.10}$$

Then following Alexandre [1] we apply a Fourier transform $\mathcal{F}$ to obtain an ordinary differential equation satisfied by $\mathcal{F}F$. Using the fact that

$$\mathcal{F}f(t,x,v) = \int_0^{2\pi} \mathcal{F}F(t,\tau,x,v)\, d\tau,$$

we obtain an ordinary differential equation satisfied by $\mathcal{F}f$. Eventually we introduce a Volterra equation as in Tartar [42, 43] and Alexandre [1] to obtain our result.

2 Scaling and qualitative study

We exhibit the important parameters playing a role when charged particles are submitted to a strong magnetic field and in view of which we provide the scaling leading to equation (1.1).

Before any scaling the evolution of the repartition function $f(t,x,v)$ representing at each time t the particle density standing in x and moving with velocity v, is given by the Vlasov equation

$$\partial_t f + v\cdot\nabla_x f + \frac{q}{m}(E + v\times B)\cdot\nabla_v f = 0. \tag{2.1}$$

We define now characteristic scales: $\overline{t}$ is the characteristic time, $\overline{L}$ the characteristic length and $\overline{v}$ the characteristic velocity; and rescaled variables: t', x' and v' by $t=\overline{t}t'$, $x=\overline{L}x'$, $v=\overline{v}v'$. We also define scaling factors for the fields $\overline{E}$ and $\overline{B}$ and rescaled fields $E'(t',x')$ and $B'(t',x')$ by $\overline{E}E'(t',x') = E(\overline{t}t',\overline{L}x')$ and $\overline{B}B'(t',x') = B(\overline{t}t',\overline{L}x')$. Lastly, defining a scaling factor $\overline{f}$ for the repartition function, without forgetting that f is a density on the phase space, we define the rescaled repartition function f' by

$$\overline{f}f'(t',x',v') = \overline{L}^3\overline{v}^3 f(\overline{t}t',\overline{L}x',\overline{v}v').$$

The new repartition function is the solution to

$$\partial_{t'} f' + \frac{\overline{v}\overline{t}}{\overline{L}} v'\cdot\nabla_{x'} f' + \Big(\frac{q\overline{E}\overline{t}}{m\overline{v}}E'(t',x') + \frac{q\overline{B}\overline{t}}{m} v'\times B'(t',x')\Big)\cdot\nabla_{v'} f = 0. \tag{2.2}$$

Let us introduce two parameters having an important physical signification: $\overline{\omega_c} = \frac{q\overline{B}}{m}$ is the characteristic cyclotron pulsation and $\overline{a_L} = \frac{\overline{v}}{\overline{\omega_c}}$ the characteristic Larmor radius. Looking at (2.2) in view of these parameters, we get

$$\partial_{t'} f' + \overline{t}\overline{\omega_c}\frac{\overline{a_L}}{\overline{L}} v'\cdot\nabla_{x'} f' + \Big(\overline{t}\overline{\omega_c}\frac{\overline{E}}{\overline{v}\overline{B}}E'(t',x') + \overline{t}\overline{\omega_c} v'\times B'(t',x')\Big)\cdot\nabla_{v'} f' = 0. \tag{2.3}$$

Now, introducing the small parameter ε, we set

$$\frac{\overline{a_L}}{\overline{L}} = \varepsilon \text{ and } \overline{\omega_c}\overline{t} = \frac{1}{\varepsilon}, \tag{2.4}$$

which means that the observation length scale and the observation time scale are respectively large in front of the Larmor radius and the cyclotronic period. This regime is relevant to describe the global behaviour of the considered particles. We also assume that the electric force is much smaller than the magnetic one. It reads:

$$\frac{\overline{E}}{\overline{v}\overline{B}} = \varepsilon. \tag{2.5}$$

Now, in order to model the fact that the particles are submitted to an oscillating electric field, we set that E' writes $E'\big(t',\frac{t'}{\varepsilon},x'\big)$. Concerning the magnetic field we assume that it is made of a constant field perturbed by an oscillating one, it gives $B' = \mathcal{M} + \varepsilon B''\big(t',\frac{t'}{\varepsilon},x'\big)$. Using those assumptions and removing subscripts $''$ and $'$, equation (2.3) leads to (1.1).

