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Contextual Epistemic Logic

MANUEL REBUSCHI

L.H.S.P. – Archives H. Poincaré

Université Nancy 2

manuel.rebuschi@univ-nancy2.fr

FRANCK LIHOREAU

Instituto de Filosofia da Linguagem

Universidade Nova de Lisboa

franck.lihoreau@fcsh.unl.pt

Abstract

One of the highlights of recent informal epistemology is its growing theoretical emphasis upon various notions of context. The present paper addresses the connections between knowledge and context within a formal approach. To this end, a “contextual epistemic logic”, CEL, is proposed, which consists of an extension of standard S5 epistemic modal logic with appropriate reduction axioms to deal with an extra contextual operator. We describe the axiomatics and supply both a Kripkean and a dialogical semantics for CEL. An illustration of how it may fruitfully be applied to informal epistemological matters is provided.

1 Introduction

The formal approach to knowledge and context that we propose in this paper was originally driven not only by formal logical concerns but also by more informal epistemological concerns.

In the last two or three decades, indeed, epistemology has seen two major “turns”:

- a “new linguistic turn”, as Ludlow [21] calls it, through the increased reliance, in contemporary epistemological debates, upon “evidence” regarding how we ordinarily *talk* about knowledge, most notably as a result of the flourishing discussions about the purported epistemological role of various notions of context and the relative merits of “contextualism” over “scepticism”, “anti-scepticism”, and “subjectivism”, *inter alia*;
- a “logical turn”, through the rising conviction that discussions in informal epistemology may benefit from formal epistemology – epistemic logic, formal learning theory, belief revision, and so on –, notably by applying the methods of epistemic logic in order to gain insights into traditional informal epistemological issues. Recent work by van Benthem [3], Hendricks [14, 15] and Hendricks & Pritchard [16], and Stalnaker [32] count as representative of this trend.

An important part of the background for the present paper consists of the project of taking advantage of the logical turn in order to record some of the main lessons to be drawn from the linguistic turn of epistemology, the acknowledgement of the possible epistemological role of context to start with.

This paper, at the junction of informal and formal epistemology, focuses on the question how to introduce the notion of context into epistemic logic. It provides a new logic, contextual epistemic logic (CEL), an extension of standard S5 epistemic modal logic with appropriate reduction axioms to deal with an extra contextual operator.

In section 2 contemporary informal epistemological discussions over context are briefly exposed, as well as three strategies available to handle context in a formal way. The authors already studied one of them in [20] and the other two seem quite natural, one of which is accounted for in the present paper. The section ends with a presentation of public announcement logic (PAL), which is technically very close to CEL yet very different from it in spirit: after a comparison of the respective objectives of the two logics we mention what must be changed in PAL to reach a logic for contexts.

In section 3 the syntax and Kripke semantics of CEL are given, and a complete proof system. In the subsequent section, we present a dialogical version of CEL: after a general presentation of dialogical logic for (multi-)S5 modal systems, we add context relativization and prove its completeness.

Finally, in section 5 we provide a connection between our formal definitions and standard epistemological positions (scepticism and anti-scepticism, contextualism, subjectivism), and we show a few applications of CEL to specifically epistemological questions.

2 Contextual Epistemology

2.1 From an informal approach to knowledge and context...

Constitutive of the linguistic turn in epistemology is contextualism. Contextualism can be viewed as an attempt to reconcile scepticism and anti-scepticism, and subjectivism as an alternative to contextualism. All these positions are better thought of as positions about what a satisfactory account for the truth of knowledge sentences like “ i knows that φ ” or “ i does not know that φ ” must look like. A way of representing what it takes for a knowledge sentence to be true is by means of an “epistemic relevance set”, i.e. a set of “epistemically relevant counter-possibilities” that an agent must be able to rule out, given the evidence available to him, for it to be true that he knows a proposition.

Thus, according to *scepticism* – also called “sceptical invariantism”, and defended by Unger [34] for instance – the epistemic standards are so extraordinarily stringent that for any (contingent) proposition, all logical counter-possibilities to this proposition will count as epistemically relevant. That is, for it to be true that an agent i knows a proposition φ , i must rule out the entire set of logical

possibilities in which not- φ ; this being an unfulfillable task for any i and φ , it is never true that anyone knows anything except, perhaps, necessary truths.¹ On the contrary, for *anti-scepticism* – also called “Moorean invariantism” and defended by Austin [1] and Moore [23] – the standards for the truth of “ i knows that φ ” being those for an ordinarily correct utterance of this sentence, they are lax enough to make (a possibly important) part of our ordinary knowledge claims turn out true: for any proposition, not all logical counter-possibilities to this proposition will count as epistemically relevant. That is, it will be true that i knows that φ only if i rules out a given (proper) subset of possibilities in which not- φ .

Scepticism and anti-scepticism are *absolutist* views about knowledge in the sense that for both of them, whether a logical counter-possibility to a proposition is epistemically relevant or not depends only on what proposition it is that is purportedly known and once this proposition is fixed, the associated epistemic relevance set is not liable to vary. According to *relativist* views, whether a logical counter-possibility to a proposition is epistemically relevant or not does not depend only on what proposition it is; it also depends on the context, for some sense of “context”.

For instance, according to *contextualism* – defended by authors like Cohen [7], De Rose [8], Lewis [19] – the context in question will be that of the possible “knowledge ascriber(s)”. Although we’d better talk of contextualism in the plural, a common contextualist assumption is that whilst the circumstances of the purported knower are fixed, the epistemic standards, therefore the epistemic relevance set, may vary with features of the context of the knowledge ascriber(s) such as what counter-possibilities they attend to, what propositions they presuppose, what is at stake in their own context, etc. In contrast, *subjectivism* – also called “subject-sensitive invariantism” or “sensitive moderate invariantism” – is defended by authors like Hawthorne [13] and Stanley [33]. This view has it that such factors – viz. attention, interests, presuppositions, stakes, etc. – are not the attention, interests, presuppositions, stakes, etc., of the knowledge ascriber, but those of the “knower” himself. That is, the epistemic standards, therefore the epistemic relevance set, may vary with the context of the purported “knower”, even if no change occurs in the circumstances he happens to be in.²

¹One might want to be a sceptic as far as contingent truths are concerned while not being a sceptic with respect to necessary truths. Such truths include logical validities as well as analytic truths. Taking the latter into account within a modal-logical framework can be done by adopting meaning postulates.

²Absent from the foregoing discussion will be another prominent view on the matter, viz. what may be coined “circumstantialism” – the view, held by Dretske [9, 10] and Nozick [24], that the relevance set depends on the objective situation the subject happens to be in. The reason for not discussing this view here is that one constraint on the formal developments in this paper was to stick to epistemological views whose associated epistemic logics (1) were normal, i.e. incorporating at least the principles of the smallest normal modal logic K – in particular the *Distribution Axiom* and the *Knowledge Generalization Rule* – and thus (2) fully characterized some class of standard Kripke models. This is not the case with circumstantialism which drops the two K-principles just mentioned. However, capturing circumstantialist epistemic logic semantically can be done in several ways compatible with a modal approach,

So, depending on whether one opts for an absolutist or a relativist view about knowledge, one will assume that a notion of context has an epistemological role to play or not. A first incursion into formalizing the possible connections between knowledge and context and the four epistemological views³ was undertaken in [20].

2.2 ... to formal approaches to reasoning about knowledge and context

The role of context in the above epistemological discussion is to impose a restriction of the relevant set of possible worlds – not to go from a possible world to another one. Contexts in a modal formalization thus cannot simply be reduced to standard modalities.

As a consequence, in order to represent contexts in epistemic logic three ways seem available:

1. use non-standard models, i.e. put the context in the metalanguage and evaluate each (standard epistemic) formula relatively to some world and context;
2. use standard models with standard modality, within an extension from basic modal logic;
3. use standard models with a non-standard interpretation for context modalities.

The first option, investigated in [20], requires that the notion of context enter the metalanguage and that the formulas of the object-language be interpreted in “contextual models”. The second track would require a kind of hybrid logic to account for the intersection of two accessibility relations. In the remainder of the subsection we briefly discuss the first two approaches before focusing on the third strategy.

two of the most straightforward being (1) by adding “awareness functions” into the standard possible worlds models as in [11], (2) or by augmenting such models with “impossible possible worlds” as in [28]. Another important epistemological view is the “assessment sensitivity” (sometimes called “relativist”) view, held by MacFarlane [22] and Richard [30] for instance, according to which the relevance set depends neither on the subject’s situation or context nor on the attributor’s context, but on the context in which a knowledge claim made by an attributor about a subject is being assessed for truth or falsity. For the same reason as before it will not be discussed here, since formalizing it would require reliance on a semantics allowing formulas to be evaluated not only relative to a world, but also to both a context of attribution and a context of assessment – i.e. a non-standard semantics, perhaps in the vicinity of [6]’s “token semantics”. These issues will not be pursued here.

³It may appear to some that scepticism does not seem to be a respectable language game with a special knowledge operator, but a defective language game with a respectable knowledge operator. This seems intuitively correct, of course. So why bother trying to capture scepticism in a formal framework anyway? Here are two reasons. First, the intuition in question might simply be an effect of a prejudice against scepticism and thus cannot be appealed to in support of a depreciative view of scepticism without begging the question; second, our commitment is to find a formal framework that makes everything “respectable” in each of the epistemological positions it is meant to capture: we want both their respective “language game” and “knowledge operator” to be treated as “respectable”.

2.2.1 Contextual models

Formalizing contexts while sticking to the standard syntax of epistemic logic implies the adoption of non-standard models. Contextual models are usual Kripke models for multi-S5 epistemic languages $\mathcal{M} = \langle W, \{\mathcal{K}_i\}_{i \in I}, V \rangle$ augmented with a pair $\langle \mathcal{C}, \mathcal{R} \rangle$, where: $\mathcal{C} = \{c_j : j \in J\} \neq \emptyset$ is a set of “contexts”, for $I \subseteq J$,⁴ and $\mathcal{R} : \mathcal{C} \rightarrow (W \rightarrow \wp(W))$ is a function of “contextual relevance” associating with each context c_j , for each world w , the set $\mathcal{R}(c_j)(w)$ of worlds relevant in that context at that world.

