An axiomatic approach to TACTIC (juin 2006)

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Abstract

Most outranking methods build a preference relation between alternatives evaluated on several attributes using the concordance / non-discordance principle. This principle leads to declaring that an alternative is "superior" to another, if the coalition of attributes supporting this proposition is "sufficiently important" (concordance condition) and if there is no attribute that "strongly rejects" it (non-discordance condition). Such a way of comparing alternatives is simple and rather natural; however, it is well-known that it may produce binary relations that do not possess any remarkable property of transitivity. This paper uses conjoint measurement techniques to obtain an axiomatic characterization of such relations that emphasizes their main distinctive feature, i.e., their very crude way to distinguish various levels of preference differences on each attribute. We focus on outranking methods, such as TACTIC, that produce an asymmetric relation, interpreted as strict preference. The results in this paper may be seen as an attempt to give such outranking methods a sound axiomatic foundation based on conjoint measurement.

Keywords: Conjoint measurement, TACTIC, outranking methods.

Résumé

Beaucoup de méthodes de surclassement reposent sur l'utilisation du principe de concordance / non discordance. Ce principe conduit à déclarer qu'une action en surclasse une autre si la coalition des critères appuyant cette proposition est « suffisamment importante » (condition de concordance) et si aucun critère ne la « rejette fortement » (condition de non discordance). Ce mode de comparaison est simple et relativement naturel. Il est bien connu qu'il peut conduire à des relations binaires non transitives. Ce texte utilise des techniques issues de la théorie du mesurage conjoint pour proposer une analyse axiomatique des relations pouvant être obtenues par application du principe de concordance / non discordance. Cette analyse met l'accent sur la caractéristique cruciale de ces relations : elles induisent sur chaque critère une relation de comparaison d'écarts de préférence très pauvre. On se concentrera sur les méthodes de surclassment, telles que TACTIC, visant à bâtir une relation interprétée comme une préférence stricte. L'objectif est de donner à ces méthodes une base axiomatique solide fondée sur la théorie du mesurage conjoint.

Mots-clés : Mesurage conjoint, TACTIC, méthodes de surclassement.

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1 Introduction

A central problem in MCDM (Multiple Criteria Decision Making) is to build a preference relation on a set of alternatives evaluated on several attributes, taking into account preferences expressed on each attribute and inter-attribute information (such as weights). The classical way of doing so (Keeney and Raiffa, 1976) is to use a *value function*. This leads to define a function v associating a real number $v(x)$ to each alternative x and to declare that x is better than y if and only if $v(x) \geq v(y)$. The number $v(x)$ depends upon the evaluation $x_1, x_2, \ldots, x_n$ of x on the n attributes, most often through an additive aggregation such that:

$$x \mathcal{P} y \Leftrightarrow \sum_{i=1}^{n} v_i(x_i) > \sum_{i=1}^{n} v_i(y_i). \qquad (A)$$

Roy and Bouyssou (1993, p. 233–237) have detailed the possible difficulties associated with such an approach. In particular, it requires a detailed analysis of the tradeoffs between the various attributes through a time-consuming and cognitively demanding interaction with the decision-maker. Facing such difficulties, the analyst may use a less demanding comparison procedure between alternatives. One such procedure, based on the concordance / non-discordance principle (henceforth, the CNDP), was first proposed by Roy (1968) and underlies most of the well-known ELECTRE methods (Roy, 1991). ELECTRE methods build a reflexive preference relation interpreted as an "at least as good as" relation between alternatives. The CNDP is also at work in outranking methods, such as TACTIC (Vansnick, 1986), that lead to an asymmetric relation interpreted as "strict preference". We focus here on the latter type of methods. It should be noted that, contrary to the ELECTRE methods, TACTIC focuses on building a strict preference relation without distinguishing indifference from incomparability. However, the analysis below can easily be extended to cover outranking methods closer to ELECTRE [1] in which such a distinction is made.

Let x and y be two alternatives evaluated on several attributes. The CNDP leads to comparing these two alternatives along the following lines:

[1] At least if the "weak preference" zone that is used is some ELECTRE methods is neglected. This zone that models an hesitation between strict preference and indifference is not easy to analyze from a theoretical point of view unless one has recourse to non-classical logic as in Tsoukiàs *et al.* (2002).

- compare the evaluations of x and y on attribute i and decide whether attribute i favors x, favors y or favors none of x and y. Repeat this operation for each attribute. This defines three disjoints subsets of attributes: those favoring x, those favoring y and those for which none of the two alternatives is favored,

- compare the set of attributes favoring x with the set of attributes favoring y in terms of "importance",

- for each attribute in the set of attributes favoring y, investigate whether this attribute "strongly" favors y,

- declare that "x is preferred to y" if the set of attributes favoring x is "more important than" the set of attributes favoring y and if there is no attribute strongly favoring y.

This way of comparing alternatives has a definite "ordinal" flavor and does not require a much detailed analysis of tradeoffs. The price to pay is that it does not always lead to preference relations, henceforth called *strict outranking relations* (SOR), having "nice" transitivity properties. On a practical level, this means that using such relations to elaborate a recommendation, e.g., through the selection of a subset containing "good" alternatives, is not easy and requires the use of specific techniques (see, e.g., Roy, 1991). On a theoretical level, such relations are quite different from the transitive structures usually studied in conjoint measurement (see Krantz *et al.*, 1971; Wakker, 1989), e.g. those representable using the additive model (A). This probably explains why outranking methods have not been much investigated at an axiomatic level. The present paper concentrates on this last aspect. Adopting a framework for conjoint measurement tolerating intransitive preferences proposed in Bouyssou and Pirlot (2002*b*) will allow us to propose an axiomatic characterization of SOR using axioms that will emphasize their main specific feature, i.e. the very crude way in which they isolate various levels of "preference differences" on each attribute. This extends the results in Bouyssou and Pirlot (2003) and Bouyssou and Pirlot (2005*c*) to include the possibility of discordance.

This paper is not the first attempt to analyze the CNDP from a theoretical perspective and two earlier attempts at doing so should be mentioned. The work of Bouyssou and Vansnick (1986) on TACTIC has the same objective as the present paper. It uses a weakening of a noncompensation condition introduced in Fishburn (1976). As detailed in Bouyssou and Pirlot (2005*a*), the use of such a condition is not entirely satisfactory since it is quite strong and does not allow to take all forms of concordance conditions into account (in particular, it excludes the possibility to include an additive threshold in the concordance condition of TACTIC). The work of Greco *et al.* (2001) is also related to the present paper. They axiomatically study a particular class of reflexive outranking relations built using the CNDP. As in the present paper, the CNDP is analyzed through its consequences on the induced relations comparing preference difference on each attribute (this idea was developed independently in Bouyssou and Pirlot, 2002*a*). Their analysis is not conducted within the framework of a general conjoint measurement model and, therefore, is not well-suited to analyze the specific features of outranking relations within a larger class of relations. Furthermore, it only deals with a very specific type of outranking relations, in which the concordance condition is close to the one used in ELECTRE I and uses some axioms that, in our view, are not especially attractive.

This paper is organized as follows. We introduce our setting in Section 2. Strict outranking relations are defined in Section 3. Our general framework for conjoint measurement allowing for nontransitive preferences is presented in Section 4. Section 5 characterizes SOR without discordance effects. Section 6 extends these results to allow for discordance. A final section discusses our results and presents directions for future research. Proofs are gathered in an appendix.

2 Definitions and Notation

In this paper we consider a set $X = \prod_{i=1}^{n} X_i$ with $n \geq 2$. Elements of X will be interpreted as *alternatives* evaluated on a set $N = \{1, 2, \ldots, n\}$ of *attributes*. When $i \in N$, we denote by X_{-i} the set $\prod_{j \in N \setminus \{i\}} X_j$. With customary abuse of notation, (x_i, y_{-i}) will denote the element $w \in X$ such that $w_i = x_i$ and $w_j = y_j$ for all $j \in N \setminus \{i\}$.

We use $\mathcal{P}$ to denote an asymmetric binary relation on X interpreted as a *strict preference relation* between alternatives. The *symmetric complement* of $\mathcal{P}$ is denoted by $\mathcal{I}$ (i.e. $x \mathrel{\mathcal{I}} y \Leftrightarrow [Not[x \mathrel{\mathcal{P}} y]$ and $Not[y \mathrel{\mathcal{P}} x]])$.

We say that attribute $i \in N$ is *influent* (for $\mathcal{P}$) if there are $x_i, y_i, z_i, w_i \in X_i$ and $a_{-i}, b_{-i} \in X_{-i}$ such that $(x_i, a_{-i}) \mathrel{\mathcal{P}} (y_i, b_{-i})$ and $Not[(z_i, a_{-i}) \mathrel{\mathcal{P}} (w_i, b_{-i})]$ and *degenerate* otherwise. It is clear that a degenerate attribute has no influence whatsoever on the comparison of the elements of X and may be suppressed from N. In order to avoid unnecessary minor complications, we suppose henceforth *all attributes in N are influent.*

3 Strict outranking relations

The following definition, building on Bouyssou and Pirlot (2005*c*), Fargier and Perny (2001) and Greco *et al.* (2001), formalizes the idea of a SOR, i.e., of a strict preference relation that has been obtained comparing alternatives by pair using the CNDP.

