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1. Introduction

The relative Chern character was defined by Atiyah and al. in [2, 5] as a map

$$\mathbf{Ch}_{X\setminus Y} : \mathbf{K}^0(X,Y) \longrightarrow \mathrm{H}^*(X,Y). \tag{1}$$

Here $Y \subset X$ are *finite* CW-complexes, $\mathbf{K}^0(X,Y)$ is the relative K-group and $\mathrm{H}^*(X,Y)$ is the singular relative cohomology group.

The relative Chern character enjoys various functorial properties. In particular, **Ch** is multiplicative: the following diagram

$$\begin{array}{ccccc} \mathbf{K}^0(X,Y) & \times & \mathbf{K}^0(X,Y') & \xrightarrow{\odot} & \mathbf{K}^0(X,Y\cup Y') \\ \downarrow{\scriptstyle \mathbf{Ch}_{X\setminus Y}} & & \downarrow{\scriptstyle \mathbf{Ch}_{X\setminus Y'}} & & \downarrow{\scriptstyle \mathbf{Ch}_{X\setminus Y\cup Y'}} \\ \mathrm{H}^*(X,Y) & \times & \mathrm{H}^*(X,Y') & \xrightarrow{\diamond} & \mathrm{H}^*(X,Y\cup Y') \end{array} \tag{2}$$

is commutative. Here $Y, Y' \subset X$ are *finite* CW-complexes. and $\odot$ and $\diamond$ denote the products. This property was extended to the case where X is a paracompact topological space and Y any *open* subset of X by Iversen in [14] (see also [12, 13]). Iversen deduces the existence of the local Chern character from functorial properties, but his construction is not explicit.

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In this article, we work in the context of manifolds and differential forms. Indeed, in this framework, Quillen [18] constructed a very natural de Rham relative cohomology class associated to a smooth morphism between vector bundles, that we call the relative Quillen Chern character. Let N be a manifold (*not necessarily compact*), and let $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ be a morphism of complex vector bundles over N. Let $\mathrm{Supp}(\sigma)$ be the *support* of σ : it is the set of points $n \in N$ where $\sigma(n)$ is not invertible. We do not suppose that $\mathrm{Supp}(\sigma)$ is compact. Quillen [18] associates to σ a couple (α, β) of differential forms, where α is given by the usual Chern-Weil construction, and β is also constructed à la Chern-Weil, via super-connections. The form α is a closed differential form on N representing the difference of Chern characters $\mathrm{Ch}(\mathcal{E}^+) - \mathrm{Ch}(\mathcal{E}^-) \in \mathcal{H}^*(N)$, and β is a differential form on $N \setminus \mathrm{Supp}(\sigma)$ such that

$$\alpha|_{N\setminus \mathrm{Supp}(\sigma)} = d\beta.$$

The couple (α, β) defines then an explicit relative de Rham cohomology class

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma) \in \mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma)).$$

The main purpose of this note is to show that Quillen's relative Chern character $\mathrm{Ch}_{\mathrm{rel}}$ is multiplicative. If σ_1, σ_2 are two morphisms on N, then the product $\sigma_1 \odot \sigma_2$ is a morphism on N with support equal to $\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)$. We prove in Section 4 that the following equality

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) = \mathrm{Ch}_{\mathrm{rel}}(\sigma_1) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2) \tag{3}$$

holds in $\mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma_1 \odot \sigma_2))$.

Our proof of this equality is direct. Denote by $[\alpha, \beta]$ the class of (α, β) in the de Rham relative cohomology. In order to prove the formula $[\alpha_1, \beta_1] \diamond [\alpha_2, \beta_2] = [\alpha_1 \wedge \alpha_2, \beta_{12}]$, we give an explicit equality (Equality (26)) between the forms β_1, β_2 and β_{12} attached by Quillen to σ_1, σ_2 and $\sigma_1 \odot \sigma_2$. We give an explicit example of the equalities involved in Subsection 5.3.3. Intuitively (and true in many analytic cases), the relative Chern class could also be represented as a current supported on $\mathrm{Supp}(\sigma)$, but currents do not usually multiply. Thus another procedure, involving a choice of partition of unity, is needed to define the product $\diamond$ of relative classes in de Rham relative cohomology.

The multiplicativity property (3) can also be deduced from the fact that Quillen's Chern character gives an explicit representative of Iversen's local Chern character, due to Schneiders functorial characterization of Iversen's class (see [19]). Our proof does not use Iversen's construction. As we need the same argument for more general situations, we feel worthwhile to present this down-to-earth proof.