Truth in contextual models is then defined relatively to a world and a context. The clauses are the usual ones for propositional connectives in the sense that contexts do not play any role on them, whereas contexts can modify the evaluation of epistemic operators in one of the following four ways:

- $\mathcal{M}, c_i, w \models \mathbf{K}_j \varphi$ iff
- (1.1) for all w' , if $\mathcal{K}_i w w'$ and $w' \in \mathcal{R}(c_i)(w)$, then $\mathcal{M}, c_i, w' \models \varphi$.
 - (1.2) for all w' , if $\mathcal{K}_i w w'$ and $w' \in \mathcal{R}(c_i)(w)$, then $\mathcal{M}, c_j, w' \models \varphi$.
 - (2.1) for all w' , if $\mathcal{K}_i w w'$ and $w' \in \mathcal{R}(c_j)(w)$, then $\mathcal{M}, c_i, w' \models \varphi$.
 - (2.2) for all w' , if $\mathcal{K}_i w w'$ and $w' \in \mathcal{R}(c_j)(w)$, then $\mathcal{M}, c_j, w' \models \varphi$.

The definitions lead to four different logics, i.e. to four interpretations of the epistemic operator \mathbf{K}_j .

Taking epistemic accessibility relations to be equivalence relations, a formal characterization of the aforementioned informal views of knowledge can be given through a proper choice among the four cases (1.1)–(2.2) or/and through proper restrictions on the contextual relevance function. The two absolutist views of knowledge – scepticism and anti-scepticism – can thus be associated with case (1.1), differing from each other only in their respective restrictions on \mathcal{R} , viz. the restriction that for all i and w , $\mathcal{R}(c_i)(w) = W$ for scepticism – hence a restriction enabling scepticism to drop the condition $w' \in \mathcal{R}(c_i)(w)$ from the truth conditions of epistemic sentences – and the minimal restriction that for all i and w , $\mathcal{R}(c_i)(w) \subsetneq W$ for anti-scepticism. In contrast, the two relativist views of knowledge can be associated with the (–.2) cases, viz. case (1.2) for contextualism and case (2.2) for subjectivism.⁵

Within this framework one can express *truth-in-a-context* for knowledge claims – an interesting result for contemporary informal epistemology whose interest in the connection between knowledge and context lies primarily in the

⁴A context c_i is thus assigned to every agent $i \in I$, but additional contexts may be added for groups of agents, conversations, and so forth.

⁵A possible worry with these formal definitions is that they end up making scepticism look like a limiting case of anti-scepticism and the characterisation of the latter unsatisfactory for anyone interested in a more substantial definition. But first, what we were after was more to contrast absolutist with relativist views than to contrast scepticism with anti-scepticism. Second, the definition we gave of anti-scepticism – viz. “for all i and w , $\mathcal{R}(c_i)(w) \subsetneq W$ ” – was meant to be broad enough so as to allow for more specific characterisations of one’s favourite version of anti-scepticism. This can be done, for instance, by listing all the worlds one thinks should be considered epistemically relevant, that is, included in $\mathcal{R}(c_i)(w)$, or by providing a general criterion for telling the worlds that are epistemically relevant from those that aren’t (in terms of “the closest” or “close enough” possible worlds, for instance).

issue of whether the *truth* of knowledge sentences should be taken as absolute or relative.

Moreover, with contextual models one can account for the logical behaviour of agents of a given epistemological type (sceptical, anti-sceptical, contextualist, or subjectivist) reasoning about the knowledge of other agents regardless of their epistemological type; that is, one can ask such things as *If contextualism is assumed, then if agent 1 knows that agent 2 knows that φ , does agent 1 know that φ ?* One can also ask such things as *If a contextualist knows that a subjectivist knows that φ , does the contextualist know that φ ?*

However, despite its merits, the contextual models framework is unlikely to win unanimous support from philosophers of language. For consider satisfaction for epistemic formulas of the form $\mathbf{K}_j\varphi$ when epistemic operator \mathbf{K}_j is taken in its (1.2) and (2.2) interpretations. In these cases, the epistemic operator \mathbf{K}_j manipulates the context parameter against which the relevant epistemic formula is evaluated: the truth of $\mathbf{K}_j\varphi$ relative to a context c_i (and world w) depends on the truth of the embedded φ relative to *another context* c_j (and world w').

Now, Kaplan [18] famously conjectured that natural language had no such devices as context-shifting operators – which he considered to be “monsters”. So, if he is right, there can be no natural language counterpart for \mathbf{K}_j in cases (1.2) and (2.2). Schlenker [31] and others have recently challenged Kaplan’s conjecture, notably by arguing that natural language allows context-shifting to occur within propositional attitudes. Nevertheless, the existence of “monsters” remains controversial amongst philosophers of language, and the question of context-shifting within attitudes is far from having been settled for that specific kind of attitude reports formed by knowledge claims.

An advantage of the contextual epistemic logic that will be introduced in the present paper is that it is immune to the charge of monstrosity: although something resembling context-shifting takes place in that framework, its language is “monster-free” since its formulas are evaluated against a world parameter only and contexts are of a purely syntactic nature.

2.2.2 Hybrid logic

The semantic intuition lying behind context-relativized attributions of knowledge is that one has to restrict the evaluation to the set of the contextually relevant worlds to check whether in this restricted set, the agent knows such and such proposition. If one wants to go back to standard Kripke models and wishes to handle contexts as standard (let us say, S5) modalities $[c_j]$, it is thus required to consider the *intersection* of two accessibility relations: an agent i will be said to know φ (at world w) relative to a context c_j – which can be formalized by: $\mathcal{M}, w \vDash [[c_j]\mathbf{K}_i]\varphi$ – if and only if $\mathcal{M}, w' \vDash \varphi$ at every world w' such that $\mathcal{R}_{j,k}ww'$, where $\mathcal{R}_{j,k}$ is the intersection of the two accessibility relations.

Now it is known that the intersection of two relations cannot be defined within basic modal logics [12]. One has to go beyond basic modal logic to express sound axioms for context-relativized knowledge. Using the hybrid nominals v_x

and operators $@_x$, we get the natural axiom schema (\mathbf{P}_i being the dual of \mathbf{K}_i): $[c_j]\mathbf{K}_i\varphi \leftrightarrow ((\mathbf{P}_i v_x \wedge \langle c_k \rangle v_x) \rightarrow @_x[c_i]\varphi)$, where $\langle k, l \rangle \in \{i, j\} \times \{i, j\}$.

Starting from Blackburn's [5] dialogical version of hybrid logics one could give a natural account of epistemic logic with standard contextual modalities. This is left for another paper.

2.2.3 Model-shifting operators

The way followed in this paper has been paved by recent works about logics of knowledge and communication. The idea, going back to [25], is to add *model-shifting operators* to basic epistemic logics.

Van Benthem et al. [4] recently accounted for model-shifting operators connected to public announcement. The idea is that the semantic contribution of an announcement, let us say φ , is to eliminate all those epistemic alternatives falsifying φ , so that a new model \mathcal{M}_φ is obtained relative to which subsequent formulas are evaluated. In this approach, context modalities are clearly not usual modalities.

In the present paper, we propose to use this Public Announcement Logic (PAL) with the required modifications. Before going further into details, let us just add that a frame such as PAL can be combined with some dynamic account of the context operators $[c_i]$. Van Benthem et al.'s paper [4] provides such a dynamic logic, using standard PDL with an epistemic interpretation. In [20] the authors of the present paper accounted for contexts using a formalism based on DRT. Anyway, the way contexts are handled (the dynamic part of the formalism) is relatively independent from their role in epistemic attributions.

2.3 From PAL to CEL

2.3.1 PAL in a nutshell

The notion of a public announcement or communication is just that of “a statement made in a conference room in which all agents are present” according to Plaza [25], and that of “an epistemic event where all agents are told simultaneously and transparently that a certain formula holds right now” according to van Benthem *et al.* [4]. The latter intend to model this informal notion by means of a modal operator $[\varphi]$, thus allowing for formulas of the form $[\varphi]\psi$, read intuitively as “ ψ holds after the announcement of φ .”

The language \mathcal{L}_{PAL} of PAL, the logic of public announcement proposed by van Benthem *et al.*, results from augmenting the basic epistemic modal language \mathcal{L} with operators of the form $[\varphi]$ where φ is any formula of \mathcal{L}_{PAL} . The semantics for the resulting language \mathcal{L}_{PAL} is based on standard Kripke models for \mathcal{L} , and the definition of \vDash for atoms, negation, conjunction and knowledge operators is as usual with such models. The more “unusual” feature of the semantics of \mathcal{L}_{PAL} is with the clause of satisfaction for $[\varphi]$, viz.:

$$\mathcal{M}, w \vDash [\varphi]\psi \quad \text{iff} \quad \text{if } \mathcal{M}, w \vDash \varphi \text{ then } \mathcal{M}_\varphi, w \vDash \psi,$$

where $\mathcal{M}_{|\varphi}$ is an “updated model” consisting of a tuple $\langle W', \mathcal{K}'_1, \dots, \mathcal{K}'_m, \mathcal{V}' \rangle$, with:

- $W' = \{v \in W : \mathcal{M}, v \models \varphi\}$,
- $\mathcal{K}'_i = \mathcal{K}_i \cap (W' \times W')$ for all agents $i \in \{1, \dots, m\}$,
- $\mathcal{V}'(p) = \mathcal{V}(p) \cap W'$ for all atomic formulas p .

This appeal to updated models, obtained by restricting a given Kripke model to those worlds where a given formula φ holds, is meant to capture the insight that the epistemic effect of a public announcement of φ at a time t is that at time $t + 1$, all the agents involved will have deleted the possible worlds where they do not know that φ .

Our proposed formalism for reasoning about knowledge and context, although it is strongly inspired by that for reasoning about public announcements, does not appeal to anything like updated models and offers a slightly more complicated semantics for the corresponding language.