Remark 2.1 The goal of this remark is to show that the considered scaling corresponds to physical situations. If the ions are potassium and the magnetic field magnitude is $1T$, the characteristic cyclotron pulsation is about 10^5 ($=\overline{\omega_c}$). In experimental situations it is realistic to consider that the ions stay 10^{-2} s in the device (we make this value as reference time $\overline{t}$). Beyond this, considering that the thermic velocity (we choose as characteristic velocity $\overline{v}$) of the ions is something like $10^3 m.s^{-1}$, the size order of their Larmor radius is the mm ($\overline{a_L} \sim 10^{-3}$m). As the device is about one meter long, we have

$$\frac{\overline{a_L}}{\overline{L}} \sim 10^{-3}, \quad \overline{\omega_c}\,\overline{t} = 10^3. \tag{2.6}$$

Yet if the magnitude of the electric field is about $1V.m^{-1}$ we also have

$$\frac{\overline{E}}{\overline{v}\,\overline{B}} = 10^{-3}. \tag{2.7}$$

As stated in Theorem 1.1, the effective behaviour of equation (1.3) involves memory effect. In order to give a way to understand why, we propose to begin by studying the following simple problem: analysing the movement of a particle (with mass and electric charge equal to 1) with no velocity component parallel to $\mathcal{M} = e_1$, and submitted to the magnetic field $\mathcal{M}/\varepsilon$. Under the mere action of this magnetic field, the particle rotates around $\mathcal{M}$ with pulsation $1/\varepsilon$. In other words, its velocity and position write

$$V(t) = r(v_0, -\frac{t}{\varepsilon}), \quad X(t) = x_0 + \varepsilon\big(r(v_0, \tfrac{\pi}{2} - \tfrac{t}{\varepsilon}) + v_0 \times \mathcal{M}\big),$$

where v_0 and x_0 are velocity and position of the particle at $t = 0$ (we assume $v_0 \cdot \mathcal{M} = 0$). As ε goes to 0, $X(t)$ tends to x_0. Yet, $V(t)$ drives the circle $|v| = |v_0|$ faster and faster. Consequently we could say that *the particle occupies the whole circle* $|v| = |v_0|$ *and has forgotten its initial direction* $\frac{v_0}{|v_0|}$. If now an oscillating electric field, writing for instance

$$E^\varepsilon(t) = r(e_2, -\frac{t}{\varepsilon}) = \begin{pmatrix} 0 \\ \cos(t/\varepsilon) \\ -\sin(t/\varepsilon) \end{pmatrix}$$

is applied to the particle in addition to the magnetic field, the movement of the particle is no more a rotation, but a spiral around the magnetic field $\mathcal{M}$. Indeed, the considered electric field does not modify the angular velocity of the particle but only the modulus $|v| = \sqrt{v_2^2 + v_3^2}$, since we have

$$V(t) = r(v_0, -\frac{t}{\varepsilon}) + tE^\varepsilon(t).$$

Observe that

$$\frac{d}{dt}|V(t)|^2 = 2(v_0 \cdot e_2 + t).$$

If $v_0 \cdot e_2 \geq 0$ the modulus $|v|$ of the velocity increases during each rotation; if $v_0 \cdot e_2 < 0$ the modulus decreases during the first rotations (until the time $t = -v_0 \cdot e_2$) and then it increases. Consequently, we could say here that *the dynamic of the particle strongly depends on the initial value* v_0. Of course this dependance is kept when ε goes to 0. But as we just saw the particle *forgets its initial value* as ε goes to 0. Because of this contradiction we need to keep additional information, which is contained in the memory term taking place in equation (1.3).

To finish this qualitative study, we shall explain the result contained in Theorem 1.5 using formal asymptotic expansion. If we assume the following ansatz of f^ε

$$f^\varepsilon(t,x,v) = F_0(t,\frac{t}{\varepsilon},x,v) + \varepsilon F_1(t,\frac{t}{\varepsilon},x,v) + \dots, \tag{2.8}$$

if we insert this in (1.1) and identify the terms at each order we get, at order 0:

$$\partial_\tau F_0 + (v \times \mathcal{M}) \cdot \nabla_v F_0 = 0, \tag{2.9}$$

and

$$\begin{aligned} &\partial_\tau F_1 + (v \times \mathcal{M}) \cdot \nabla_v F_1 \\ &\qquad = -\big(\partial_t F_0 + v \cdot \nabla_x F_0 + \big(E(t,\tau,x) + v \times B(t,\tau,x)\big) \cdot \nabla_v F_0\big). \end{aligned} \tag{2.10}$$

We then see that the first term F_0 is nothing but the two scale limit F and that equation (2.9) is nothing but (1.10a). The second equation (1.10b) is given as a compatibility condition on $F_0 = F$ in order that (2.10) has solutions.

3 Equation satisfied by the 2-scale limit.

In this section we follow the procedure of Frénod and Sonnendrücker [15] to prove Theorem 1.5. We only sketch the proof and refer the reader to [15] for details.