Definition 1 (Strict outranking relations) *Let $\mathcal{P}$ be an asymmetric binary relation on $X = \prod_{i=1}^{n} X_i$. We say that $\mathcal{P}$ is a SOR if there are:*

- asymmetric *binary relations P_i and V_i on each X_i such that $V_i \subseteq P_i$ $(i = 1, 2, \ldots, n)$,*
- *a binary relation $\rhd$ between disjoint subsets of N that is* monotonic *w.r.t. inclusion, i.e. such that for all $A, B, C, D \subseteq N$ with $A \cap B = \varnothing$ and $C \cap D = \varnothing$,*

$$\left.\begin{array}{c} A \rhd B \\ C \supseteq A \text{ and } B \supseteq D \end{array}\right\} \Rightarrow C \rhd D,$$

such that, for all $x, y \in X$,

$$x \; \mathcal{P} \; y \Leftrightarrow [P(x, y) \rhd P(y, x) \text{ and } V(y, x) = \varnothing],$$

where $P(x, y) = \{i \in N : x_i \; P_i \; y_i\}$ and $V(y, x) = \{i \in N : y_i \; V_i \; x_i\}$.

Hence, when $\mathcal{P}$ is a SOR, the preference between x and y only depends on the comparison in terms of "importance" of the subsets of attributes favoring x or y in terms of the asymmetric relation P_i (concordance) and on the absence of any attribute such that $y_i \; V_i \; x_i$ (non-discordance). It is useful to interpret $x_i \; P_i \; y_i$ as "x_i is strictly preferred to y_i", $x_i \; V_i \; y_i$ as "x_i is strongly preferred to y_i" and $A \rhd B$ as "the coalition A of attributes is more important than the coalition B of attributes".

The main objective of this paper is to characterize SOR within a general framework of conjoint measurement, using conditions that will allow us to isolate their specific features.

A well-known example of an outranking method leading to a SOR is TACTIC (Vansnick, 1986). This method builds an asymmetric preference relation $\mathcal{P}$ on X letting, for all $x, y \in X$,

$$x \,\mathcal{P}\, y \Leftrightarrow \begin{cases} \sum_{i \in P(x,y)} w_i > \rho \sum_{j \in P(y,x)} w_j \ + \ \varepsilon \\ \text{and} \\ Not[y_j \ V_j \ x_j], \text{ for all } j \in P(y,x) \end{cases}$$

where P_i and V_i are semiorders on X_i such that $V_i \subseteq P_i$, $w_i > 0$ is the weight assigned to attribute $i \in N$ and $\rho > 1$ and $\varepsilon \geq 0$ are thresholds.

Defining $\rhd$ letting, for all $A, B \subseteq N$ such that $A \cap B = \varnothing$:

$$A \rhd B \Leftrightarrow \sum_{i \in A} w_i > \rho \sum_{j \in B} w_j + \varepsilon,$$

it is easy to see that $\mathcal{P}$ as defined in TACTIC is a SOR. Simple examples inspired by Condorcet's paradox show that the relation $\mathcal{P}$ in TACTIC is not always transitive and may even have cycles. The main differences between a preference relation obtained with TACTIC and a SOR are that, in a SOR, it is neither supposed that the importance relation between coalitions can be represented additively using weights nor that P_i and V_i are strict semiorders.

4 A general framework for nontransitive conjoint measurement

This section follows the analysis in Bouyssou and Pirlot (2002b) using asymmetric relations instead of reflexive relations. We envisage here binary relations $\mathcal{P}$ on X that can be represented as:

$$x \,\mathcal{P}\, y \Leftrightarrow F(p_1(x_1, y_1), \ldots, p_n(x_n, y_n)) > 0, \tag{M}$$

where p_i are real-valued functions on X_i^2 that are *skew symmetric* (i.e. such that $p_i(x_i, y_i) = -p_i(y_i, x_i)$, for all $x_i, y_i \in X_i$) and F is a real-valued function on $\prod_{i=1}^{n} p_i(X_i^2)$ being *odd* (i.e. such that $F(\mathbf{x}) = -F(-\mathbf{x})$, abusing notation in an obvious way) and *nondecreasing* in all its arguments.

The intuition underlying model (M) is that functions p_i measure (positive or negative) preference differences and that the function F synthesizes these measures. Indeed, the analysis of model (M) is based on relations comparing preference differences on each attribute induced by $\mathcal{P}$.

Definition 2 (Relations $\succsim_i^*$ and $\succsim_i^{**}$) *Let $\mathcal{P}$ be a binary relation on a set $X = \prod_{i=1}^n X_i$. We define the binary relations $\succsim_i^*$ and $\succsim_i^{**}$ on X_i^2 letting, for all $x_i, y_i, z_i, w_i \in X_i$,*

$$
\begin{gathered}
(x_i, y_i) \succsim_i^* (z_i, w_i) \Leftrightarrow \\
[\textit{for all } a_{-i}, b_{-i} \in X_{-i}, \\
(z_i, a_{-i}) \ \mathcal{P} \ (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \ \mathcal{P} \ (y_i, b_{-i})],
\end{gathered}
$$

$$
\begin{gathered}
(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow \\
[(x_i, y_i) \succsim_i^* (z_i, w_i) \textit{ and } (w_i, z_i) \succsim_i^* (y_i, x_i)] .
\end{gathered}
$$

The definition of $\succsim_i^*$ suggests that $(x_i, y_i) \succsim_i^* (z_i, w_i)$ can be interpreted as saying that the preference difference between x_i and y_i is at least as large as the preference difference between z_i and w_i. Indeed, as soon as $(z_i, a_{-i}) \ \mathcal{P} \ (w_i, b_{-i})$, $(x_i, y_i) \succsim_i^* (z_i, w_i)$ implies $(x_i, a_{-i}) \ \mathcal{P} \ (y_i, b_{-i})$. However, the definition of $\succsim_i^*$ does not imply that the two "opposite" differences (x_i, y_i) and (y_i, x_i) are linked. This is at variance with the usual intuition concerning preference differences and motivates the introduction of the relation $\succsim_i^{**}$. We have $(x_i, y_i) \succsim_i^{**} (z_i, w_i)$ when both $(x_i, y_i) \succsim_i^* (z_i, w_i)$ and $(w_i, z_i) \succsim_i^* (y_i, x_i)$ hold. This implies that $\succsim_i^{**}$ is *reversible*, i.e. $(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow (w_i, z_i) \succsim_i^{**} (y_i, x_i)$. The asymmetric and symmetric parts of $\succsim_i^*$ are respectively denoted by $\succ_i^*$ and $\sim_i^*$, a similar convention holding for $\succsim_i^{**}$.

By construction, $\succsim_i^*$ and $\succsim_i^{**}$ are reflexive and transitive. Therefore, $\sim_i^*$ and $\sim_i^{**}$ are equivalence relations. It is important to notice that $\succsim_i^*$ and $\succsim_i^{**}$ may not be complete. Interesting consequences obtain when this is the case. This motivates the introduction of the following two conditions.

Definition 3 (Conditions $\boldsymbol{ARC1}$ and $\boldsymbol{ARC2}$) *Let $\mathcal{P}$ be a binary relation on a set $X = \prod_{i=1}^n X_i$. This relation is said to satisfy:*
$ARC1_i$ if

$$
\left.
\begin{array}{c}
(x_i, a_{-i}) \ \mathcal{P} \ (y_i, b_{-i}) \\
\textit{and} \\
(z_i, c_{-i}) \ \mathcal{P} \ (w_i, d_{-i})
\end{array}
\right\} \Rightarrow
\left\{
\begin{array}{c}
(x_i, c_{-i}) \ \mathcal{P} \ (y_i, d_{-i}) \\
\textit{or} \\
(z_i, a_{-i}) \ \mathcal{P} \ (w_i, b_{-i}),
\end{array}
\right.
$$

$ARC2_i$ if

$$
\left.
\begin{array}{c}
(x_i, a_{-i}) \ \mathcal{P} \ (y_i, b_{-i}) \\
\textit{and} \\
(y_i, c_{-i}) \ \mathcal{P} \ (x_i, d_{-i})
\end{array}
\right\} \Rightarrow
\left\{
\begin{array}{c}
(z_i, a_{-i}) \ \mathcal{P} \ (w_i, b_{-i}) \\
\textit{or} \\
(w_i, c_{-i}) \ \mathcal{P} \ (z_i, d_{-i}),
\end{array}
\right.
$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $\mathcal{P}$ satisfies $ARC1$ (resp. $ARC2$) if it satisfies $ARC1_i$ (resp. $ARC2_i$) for all $i \in N$.