2.3.2 Different means for different goals

Let us first compare the objectives of the two formalisms. **PAL** is specifically designed to account for shifts of common knowledge following public announcements. As a successful announcement is made, its content φ becomes common knowledge and the model is restricted to all and only those possible worlds where φ is true.

The situation is different with **CEL**: here sentences are presupposed (and as a consequence, known by each agent) as long as they remain implicit; when a presupposition is made explicit – asserted, rejected, questioned. . . – it is removed from the “context”. Such a context does not determine the set of epistemic possibilities, but it determines a set of epistemically relevant possibilities.

For instance, in usual (non-sceptical) contexts it is assumed that b , “we are all brains in vats”, is false; so $\neg b$ belongs to the context. Now if a provocative Sceptic enters the scene and asks whether we are sure that we are not brains in vats, or, better, asserts that we actually are brains in vats, then the context changes: b becomes epistemically relevant. It does not mean that b is true at every accessible world / epistemic possibility, since such a “public announcement” is not necessarily approved by all the agents; but b is now a (counterfactual) possibility that needs to be rejected to grant knowledge.⁶

Hence the modifications of contexts according to **CEL** do not yield any modification of the accessibility relation: it expands or restricts the whole set of possible worlds. Whereas **PAL** is concerned with changes of (common) knowledge of agents, **CEL** aims at accounting for changes of knowledge attributions. The contrast between the two theories is summarized in the following table:

⁶ As a simplification, we consider only literals as presuppositions; it means that an assertion whose content is a complex proposition will modify the context by removing all the literals occurring in it.

PAL	CEL
Explicit Public announcement [φ]	Implicit (Proto-)Context [c_i]
Any (complex) formula	Conjunction of literals
What is <i>a posteriori</i> known: Set of epistemic possibilities	What is <i>a priori</i> assumed: Set of counterfactual possibilities ("epistemically relevant possibilities")
Restriction of the accessibility relation: Removing worlds	Restriction or expansion of the set of relevant possible worlds: Removing or adding worlds
Knowledge shifting	Knowledge attribution shifting

2.3.3 Consequences

PAL is useful to reach CEL for it provides a formalization of a logic with a model-shifting operator. However, a few changes are required to adapt PAL to a logic with contexts:

- Model-shifting operators [φ] of PAL are interpreted so that sizes of models are systematically decreasing; in CEL, model-shifting operators [c_i] must allow increasing as well as decreasing sizes of models.
- Whereas the announcements of PAL are obviously linguistic, the contexts of CEL need not be linguistic in nature; what is required is that they be depicted by some linguistic sentence.
- The outcome of a public announcement for the agents' knowledge is straightforward, as is the corresponding axiom schema of PAL. By contrast, several kinds of subtle interactions between context operators and epistemic modalities can be grasped in CEL.

As will be shown in the next section, these slight divergences lead to an important dissimilarity at the semantic level: CEL semantics is much more inelegant than PAL semantics. This will be a good reason enough to go over to the dialogical framework.

3 Contextual Epistemic Logic

In this section we first introduce the syntax and semantics of CEL, then a sound and complete proof system. As was explained above, CEL is strongly inspired by the model of PAL.

3.1 CEL: Syntax and Kripke semantics

3.1.1 Standard epistemic logic

The standard formal approach to reasoning about *knowledge* starts with the choice of a basic epistemic modal language. This language $\mathcal{L}_{\mathbf{K}}^m(\mathcal{A}t)$ consists of

the set of formulas over a finite or infinite set $\mathcal{At} = \{p_0, p_1, \dots\}$ of atomic formulas and a set $J = \{1, \dots, m\}$ of m agents, given by the following form (F standing for a formula of $\mathcal{L}_{\mathbf{K}}^m(\mathcal{At})$):

$$F ::= \mathcal{At} \mid \neg F \mid (F \wedge F) \mid \mathbf{K}_j(F).$$

This language is then interpreted in Kripke models consisting of tuples $\mathcal{M} = \langle W, \mathcal{R}_1, \dots, \mathcal{R}_m, \mathcal{V} \rangle$, where: $W \neq \emptyset$ is a set of “worlds”; $\mathcal{R}_j \subseteq W \times W$ is a relation of “epistemic accessibility”, for all $j \in J$; $\mathcal{V} : \mathcal{At} \rightarrow \wp(W)$ is a “valuation” mapping each atomic formula onto a set of worlds.

3.1.2 CEL Syntax

In order to add contexts to basic epistemic languages, one can choose between two equivalent notations (see [2]). For convenience – especially for the dialogical approach – we will not use the modal contextual operator prefixing formulas $[c_i]\varphi$ but a syntactic relativization of formulas: $(\varphi)^{c_i}$. Moreover, each context c_i is characterized by a conjunction of literals (i.e. atoms and negation of atoms); the “context formula” characterizing a context c_i will be referred to by the same symbol, c_i .⁷ So “context formulas” c can be defined as follows:

$$c ::= \mathcal{At} \mid \neg \mathcal{At} \mid \top \mid \perp \mid c \wedge c.$$

We thus recapitulate the syntax of epistemic languages with context $\mathcal{L}_{\mathbf{K}\mathbf{C}}^m(\mathcal{At})$ (G standing for a formula of $\mathcal{L}_{\mathbf{K}\mathbf{C}}^m(\mathcal{At})$), built upon a set of contexts $\mathbf{C} = \{c_i\}_{i \in I}$ ($J \subseteq I$, J being the set of agents):

$$G ::= \mathcal{At} \mid \neg G \mid (G \wedge G) \mid \mathbf{K}_j(G) \mid (G)^{c_i}.$$

Formulas containing at least one operator \mathbf{K}_j are called *epistemic formulas*; formulas containing at least one subformula of the kind $(\varphi)^{c_i}$ are said *context-relativized*, and formulas being not context-relativized are said *absolute*. As is immediately seen from the definition, **EL** is a syntactic fragment of **CEL**.

3.1.3 CEL Kripke semantics

The contextual epistemic language $\mathcal{L}_{\mathbf{K}\mathbf{C}}^m(\mathcal{At})$ is interpreted relatively to the same Kripke models as $\mathcal{L}_{\mathbf{K}}^m(\mathcal{At})$. The satisfaction of atoms, absolute negations and conjunctions is defined as usual. For epistemic operators and contextual relativization, the semantics is a bit more complicated:

⁷Contexts need not be *identified* with context formulas; they can be partial models or incomplete possible worlds for instance.

$\mathcal{M}, w \vDash p$	iff	$w \in \mathcal{V}(p)$	(p being atomic)
$\mathcal{M}, w \vDash \neg\chi$	iff	$\mathcal{M}, w \not\vDash \chi$	
$\mathcal{M}, w \vDash \chi \wedge \xi$	iff	$\mathcal{M}, w \vDash \chi$ and $\mathcal{M}, w \vDash \xi$	
$\mathcal{M}, w \vDash \mathbf{K}_j\varphi$	iff	for every $w' \in W$ such that R_jww' : $\mathcal{M}, w' \vDash \varphi$	
$\mathcal{M}, w \vDash (p)^{c_i}$	iff	$\mathcal{M}, w \not\vDash c_i$ or $\mathcal{M}, w \vDash p$	(p being atomic)
$\mathcal{M}, w \vDash ((\varphi)^{c_k})^{c_i}$	iff	$\mathcal{M}, w \not\vDash c_i$ or $\mathcal{M}, w \vDash (\varphi)^{c_k}$	
$\mathcal{M}, w \vDash (\neg\varphi)^{c_i}$	iff	$\mathcal{M}, w \not\vDash c_i$ or $\mathcal{M}, w \not\vDash (\varphi)^{c_i}$	
$\mathcal{M}, w \vDash (\varphi \wedge \psi)^{c_i}$	iff	$\mathcal{M}, w \vDash (\varphi)^{c_i}$ and $\mathcal{M}, w \vDash (\psi)^{c_i}$	
$\mathcal{M}, w \vDash (\mathbf{K}_j\varphi)^{c_i}$	iff	$\mathcal{M}, w \not\vDash c_x$ or $\mathcal{M}, w \vDash \mathbf{K}_j(\varphi)^{c_y}$	where $\langle x, y \rangle$ is chosen in $\{i, j\} \times \{i, j\}$

There are four different clauses hidden in the last one, depending on the values of x and y . Each choice regiments a specific position about the interaction between knowledge and context. We will refer to these four possibilities as (1.1), (1.2), (2.1), and (2.2), sometimes adding an explicit exponent to the knowledge operator – $\mathbf{K}_j^{1.1}$, $\mathbf{K}_j^{1.2}$, $\mathbf{K}_j^{2.1}$, and $\mathbf{K}_j^{2.2}$ respectively – with the following understanding:

$\mathcal{M}, w \vDash (\mathbf{K}_j^{1.1}\varphi)^{c_i}$	iff	$\mathcal{M}, w \not\vDash c_i$ or $\mathcal{M}, w \vDash \mathbf{K}_j^{1.1}(\varphi)^{c_i}$
$\mathcal{M}, w \vDash (\mathbf{K}_j^{1.2}\varphi)^{c_i}$	iff	$\mathcal{M}, w \not\vDash c_i$ or $\mathcal{M}, w \vDash \mathbf{K}_j^{1.2}(\varphi)^{c_j}$
$\mathcal{M}, w \vDash (\mathbf{K}_j^{2.1}\varphi)^{c_j}$	iff	$\mathcal{M}, w \not\vDash c_j$ or $\mathcal{M}, w \vDash \mathbf{K}_j^{2.1}(\varphi)^{c_i}$
$\mathcal{M}, w \vDash (\mathbf{K}_j^{2.2}\varphi)^{c_i}$	iff	$\mathcal{M}, w \not\vDash c_j$ or $\mathcal{M}, w \vDash \mathbf{K}_j^{2.2}(\varphi)^{c_j}$

Remarks:

- In the notation $\mathbf{K}_j^{u,v}$, the first superscript u corresponds to the contextual condition, and the second one v to the context according to which the evaluation is to be continued; the possible values of u and v are 1 for the current context, and 2 for the agent.
- It is worth noticing that no restriction of the model \mathcal{M} is required for the evaluation of context-relativized formulas.