Let (f^ε) be a sequence of solutions of (1.1). As the sequence is bounded in $L^\infty(0,T;L^2(\Omega))$, by Theorem 1.4 there exists a 2π-periodic in τ profile $F(t,\tau,x,v)$ in $L^\infty(0,T;L^\infty(\mathbb{R}_\tau;L^2(\Omega)))$ such that for every $\psi(t,\tau,x,v)$ regular, compactly supported with respect to (t,x,v) and 2π-periodic with respect to τ, we have, up to a subsequence,

$$\int_{\mathcal{Q}} f^\varepsilon \psi^\varepsilon \; dt\; dx\; dv \to \int_{\mathcal{Q}} \int_0^{2\pi} F\psi \, d\tau\; dt\; dx\; dv \tag{3.1}$$

The proof of Theorem 1.5 is led in three steps:
Step 1: First we use a weak formulation of (1.1) in $\mathcal{D}'(\mathcal{Q})$ with functions $\psi^\varepsilon(t,x,v) = \psi(t,\frac{t}{\varepsilon},x,v)$ where ψ is regular with compact support in (t,x,v)

and 2π-periodic in τ, which writes:

$$\int_{\mathcal{Q}} f^\varepsilon \big(\partial_t \psi^\varepsilon + v\cdot\nabla_x\psi^\varepsilon + \big(E^\varepsilon + v\times(B^\varepsilon + \frac{\mathcal{M}}{\varepsilon})\big)\cdot\nabla_v\psi^\varepsilon\big)\,dt\,dx\,dv$$
$$= -\int_\Omega f_0(x,v)\psi^\varepsilon(0,x,v)\,dx\,dv. \quad (3.2)$$

Notice that $\partial_t(\psi^\varepsilon) = \big(\partial_t\psi + \frac{1}{\varepsilon}\partial_\tau\psi\big)^\varepsilon$. Multiply (3.2) by ε, then let ε tend to 0 and apply the two scale convergence (3.1) to deduce that F belongs to the kernel of the singular perturbation appearing in (1.1), in other words:

$$\partial_\tau F + (v\times\mathcal{M})\cdot\nabla_v F = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}_\tau\times\mathbb{R}^3_v) \quad (3.3)$$

for almost every (x,t) in $[0,T[\times\mathbb{R}^3_x$.

Step 2: Next we use the following lemma (Lemma 2.3 of [15]), which characterizes the kernel of the singular perturbation.

Lemma 3.1 *A function $F(\tau,v)\in L^\infty(\mathbb{R}_\tau, L^2(\mathbb{R}^3_v))$ 2π-periodic in τ satisfies*

$$\partial_\tau F + (v\times\mathcal{M})\cdot\nabla_v F = 0 \quad \textit{in } \mathcal{D}'(\mathbb{R}_\tau\times\mathbb{R}^3_v)$$

if and only if there exists a function $G\in L^2(\mathbb{R}^3_u)$ such that $F(\tau,v) = G(r(v,\tau))$.

According to this lemma there exists a function G in $L^\infty(0,T;L^2(\mathbb{R}^3_x\times\mathbb{R}^3_u))$ such that

$$F(t,\tau,x,v) = G(t,x,r(v,\tau)). \quad (3.4)$$

Step 3: We denote here $\Omega' = \mathbb{R}^3_x\times\mathbb{R}^3_u$, $\mathcal{Q}' = [0,T[\times\Omega'$. The goal of this step is to project equation (1.1) on the orthogonal of the kernel we identified above and to pass again to the limit. In order to achieve this we build test functions belonging to the kernel in the following way. For every regular compactly supported $\varphi(t,x,u)$ we consider the 2π-periodic in τ function $\psi(t,\tau,x,v) = \varphi(t,x,r(v,\tau))$, which in view of Lemma 3.1 satisfies

$$\partial_\tau\psi + (v\times\mathcal{M})\cdot\nabla_v\psi = 0.$$

We take $\psi^\varepsilon(t,x,v) = \psi(t,\frac{t}{\varepsilon},x,v)$ in the weak formulation of the Vlasov equation (3.2) obtained in the first step so that

$$\int_{\mathcal{Q}} f^\varepsilon \big((\partial_t\psi)^\varepsilon + (v\cdot\nabla_x\psi)^\varepsilon + \big((E+v\times B)\cdot\nabla_v\psi\big)^\varepsilon\big)\,dt\,dx\,dv$$
$$= -\int_\Omega f_0(x,v)\psi(0,0,x,v)\,dx\,dv.$$

We let ε tend to 0, use the two scale convergence (3.1) and the equality (3.4) to deduce that

$$\int_{\mathcal{Q}}\int_0^{2\pi} G\big(t,x,r(v,\tau)\big)\big(\partial_t\varphi + v\cdot\nabla_x\varphi + r\big(E+v\times B,\tau\big)\cdot\nabla_u\varphi\big)\,d\tau\ dt\ dx\ dv$$
$$= -\int_\Omega f_0(x,v)\varphi(0,x,v)\ dx\ dv.$$