Condition $ARC1_i$ suggests that either the difference (x_i, y_i) is at least as large as the difference (z_i, w_i) or *vice versa*. Condition $ARC2_i$ suggests that the preference difference (x_i, y_i) is linked to the "opposite" preference difference (y_i, x_i). Taking $x_i = y_i$, $z_i = w_i$, $a_{-i} = c_{-i}$ and $b_{-i} = d_{-i}$ shows that $ARC2$ implies that $\mathcal{P}$ is independent, i.e. that $(x_i, a_{-i})\ \mathcal{P}\ (x_i, b_{-i})$ implies $(y_i, a_{-i})\ \mathcal{P}\ (y_i, b_{-i})$, for all $i \in N$, all $x_i, y_i \in X_i$ and all $a_{-i}, b_{-i} \in X_{-i}$.

The consequences of $ARC1_i$ and $ARC2_i$ on our two relations comparing preference differences on each attribute are noted below.

Lemma 1

1. *$ARC1_i \Leftrightarrow [\succsim^*_i$ is complete$]$,*
2. *$ARC2_i \Leftrightarrow$
   $[$for all $x_i, y_i, z_i, w_i \in X_i$, $Not[(x_i, y_i) \succsim^*_i (z_i, w_i)] \Rightarrow (y_i, x_i) \succsim^*_i (w_i, z_i)]$,*
3. *$[ARC1_i$ and $ARC2_i] \Leftrightarrow [\succsim^{**}_i$ is complete$]$.*
4. *In the class of asymmetric relations, $ARC1$ and $ARC2$ are independent conditions.*

These two conditions allow to characterize model (M) when X is finite or countably infinite (see Bouyssou and Pirlot, 2003, Theorem 1).

Theorem 1 *Let $\mathcal{P}$ be a binary relation on finite or countably infinite set $X = \prod_{i=1}^n X_i$. Then $\mathcal{P}$ has a representation (M) iff it is asymmetric and satisfies $ARC1$ and $ARC2$.*

It should be observed at this point that model (M) is sufficiently general to contain as particular cases most conjoint measurement models, when interpreted in terms of an asymmetric binary relation, including the classical additive model (A) (see Krantz *et al.*, 1971; Wakker, 1989), and the additive difference model (see Tversky, 1969). We show below that SOR form a subclass of the binary relations having a representation in model (M).

5 Strict concordance relations

In Bouyssou and Pirlot (2005c), we showed that *strict concordance relations*, i.e., SOR in which all relations V_i are empty, are exactly the binary relations having a representation in model (M) with all functions p_i taking at most *three* distinct values. We briefly recall here the main points in this analysis.

Definition 4 (Conditions $\boldsymbol{MAJ1}$, $\boldsymbol{MAJ2}$) *Let $\mathcal{P}$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. This relation is said to satisfy:*
$MAJ1_i$ if

$$\left.\begin{array}{c}(x_i, a_{-i})\ \mathcal{P}\ (y_i, b_{-i})\\ \textit{and}\\ (z_i, a_{-i})\ \mathcal{P}\ (w_i, b_{-i})\\ \textit{and}\\ (z_i, c_{-i})\ \mathcal{P}\ (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i})\ \mathcal{P}\ (x_i, b_{-i})\\ \textit{or}\\ (x_i, c_{-i})\ \mathcal{P}\ (y_i, d_{-i}),\end{array}\right.$$

$MAJ2_i$ if

$$\left.\begin{array}{c}(x_i, a_{-i})\ \mathcal{P}\ (y_i, b_{-i})\\ \textit{and}\\ (w_i, a_{-i})\ \mathcal{P}\ (z_i, b_{-i})\\ \textit{and}\\ (y_i, c_{-i})\ \mathcal{P}\ (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i})\ \mathcal{P}\ (x_i, b_{-i})\\ \textit{or}\\ (z_i, c_{-i})\ \mathcal{P}\ (w_i, d_{-i}),\end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $\mathcal{P}$ satisfies $MAJ1$ (resp. $MAJ2$) if it satisfies $MAJ1_i$ (resp. $MAJ2_i$) for all $i \in N$.

Intuitively, the role of $MAJ1_i$ is to limit the number of possible equivalence classes of $\succsim_i^*$ and, hence, $\succsim_i^{**}$. Indeed, suppose that we have $(x_i, a_{-i})\ \mathcal{P}\ (y_i, b_{-i})$ and $Not[(y_i, a_{-i})\ \mathcal{P}\ (x_i, b_{-i})]$. This means that the preference difference between y_i and x_i is not larger than the preference difference between x_i and y_i. In a strict concordance relation, this is only possible if $x_i\ P_i\ y_i$. But this means that no preference difference is larger than the preference difference between x_i and y_i. Hence if $(z_i, c_{-i})\ \mathcal{P}\ (w_i, d_{-i})$, we must have that $(x_i, c_{-i})\ \mathcal{P}\ (y_i, d_{-i})$, as required by $MAJ1_i$. The interpretation of $MAJ2_i$ is dual: if the preference difference between y_i and x_i is not larger than the preference difference between x_i and y_i, then it is the smallest possible preference difference. The impact of these two conditions is made precise in the following:

Lemma 2 *$ARC1_i$, $ARC2_i$, $MAJ1_i$ and $MAJ2_i$ hold iff the relation $\succsim_i^{**}$ is a weak order having at most three distinct equivalence classes.*

Relations $\mathcal{P}$ for which all relations $\succsim_i^{**}$ have at most three distinct equivalence classes are intuitively quite close from strict concordance relations. In such relations, the preference between x and y only depends on the subsets of attributes favoring x or y in terms of the asymmetric relation P_i. It does not depend on preference differences between the various levels on each attribute besides the distinction between levels indicated by P_i. It is not difficult to show that, if a relation $\mathcal{P}$ has a representation in model (M) and is such that all $\succsim_i^{**}$ have only three distinct equivalence classes, it will have a representation in model (M) in which all functions p_i take at most three distinct values. Using such a representation, the relations P_i defined letting $x_i \; P_i \; y_i \Leftrightarrow p_i(x_i, y_i) > 0$ are indeed asymmetric and capture all the information contained in the relations $\succsim_i^{**}$. This is at the heart of the characterization of strict concordance relations proposed in Bouyssou and Pirlot (2005*c*). We have:

Theorem 2 *Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$. Then $\mathcal{P}$ is a strict concordance relation iff it is asymmetric and satisfies $ARC1$, $ARC2$, $MAJ1$ and $MAJ2$. Furthermore, conditions $ARC1$, $ARC2$, $MAJ1$ and $MAJ2$ are independent in the class of all asymmetric relations on X.*

We have shown in Bouyssou and Pirlot (2005*a*) that this approach to concordance relations was more general than the one based on the use of a "noncompensation" condition as in, e.g., Bouyssou and Vansnick (1986) or Fargier and Perny (2001).

6 Strict outranking relations

Theorem 2 characterizes SOR in which discordance plays no role. In order to allow for possible discordance effects, some of the conditions used in Theorem 2 have to be weakened. It is not difficult to see that a SOR always satisfies conditions $ARC1$, $ARC2$, $MAJ1$. Hence, condition $MAJ2$ is the natural candidate for such a weakening. Indeed, condition $MAJ2_i$ may be violated due to discordance effects. Suppose that $(x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i})$, $(w_i, a_{-i}) \; \mathcal{P} \; (z_i, b_{-i})$, $(y_i, c_{-i}) \; \mathcal{P} \; (x_i, d_{-i})$, with $x_i \; P_i \; y_i$, $w_i \; P_i \; z_i$ while $Not[x_i \; V_i \; y_i]$ and $w_i \; V_i \; z_i$. It may happen that we have $Not[(y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i})]$. But, since $w_i \; V_i \; z_i$, it is also impossible that $(z_i, c_{-i}) \mathcal{P} (w_i, d_{-i})$ and $MAJ2_i$ is violated. It is easy to convince oneself that such violations may only occur if $w_i \; V_i \; z_i$. Therefore, if we require that $(z_i, e_{-i}) \mathcal{P} (w_i, f_{-i})$, for some $e_{-i}, f_{-i} \in X_{-i}$, we are sure that this will never happen. This is precisely what condition $MAJ3_i$ requires.

Definition 5 (Condition *MAJ*3) *Let $\mathcal{P}$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. This relation is said to satisfy $MAJ3_i$ if*

$$\left.\begin{array}{c} (x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i}) \\ \textit{and} \\ (w_i, a_{-i}) \; \mathcal{P} \; (z_i, b_{-i}) \\ \textit{and} \\ (y_i, c_{-i}) \; \mathcal{P} \; (x_i, d_{-i}) \\ \textit{and} \\ (z_i, e_{-i}) \; \mathcal{P} \; (w_i, f_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i}) \\ \textit{or} \\ (z_i, c_{-i}) \; \mathcal{P} \; (w_i, d_{-i}), \end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i}, e_{-i}, f_{-i} \in X_{-i}$. We say that $\mathcal{P}$ satisfies $MAJ3$ if it satisfies $MAJ3_i$ for all $i \in N$.