According to our definition the context relativization $(\varphi)^{c_i}$ of a non-epistemic formula φ is nothing but a notational variant for the conditional $c_i \rightarrow \varphi$. Since such a relativization has no further impact on epistemic $\mathbf{K}^{1.1}$ formulas, the notion of “context” carried by our formalism will appear to be non-committing for supporters of absolutist conceptions of knowledge. However, this innocent account of context will turn out to be sufficient to model contextualist and subjectivist epistemologies. With **CEL** context friends and context enemies can thus be put together onto the same neutral field.

3.2 CEL: Proof system

3.2.1 Proof systems

There are several proof systems depending on the interaction between epistemic operators and context. The differences are given through axioms of Contextual knowledge. Each proof systems for **CEL** is that for multi-modal **S5** epistemic logic **EL** plus the following schemas:

Atoms	$\vdash (p)^{c_i} \leftrightarrow (c_i \rightarrow p)$
Contextual negation	$\vdash (\neg\varphi)^{c_i} \leftrightarrow (c_i \rightarrow \neg(\varphi)^{c_i})$
Contextual conjunction	$\vdash (\varphi \wedge \psi)^{c_i} \leftrightarrow ((\varphi)^{c_i} \wedge (\psi)^{c_i})$
Context iteration	$\vdash ((\varphi)^{c_k})^{c_i} \leftrightarrow (c_i \rightarrow (\varphi)^{c_k})$
Contextual Knowledge $(\langle x, y \rangle \in \{i, j\} \times \{i, j\})$	$\vdash (\mathbf{K}_j\varphi)^{c_i} \leftrightarrow (c_x \rightarrow \mathbf{K}_j(\varphi)^{c_y})$

as well as the following rule of inference:

Context generalization From $\vdash \varphi$, infer $\vdash (\varphi)^{c_i}$.

Like for the semantics of **CEL**, the schema for Contextual Knowledge can be made more explicit through the following versions:

1.1-Contextual Knowledge	$\vdash (\mathbf{K}_i^{1.1}\varphi)^{c_i} \leftrightarrow (c_i \rightarrow \mathbf{K}_i^{1.1}(\varphi)^{c_i})$
1.2-Contextual Knowledge	$\vdash (\mathbf{K}_j^{1.2}\varphi)^{c_i} \leftrightarrow (c_i \rightarrow \mathbf{K}_j^{1.2}(\varphi)^{c_j})$
2.1-Contextual Knowledge	$\vdash (\mathbf{K}_j^{2.1}\varphi)^{c_i} \leftrightarrow (c_j \rightarrow \mathbf{K}_j^{2.1}(\varphi)^{c_i})$
2.2-Contextual Knowledge	$\vdash (\mathbf{K}_j^{2.2}\varphi)^{c_i} \leftrightarrow (c_j \rightarrow \mathbf{K}_j^{2.2}(\varphi)^{c_j})$

3.2.2 Soundness and completeness

As **CEL** axioms are reduction axioms, each **CEL**-formula φ can be translated into a standard epistemic formula φ' such that for every model \mathcal{M} and world w : $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, w \models \varphi'$. Completeness of **CEL** axiomatics relative to the set of formulas valid in the class \mathcal{M}^{rst} of reflexive, symmetric and transitive models is thus inherited from that of usual epistemic logic.

Theorem 1 *For formulas in $\mathcal{L}_{\mathbf{KC}}^m(\mathcal{At})$, **CEL** is a sound and complete axiomatization w.r.t. \mathcal{M}^{rst} .*

Proof. From any **CEL** formula $\varphi \in \mathcal{L}_{\mathbf{KC}}^m(\mathcal{At})$, one can build a tuple $\langle \varphi_0, \dots, \varphi_n \rangle$ of **CEL** formulas and reach a formula $\varphi' \in \mathcal{L}_{\mathbf{K}}^m(\mathcal{At})$ such that: $\varphi_0 = \varphi$, $\varphi_n = \varphi'$, and every formula $(\varphi_m \leftrightarrow \varphi_{m+1})$ ($0 \leq m \leq n-1$) is an instantiation of an axiom schema. As a consequence:

$$\vdash_{\mathbf{CEL}} \varphi \text{ iff } \vdash_{\mathbf{CEL}} \varphi_1 \text{ iff } \dots \text{ iff } \vdash_{\mathbf{CEL}} \varphi_{n-1} \text{ iff } \vdash_{\mathbf{CEL}} \varphi'$$

Since $\varphi' \in \mathcal{L}_{\mathbf{K}}^m(\mathcal{At})$, $\vdash_{\mathbf{CEL}} \varphi'$ is equivalent to $\vdash_{\mathbf{EL}} \varphi'$. Now, **EL** is sound and complete, so:

$$\vdash_{\mathbf{CEL}} \varphi' \text{ iff } \mathcal{M}^{rst} \models \varphi' \quad (\text{i.e. iff } \varphi' \text{ is valid w.r.t. } \mathcal{M}^{rst})$$

Hence in order to prove soundness and completeness of **CEL**, it suffices to prove that for each reduction axiom schema: $\vdash_{\mathbf{CEL}} \psi \leftrightarrow \psi'$, one gets: $\mathcal{M}^{rst} \models \psi$ iff $\mathcal{M}^{rst} \models \psi'$. This obtains immediately by virtue of the definitions. \blacksquare

4 Dialogical CEL

In this section, we present a dialogical version of **CEL**. We first introduce dialogical logic for (multi-)S5, then we extend it to context-relativized formulas.

4.1 Dialogical (multi-)modal logic

In a dialogical game, two players argue about a thesis (a formula): The proponent **P** defends it against the attacks of the opponent **O**. For any set of game rules **Dial** Σ associated with some logical theory Σ , we will use the notation **Dial** $\Sigma \models \varphi$ to say that there is a winning strategy for the proponent in the dialogical game $G_\Sigma(\varphi)$, i.e. if playing according to the rules of **Dial** Σ , she can defend the formula φ against any attack from the opponent – owning a winning strategy for a game enables a player to win any play of the game. The game rules are defined such that a formula φ is valid or logically true in Σ ($\models_\Sigma \varphi$) if and only if **Dial** $\Sigma \models \varphi$.

The rules belong to two categories: particle rules and structural rules. In the remainder of the section, we just give and briefly explain the rules for games reaching multi-S5 valid formulas.⁸

4.1.1 Worlds numbering

The thesis of the dialogue is uttered at a given world w , as well as the subsequent formulas. This world relativization is explicit in dialogue games: we will use labelled formulas of the kind “ $w : \varphi$ ”, like in explicit tableau systems. For that purpose, we need a system of world numbering that reflects syntactically the accessibility relation. We will use the following principles, inspired from Fitting numbers for mono-modal logic:

- The initial world is numbered 1. The n immediate successors of w according to the agent j are numbered $wj1, wj2, \dots, wjn$.
- An immediate successor wju of a world w is said to be of rank +1 relative to w , and w is said to be of rank –1 relative to its immediate successors. A successor $wjuv$ of a world w is said to be of rank +2 relative to w , etc.

So for instance, a play on a thesis “ $1 : \mathbf{K}_i \mathbf{K}_j \varphi$ ” can reach the following labelled formula: “ $1i1j2 : \varphi$.”

4.1.2 Particle Rules

The meaning of each logical constant is given through a particle rule which determines how to attack and defend a formula whose main connective is the constant in question. The set, **EL^mPartRules**, of particle rules for disjunction, conjunction, subjunction, negation, and epistemic operators is recapitulated in the following table:

⁸ Dialogical modal logics were first introduced in Rahman & Rückert’s paper [27]. Readers interested by some more detailed account of both non-modal and modal logics should refer to that paper. For a presentation of game rules close to the present one, see [29].

	Attack	Defence
$w : \varphi \vee \psi$	$w : ?$	$w : \varphi$, or $w : \psi$ (The defender chooses)
$w : \varphi \wedge \psi$	$w : ?_L$, or $w : ?_R$ (The attacker chooses)	$w : \varphi$, or $w : \psi$ (respectively)
$w : \varphi \rightarrow \psi$	$w : \varphi$	$w : \psi$
$w : \neg\varphi$	$w : \varphi$	\otimes (No possible defence)
$w : \mathbf{K}_i\varphi$	$w : ?_{\mathbf{K}_i/wi w'}$ (the attacker chooses an available world $wi w'$)	$wi w' : \varphi$
$w : \mathbf{P}_i\varphi$	$w : ?_{\mathbf{P}_i}$	$wi w' : \varphi$ (the available world $wi w'$ being chosen by the defender)

The idea for disjunction is that the proposition $\varphi \vee \psi$, when asserted (at world w) by a player, is challenged by the question “*Which one?*”; the defender has then to choose one of the disjuncts and to defend it against any new attack. The rule is the same for the conjunction $\varphi \wedge \psi$, except that the choice is now made by the attacker: “*Give me the left conjunct ($?_L$)*” or “*Give me the right one ($?_R$)*”, and the defender has to assume the conjunct chosen by his or her challenger. For the conditional $\varphi \rightarrow \psi$, the attacker assumes the antecedent φ and the defender continues with ψ . Negated formulas are attacked by the cancellation of negation, and cannot be defended; the defender in this case can thus only counterattack (if she can).

The particle rules for each epistemic operator \mathbf{K}_i and its dual \mathbf{P}_i ($i \in \{1, \dots, m\}$) enable the players to change the world. They are defined in a way analogous to conjunction and disjunction respectively, regarding the player (challenger or defender) expected to make the relevant choice.

4.1.3 Structural Rules

In addition to the particle rules connected to each logical constant, one also needs structural rules to be able to play in such and such a way at the level of the whole game. The first five of them yield games for classical propositional logic, and the last two rules regiment the modal part of epistemic logic.