Now we make the change of variables $u = r(v, \tau)$ and perform the integration with respect to τ over $[0, 2\pi]$. We obtain in this way the equation satisfied by G:

Lemma 3.2 *The function $G(t, x, u)$ linked to the 2π-periodic profile F by (3.4) is the unique solution of*

$$\begin{aligned} \partial_t G + u_1 \cdot \partial_{x_1} G + (\tilde{E} + u \times \tilde{B}) \cdot \nabla_u G = 0 \\ G(0, x, u) = \frac{1}{2\pi} f_0(x, u). \end{aligned} \tag{3.5}$$

The uniqueness of the solution of (3.5) enables us to deduce that the whole sequence f^ε two-scale converges to F and, because of the link between F and f, weak-$\star$ converges to f (recall that $\int_0^{2\pi} F(t, \tau, x, v)\, d\tau = f(t, x, v)$).

To prove Theorem 1.5 we rewrite the equation (3.5) in terms of F using the equality $F(t, \tau, x, v) = G(t, x, r(v, \tau))$ and thus obtain the equation satisfied by F since

$$r\big(\nabla_v F(t, \tau, x, v), \tau\big) = \nabla_u G\big(t, x, r(v, \tau)\big)$$

and

$$\big(r(v, \tau) \times \tilde{B}\big) \cdot r(\nabla_v F, \tau) = \big(v \times r(\tilde{B}, -\tau)\big) \cdot \nabla_v F.$$

The Theorem is then proved. □

Remark 3.1 At this stage we can integrate in τ over $[0, 2\pi]$ the equation

$$\begin{aligned} \partial_t F + v_1 \cdot \partial_{x_1} F + \big(r(\tilde{E}, -\tau) + v \times r(\tilde{B}, -\tau)\big) \cdot \nabla_v F = 0 \\ F(0, \tau, x, v) = \frac{1}{2\pi} f_0\big(x, r(v, \tau)\big) \end{aligned}$$

and use the equality $f(t, x, v) = \int_0^{2\pi} F(t, \tau, x, v)\, d\tau$ to get the following equation which is satisfied by f:

$$\begin{aligned} &\partial_t f + v_1 \cdot \partial_{x_1} f + \big(\tilde{E}_\parallel + v \times \tilde{B}_\parallel\big) \nabla_v f \\ &\quad + \big(\tilde{E}_\perp + v \times \tilde{B}_\perp\big) \cdot \nabla_v a + \big(\tilde{E} \times \mathcal{M} + v \times (\tilde{B} \times \mathcal{M})\big) \cdot \nabla_v b = 0, \end{aligned}$$

where $a(t, x, v) = \int_0^{2\pi} F(t, \tau, x, v) \cos\tau\, d\tau, \quad b(t, x, v) = \int_0^{2\pi} F(t, \tau, x, v) \sin\tau\, d\tau.$

4 Equation satisfied by the weak-* limit.

In this section we prove our main results Theorems 1.1, 1.2 and 1.3 by using Fourier transform. Recall that under the assumptions of Theorem 1.1 the 2-scale limit F satisfies the equation

$$\begin{aligned} \partial_t F + v_1 \cdot \partial_{x_1} F + r(\tilde{E}, -\tau) \cdot \nabla_v F = 0 \\ F(0, \tau, x, v) = \frac{1}{2\pi} f_0(x; v_\parallel, |v_\perp|), \end{aligned} \tag{4.1}$$

with

$$r(\tilde{E}(t,x_\perp),-\tau) = \left| \begin{array}{l} 0 \\ \tilde{E}_2(t,x_\perp)\cos\tau + \tilde{E}_3(t,x_\perp)\sin\tau \\ -\tilde{E}_2(t,x_\perp)\sin\tau + \tilde{E}_3(t,x_\perp)\cos\tau \end{array} \right.$$

Thanks to the hypothesis the only derivatives of F involved in the equation are with respect to x_1, v_2, v_3 and the coefficients only depend on v_1, x_2, x_3. The transversality assumption required for non local homogenization methods is realized. For convenience we rename the variables by

$$z = \left| \begin{array}{l} x_1 \\ v_2 \\ v_3 \end{array} \right. , \quad y = \left| \begin{array}{l} v_1 \\ x_2 \\ x_3 \end{array} \right. .$$

and define

$$\begin{gathered} H(t,\tau,z,y) = F(t,\tau,x,v), \\ h(t,z,y) = f(t,x,v), \\ h_0(z,y) = h(0,z,y) = f(0,x,v) = f_0(x,v), \\ a(t,\tau,y) = \left| \begin{array}{l} y_1 \\ \tilde{E}_2\cos\tau + \tilde{E}_3\sin\tau \\ -\tilde{E}_2\sin\tau + \tilde{E}_3\cos\tau \end{array} \right. , \end{gathered}$$

so that (4.1) becomes

$$\begin{gathered} \partial_t H(t,\tau,z,y) + a(t,\tau,y)\cdot\nabla_z H(t,\tau,z,y) = 0, \\ H(0,\tau,z,y) = \frac{1}{2\pi} h_0(z,y). \end{gathered} \tag{4.2}$$