It is clear that $MAJ3_i$ weakens $MAJ2_i$. The intuition behind this condition is that allowing for discordance effects possibly creates two new equivalence classes of $\succsim_i^{**}$. When $\succsim_i^{**}$ has five equivalence classes, the lowest equivalence class will contain the pairs (x_i, y_i) such that $y_i \; V_i \; x_i$. On the contrary, the highest possible class will contain the pairs (x_i, y_i) such that $x_i \; V_i \; y_i$ (they will not play a role that is different from the pairs in the second equivalence class, i.e., the pairs (z_i, w_i) such that $z_i \; P_i \; w_i$ but $Not[z_i \; V_i \; w_i]$). Condition $MAJ3_i$ formalizes an idea exposed in Bouyssou and Pirlot (2002*a*). It is similar in spirit to a condition used in Greco *et al.* (2001).

In presence of the other conditions, going from $MAJ2_i$ to $MAJ3$ allows to go from a relation $\succsim_i^{**}$ having at most three equivalence classes to a relation $\succsim_i^{**}$ than can have up to five, the last equivalence class modeling discordance effects. The main result in this paper says that weakening $MAJ2$ to $MAJ3$ in Theorem 2 is exactly what is needed to characterize SOR. We have:

Theorem 3 *Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$. Then $\mathcal{P}$ is a SOR iff it is asymmetric and satisfies $ARC1$, $ARC2$, $MAJ1$ and $MAJ3$. Furthermore, conditions $ARC1$, $ARC2$, $MAJ1$ and $MAJ3$ are independent in the class of all asymmetric relations on X.*

As we shall see below, it is not difficult to strengthen this result imposing additional conditions that will imply that both P_i and V_i are strict semiorders.

If we neglect the question of the additive representation of the importance relation between coalitions of attributes using weights, the above theorem may therefore be seen as giving an axiomatic characterization of the binary relations that can be obtained with TACTIC. Within the framework of model (M), their main distinctive characteristics lie in conditions $MAJ1$ and $MAJ3$ implying that, on each attribute, only very few classes of preference differences are taken into account.

7 Discussion

The main contribution of this paper was to propose a characterization of SOR within the framework of model (M) that includes many different types of aggregation models as particular cases. Our characterization has emphasized the main specific feature of SOR with model (M), i.e., the option not to distinguish a rich preference difference relation on each attribute: only five distinct classes of preference differences on each attribute, with the lowest class playing a very particular role, are allowed in a SOR. This is in line with the common view of outranking methods as being "more ordinal" than the additive model (A).

Theorem 3 can be seen as giving a sound theoretical foundation to outranking methods aiming at building a strict preference relation using the CNDP, as in the TACTIC method (Vansnick, 1986). The results in this paper can be extended in several directions.

First, our definition of SOR does not require the relations P_i and V_i to possess any remarkable property besides asymmetry and the fact that $V_i \subseteq P_i$. This is at variance with what is done in most outranking methods. (remember, e.g., that in TACTIC both P_i and V_i are strict semiorders). Following the analysis in Bouyssou and Pirlot (2004), it is not difficult to tackle this case replacing in model (M), the terms $p_i(x_i, y_i)$ by terms $\varphi_i(u_i(x_i), u_i(y_i))$ where u_i are real-valued functions on X_i, φ_i are real-valued functions on $u_i(X_i)^2$ that are *skew symmetric* and *nondecreasing* in their first argument. Adding conditions $MAJ1$ and $MAJ3$ to the conditions underlying such a model allows to characterize SOR in which P_i and V_i are strict semiorders.

Second, we restricted our attention here to an asymmetric relation $\mathcal{P}$ interpreted as strict preference. It is not difficult to extend our analysis to reflexive relations, interpreted as "at least as good as" relations, for which: $x \; \mathcal{S} \; y \Leftrightarrow [S(x, y) \trianglerighteq S(y, x) \text{ and } V(y, x) = \varnothing]$, where $\mathcal{S}$ is a reflexive binary relation on X, S_i is a *complete* binary relation on X_i, $\trianglerighteq$ a relation on 2^N that is compatible with set inclusion and $S(x, y) = \{i \in N : x_i \, S_i \, y_i\}$ and V_i is included in the asymmetric part of S_i. Such an analysis, requires to distinguish "indifference" from "incomparability" and raises interesting duality questions. It is conducted in Bouyssou and Pirlot (2005a) when all relations V_i are empty. This is generalized in Bouyssou and Pirlot (2005b) to cope with discordance effects.

Appendix: proofs

Lemma 1 and Theorem 1

See Bouyssou and Pirlot (2003, Lemma 4) and Bouyssou and Pirlot (2003, Theorem 1).

Lemma 2

The proof of Lemma 2 will use the following definitions and results.

Definition 6 *Let $\mathcal{P}$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. This relation is said to satisfy:*
UC_i if

$$\left.\begin{array}{c} (x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \; \mathcal{P} \; (w_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i}) \\ \text{or} \\ (x_i, c_{-i}) \; \mathcal{P} \; (y_i, d_{-i}), \end{array}\right.$$

LC_i if

$$\left.\begin{array}{c} (x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i}) \\ \text{and} \\ (y_i, c_{-i}) \; \mathcal{P} \; (x_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i}) \\ \text{or} \\ (z_i, c_{-i}) \; \mathcal{P} \; (w_i, d_{-i}), \end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $\mathcal{P}$ satisfies UC (resp. LC) if it satisfies UC_i (resp. LC_i) for all $i \in N$.

Lemma 3

1. *$UC_i \Leftrightarrow [Not[(y_i, x_i) \succsim_i^* (x_i, y_i)] \Rightarrow (x_i, y_i) \succsim_i^* (z_i, w_i)$, for all $x_i, y_i, z_i, w_i \in X_i]$.*

2. *$LC_i \Leftrightarrow [Not[(y_i, x_i) \succsim_i^* (x_i, y_i)] \Rightarrow (z_i, w_i) \succsim_i^* (y_i, x_i)$, for all $x_i, y_i, z_i, w_i \in X_i]$.*

3. $[ARC1_i, ARC2_i, UC_i$ *and* $LC_i] \Rightarrow [\succsim_i^{**}$ *is a weak order having at most three equivalence classes*$]$.

PROOF
See Bouyssou and Pirlot (2003, Lemma 11). □

Lemma 4 *Let* $\mathcal{P}$ *be a binary relation on* $X = \prod_{i=1}^n X_i$. *If* $\succsim_i^{**}$ *is a weak order having at most three equivalence classes then* $\mathcal{P}$ *satisfies* $ARC1_i$, $ARC2_i$, UC_i *and* LC_i.

PROOF
The necessity of $ARC1_i$ and $ARC2_i$ follows from Part 3 of Lemma 1. Suppose that UC_i is violated so that $(x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i})$, $(z_i, c_{-i}) \; \mathcal{P} \; (w_i, d_{-i})$, $Not[(y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i})]$ and $Not[(x_i, c_{-i}) \; \mathcal{P} \; (y_i, d_{-i})]$. This implies $(x_i, y_i) \succ_i^* (y_i, x_i)$ and $(z_i, w_i) \succ_i^* (x_i, y_i)$. Using $ARC2_i$, we easily obtain $(z_i, w_i) \succ_i^{**} (x_i, y_i) \succ_i^{**} (x_i, x_i) \succ_i^{**} (y_i, x_i) \succ_i^{**} (w_i, z_i)$, so that $\succsim_i^{**}$ has five equivalence classes. The proof with LC_i is similar. □

Lemma 5

1. $UC_i \Rightarrow MAJ1_i$,

2. $LC_i \Rightarrow MAJ2_i$,

3. $[ARC1_i$ *and* $MAJ1_i] \Rightarrow UC_i$,

4. $[ARC2_i$ *and* $MAJ2_i] \Rightarrow LC_i$.

PROOF
Parts 1 and 2 are obvious since $MAJ1_i$ (resp. $MAJ2_i$) amounts to adding a premise to UC_i (resp. LC_i).

Part 3. Suppose that UC_i is violated so that $(x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i})$, $(z_i, c_{-i}) \; \mathcal{P} \; (w_i, d_{-i})$, $Not[(y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i})]$ and $Not[(x_i, c_{-i}) \; \mathcal{P} \; (y_i, d_{-i})]$, for some $x_i, y_i, z_i, w_i \in X_i$ and some $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$.

We distinguish two cases.

- If $(z_i, a_{-i}) \; \mathcal{P} \; (w_i, b_{-i})$, then $(x_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i})$, $(z_i, a_{-i}) \; \mathcal{P} \; (w_i, b_{-i})$ and $(z_i, c_{-i}) \; \mathcal{P} \; (w_i, d_{-i})$ imply, using $MAJ1_i$, $(y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i})$ or $(x_i, c_{-i}) \; \mathcal{P} \; (y_i, d_{-i})$, a contradiction.