- (PL-0) **Starting Rule:** The initial formula (the *thesis* of the dialogical game) is asserted by **P** at world 1. Moves are numbered and alternatively uttered by **P** and **O**. Each move after the initial utterance is either an attack or a defence.
 - (PL-1) **Winning Rule:** Player **X** wins iff it is **Y**'s turn to play and **Y** cannot perform any move.
 - (PL-2) **No Delaying Tactics Rule:** Both players can only perform moves that change the situation.
 - (PL-3) **Formal Rule for Atoms:** *At a given world P cannot introduce any new atomic formula; new atomic formulas must be stated by O first. Atomic formulas can never be attacked.*
 - (PL-4c) **Classical Rule:** In any move, each player may attack a complex formula uttered by the other player or defend him/herself against *any attack* (including those that have already been defended).^a
- A world w is said to be *introduced* by a move in a play when w is first mentioned either through an asserted labeled formula ($w : \varphi$), or through a non-assertive attack ($?_{\kappa_i/w}$).
- (ML-frw) **Formal Rule for Worlds:** *P cannot introduce a new world; new worlds must be introduced by O first.*
 - (ML-S5) **S5 Rule:** **P** can choose any (given) world.^b

^aThis rule can be replaced by the following one to get games for intuitionistic logic:

Intuitionistic Rule: In any move, each player may attack a complex formula uttered by the other player or defend him/herself against *the last attack that has not yet been defended*.

^bOther structural rules could define other usual modal systems (K, D, T, S4, etc.). See [27].

Now we can build the set of rules for multi-S5 epistemic logic (**EL**):

$$\text{DialEL} := \mathbf{EL}^m \text{PartRules} \cup \{\text{PL-0, PL-1, PL-2, PL-3, PL-4c, ML-frc, ML-S5}\}$$

It is assumed that this dialogical system is sound and complete, i.e.:

$$\text{DialEL} \models \varphi \text{ iff } \models_{\text{EL}} \varphi.$$

This is shown using strategic tableaux analogous which are similar to usual semantic tableaux, after a reinterpretation of the players' roles.⁹

4.1.4 Examples

Example 2 *Let us consider a substitution instance of the Positive Introspection Property (also known as Axiom 4): $\mathbf{K}_i\phi \rightarrow \mathbf{K}_i\mathbf{K}_i\phi$. As our dialogical rules*

⁹The proofs of soundness and completeness of Dialogical **EL** are non-trivial. They are not explicitly given in [27], even though a halfway point is reached there.

correspond to **S5**, the proponent is expected to have a winning strategy in the corresponding game.

<i>O</i>		<i>P</i>	
		1 : $\mathbf{K}_i a \rightarrow \mathbf{K}_i \mathbf{K}_i a$	(0)
(1)	1 : $\mathbf{K}_i a$	0	1 : $\mathbf{K}_i \mathbf{K}_i a$
(3)	1 : $?_{\mathbf{K}_i/1i1}$	2	1i1 : $\mathbf{K}_i a$
(5)	1i1 : $?_{\mathbf{K}_i/1i1i}$	4	1i1i1 : a
(7)	1i1i1 : a	1	1 : $?_{\mathbf{K}_i/1i1i}$

P wins the play

The numerals within brackets in the external column indicate the moves and the corresponding arguments (here from (0) to (8)); when a move is an attack, the internal array indicates the argument which is under attack; the corresponding defence is written on the same line, even if the move is made later in the play.

Let us comment this particular play in detail. It starts at move (0) with the utterance of the thesis by *P* at world 1 (PL-0). The formula is challenged by *O* at move (1), using the particle rule for implication; at move (2) *P* immediately defends his initial argument. *O* then attacks the epistemic formula at move (3), and using the correlated particle rule as well as (ML-frw), he introduces a new world, 1i1: in dialogical games the opponent is considered as using the best available strategy; he thus jumps from one world to another as much as possible, to prevent the proponent to use his concessions (atomic utterances) at a given world. At move (4) *P* defends her formula using (ML-S5). Then in (5), *O* attacks the epistemic operator introducing once more a new world, 1i1i1. Now the proponent cannot immediately defend her utterance, because it would lead her to utter an atomic formula (*a*) which has not been previously introduced by *O* at 1i1i1. At move (6), *P* thus counterattacks (1) using the new world introduced by *O*, asking him to utter *a* at this world (ML-S5); in (7) *O* defends himself uttering *a* at 1i1i1: the atomic formula is now available for *P*, who can win the play at move (8). As the opponent could not play better – actually, he could not play differently than he did in this play –, this play shows that there is a winning strategy for *P* in the game. ■

Example 3 Now we consider a formula with two epistemic operators: $\mathbf{K}_i \mathbf{K}_j a \rightarrow (\mathbf{K}_i a \wedge \mathbf{K}_j a)$. Here it is not enough to consider one play: after move (2), the opponent can choose either the left or the right conjunct. Depending on this choice, the remainder of the play will not be the same. So after checking that the proponent has a winning strategy in plays where *O* chooses the left conjunct, one cannot conclude that she has a winning strategy at all: it must be verified that she can also systematically win against *O* when he chooses the right conjunct.

<i>O</i>		<i>P</i>	
		1 :	$\mathbf{K}_i \mathbf{K}_j a \rightarrow (\mathbf{K}_i a \wedge \mathbf{K}_j a)$ (0)
(1)	1 :	0	$\mathbf{K}_i \mathbf{K}_j a$ (2)
(3)	1 :	2	$?_L a$ (4)
(5)	1 :	4	$?_{\mathbf{K}_i/1i1} a$ (10)
(7)	1i1 :	1	$\mathbf{K}_j a$ (6)
(9)	1i1 :	7	$?_{\mathbf{K}_i/1i1} a$ (8)

P wins the play

<i>O</i>		<i>P</i>	
		1 :	$\mathbf{K}_i \mathbf{K}_j a \rightarrow (\mathbf{K}_i a \wedge \mathbf{K}_j a)$ (0)
(1)	1 :	0	$\mathbf{K}_i \mathbf{K}_j a$ (2)
(3*)	1 :	2	$?_R a$ (4*)
(5*)	1 :	4	$?_{\mathbf{K}_i/1j1} a$ (10*)
(7*)	1 :	1	$\mathbf{K}_j a$ (6*)
(9*)	1j1 :	7	$?_{\mathbf{K}_i/1j1} a$ (8*)

P wins the play

As expected, there is a winning strategy for the proponent in each case. The formula is thus proved EL valid. ■

4.2 Adding contextual relativization

4.2.1 Particle rules

The table below gives the particle rules for context-relativized formulas: the rules follow the reduction axioms in a natural way. This new set of particle rules will be referred to as **CEL^mPartRules**.

	Attack	Defence
$w : (p)^{c_i}$ (<i>p</i> being an atom)	$w : c_i$	$w : p$
$w : ((\varphi)^{c_k})^{c_i}$	$w : c_i$	$w : (\varphi)^{c_k}$
$w : (\varphi \vee \psi)^{c_i}$	$w : c_i$	$w : (\varphi)^{c_i} \vee (\psi)^{c_i}$
$w : (\varphi \wedge \psi)^{c_i}$	$w : ?_L$, or $w : ?_R$ (The attacker chooses)	$w : (\varphi)^{c_i}$, or $w : (\psi)^{c_i}$ (respectively)
$w : (\varphi \rightarrow \psi)^{c_i}$	$w : c_i$	$w : (\varphi)^{c_i} \rightarrow (\psi)^{c_i}$
$w : (\neg\varphi)^{c_i}$	$w : c_i$	$w : \neg(\varphi)^{c_i}$
$w : (\mathbf{K}_j \varphi)^{c_i}$	$w : c_x$	$w : \mathbf{K}_j(\varphi)^{c_y}$

Here again, we can give a more explicit version of the last particle rule depending on the kind of the epistemic operator:

	Attack	Defence
$w : (\mathbf{K}_j^{1,1}\varphi)^{c_i}$	$w : c_i$	$w : \mathbf{K}_j^{1,1}(\varphi)^{c_i}$
$w : (\mathbf{K}_j^{1,2}\varphi)^{c_i}$	$w : c_i$	$w : \mathbf{K}_j^{1,2}(\varphi)^{c_j}$
$w : (\mathbf{K}_j^{2,1}\varphi)^{c_i}$	$w : c_j$	$w : \mathbf{K}_j^{2,1}(\varphi)^{c_i}$
$w : (\mathbf{K}_j^{2,2}\varphi)^{c_i}$	$w : c_j$	$w : \mathbf{K}_j^{2,2}(\varphi)^{c_j}$

4.2.2 Structural Rule

There is only one structural rule to add to the listing of ML: the rule that states which player can introduce a context c_i , by asserting its characteristic formula c_i . The idea is that a context should not be assumed in any formal proof.

- (ML-*frc*) **Formal Rule for contexts:** **P** cannot introduce a new context c in a given world w by playing $w : c$; new contexts must be introduced by **O** first.

The intuition behind such a rule is that a context can be any formula, including atomic ones; so the proponent should not have powers regarding contexts she does not already have for atoms.

Now we have the set of rules for CEL:

$$\text{DialCEL} := \text{DialEL} \cup \text{CEL}^{\text{m}}\text{PartRules} \cup \{\text{ML-*frc*\}\}$$

Soundness and completeness will be handled after the following examples.

4.2.3 Examples

In the following two tables, we consider the validity of a contextually modified version of positive introspection, where the consequent is evaluated relative to another context than the antecedent. As easily appears through the games, the upshot depends on the chosen position: the formula is valid according to (2.2), but not valid according to (1.2).

Example 4 Using explicit exponents for operators, the formula to be played is the following one: $(\mathbf{K}_i^{1,2}a)^{c_i} \rightarrow (\mathbf{K}_i^{1,2}\mathbf{K}_i^{1,2}a)^{c_j}$.