Applying a Fourier transform in the z variable we get the ordinary differential equation

$$\begin{gathered} \partial_t \mathcal{F}H(t,\tau,k,y) + ik\cdot a(t,\tau,y)\mathcal{F}H(t,\tau,k,y) = 0, \\ \mathcal{F}H(0,\tau,k,y) = \frac{1}{2\pi}\mathcal{F}h_0(k,y), \end{gathered} \tag{4.3}$$

which has the explicit solution

$$\begin{aligned} \mathcal{F}H(t,\tau,k,y) &= \frac{1}{2\pi}\mathcal{F}h_0(k,y)\exp\big(-ik\cdot\int_0^t a(\sigma,\tau,y)\,d\sigma\big) \\ &= \frac{1}{2\pi}\mathcal{F}h_0(k,y)\exp(-ik_1y_1t)\exp\big(-ik\cdot\int_0^t r\big(\tilde{E}(\sigma,y),-\tau\big)\,d\sigma\big). \end{aligned}$$

As

$$f(t,x,v) = \int_0^{2\pi} F(t,\tau,x,v)\,d\tau$$

we know that

$$\mathcal{F}h(t,k,y) = \int_0^{2\pi} \mathcal{F}H(t,\tau,k,y)\,d\tau,$$

hence

$$\mathcal{F}h(t,k,y) = \mathcal{F}h_0(k,y)\exp(-ik_1y_1t)\frac{1}{2\pi}\int_0^{2\pi} \exp\big(-ik\cdot\int_0^t r\big(\tilde{E}(\sigma,y),-\tau\big)d\sigma\big)\,d\tau.$$

We introduce the quantities

$$L(s,t,y) = \int_s^t \tilde{E}(\sigma,y)d\sigma, \tag{4.4}$$

$$A(s,t,y,Z) = \frac{1}{2\pi}\int_0^{2\pi} \exp\big(-Z\cdot\int_s^t r\big(\tilde{E}(\sigma,y),-\tau\big)\,d\sigma\big)\,d\tau, \tag{4.5}$$

so that

$$\mathcal{F}h(t,k,y) = \mathcal{F}h_0(k,y)\exp(-ik_1y_1t)A(0,t,y,ik),$$

and differentiating with respect to t, the following equation satisfied by $\mathcal{F}h$:

$$\begin{gathered}\partial_t\mathcal{F}h(t,k,y) + ik_1y_1\mathcal{F}h(t,k,y) = \mathcal{F}h_0(k,y)\exp(-ik_1y_1t)\partial_tA(0,t,y,ik),\\ \mathcal{F}h(0,k,y) = \mathcal{F}h_0(k,y).\end{gathered} \tag{4.6}$$

Now we use the method of Tartar [42, 43] (see also Alexandre [1]) to form an integro-differential operator. We define

$$C(s,t,y,Z) = -\partial_tA(s,t,y;Z_2,Z_3)\exp(-Z_1y_1(t-s)) \quad \text{for } Z\in\mathbb{C}^3,$$

so that

$$\partial_t\mathcal{F}h(t,k,y) + ik_1y_1\mathcal{F}h(t,k,y) = -\mathcal{F}h_0(k,y)C(0,t,y,ik). \tag{4.7}$$

We denote by $D(s,t,y,Z)$ the solution of the Volterra-Green equation

$$D(s,t,y,Z) - \int_s^t C(s,\sigma,y,Z)D(\sigma,t,y,Z)\,d\sigma = C(s,t,y,Z). \tag{4.8}$$

We replace $C(0,t,y,ik)$ in equation (4.7) by its integral form, which gives

$$\begin{aligned}&\partial_t\mathcal{F}h(t,k,y) + ik_1y_1\mathcal{F}h(t,k,y)\\ &= -\mathcal{F}h_0(k,y)D(0,t,y,ik) + \int_0^t \mathcal{F}h_0(k,y)C(0,\sigma,y,ik)D(\sigma,t,y,ik)\,d\sigma\\ &= -\mathcal{F}h_0(k,y)D(0,t,y,ik) - \int_0^t \partial_t\mathcal{F}h(\sigma,k,y)D(\sigma,t,y,ik)\,d\sigma\\ &\quad - ik_1y_1\int_0^t \mathcal{F}h(\sigma,k,y)D(\sigma,t,y,ik)\,d\sigma.\end{aligned}$$

Integrating by parts in the second term, observe that

$$A(t,t,y,Z) = 1, D(t,t,y,Z) = -\partial_tA(t,t,y,Z) = 0,$$

and we obtain eventually that

$$\begin{aligned} \partial_t \mathcal{F}h(t,k,y) + ik_1y_1\mathcal{F}h(t,k,y) &= \int_0^t K(\sigma,t,y,ik)\mathcal{F}h(\sigma,k,y)\,d\sigma, \\ \mathcal{F}h(0,k,y) &= \mathcal{F}h_0(k,y), \end{aligned} \tag{4.9}$$

where $K(s,t,y,Z) = \partial_s D(s,t,y,Z) - ik_1y_1D(s,t,y,Z)$.