- If $Not[(z_i, a_{-i}) \mathcal{P} (w_i, b_{-i})]$. Using $ARC1_i$, $(x_i, a_{-i}) \mathcal{P} (x_i, b_{-i})$ and $(z_i, c_{-i}) \mathcal{P} (w_i, d_{-i})$ imply $(z_i, a_{-i}) \mathcal{P} (w_i, b_{-i})$ or $(x_i, c_{-i}) \mathcal{P} (y_i, d_{-i})$. Hence, we must have $(x_i, c_{-i}) \mathcal{P} (y_i, d_{-i})$, a contradiction.

Part 4. Suppose that LC_i is violated so that $(x_i, a_{-i}) \mathcal{P} (y_i, b_{-i})$, $(y_i, c_{-i}) \mathcal{P} (x_i, d_{-i})$, $Not[(y_i, a_{-i}) \mathcal{P} (x_i, b_{-i})]$ and $Not[(z_i, c_{-i}) \mathcal{P} (w_i, d_{-i})]$, for some $x_i, y_i, z_i, w_i \in X_i$ and some $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$.

We distinguish two cases.

- If $(w_i, a_{-i}) \mathcal{P} (z_i, b_{-i})$, then $(x_i, a_{-i}) \mathcal{P} (y_i, b_{-i})$, $(w_i, a_{-i}) \mathcal{P} (z_i, b_{-i})$ and $(y_i, c_{-i}) \mathcal{P} (x_i, d_{-i})$ imply, using $MAJ2_i$, $(y_i, a_{-i}) \mathcal{P} (x_i, b_{-i})$ or $(z_i, c_{-i}) \mathcal{P} (w_i, d_{-i})$, a contradiction.
- If $Not[(w_i, a_{-i}) \mathcal{P} (z_i, b_{-i})]$. Using $ARC2_i$, $(x_i, a_{-i}) \mathcal{P} (x_i, b_{-i})$ and $(y_i, c_{-i}) \mathcal{P} (x_i, d_{-i})$ imply $(w_i, a_{-i}) \mathcal{P} (z_i, b_{-i})$ or $(z_i, c_{-i}) \mathcal{P} (w_i, d_{-i})$. Hence, we must have $(z_i, c_{-i}) \mathcal{P} (w_i, d_{-i})$, a contradiction. □

PROOF (LEMMA 2)
Necessity. The necessity of $ARC1$ and $ARC2$ follows from Lemma 4. The necessity of $MAJ1$ and $MAJ2$ follows from Lemma 4 and Parts 1 and 2 of Lemma 5.

Sufficiency. Parts 3 and 4 of Lemma 5 imply that UC and LC hold. Sufficiency results from Part 3 of Lemma 3. □

Theorem 2

The proof of Theorem 2 uses the following results.

Lemma 6 *Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$. If $\mathcal{P}$ is a strict concordance relation then all relations $\succsim_i^{**}$ are weak orders having at most three equivalence classes.*

PROOF
See Bouyssou and Pirlot (2003, Lemma 6). □

Lemma 7 *Let $\mathcal{P}$ be an asymmetric binary relation on $X = \prod_{i=1}^{n} X_i$. If all relations $\succsim_i^{**}$ are weak orders having at most three equivalence classes then $\mathcal{P}$ is a strict concordance relation.*

PROOF
See Bouyssou and Pirlot (2003, Lemma 7). □

Proof (Theorem 2)
The necessity of $ARC1$, $ARC2$, $MAJ1$ and $MAJ2$ follows from Lemma 6 and Lemma 2. Sufficiency follows from Lemma 2 and Lemma 7.

The following examples show that, in the class of asymmetric relations, conditions $ARC1$, $ARC2$, $MAJ1$ and $MAJ2$ are independent.

Example 1 (ARC1, MAJ1, MAJ2, Not[ARC2$_i$])
Let $X = \{a, b\} \times \{x, y\}$. Consider the asymmetric relation $\mathcal{P}$ on X containing only the two relations $(a, x)\ \mathcal{P}\ (b, y)$ and $(a, y)\ \mathcal{P}\ (b, x)$. We have, abusing notation:

- $(a, b) \succ_1^* [(a, a), (b, b), (b, a)]$ and
- $[(x, y), (y, x)] \succ_2^* [(x, x), (y, y)]$.

$ARC2_2$ is violated since $Not[(x, x) \succsim_2^* (x, y)]$ and $Not[(x, x) \succsim_2^* (y, x)]$. It is clear that $ARC1$ and $ARC2_1$ hold. It is easy to see, using Parts 1 and 2 of Lemma 3, that UC and LC hold. Hence $MAJ1$ and $MAJ2$ hold in view of Parts 3 and 4 of Lemma 5. $\diamond$

Example 2 (ARC1, ARC2, MAJ1,Not[MAJ2$_i$])
Let $X = \{a, b\} \times \{x, y, z\}$ and $\mathcal{P}$ on X be identical to the strict linear order (abusing notation in an obvious way):

$$(a, x)\ \mathcal{P}\ (a, y)\ \mathcal{P}\ (a, z)\ \mathcal{P}\ (b, x)\ \mathcal{P}\ (b, y)\ \mathcal{P}\ (b, z),$$

except that $(a, z)\ \mathcal{I}\ (b, x)$. It is easy to see that $\mathcal{P}$ is asymmetric. We have, abusing notation,

- $(a, b) \succ_1^* [(a, a), (b, b)] \succ_1^* (b, a)$ and
- $[(x, y), (x, z), (y, z)] \succ_2^* [(x, x), (y, y), (z, z), (y, x), (z, y)] \succ_2^* (z, x)$.

Using Lemma 1, it is easy to check that $\mathcal{P}$ satisfies $ARC1$ and $ARC2$. It is clear that UC_1, LC_1 and UC_2 hold. This shows that $MAJ1_1$, $MAJ1_2$ and $MAJ2_1$ hold. $MAJ2_2$ is violated since we have $(a, x)\ \mathcal{P}\ (a, y)$, $(a, x)\ \mathcal{P}\ (a, z)$, $(a, y)\ \mathcal{P}\ (b, x)$ but neither $(a, y)\ \mathcal{P}\ (a, x)$ nor $(a, z)\ \mathcal{P}\ (b, x)$. $\diamond$

Example 3 (ARC1, ARC2, MAJ2, Not[MAJ1$_i$])
Let $X = \{a, b\} \times \{x, y, z\}$ and $\mathcal{P}$ on X be identical to the strict linear order (abusing notation in an obvious way):

$$(a, x)\ \mathcal{P}\ (b, x)\ \mathcal{P}\ (a, y)\ \mathcal{P}\ (b, y)\ \mathcal{P}\ (a, z)\ \mathcal{P}\ (b, z),$$

except that $(b,x)\ \mathcal{I}\ (a,y)$. It is easy to see that $\mathcal{P}$ is asymmetric. We have, abusing notation:

- $(a,b) \succ_1^* [(a,a),(b,b)] \succ_1^* (b,a)$ and
- $[(x,z),(y,z)] \succ_2^* (x,y) \succ_2^* [(x,x),(y,y),(z,z)] \succ_2^* [(y,x),(z,x),(z,y)]$.

Using Lemma 1, it is easy to check that $\mathcal{P}$ satisfies $ARC1$ and $ARC2$. It is clear that UC_1, LC_1 and LC_2 hold. This shows that $MAJ1_1$, $MAJ2_1$ and $MAJ2_2$ hold. $MAJ1_2$ is violated since $(a,x)\ \mathcal{P}\ (a,y)$, $(a,x)\ \mathcal{P}\ (a,z)$, $(b,x)\ \mathcal{P}\ (a,z)$ but neither $(a,y)\ \mathcal{P}\ (a,x)$ nor $(b,x)\ \mathcal{P}\ (a,y)$. $\diamond$

Example 4 ($ARC2$, $MAJ2$, $MAJ1$, $Not[ARC1_i]$)
Let $X = \{x,y,z\} \times \{a,b,c\} \times \{p,q,r\}$ and $\mathcal{P}$ on X be empty except that: $(x,a,r)\ \mathcal{P}\ (y,b,p)$, $(z,a,r)\ \mathcal{P}\ (x,b,p)$, $(x,a,p)\ \mathcal{P}\ (y,b,q)$, $(z,c,r)\ \mathcal{P}\ (x,a,p)$, $(y,a,r)\ \mathcal{P}\ (x,b,p)$, $(x,a,r)\ \mathcal{P}\ (z,b,p)$, $(x,a,r)\ \mathcal{P}\ (x,b,p)$, $(y,a,r)\ \mathcal{P}\ (y,b,p)$, $(z,a,r)\ \mathcal{P}\ (z,b,p)$, $(z,a,r)\ \mathcal{P}\ (y,b,p)$, $(y,a,r)\ \mathcal{P}\ (z,b,p)$, $(z,a,r)\ \mathcal{P}\ (x,a,p)$, $(z,b,r)\ \mathcal{P}\ (x,b,p)$, $(z,c,r)\ \mathcal{P}\ (x,c,p)$, $(z,b,r)\ \mathcal{P}\ (x,a,p)$, $(z,a,r)\ \mathcal{P}\ (x,c,p)$, $(z,b,r)\ \mathcal{P}\ (x,c,p)$, $(z,c,r)\ \mathcal{P}\ (x,b,p)$, $(x,a,p)\ \mathcal{P}\ (y,b,p)$, $(x,a,q)\ \mathcal{P}\ (y,b,q)$, $(x,a,r)\ \mathcal{P}\ (y,b,r)$, $(x,a,p)\ \mathcal{P}\ (y,b,r)$, $(x,a,q)\ \mathcal{P}\ (y,b,p)$, $(x,a,q)\ \mathcal{P}\ (y,b,r)$, $(x,a,r)\ \mathcal{P}\ (y,b,q)$.