1.2	O		P
			1 : $(\mathbf{K}_i a)^{c_i} \rightarrow (\mathbf{K}_i \mathbf{K}_i a)^{c_j}$ (0)
(1)	1 : $(\mathbf{K}_i a)^{c_i}$	0	1 : $(\mathbf{K}_i \mathbf{K}_i a)^{c_j}$ (2)
(3)	1 : c_j	2	1 : $\mathbf{K}_i (\mathbf{K}_i a)^{c_i}$ (4)
(5)	1 : $?_{\mathbf{K}_i/1i1}$	4	1i1 : $(\mathbf{K}_i a)^{c_i}$ (6)
(7)	1i1 : c_i	6	1i1 : $\mathbf{K}_i (a)^{c_i}$ (8)
(9)	1i1 : $?_{\mathbf{K}_i/1i1i1}$	8	1i1i1 : $(a)^{c_i}$ (10)
(11)	1i1i1 : c_i	10	

O wins the play

After move (11), P cannot answer a for it has not been yet introduced by O at world $1i1i1$. The only possible solution for P would be to attack (1) to force O to utter $\mathbf{K}_i a$ at world 1, then to force him to utter a at $1i1i1$. But she cannot, since the opponent never introduced c_i at world 1. So she loses the play. ■

Example 5 Let us now consider the (2.2) version of the same formula:
 $(\mathbf{K}_i^{2.2} a)^{c_i} \rightarrow (\mathbf{K}_i^{2.2} \mathbf{K}_i^{2.2} a)^{c_i}$.

2.2	O		P			
			1 :	$(\mathbf{K}_i a)^{c_i} \rightarrow (\mathbf{K}_i \mathbf{K}_i a)^{c_i}$ (0)		
(1)	1 :	$(\mathbf{K}_i a)^{c_i}$	0	1 :	$(\mathbf{K}_i \mathbf{K}_i a)^{c_i}$ (2)	
(3)	1 :	c_i	2	1 :	$\mathbf{K}_i (\mathbf{K}_i a)^{c_i}$ (4)	
(5)	1 :	$?_{\mathbf{K}_i/1i1}$	4	$1i1 :$	$(\mathbf{K}_i a)^{c_i}$ (6)	
(7)	$1i1 :$	c_i	6	$1i1 :$	$\mathbf{K}_i (a)^{c_i}$ (8)	
(9)	$1i1 :$	$?_{\mathbf{K}_i/1i1i1}$	8	$1i1i1 :$	$(a)^{c_i}$ (10)	
(11)	$1i1i1 :$	c_i	10	$1i1i1 :$	a (18)	
(13)	1 :	$\mathbf{K}_i (a)^{c_i}$		1	1 :	c_i (12)
(15)	$1i1i1 :$	$(a)^{c_i}$		13	1 :	$?_{\mathbf{K}_i/1i1i1}$ (14)
(17)	$1i1i1 :$	a		15	$1i1i1 :$	c_i (16)

P wins the play

Here the proponent has a winning strategy, thanks to the utterance of c_i by O at world 1 in the third move. ■

4.2.4 Soundness and completeness

Theorem 6 Assuming that DialEL is sound and complete w.r.t. \mathcal{M}^{rst} , DialCEL defines a sound and complete dialogics w.r.t. \mathcal{M}^{rst} , i.e. for every CEL formula $\varphi \in \mathcal{L}_{\mathbf{KC}}^m(\mathcal{A}t)$, the following equivalence holds: DialCEL $\models \varphi$ iff $\mathcal{M}^{rst} \models \varphi$.

The proof is analogous to that of Theorem 1: we use the reduction axioms to translate CEL formulas into EL formulas, and assuming that dialogical logic DialEL is sound and complete for EL, we concentrate on the axiom schemas of CEL.

Proof. From any CEL formula $\varphi \in \mathcal{L}_{\mathbf{KC}}^m(\mathcal{A}t)$, one can build a tuple $\langle \varphi_0, \dots, \varphi_n \rangle$ of CEL formulas and reach a formula $\varphi' \in \mathcal{L}_{\mathbf{K}}^m(\mathcal{A}t)$ such that: $\varphi_0 = \varphi$, $\varphi_n = \varphi'$, and every formula $(\varphi_m \leftrightarrow \varphi_{m+1})$ ($0 \leq m \leq n-1$) is an instantiation of an axiom schema. As a consequence:

$$\vdash_{\text{CEL}} \varphi \text{ iff } \vdash_{\text{CEL}} \varphi_1 \text{ iff } \dots \text{ iff } \vdash_{\text{CEL}} \varphi_{n-1} \text{ iff } \vdash_{\text{CEL}} \varphi'$$

According to Theorem 1, CEL is complete so:

$$\mathcal{M}^{rst} \models \varphi \text{ iff } \mathcal{M}^{rst} \models \varphi_1 \text{ iff } \dots \text{ iff } \mathcal{M}^{rst} \models \varphi_{n-1} \text{ iff } \mathcal{M}^{rst} \models \varphi'$$

Since $\varphi' \in \mathcal{L}_{\mathbf{K}}^m(\mathcal{A}t)$, as DialEL is sound and complete, we have:

$\text{DialCEL} \models \varphi'$ iff $\text{DialEL} \models \varphi'$ iff $\mathcal{M}^{rst} \vDash \varphi'$.

Hence in order to prove soundness and completeness of **DialCEL**, it suffices to prove that for each reduction axiom schema: $\vdash_{\text{CEL}} \psi \leftrightarrow \psi'$, the following holds: $\text{DialCEL} \models \psi \rightarrow \psi'$ and $\text{DialCEL} \models \psi' \rightarrow \psi$.

Let us consider the axiom for Negation. We thus have two (kinds of) games.

O				P			
				1:	$(\neg\varphi)^{c_i} \rightarrow (c_i \rightarrow \neg(\varphi)^{c_i})$	(0)	
(1)	1:	$(\neg\varphi)^{c_i}$	0	1:	$c_i \rightarrow \neg(\varphi)^{c_i}$	(2)	
(3)	1:	c_i	2	1:	$\neg(\varphi)^{c_i}$	(4)	
(5)	1:	$(\varphi)^{c_i}$	4		\otimes		
(7)	1:	φ		5	1:	c_i	(6)
(9)	1:	$\neg\varphi$		1	1:	c_i	(8)
		\otimes		9	1:	φ	(10)

O				P			
				1:	$(c_i \rightarrow \neg(\varphi)^{c_i}) \rightarrow (\neg\varphi)^{c_i}$	(0)	
(1)	1:	$c_i \rightarrow \neg(\varphi)^{c_i}$	0	1:	$(\neg\varphi)^{c_i}$	(2)	
(3)	1:	c_i	2	1:	$\neg\varphi$	(4)	
(5)	1:	φ	4		\otimes		
(7)	1:	$\neg(\varphi)^{c_i}$		1	1:	c_i	(6)
		\otimes		7	1:	$(\varphi)^{c_i}$	(8)
(9)		c_i	8	1:	φ	(10)	

In both plays, the opponent could attack φ after move (10), but as he has already uttered the same formula before, and as the proponent can attack the same argument several times, any strategy deployed by **O** will be turned back as a winning strategy by **P**.

The other implications obtain similarly from the definitions. ■

5 Epistemological Applications

5.1 Epistemological positions formalized

Dialogical CEL provides us with powerful tools for gaining insights into the informal epistemological debate over the context-relativity of knowledge claims presented in section 2. To illustrate this, let us first show how the four epistemological positions alluded to earlier – scepticism, antiscepticism, contextualism and subjectivism – can be captured within our contextual logico-epistemic framework.

In section 2 the four epistemological positions were introduced in terms of the “ruling out” of “epistemically relevant counter-possibilities”, which it is quite

natural to understand in terms of S5 epistemic accessibility relations. For if we take it that $\mathcal{R}_i w w'$ iff agent i cannot tell w from w' from all the information available to him at w , we can read “ i can rule out w' on the basis of the information he has in w ” as “ w' is not epistemically accessible for i from w ”. Moreover, the appropriate epistemic accessibility relations must be equivalence relations, for it is quite natural to think that a possible world is ruled out by a subject as soon as it is not *exactly the same as* the actual world with respect to the totality of the subject’s evidence or information, that is, if it differs, were it only in a minimal way, from the actual world in that respect;¹⁰ and *being exactly the same as* is an equivalence relation.

Now, the four epistemological positions are positions for or against the relativity of knowledge claims to a given type of context – relativity to the “knowledge ascriber’s” or “attributor’s” context for contextualism, to the “knower’s” or “knowing subject’s” context for subjectivism, and no relativity to any context for scepticism and anti-scepticism. In order to capture these differences within our contextual logico-epistemic framework, let us opt for the following conventions:

- When dealing with a contextual operator $(\cdot)^{c_i}$, we shall take it that the context c_i stands for the set of formulas that are being presupposed or taken for granted by the agent or group of agents i . So we shall keep in mind that a context c_i may be understood as the result or/and background of a conversation between several agents – what they all take for granted for the purpose of their linguistic interactions –, if i stands for a group of such agents, as well as it may be understood as the result or/and background of an agent’s “conversation” with himself – e.g. what he takes for granted for the purpose of his current reflections –, if i is a single agent;
- In general, $(\varphi)^{c_i}$ shall be read as “it follows from context c_i that φ ”. In particular, if $(\mathbf{K}_j \varphi)^{c_i}$ can be read as “it follows from context c_i that j knows that φ ”, it may better be read as “in context c_i , j counts as knowing that φ ”, that is “given what is taken for granted by i . . .”. Moreover, when we have a formula of the form $(\mathbf{K}_j \varphi)^{c_i}$, we shall take it that the agent (or group) i is to be called the “attributor” and c_i “attributor i ’s context”, and the agent (or group) j is to be called the “subject”. On the other hand, if we need to reflect on a c_x , depending on whether $x = i$ or $x = j$, we shall speak of c_x as of the “attributor’s context” or as of the “subject’s context”.

With these conventions in hand, we can now establish correspondences between the explicitly exponented knowledge operators we distinguished and the different epistemological positions.

¹⁰Two worlds may differ in numerous respects, and yet be exactly the same with respect to the evidence (conceived of in internalistic terms) at an agent’s disposal. For instance, an Evil Genius world would be very different from what we take our world to be, but as the sceptical argument goes, in it we would have exactly the same evidence as we have in our world.