To simplify the expression of K we define $\tilde{D}$ by

$$\tilde{D}(s,t,y,Z) = \exp\big(Z_1y_1(t-s)\big)D(s,t,y,Z).$$

Then replacing D in the expression of K we obtain that

$$K(s,t,y,Z) = \exp\big(-Z_1y_1(t-s)\big)\tilde{K}(s,t,y,Z), \tag{4.10}$$

which is equation (1.4), with

$$\tilde{K}(s,t,y,Z) = \partial_s\tilde{D}(s,t,y,Z), \tag{4.11}$$

and replacing D in the Volterra equation (4.8) we obtain that $\tilde{D}$ is the solution of the Volterra equation

$$\tilde{D}(s,t,y,Z) + \int_s^t \partial_t A(s,\sigma,y,Z)\tilde{D}(\sigma,t,y,Z)\,d\sigma = -\partial_t A(s,t,y,Z). \tag{4.12}$$

We observe that $\tilde{D}$ does not depend on Z_1. Performing an integration by parts we get the following equation satisfied by $\tilde{K} = \partial_s\tilde{D}$:

$$\int_s^t A(s,\sigma,y,Z)\tilde{K}(\sigma,t,y,Z)\,d\sigma = \partial_t A(s,t,y,Z). \tag{4.13}$$

Let us now simplify the (4.5) expression of A: for fixed s,t,y,k, we choose α and β in $[0,2\pi]$ such that

$$\begin{aligned} r(L(s,t,y),\alpha) &= |L(s,t,y)|e_2, \\ k_\perp &= |k_\perp| r(e_2,\beta). \end{aligned}$$

Then

$$\begin{aligned} A(s,t,y,ik) &= \frac{1}{2\pi}\int_0^{2\pi} \exp\Big(-ik\cdot\int_s^t r\big(\tilde{E}(\sigma,y),-\tau\big)\,d\sigma\Big)\,d\tau \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp\Big(-ik\cdot r\big(L(s,t,y),-\tau\big)\Big)\,d\tau \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp\Big(-i|k_\perp||L(s,t,y)|r(e_2,\beta+\alpha+\tau)\cdot e_2\Big)\,d\tau \\ &= \frac{1}{2\pi}\int_0^{2\pi} \exp\Big(-i|k_\perp||L(s,t,y)|\cos\tau\Big)\,d\tau \\ &= \frac{1}{2\pi}\int_0^{2\pi} \cos\Big(|k_\perp||L(s,t,y)|\cos\tau\Big)\,d\tau, \end{aligned}$$

so that
$$A(s,t,y,ik) = J_0\big(|k_\perp||L(s,t,y)|\big). \tag{4.14}$$
Differentiating with respect to t, we obtain
$$\partial_t A(s,t,y,ik) = -\frac{L(s,t,y)\cdot \tilde{E}(t,y)}{|L(s,t,y)|}|k_\perp|J_1\big(|k_\perp||L(s,t,y)|\big). \tag{4.15}$$
Hence $\tilde{K}(s,t,y,ik)$ is the unique solution of the equation
$$\begin{aligned}\int_s^t J_0\big(|k_\perp||L(s,\sigma,y)|\big)\tilde{K}(\sigma,t,y,ik)\,d\sigma \\ = -\frac{L(s,t,y)\cdot\tilde{E}(t,y)}{|L(s,t,y)|}|k_\perp|J_1\big(|k_\perp||L(s,t,y)|\big)\end{aligned} \tag{4.16}$$
for every $s,t\in[0,T]$, that is (1.5).