It is easy to check that $\mathcal{P}$ is asymmetric. We have, abusing notation:

- $(x,y) \succ_1^* [(x,x),(y,y),(z,z),(y,x),(x,z),(y,z),(z,y)]$ and $(z,x) \succ_1^* [(x,x),(y,y),(z,z),(y,x),(x,z),(y,z),(z,y)]$, while (x,y) and (z,x) are incomparable in terms of $\succsim_1^*$.
- $(a,b) \succ_2^* [(a,a),(b,b),(c,c),(b,a),(b,c),(c,b),(a,c),(c,a)]$,
- $(r,p) \succ_3^* [(p,p),(q,q),(r,r),(p,q),(q,p),(p,r),(q,r),(r,q)]$.

For $j \in \{2,3\}$, $ARC1_j$, $ARC2_j$, UC_j and LC_j are clearly satisfied. On attribute 1, it is easy to check that $ARC2_1$ is satisfied, while $ARC1_1$ is violated. Condition LC_1 holds so that $MAJ2_1$ holds.

It remains to check that condition $MAJ1_1$ holds. Consider the first premise of $MAJ1_i$ $(x_i,a_{-i})\ \mathcal{P}\ (y_i,b_{-i})$. One of the two possible conclusions of $MAJ1_i$ is that $(y_i,a_{-i})\ \mathcal{P}\ (x_i,b_{-i})$.

Since $(x,y) \succ_1^* (y,x)$, it is clear that $MAJ1_1$ cannot be violated if $x_i = y$ and $y_i = x$. Similarly, since $(z,x) \succ_1^* (x,z)$, $MAJ1_1$ cannot be violated if $x_i = x$ and $y_i = z$.

Given the structure of $\succsim_1^*$, there are several cases to examine.

Suppose first that (x_i, y_i) in the first premise is equal to (x, y). If the first premise is taken to be (x, a, r) $\mathcal{P}$ (y, b, p) then $MAJ1_1$ cannot be violated since we have (y, a, r) $\mathcal{P}$ (x, b, p). Now if any other premise is taken, there will not be any (z_i, w_i) satisfying the second premise. A similar reasoning shows that if (x_i, y_i) is equal to (z, x), then $MAJ1_1$ cannot be violated.

Suppose now that (x_i, y_i) in the first premise is distinct from (x, y), (y, x), (z, x) and (x, z). Then the only possibility is take $a_{-i} = (a, r)$ and $b_{-i} = (b, p)$. Therefore $MAJ1_1$ cannot be violated because, for all $\alpha, \beta \in X_1$, we have (α, a, r) $\mathcal{P}$ (β, b, p) and (β, a, r) $\mathcal{P}$ (α, b, p). Hence, $MAJ1_1$ holds. $\diamond$

$\square$

Theorem 3

The proof will use the following results.

Lemma 8 *Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$. If $\mathcal{P}$ is a SOR then it satisfies $ARC1$, $ARC2$, $MAJ1$ and $MAJ3$.*

Proof

[$ARC1$] Let $\langle \rhd, P_i, V_i \rangle$ be a representation of $\mathcal{P}$ as a SOR. Suppose that (x_i, a_{-i}) $\mathcal{P}$ (y_i, b_{-i}), (z_i, c_{-i}) $\mathcal{P}$ (w_i, d_{-i}). This implies that $Not[y_i \; V_i \; x_i]$ and $Not[w_i \; V_i \; z_i]$. Suppose that $y_i \; P_i \; x_i$. The definition of a SOR implies that (z_i, a_{-i}) $\mathcal{P}$ (w_i, b_{-i}). If $x_i \; P_i \; y_i$, the definition of a SOR implies (x_i, c_{-i}) $\mathcal{P}$ (y_i, d_{-i}). If $x_i \; I_i \; y_i$ and $z_i \; S_i \; w_i$, the definition of a SOR implies (z_i, a_{-i}) $\mathcal{P}$ (w_i, b_{-i}). If $x_i \; I_i \; y_i$ and $w_i \; S_i \; z_i$, the definition of a SOR implies (x_i, c_{-i}) $\mathcal{P}$ (y_i, d_{-i}).

[$ARC2$] Suppose that (x_i, a_{-i}) $\mathcal{P}$ (y_i, b_{-i}), (y_i, c_{-i}) $\mathcal{P}$ (x_i, d_{-i}). This implies that $Not[y_i \; V_i \; x_i]$ and $Not[x_i \; V_i \; y_i]$.

Suppose that $x_i \; P_i \; y_i$. If $z_i \; P_i \; w_i$, we know that $Not[w_i \; V_i \; z_i]$ and the definition of a SOR leads to (z_i, a_{-i}) $\mathcal{P}$ (w_i, b_{-i}). If $w_i \; P_i \; z_i$ or $w_i \; I_i \; z_i$, we know that $Not[z_i \; V_i \; w_i]$. The definition of a SOR leads to (w_i, c_{-i}) $\mathcal{P}$ (z_i, d_{-i}). The proof is similar if we suppose that $y_i \; P_i \; x_i$.

Suppose now that $x_i \; I_i \; y_i$. If $z_i \; P_i \; w_i$ or $z_i \; I_i \; w_i$, we know that $Not[w_i \; V_i \; z_i]$ and the definition of a SOR implies that (z_i, a_{-i}) $\mathcal{P}$ (w_i, b_{-i}). If $w_i \; P_i \; z_i$, we know that $Not[z_i \; V_i \; w_i]$ and the definition of a SOR implies that (w_i, c_{-i}) $\mathcal{P}$ (z_i, d_{-i}).

[$MAJ1$] Suppose that (x_i, a_{-i}) $\mathcal{P}$ (y_i, b_{-i}), (z_i, a_{-i}) $\mathcal{P}$ (w_i, b_{-i}) and (z_i, c_{-i}) $\mathcal{P}$ (w_i, d_{-i}). This implies that $Not[y_i \; V_i \; x_i]$ and $Not[w_i \; V_i \; z_i]$.

If $y_i \; P_i \; x_i$, we know that $Not[x_i \; V_i \; y_i]$ so that we have $(y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i})$, using the definition of a SOR. If $x_i \; P_i \; y_i$, we know that $Not[y_i \; V_i \; x_i]$ so that we have $(x_i, c_{-i}) \; \mathcal{P} \; (y_i, d_{-i})$, using the definition of a SOR. If $x_i \; I_i \; y_i$, the definition of a SOR implies $(y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i})$.

$[MAJ3]$ Suppose that $(x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i})$, $(w_i, a_{-i}) \; \mathcal{P} \; (z_i, b_{-i})$, $(y_i, c_{-i}) \; \mathcal{P} \; (x_i, d_{-i})$ and $(z_i, e_{-i}) \; \mathcal{P} \; (w_i, f_{-i})$. By construction, we know that $Not[x_i \; V_i \; y_i]$, $Not[y_i \; V_i \; x_i]$, $Not[z_i \; V_i \; w_i]$ and $Not[w_i \; V_i \; z_i]$. If $y_i \; P_i \; x_i$, the definition of a SOR implies $(y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i})$. If $x_i \; I_i \; y_i$, the definition of a SOR implies $(y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i})$. If $x_i \; P_i \; y_i$, the definition of a SOR implies $(z_i, c_{-i}) \; \mathcal{P} \; (w_i, d_{-i})$. □

Lemma 9 *Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$. If $\mathcal{P}$ satisfies $ARC1$, $ARC2$, $MAJ1$ and $MAJ3$, then, for all $x_i, y_i, z_i, w_i, r_i, s_i \in X_i$,*

1. *$(x_i, y_i) \succ_i^* (y_i, x_i) \Rightarrow (x_i, y_i) \succsim_i^* (z_i, w_i)$.*
2. *$[(x_i, y_i) \succ_i^* (y_i, x_i) \succ_i^* (z_i, w_i)] \Rightarrow (r_i, s_i) \succsim_i^* (z_i, w_i)$. Furthermore, we have $Not[(z_i, a_{-i}) \; \mathcal{P} \; (w_i, b_{-i})]$, for all $a_{-i}, b_{-i} \in X_{-i}$.*

Proof

Part 1 follows from Lemma 3, since $ARC1_i$ and $MAJ1_i$ imply UC_i.