5.1.1 Scepticism and Anti-scepticism

The two *absolutist* views – scepticism and anti-scepticism – can both be associated with the knowledge operator $\mathbf{K}_j^{1.1}$, whose behaviour with respect to contextual relativization was fully characterized by the following reduction axiom in CEL:

$$\vdash_{\text{CEL}} (\mathbf{K}_j^{1.1}\varphi)^{c_i} \leftrightarrow (c_i \rightarrow \mathbf{K}_j^{1.1}(\varphi)^{c_i})$$

This axiom says that for subject j to count as knowing that φ in attributor i 's context c_i , it must follow from this context that j knows that φ holds in that same context. So it is always the very same attributor's context against which an agent will count as knowing something. This is precisely what the two absolutist views about knowledge claims have in common: epistemic standards and relevance sets are not contextually variable but constant matters. The only difference between scepticism and anti-scepticism is that for the former the epistemic standards are too stringent to ever be met while for the latter they are lax enough to be met very often and possibly most of the time. So, the appropriate translation from the informal sceptic/anti-sceptic talk into the formal $\mathbf{K}_j^{1.1}$ talk simply consists in substituting the appropriate contexts – c_{scep} and c_{anti} respectively – for c_i in the axiom above, and in adding the following condition:

$$c_{\text{anti}} \rightarrow c_{\text{scep}},$$

but not *vice versa*. This is the most minimal way to capture the idea that while c_{anti} excludes all the far-fetched possibilities arising from radical sceptical concerns and encapsulates a rather large set of presuppositions shared by most agents in their ordinary talk about knowledge, c_{scep} excludes all such presuppositions and encapsulates a rather large set of far-fetched possibilities.¹¹ For instance, the sceptic about contingent truths may not be a sceptic about necessary truths and thus may take the proposition that $2 \times 2 = 4$ into his c_{scep} , just as the anti-sceptic has this in his c_{anti} ; but whilst the anti-sceptic will also have the proposition that he is not a victim of an Evil Genius in his c_{anti} , the sceptic will not let this into his c_{scep} . Actually, c_{scep} can be identified with \top , which means that context-relativization according to scepticism is no relativization at all.

5.1.2 Contextualism and subjectivism

Contrary to the absolutist views, the two relativist views, according to which epistemic standards and relevance sets are contextually variable, fall under different knowledge operators. Contextualism is the view that the variability in question is a variability according to the attributor's context, not the subject's.

¹¹What we said in footnote 5 about the definition of anti-scepticism in the contextual model framework holds *mutatis mutandis* in the current framework. One can work with the version of anti-scepticism they favor simply by specifying what, according to their version, can be considered the constant set of epistemically relevant presuppositions, that is, the constant set of literals that constitute c_{anti} .

It is thus natural to associate contextualism with operator $\mathbf{K}_j^{1.2}$, whose behaviour with respect to contextual relativization was fully characterized by:

$$\vdash_{\text{CEL}} (\mathbf{K}_j^{1.2}\varphi)^{c_i} \leftrightarrow (c_i \rightarrow \mathbf{K}_j^{1.2}(\varphi)^{c_i}),$$

for this axiom says that for subject j to count as knowing that φ in attributor i 's context c_i , it must follow from *this* context that j knows that φ holds in j 's own context, that is, in the context whose attributor is subject j himself. So it is not always the same attributor's context against which something will count as being known. Relativity to the attributor's context is thus encapsulated in knowledge operator $K_j^{1.2}$. In contrast, relativity to the subject's context is encapsulated in knowledge operator $K_j^{2.2}$, since the reduction axiom characterizing its behaviour when contextualized was:

$$\vdash_{\text{CEL}} (\mathbf{K}_j^{2.2}\varphi)^{c_i} \leftrightarrow (c_j \rightarrow \mathbf{K}_j^{2.2}(\varphi)^{c_i}),$$

which says that for subject j to count as knowing that φ in attributor i 's context c_i , it must follow from subject j 's own context that j knows that φ holds in the same context, that is, subject j 's context. We can thus naturally associate the subjectivist view, according to which epistemic standards and relevance sets shift with the subject's context, with operator $\mathbf{K}_j^{2.2}$.

5.1.3 What about the remaining operator?

One may ask which epistemological view could match the $\mathbf{K}^{2.1}$ -operator. In our opinion the main interest of this operator does not lie so much in its possible correspondence with a view to be found in the epistemological literature as in the means it would offer us to handle indexicality phenomena, for instance if we wanted to incorporate personal pronouns like 'I' and 'he'. Consider a true utterance of 'He knows that I am here'. To formally account for its truth, we would need a knowledge operator whose logical behavior would match the following equivalence: $(\mathbf{K}_{he}I \text{ am here})^{c_I} \leftrightarrow (c_{he} \rightarrow \mathbf{K}_{he}(I \text{ am here})^{c_I})$, where c_I is the context of the agent who uses 'I' and c_{he} the context of the agent referred to by 'he'. Of course, we would not thereby have accounted for even a bit of the great complexity of natural language indexicality, and it is not our intention to do so at all.

5.2 Knowledge features uncovered

This formal translation of the four epistemological positions can now be exploited within our contextual logico-epistemic framework. In the remainder of this section we will use dialogical CEL to illustrate how differences between the four epistemological positions that have been pointed out in the contemporary philosophical literature on knowledge can be recovered within our formal framework, as well as to illustrate how differences that have not been touched upon in the literature can be discovered through that framework. We will give four such illustrations.

5.2.1 Normality

One thing to note is that the three operators $\mathbf{K}_j^{1,1}$, $\mathbf{K}_j^{1,2}$, $\mathbf{K}_j^{2,2}$, behave in the same manner with respect to the contextualized version of the \mathbf{K} axiom for knowledge, viz.:

$$\vdash_{\text{CEL}} (\mathbf{K}_j^{u,v}\varphi \wedge \mathbf{K}_j^{u,v}(\varphi \rightarrow \psi)) \rightarrow \mathbf{K}_j^{u,v}\psi)^{c_i}.$$

For any value of φ , ψ , u and v , this is a theorem of CEL and it is valid with respect to the class \mathcal{M}^{rst} of Kripke models with equivalence accessibility relations. Although it is a trivial result from a logico-epistemic point of view, it is not from an epistemological one. One can tell from the literature that it is important for advocates of the four identified epistemological positions that knowledge closure under known material implication holds. This is crucial to both the famous sceptical argument from ignorance and to the famous Moorean anti-sceptical response to it. Even those who take it that knowledge claims are context-relative admit that closure holds while insisting that it holds only within contexts and not across contexts (see [19] for the contextualist case and [13] for the subjectivist case, for instance).

5.2.2 Factivity (or not)

A more interesting result is that our framework clearly establishes a difference between case (1.1) – absolutists – and cases (1.2) and (2.2) – relativists – with respect to the contextualized version of the \mathbf{T} axiom – call it $(T)^c$:

$$(\mathbf{K}_j^{u,v}\varphi \rightarrow \varphi)^{c_i}.$$

For consider the following example of CEL-dialogical games for $(T)^c$, where φ is an epistemic formula:

Example 7 *In what follows, we compare the factivity of contextual knowledge according to 1.1, 1.2 and 2.2. In particular, we will check whether $(\mathbf{K}_j\mathbf{K}_k p \rightarrow \mathbf{K}_k p)^{c_i}$ is valid or not.*

$$1.1\text{-version: } (\mathbf{K}_j^{1,1}\mathbf{K}_k^{1,1}p \rightarrow \mathbf{K}_k^{1,1}p)^{c_i}$$

1.1		O	P		
			1:	$(\mathbf{K}_j\mathbf{K}_k p \rightarrow \mathbf{K}_k p)^{c_i}$ (0)	
(1)	1:	c_i	0	1:	$(\mathbf{K}_j\mathbf{K}_k p)^{c_i} \rightarrow (\mathbf{K}_k p)^{c_i}$ (2)
(3)	1:	$(\mathbf{K}_j\mathbf{K}_k p)^{c_i}$	2	1:	$(\mathbf{K}_k p)^{c_i}$ (4)
(5)	1:	c_i	4	1:	$\mathbf{K}_k(p)^{c_i}$ (6)
(7)	1:	$?_{\mathbf{K}_k/1k1}$	6	1k1:	$(p)^{c_i}$ (8)
(9)	1k1:	c_i	8	1k1:	p (20)
(11)	1:	$\mathbf{K}_j(\mathbf{K}_k p)^{c_i}$	3	1:	c_i (10)
(13)	1:	$(\mathbf{K}_k p)^{c_i}$	11	1:	$?_{\mathbf{K}_j/1}$ (12)
(15)	1:	$\mathbf{K}_k(p)^{c_i}$	13	1:	c_i (14)
(17)	1k1:	$(p)^{c_i}$	15	1:	$?_{\mathbf{K}_k/1k1}$ (16)
(19)	1k1:	p	17	1k1:	c_i (18)

P wins the play

1.2-version: $(\mathbf{K}_j^{1,2}\mathbf{K}_k^{1,2}p \rightarrow \mathbf{K}_k^{1,2}p)^{c_i}$

1.2	O		P	
			1 :	$(\mathbf{K}_j\mathbf{K}_k p \rightarrow \mathbf{K}_k p)^{c_i}$ (0)
(1)	1 :	c_i 0	1 :	$(\mathbf{K}_j\mathbf{K}_k p)^{c_i} \rightarrow (\mathbf{K}_k p)^{c_i}$ (2)
(3)	1 :	$(\mathbf{K}_j\mathbf{K}_k p)^{c_i}$ 2	1 :	$(\mathbf{K}_k p)^{c_i}$ (4)
(5)	1 :	c_i 4	1 :	$\mathbf{K}_k(p)^{c_k}$ (6)
(7)	1 :	$?_{\mathbf{K}_k/1k1}$ 6	1k1 :	$(p)^{c_k}$ (8)
(9)	1k1 :	c_k 8		
(11)	1 :	$\mathbf{K}_j(\mathbf{K}_k p)^{c_j}$	3	1 : c_i (10)
(13)	1 :	$(\mathbf{K}_k p)^{c_j}$	11	1 : $?_{\mathbf{K}_j/1}$ (12)

O wins the play

Short comment: After move (13) P would have to utter “1 : c_j ”; but this context has not been previously introduced by O , so she cannot.