Applying formally the inverse Fourier transform to (4.9) we find that
$$\partial_t h(t,z,y) + y_1\cdot\partial_{z_1} h(t,z,y) = \int_0^t K(\sigma,t,y;D)h(\sigma,z,y)\,d\sigma.$$
Now
$$h(t,z,y) = \int_0^{2\pi} H(t,\tau,z,y)\,d\tau = \int_0^{2\pi} F(t,\tau,x,v)\,d\tau = f(t,x,v),$$
from which we deduce (1.3) satisfied by f, and completes the proof of Theorem 1.1. □

We shall now treat the particular case of Theorem 1.3 where E is independent of time: in this case $\tilde{E} = \tilde{E}_\perp$ is independant of time too and we have
$$L(s,t,y) = (t-s)\tilde{E}_\perp(y),$$
so that using the parity of J_0 and the imparity of J_1 the last equation becomes
$$\begin{aligned}\int_s^t J_0\big(|k_\perp||\tilde{E}_\perp(y)|(\sigma-s)\big)\tilde{K}(\sigma,t,y,ik)\,d\sigma \\ = -|\tilde{E}_\perp(y)||k_\perp|J_1\big(|k_\perp||\tilde{E}(y)|(t-s)\big) \quad \text{for every } s,t\in[0,T].\end{aligned}$$
For sake of clarity we fix for the moment y and k and we set
$$C = |k_\perp||\tilde{E}_\perp(y)|.$$
It is easy to see that for every $h\in]-T,+T[$, the function $(s,t)\mapsto\tilde{K}(s+h,t+h)$ is solution of the equation on $[-h,T-h]\times[-h,T-h]$, so that $\tilde{K}(s+h,t+h) = \tilde{K}(s,t)$ for every s,$t\in[0,T]\cap[-h,T-h]$ by uniqueness of the solution. So we can set
$$G(t-s) = \tilde{K}(s,t).$$

Replacing in the equation, performing a change of variable and setting $x = t - s$, one gets

$$\int_0^x J_0\big(C(x-u)\big)G(u)\,du = -CJ_1(Cx) \quad \text{for every } x \in [-T,T].$$

As G is easily seen to be even this is equivalent to

$$\int_0^x J_0\big(C(x-u)\big)G(u)\,du = -CJ_1(Cx) \quad \text{for every } x \in [0,T]. \tag{4.17}$$

Performing a Laplace transformation in x gives the equality of functions

$$\mathcal{L}J_0(C\cdot) \times \mathcal{L}G = \mathcal{L}(-CJ_1(C\cdot)),$$

from which we deduce the Laplace transform $\mathcal{L}G$ of G, and by the inverse Laplace transform (see for instance the formulas for Laplace transformations in [46]) we obtain that

$$G(x) = -\frac{CJ_1(Cx)}{x} \text{ on } [0,T],$$

hence on $[-T,T]$ by parity, and eventually one gets

$$\tilde{K}(s,t) = -\frac{CJ_1(C(t-s))}{t-s} \text{ for every } s,t \in [0,T], \tag{4.18}$$

giving Theorem 1.3. □

Concerning the growth in terms of power of k, we see that

$$|\tilde{K}(s,t,y,ik)| \le |k_\perp|^2 \|E\|_\infty \|J_1'\|_\infty \le (1+|k|^2)\|E\|_\infty.$$

Now we shall prove Theorem 1.2. Here E can be written in the form

$$E_\perp(t,\tau,x_\perp) = \mathcal{E}(t,x_\perp)g(\tau,x_\perp),$$

with $\mathcal{E}$ $\mathbb{R}$-valued and g $\mathbb{R}^2$-valued. For simplicity we fix x and we write

$$E_\perp(t,\tau,x_\perp) = E_\perp(t,\tau) = \mathcal{E}(t)g(\tau).$$

Then $\tilde{E}(t) = \mathcal{E}(t)V$, where V is the vector $V = \frac{1}{2\pi}\int_0^{2\pi} r\big(g(\tau),\tau\big)\,d\tau$, and

$$L_\perp(s,t) = V\int_s^t \mathcal{E}(\sigma)\,d\sigma = V\big(\Phi(t)-\Phi(s)\big)$$

where we note Φ a primitive of $\mathcal{E}$. The equation (4.16) becomes

$$\int_s^t J_0\big(|k_\perp||V||\Phi(\sigma)-\Phi(s)|\big)\tilde{K}(\sigma,t,ik)\,d\sigma \\ = -\frac{\Phi(t)-\Phi(s)}{|\Phi(t)-\Phi(s)|}|k_\perp||V|\mathcal{E}(t)J_1\big(|k_\perp||V||\Phi(t)-\Phi(s)|\big),$$

in other words

$$\int_s^t J_0\big(|k_\perp||V|(\Phi(\sigma)-\Phi(s))\big)\tilde K(\sigma,t,ik)\,d\sigma = -|k_\perp||V|\mathcal{E}(t)J_1\big(|k_\perp||V|(\Phi(t)-\Phi(s))\big) \quad (4.19)$$

for every $s,t\in[0,T]$. Now if $\mathcal{E}(t)$ is strictly positive (respectively strictly negative) for every $t\in]a,b[$, we have $\Phi'(t)=\mathcal{E}(t)>0$ (respectively <0) hence Φ is bijective from $[a,b]$ onto the interval I of extremities $\Phi(a)$, $\Phi(b)$. We make the change of variables $u=\Phi(\sigma)$ in the integral and we obtain