Part 2. Suppose that, for some $x_i, y_i, z_i, w_i, r_i, s_i \in X_i$, we have $(x_i, y_i) \succ_i^* (y_i, x_i) \succ_i^* (z_i, w_i)$ and $(z_i, w_i) \succ_i^* (r_i, s_i)$. This implies $(x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i})$, $Not[(y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i})]$, $(y_i, c_{-i}) \; \mathcal{P} \; (x_i, d_{-i})$, $Not[(z_i, c_{-i}) \; \mathcal{P} \; (w_i, d_{-i})]$ and $(z_i, e_{-i}) \; \mathcal{P} \; (w_i, f_{-i})$, for some $a_{-i}, b_{-i}, c_{-i}, d_{-i}, e_{-i}, f_{-i} \in X_{-i}$.

Using $ARC2$, we know that $(w_i, z_i) \succsim_i^* (x_i, y_i)$ so that $(w_i, a_{-i}) \; \mathcal{P} \; (z_i, b_{-i})$. Using $MAJ3_i$, $(x_i, a_{-i}) \; \mathcal{P} \; (y_i, b_{-i})$, $(w_i, a_{-i}) \; \mathcal{P} \; (z_i, b_{-i})$, $(y_i, c_{-i}) \; \mathcal{P} \; (x_i, d_{-i})$, and $(z_i, e_{-i}) \; \mathcal{P} \; (w_i, f_{-i})$ imply $(y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i})$ or $(z_i, c_{-i}) \; \mathcal{P} \; (w_i, d_{-i})$, a contradiction. Note that the contradiction is obtained as soon as $(z_i, e_{-i}) \; \mathcal{P} \; (w_i, f_{-i})$, for some $e_{-i}, f_{-i} \in X_{-i}$. This proves the second part of the assertion. □

Lemma 10 *Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^{n} X_i$ satisfying $ARC1$, $ARC2$, $MAJ1$ and $MAJ3$. Define the relation P_i on X_i defined letting, for all $x_i, y_i \in X_i$,*

$$x_i \; P_i \; y_i \Leftrightarrow (x_i, y_i) \succ_i^* (y_i, x_i).$$

We have:

1. *P_i is asymmetric and nonempty,*

2. $z_i \; P_i \; w_i$ *and* $x_i \; P_i \; y_i$ *imply* $(z_i, w_i) \sim_i^* (x_i, y_i)$,

3. $x_i \; I_i \; y_i$ *and* $z_i \; I_i \; w_i$ *imply* $(x_i, y_i) \sim_i^* (z_i, w_i) \sim_i^* (y_i, x_i) \sim_i^* (w_i, z_i) \sim_i^* (a_i, a_i)$,

4. $z_i \; P_i \; w_i$ *and* $x_i \; I_i \; y_i$ *imply* $(z_i, w_i) \succsim_i^* (x_i, y_i)$ *and* $(x_i, y_i) \succsim_i^* (w_i, z_i)$.

Proof

Part 1. We have $x_i \; P_i \; y_i$ iff $(x_i, y_i) \succ_i^* (y_i, x_i)$. Since $\succ_i^*$ is asymmetric, it follows that P_i is asymmetric. The influence of $i \in N$ implies that there are $x_i, y_i, z_i, w_i \in X_i$ such that $(x_i, y_i) \succ_i^* (z_i, w_i)$. If $(z_i, w_i) \succsim_i^* (y_i, x_i)$, we obtain $(x_i, y_i) \succ_i^* (y_i, x_i)$ so that $x_i \; P_i \; y_i$. Suppose that $(y_i, x_i) \succ_i^* (z_i, w_i)$. It is impossible that $(z_i, w_i) \succsim_i^* (x_i, x_i)$ since, using the fact that $\succsim_i^*$ is a weak order, this would imply $(x_i, y_i) \succ_i^* (x_i, x_i)$ and $(y_i, x_i) \succ_i^* (x_i, x_i)$, violating $ARC2$. Hence we have $(x_i, x_i) \succ_i^* (z_i, w_i)$. Using $ARC2$, this implies $(w_i, y_i) \succsim_i^* (x_i, x_i)$ so that $(w_i, y_i) \succ_i^* (z_i, w_i)$ and $w_i \; P_i \; z_i$. Hence, P_i is not empty.

Part 2. Suppose that $z_i \; P_i \; w_i$ and $x_i \; P_i \; y_i$. Using lemma 9, we have $(z_i, w_i) \succsim_i^* (x_i, y_i)$ and $(x_i, y_i) \succsim_i^* (z_i, w_i)$ so that $(z_i, w_i) \sim_i^* (x_i, y_i)$.

Part 3. Suppose that $x_i \; I_i \; y_i$ and $z_i \; I_i \; w_i$. Using the definition of P_i and $ARC1$, this implies $(x_i, y_i) \sim_i^* (y_i, x_i)$ and $(z_i, w_i) \sim_i^* (w_i, z_i)$. The conclusion follows from $ARC2$.

Part 4. Suppose that $z_i \; P_i \; w_i$ and $x_i \; I_i \; y_i$, so that $(x_i, y_i) \sim_i^* (y_i, x_i)$ and $(z_i, w_i) \succ_i^* (w_i, z_i)$. If $(x_i, y_i) \succ_i^* (z_i, w_i)$, $ARC2_i$ implies $(w_i, z_i) \succsim_i^* (y_i, x_i)$, a contradiction. Similarly if $(w_i, z_i) \succ_i^* (x_i, y_i)$, $ARC2_i$ implies $(y_i, x_i) \succsim_i^* (z_i, w_i)$, a contradiction. □

Lemma 11 *Let $\mathcal{P}$ be a binary relation on $X = \prod_{i=1}^n X_i$ satisfying $ARC1$, $ARC2$, $MAJ1$ and $MAJ3$. Define the relation V_i on X_i letting, for all $x_i, y_i \in X_i$,*

$$x_i \; V_i \; y_i \Leftrightarrow \left[(z_i, w_i) \succ_i^* (w_i, z_i) \succ_i^* (y_i, x_i) \text{ for some } z_i, w_i \in X_i\right].$$

We have:

1. V_i *is included in* P_i.

2. $z_i \; V_i \; w_i$ *and* $x_i \; V_i \; y_i$ *imply* $(w_i, z_i) \sim_i^* (y_i, x_i)$.

3. $z_i \; P_i \; w_i$, *Not*$[z_i \; V_i \; w_i]$, $x_i \; P_i \; y_i$ *and Not*$[x_i \; V_i \; y_i]$, *imply* $(w_i, z_i) \sim_i^* (y_i, x_i)$.

4. $z_i \; V_i \; w_i$*,* $x_i \; P_i \; y_i$*,* $Not[x_i \; V_i \; y_i]$ *imply* $(y_i, x_i) \succ_i^* (w_i, z_i)$*.*

PROOF
Part 1. We have $x_i \; V_i \; y_i$ iff $(z_i, w_i) \succ_i^* (w_i, z_i) \succ_i^* (y_i, x_i)$. Using $ARC2_i$, $(w_i, z_i) \succ_i^* (y_i, x_i)$ implies $(x_i, y_i) \succsim_i^* (z_i, w_i)$, so that $(x_i, y_i) \succ_i^* (y_i, x_i)$ and $x_i \; P_i \; y_i$.

Part 2. Suppose that $x_i \; V_i \; y_i$ and $z_i \; V_i \; w_i$. By definition, we have $(a_i, b_i) \succ_i^* (b_i, a_i) \succ_i^* (y_i, x_i)$ and $(c_i, d_i) \succ_i^* (d_i, c_i) \succ_i^* (w_i, z_i)$ Using lemma 9, we know that, for all $a_{-i}, b_{-i} \in X_{-i}$, we have $Not[(y_i, a_{-i}) \; \mathcal{P} \; (x_i, b_{-i})]$ and $Not[(w_i, a_{-i}) \; \mathcal{P} \; (z_i, b_{-i})]$, so that $(y_i, x_i) \sim_i^* (w_i, z_i)$.

Part 3. Suppose that $z_i \; P_i \; w_i$ and $x_i \; P_i \; y_i$ By definition, we have $(x_i, y_i) \succ_i^* (y_i, x_i)$ and $(z_i, w_i) \succ_i^* (w_i, z_i)$. Suppose that $(y_i, x_i) \succ_i^* (w_i, z_i)$. This would imply $(x_i, y_i) \succ_i^* (y_i, x_i) \succ_i^* (w_i, z_i)$, contradicting the fact that $Not[z_i \; V_i \; w_i]$. Similarly it is impossible that $(w_i, z_i) \succ_i^* (y_i, x_i)$. Hence, we have $(y_i, x_i) \sim_i^* (w_i, z_i)$.

Part 4. Suppose that $z_i \; V_i \; w_i$, $x_i \; P_i \; y_i$ and $Not[x_i \; V_i \; y_i]$. We have $(a_i, b_i) \succ_i^* (b_i, a_i) \succ_i^* (w_i, z_i)$ and $(x_i, y_i) \succ_i^* (y_i, x_i)$. Supposing that $(b_i, a_i) \succ_i^* (y_i, x_i)$ would contradict the fact that $Not[x_i \; V_i \; y_i]$. Hence, we have $(y_i, x_i) \succsim_i^* (b_i, a_i)$, so that $(y_i, x_i) \succ_i^* (w_i, z_i)$. □

PROOF (THEOREM 3)
Necessity follows from Lemmas 8. We show sufficiency. Define the relation P_i and V_i on X_i as in Lemmas 10 and 11. We have shown that P_i is asymmetric and nonempty and that V_i is included in P_i.