2.2-version: $(\mathbf{K}_j^{2,2}\mathbf{K}_k^{2,2}p \rightarrow \mathbf{K}_k^{2,2}p)^{c_i}$

2.2	O		P	
			1 :	$(\mathbf{K}_j\mathbf{K}_k p \rightarrow \mathbf{K}_k p)^{c_i}$ (0)
(1)	1 :	c_i 0	1 :	$(\mathbf{K}_j\mathbf{K}_k p)^{c_i} \rightarrow (\mathbf{K}_k p)^{c_i}$ (2)
(3)	1 :	$(\mathbf{K}_j\mathbf{K}_k p)^{c_i}$ 2	1 :	$(\mathbf{K}_k p)^{c_i}$ (4)
(5)	1 :	c_k 4	1 :	$\mathbf{K}_k(p)^{c_k}$ (6)
(7)	1 :	$?_{\mathbf{K}_k/1k1}$ 6	1k1 :	$(p)^{c_k}$ (8)
(9)	1k1 :	c_k 8		

O wins the play

Short comment: Here P cannot even attack (3): with (2.2), she would have to utter “1 : c_j ”, which has not been introduced by O . ■

What this shows is that absolutist contextual knowledge is always factive while relativist contextual knowledge is not always factive. More specifically, $(T)^c$ holds for relativist knowledge operators $\mathbf{K}_j^{1,2}$ and $\mathbf{K}_j^{2,2}$ when they bear on “absolute” non-epistemic formulas, but not in the general case.

A precision is required here: There is a possible loss of factivity for contextualism or subjectivism on absolute formulas, but this would be a trivial one like the loss of factivity possibly occurring in standard multi-modal epistemic logic: the formula $\mathbf{K}_j p \rightarrow p$ is trivially falsified at any world w^* lying beyond the scope of the accessibility relation \mathcal{R}_j where p is false. Analogous cases in CEL are formulas like $(\mathbf{K}_j p \rightarrow p)^{c_i}$, which are falsified at worlds where c_i is true while c_j and p are false; but of course, c_i is as irrelevant for the agent i 's context-relativized knowledge as is the world w^* for his absolute knowledge. However, the situation is different with epistemic formulas like the ones just evaluated through dialogical games: here the context used to falsify the formulas is perfectly relevant to evaluate the agents' knowledge, as is seen in the Figure below:

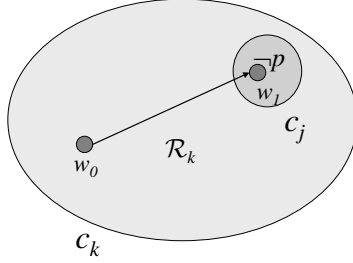


Figure 1: A counter-model to $(\mathbf{K}_j^2 \mathbf{K}_k^2 p \rightarrow \mathbf{K}_k^2 p)^{c_i}$

To our knowledge this result that factivity for contextualism and subjectivism is restricted to knowledge of the world (*versus* of other people's knowledge) has never been highlighted in the epistemological literature¹².

5.2.3 Context-relativized introspection

Another first for epistemological discussions over knowledge and context is the following difference between contextualism and subjectivism: if subjectivism is true, what one knows in one's own context, one also knows that one knows it in anyone else's context; whereas if contextualism is true, what one knows in one's own context, one may not know that one knows it in anyone else's context. This is clear from the dialogical games in examples 4 and 5 in the previous section, which made it explicit that:

- $\not\vdash_{\text{CEL}} (\mathbf{K}_i^{1.2} p)^{c_i} \rightarrow (\mathbf{K}_i^{1.2} \mathbf{K}_i^{1.2} p)^{c_i}$
- $\vdash_{\text{CEL}} (\mathbf{K}_i^{2.2} p)^{c_i} \rightarrow (\mathbf{K}_i^{2.2} \mathbf{K}_i^{2.2} p)^{c_i}$

What this means is that for a subjectivist agent i – case (2.2) –, if it follows from the context c_i of which he is the attributor that he knows that p , then it follows from any other attributor j 's context that he knows that p ; while for a contextualist agent, this is not true. This can be explained informally as follows:

- (Subjectivist case) If subjectivism is true, then for agent i to know that p , he must meet the standards in vigour in his own context c_i for knowing that p . But he will then *ipso facto* meet the standards for his knowing that he knows that p . For if he did not know that he knows that p , that would be because he considers it possible that he does not know that p ; but he cannot consider this a serious possibility if he already knows that p . And if i counts as knowing that he knows that p in his own context, and since we are dealing with subjectivism, i will count as knowing that he knows that p in any other agent j 's context.

¹²However, see Stanley [33] for a discussion of factivity and related matters in a subjectivist setting.

- (Contextualist case) If contextualism is true, then the relevant standards for knowing a proposition may vary from one attributor’s context to that of another. This being so, agent i may count as knowing that p relative to the context c_i of which he is the attributor, whilst not relative to the context c_j of another attributor j associated with more stringent standards than in c_i . In this case, since one cannot know what is false and since it is false in c_j that i knows that p , it is false in this same c_j that i knows that he knows that p .

It is an advantage of our formal framework that it can capture this informal difference between contextualism and subjectivism.

5.2.4 Mixing agents

Another interesting feature of that framework is that it allows us to reason about knowledge in a group of epistemologically heterogeneous agents and to answer formally such informal questions as “If a contextualist knows that a subjectivist knows this or that, does the contextualist know this or that?”¹³ Here we give only one example we find interesting of that feature through the following question: if an absolutist agent knows that a subjectivist agent knows a proposition relative to a context, does the subjectivist know that proposition relative to that context? In our framework this question becomes that of deciding whether the following formula is a theorem of CEL:

$$(\mathbf{K}_j^{1,1}\mathbf{K}_k^{2,2}p)^{c_i} \rightarrow (\mathbf{K}_k^{2,2}p)^{c_i}.$$

Now, this question is easily settled by means of the following CEL-dialogical game:

Example 8 *In the following example, we consider simultaneously agents of different kinds. The upshot is a kind of failure of Axiom T for an absolutist knower.*

O		P	
		1 :	$(\mathbf{K}_j^{1,1}\mathbf{K}_k^{2,2}p)^{c_i} \rightarrow (\mathbf{K}_k^{2,2}p)^{c_i}$ (0)
(1)	1 :	$(\mathbf{K}_j^{1,1}\mathbf{K}_k^{2,2}p)^{c_i}$	0 (2)
(3)	1 :	c_k	2 (4)
(5)	1 :	$?_{\mathbf{K}_k/1k1}$	6 (8)
(9)	1k1 :	c_k	8

O wins the play

¹³Suppose that agent i is a F -ist (a sceptic, an anti-sceptic, a contextualist, or a subjectivist). If i is coherent with his own theory of knowledge (ascriptions), which he intends to be the only correct one for *any* agent’s knowledge, he ought to reason accordingly not only about his own knowledge but also about other agent’s knowledge. This amounts to saying *both* that his reasoning about knowledge ought to meet the subjectivist expectations and that he ought to expect other people to meet the same expectations when they reason about knowledge. That is our motivation for talking about sceptical, anti-sceptical, contextualist, and subjectivist *agents* and for asking what they can know about what other types of agents know.

Short comment: P cannot go on and attack (1), for it would require that she utter “ $1 : c_i$ ”, which has not been introduced by O . ■

Thus, the formula is not a principle of our contextual epistemic logic. This comes as an amendment to, or better as a complement to what we said earlier about the factivity of absolutist knowledge. For the lesson to be drawn from this is, roughly, that an absolutist agent (a sceptic, an anti-sceptic) may know something without this something being true when this something is about what a subjectivist agent knows.

Incidentally, we may notice that the following formula, differing from the previous one in that the subscripted agent is now the same for each occurrence of a knowledge operator, is not a principle of our contextual epistemic logic either:

$$(\mathbf{K}_j^{1.1}\mathbf{K}_j^{2.2}p)^{c_i} \rightarrow (\mathbf{K}_j^{2.2}p)^{c_i}.$$

Funnily enough, this could be interpreted in terms of “epistemically schizophrenic” agents whose knowledge is compartmented in the sense that they know different things when they are in a subjectivist or in a sceptical mood from what they know when they are in an anti-sceptical or in a contextualist mood, or in the sense that the subjectivist or the sceptical part of them knows different things from what the anti-sceptical or the contextualist part of them knows. Then the lesson to be drawn from the result in question would be that what your sceptic or anti-sceptic compartment knows about your subjectivist compartment, your subjectivist compartment may not know of itself.

6 Conclusion

Our primary goal in this paper was to investigate the relationships between knowledge and context in the formal framework of epistemic modal logic. We thus provided an epistemic logic with context relativization, CEL, together with its dialogical semantics, and applied it to epistemological issues.

The subsequent results can be interesting both from a logical and from an epistemological point of view. From the former point of view, the interesting upshot is that the logic of public announcements can be translated into a logic for context. The interaction between knowledge and context is slightly more subtle than that of knowledge with announcements, but the result is really close to PAL.

From the epistemological point of view, this time, the interesting result is that CEL provides us with a powerful formal tool not only for capturing informal views about knowledge and context, but also for gaining new insights into the debates over their possible interconnections, contributing thereby to the current research program in formal epistemology, at the interface of logic and the theory of knowledge.

Let us just add that two typical “Rahmanian issues” came out from this work. First, the dialogical version of PAL and its application to epistemology constitute a new confirmation of the fruitfulness of dialogical logic as a framework to

combine different logics. Second, we considered diverging agents mutually reasoning about their respective knowledge; this strongly echoes Shahid’s recent work [26] about non-normal logics, classical agents reasoning about intuitionistic ones. Of course, all our agents are normal (and even S5) knowers; anyway as epistemologists, some of them appear to be strange, at least.

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