$$\int_\alpha^\beta J_0\big(|k_\perp||V|(u-\alpha)\big)\tilde K(\Phi^{-1}(u),\Phi^{-1}(\beta),ik)\ \frac{du}{\mathcal{E}(\Phi^{-1}(u))} = -|k_\perp||V|\mathcal{E}(\Phi^{-1}(\beta))J_1\big(|k_\perp||V|(\beta-\alpha)\big) \text{ for every } \alpha,\beta\in I,$$

where we have set $\alpha=\Phi(s)$ and $\beta=\Phi(t)$. We set

$$\Gamma(\alpha,\beta)=\frac{\tilde K(\Phi^{-1}(u),\Phi^{-1}(\beta))}{\mathcal{E}(\Phi^{-1}(\alpha))\mathcal{E}(\Phi^{-1}(\beta))}$$

and then we have

$$\int_\alpha^\beta J_0\big(|k_\perp||V|(u-\alpha)\big)\Gamma(u,\beta)\,du = -|k_\perp||V|J_1\big(|k_\perp||V|(\beta-\alpha)\big) \text{ for every } \alpha,\beta\in I.$$

We met the same equation in the preceding case. The solution is

$$\Gamma(\alpha,\beta)=-\frac{|k_\perp||V|J_1\big(|k_\perp||V|(\beta-\alpha)\big)}{\beta-\alpha},$$

which gives

$$\tilde K(s,t)=-\mathcal{E}(t)\mathcal{E}(s)\frac{|k_\perp||V|J_1\big(|k_\perp||V|(\Phi(t)-\Phi(s))\big)}{\Phi(t)-\Phi(s)} \text{ for every } s,t\in[a,b]. \quad (4.20)$$

At this point we know $\tilde K(s,t)$ for s, t both in an interval where $\mathcal{E}$ is strictly positive or strictly negative. For such s, t we have, replacing in (4.19), the equality

$$\int_s^t J_0\big(|k_\perp||V|(\Phi(\sigma)-\Phi(s))\big)\mathcal{E}(\sigma)\frac{J_1\big(|k_\perp||V|(\Phi(t)-\Phi(\sigma))\big)}{\Phi(t)-\Phi(\sigma)}\,d\sigma = J_1\big(|k_\perp||V|(\Phi(t)-\Phi(s))\big).$$

We can write

$$J_0\big(|k_\perp||V|(\Phi(\sigma)-\Phi(s))\big) = \sum_{n=0}^\infty \frac{J_0^{(n)}\big(|k_\perp||V|(\Phi(t)-\Phi(s))\big)}{n!}(-1)^n|k_\perp|^n|V|^n(\Phi(t)-\Phi(\sigma))^n,$$

$$\frac{J_1\big(|k_\perp||V|(\Phi(t)-\Phi(\sigma))\big)}{\Phi(t)-\Phi(\sigma)} = \sum_{n=1}^{\infty} \frac{J_1^{(n)}(0)}{n!}|k_\perp|^n|V|^n(\Phi(t)-\Phi(\sigma))^{n-1},$$

and write the product of the two series as

$$\begin{aligned} J_0\big(|k_\perp||V|(\Phi(\sigma)-\Phi(s))\big)\frac{J_1\big(|k_\perp||V|(\Phi(t)-\Phi(\sigma))\big)}{\Phi(t)-\Phi(\sigma)} \\ = \sum_{n=0}^{\infty} P_n(s,t)(\Phi(t)-\Phi(\sigma))^n. \end{aligned}$$

Integrating term by term we get

$$\int_s^t \sum_{n=0}^{\infty} P_n(s,t)(\Phi(t)-\Phi(\sigma))^n \mathcal{E}(\sigma)\,d\sigma = \sum_{n=0}^{\infty} P_n(s,t)\frac{(\Phi(t)-\Phi(s))^{n+1}}{n+1},$$

so that the following equality holds:

$$\sum_{n=0}^{\infty} P_n(s,t)\frac{(\Phi(t)-\Phi(s))^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{J_1^{(n)}(0)}{n!}|k_\perp|^n|V|^n(\Phi(t)-\Phi(s))^n.$$

As this is an equality of series which has nothing to do with the sign of $\mathcal{E}$ we deduce that the formula (4.20) giving $\tilde{K}(s,t)$ is valid for every s, t in $[0,T]$. The inequality

$$|K(s,t,y,ik)| \le \|\mathcal{E}\|_\infty |\tilde{g}(y)|^2(1+|k|^2)$$

is immediate. This completes the proof of Theorem 1.2.

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