Consider any two disjoint subsets $A, B \subseteq N$ and let:

$$A \rhd B \Leftrightarrow$$
$$[x \; \mathcal{P} \; y, \text{ for some } x, y \in X \text{ such that } P(x, y) = A \text{ and } P(y, x) = B].$$

Suppose that $x \; \mathcal{P} \; y$. Using Part 2 of Lemma 9, we know that $V(y, x) = \varnothing$. By construction, we have $P(x, y) \rhd P(y, x)$.

Suppose now that $V(y, x) = \varnothing$ and $P(x, y) \rhd P(y, x)$ and let us show that we have $x \; \mathcal{P} \; y$. By construction, $P(x, y) \rhd P(y, x)$ implies that there are $z, w \in X$ such that $z \; \mathcal{P} \; w$, $P(x, y) = P(z, w)$ and $P(y, x) = P(w, z)$.

For all $i \in N$ such that $z_i \; I_i \; w_i$, we have, by construction, $x_i \; I_i \; y_i$ so that, using Lemma 10, $(x_i, y_i) \sim_i^* (z_i, w_i)$.

For all $i \in N$ such that $z_i \; P_i \; w_i$ we have $x_i \; P_i \; y_i$ so that, using lemma 10, $(x_i, y_i) \sim_i^* (z_i, w_i)$.

For all $i \in N$ such that $w_i \; P_i \; z_i$, we have $y_i \; P_i \; x_i$. By hypothesis, we have $Not[y_i \; V_i \; x_i]$. Because $z \; \mathcal{P} \; w$, we have $Not[w_i \; V_i \; z_i]$. Using lemma 10, we know that $(x_i, y_i) \sim_i^* (z_i, w_i)$.

Hence, we have $(x_i, y_i) \sim_i^* (z_i, w_i)$, for all $i \in N$ so that $z \; \mathcal{P} \; w$ implies $x \; \mathcal{P} \; y$.

It remains to show that $\rhd$ is monotonic. Suppose that $A \rhd B$, so that, for some $x, y \in X$, $P(x, y) = A$, $P(y, x) = B$ and $x \; \mathcal{P} \; y$. Since $x \; \mathcal{P} \; y$, we know that $Not[y_i \; V_i \; x_i]$, for all $i \in N$. Suppose that $C \supseteq A$, $B \supseteq D$ and $C \cap D = \varnothing$. Let $E = C \cap B$, $F = C \setminus [A \cup B]$, so that $C = A \cup E \cup F$. Let $G = B \setminus [D \cup C]$, so that $B = D \cup F \cup G$. Let $H = N \setminus [C \cup B]$. Since P_i is nonempty, choose on each $i \in N$, $a_i, b_i \in X_i$ such that $a_i \; P_i \; b_i$.

Consider the following alternatives:

	A	E	F	D	G	H
x	x_i	x_i	x_i	x_i	x_i	x_i
y	y_i	y_i	y_i	y_i	y_i	y_i
z	x_i	a_i	a_i	x_i	a_i	x_i
w	y_i	b_i	b_i	y_i	a_i	y_i

For all $i \in A \cup D \cup H$, we clearly have $(z_i, w_i) \sim_i^* (x_i, y_i)$. For all $i \in E \cup F$, we have $z_i \; P_i \; w_i$ so that, using Lemma 9, $(z_i, w_i) \succsim_i^* (x_i, y_i)$. For all $i \in G$, we have $y_i \; P_i \; x_i$ and $z_i \; I_i \; w_i$. Using Lemma 10, this implies $(z_i, w_i) \succsim_i^* (x_i, y_i)$. Hence, for all $i \in N$, we have $(z_i, w_i) \succsim_i^* (x_i, y_i)$. Therefore, $x \; \mathcal{P} \; y$ implies $z \; \mathcal{P} \; w$, so that we obtain $C \rhd D$.

It is clear that $MAJ2_i$ implies $MAJ3_i$. Hence, in view of Theorem 2, the independence of the conditions will be established if we give an example of relation satisfying $ARC1$, $ARC2$, $MAJ1$ and $MAJ3_i$ on all but one attribute.

Example 5 (ARC1, ARC2, MAJ1, Not[MAJ3$_i$])

Let $X = \{x, y, z\} \times \{a, b\} \times \{p, q\}$ and $\mathcal{P}$ on X be empty except that: $(x, a, p) \; \mathcal{P} \; (x, b, p)$, $(x, a, p) \; \mathcal{P} \; (x, b, q)$, $(x, a, p) \; \mathcal{P} \; (y, a, p)$, $(x, a, p) \; \mathcal{P} \; (y, a, q)$, $(x, a, p) \; \mathcal{P} \; (y, b, p)$, $(x, a, p) \; \mathcal{P} \; (y, b, q)$, $(x, a, p) \; \mathcal{P} \; (z, a, p)$, $(x, a, p) \; \mathcal{P} \; (z, a, q)$, $(x, a, p) \; \mathcal{P} \; (z, b, p)$, $(x, a, p) \; \mathcal{P} \; (z, b, q)$, $(x, a, q) \; \mathcal{P} \; (x, b, q)$, $(x, a, q) \; \mathcal{P} \; (y, a, q)$, $(x, a, q) \; \mathcal{P} \; (y, b, q)$, $(x, a, q) \; \mathcal{P} \; (z, a, q)$, $(x, a, q) \; \mathcal{P} \; (z, b, q)$, $(x, b, p) \; \mathcal{P} \; (y, b, p)$, $(x, b, p) \; \mathcal{P} \; (y, b, q)$, $(x, b, p) \; \mathcal{P} \; (z, b, p)$, $(x, b, p) \; \mathcal{P} \; (z, b, q)$, $(x, b, q) \; \mathcal{P} \; (y, b, q)$, $(x, b, q) \; \mathcal{P} \; (z, b, q)$, $(y, a, p) \; \mathcal{P} \; (x, b, p)$, $(y, a, p) \; \mathcal{P} \; (x, b, q)$, $(y, a, p) \; \mathcal{P} \; (y, b, p)$, $(y, a, p) \; \mathcal{P} \; (y, b, q)$, $(y, a, p) \; \mathcal{P} \; (z, a, p)$, $(y, a, p) \; \mathcal{P} \; (z, a, q)$, $(y, a, p) \; \mathcal{P} \; (z, b, p)$, $(y, a, p) \; \mathcal{P} \; (z, b, q)$, $(y, a, q) \; \mathcal{P} \; (x, b, q)$, $(y, a, q) \; \mathcal{P} \; (y, b, q)$, $(y, a, q) \; \mathcal{P} \; (z, a, q)$, $(y, a, q) \; \mathcal{P} \; (z, b, q)$, $(y, b, p) \; \mathcal{P} \; (z, b, p)$, $(y, b, p) \; \mathcal{P} \; (z, b, q)$, $(y, b, q) \; \mathcal{P} \; (z, b, q)$,

$(z,a,p)\ \mathcal{P}\ (x,b,q)$, $(z,a,p)\ \mathcal{P}\ (y,b,p)$, $(z,a,p)\ \mathcal{P}\ (y,b,q)$, $(z,a,p)\ \mathcal{P}\ (z,b,p)$, $(z,a,p)\ \mathcal{P}\ (z,b,q)$, $(z,a,q)\ \mathcal{P}\ (y,b,q)$, $(z,a,q)\ \mathcal{P}\ (z,b,q)$.

It is easy to check that $\mathcal{P}$ is asymmetric. It is not difficult to see that we have, abusing notation,

- $[(x,y),(x,z),(y,z)] \succ_1^* [(x,x),(y,y),(z,z),(y,x),(z,y)] \succ_1^* (z,x)$.
- $(a,b) \succ_2^* [(a,a),(b,b)] \succ_2^* (b,a)$ and
- $(p,q) \succ_3^* [(p,p),(q,q)] \succ_3^* (q,p)$.

This shows that $ARC1$, $ARC2$ and $MAJ1$ hold. It is easy to see that $MAJ2_2$ and $MAJ2_3$ hold so that $MAJ3_2$ and $MAJ3_3$ are satisfied. Condition $MAJ3_1$ is violated since $(x,a,p)\ \mathcal{P}\ (y,a,p)$, $(x,a,p)\ \mathcal{P}\ (z,a,p)$, $(y,a,p)\ \mathcal{P}\ (x,b,p)$ and $(z,a,p)\ \mathcal{P}\ (x,b,q)$ but neither $(y,a,p)\ \mathcal{P}\ (x,a,p)$ nor $(z,a,p)\ \mathcal{P}\ (x,b,p)$. $\diamond$ $\square$

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