When $\mathrm{Supp}(\sigma)$ is *compact*, there is a natural homomorphism from $\mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma))$ into the compactly supported cohomology algebra $\mathcal{H}^*_c(N)$ and the image of the Quillen relative Chern character $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ is the Chern character $\mathrm{Ch}_c(\sigma)$ with compact support. The equality (3) implies the relation

$$\mathrm{Ch}_c(\sigma_1 \odot \sigma_2) = \mathrm{Ch}_c(\sigma_1) \wedge \mathrm{Ch}_c(\sigma_2) \quad \text{in} \quad \mathcal{H}^*_c(N). \tag{4}$$

This last relation is well known and follows also from the fact that $\mathrm{Ch}_c(\sigma)$ is the Chern character of a difference bundle on a compactification of N.

In Theorem 5.5, we refine this relation to the case where $\mathrm{Supp}(\sigma_1)$ and $\mathrm{Supp}(\sigma_2)$ are not necessarily compact by associating to the relative class $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ a closed differential form on N supported very near the closed set $\mathrm{Supp}(\sigma)$. Here the product

is given by the natural wedge product of forms, but it is at the expense of fattening the support of σ.

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2. Cohomological structures

Let N be a manifold. We denote by $\mathcal{A}^*(N)$ the algebra of differential forms on N and by $\mathcal{H}^*(N)$ the de Rham cohomology algebra of N with complex coefficients. We denote by $\mathcal{H}^*_c(N)$ its compactly supported cohomology algebra. There is a natural map $\mathcal{H}^*_c(N) \to \mathcal{H}^*(N)$.

2.1. **Relative cohomology.** Let F be a closed subset of N. To a cohomology class on N vanishing on $N \setminus F$, we associate a relative cohomology class. Let us explain the construction (see [9]). Consider the complex $\mathcal{A}^*(N, N \setminus F)$ with

$$\mathcal{A}^k(N, N \setminus F) := \mathcal{A}^k(N) \oplus \mathcal{A}^{k-1}(N \setminus F)$$

and differential $d_{\mathrm{rel}}(\alpha, \beta) = \big(d\alpha, \alpha|_{N\setminus F} - d\beta\big)$.

Definition 2.1. *The cohomology of the complex $(\mathcal{A}^*(N, N \setminus F), d_{\mathrm{rel}})$ is the relative cohomology space $\mathcal{H}^*(N, N \setminus F)$.*

The class defined by a d_{rel}-closed element $(\alpha, \beta) \in \mathcal{A}^*(N, N \setminus F)$ will be denoted $[\alpha, \beta]$. We review the basic facts of the relative cohomology groups :

- There is a natural map $\mathcal{H}^*(N, N \setminus F) \to \mathcal{H}^*(N)$.
- If $F \subset F'$ are closed subsets of N, there is a natural restriction map $\mathcal{H}^*(N, N \setminus F) \to \mathcal{H}^*(N, N \setminus F')$.
- If F_1 and F_2 are closed subsets of N, there is a natural product

$$\begin{array}{ccc} \mathcal{H}^*(N, N \setminus F_1) \times \mathcal{H}^*(N, N \setminus F_2) & \longrightarrow & \mathcal{H}^*(N, N \setminus (F_1 \cap F_2)) \\ (\quad a \quad , \quad b \quad) & \longmapsto & a \diamond b\ . \end{array} \tag{5}$$

We will use an explicit formula for $\diamond$ that we need to recall.

Let F_1, F_2 be two closed subsets of N. Let $U_1 := N \setminus F_1$, $U_2 := N \setminus F_2$ so that $U := N \setminus (F_1 \cap F_2) = U_1 \cup U_2$. Let $\Phi := (\Phi_1, \Phi_2)$ be a partition of unity subordinate to the covering $U_1 \cup U_2$ of U. As $\Phi_1 + \Phi_2 = 1$, we have $d\Phi_1 + d\Phi_2 = 0$. With the help of Φ, we define a bilinear map

$$\diamond_\Phi : \mathcal{A}^*(N, N \setminus F_1) \times \mathcal{A}^*(N, N \setminus F_2) \to \mathcal{A}^*(N, N \setminus (F_1 \cap F_2))$$

as follows.

Let $a_i := (\alpha_i, \beta_i) \in \mathcal{A}^{k_i}(N) \oplus \mathcal{A}^{k_i-1}(N \setminus F_i)$, $i = 1, 2$. Define

$$a_1 \diamond_\Phi a_2 := \Big(\alpha_1 \wedge \alpha_2, \Phi_1\beta_1 \wedge \alpha_2 + (-1)^{k_1}\alpha_1 \wedge \Phi_2\beta_2 - (-1)^{k_1} d\Phi_1 \wedge \beta_1 \wedge \beta_2\Big).$$

Remark that all forms $\Phi_1\beta_1 \wedge \alpha_2$, $\alpha_1 \wedge \Phi_2\beta_2$ and $d\Phi_1 \wedge \beta_1 \wedge \beta_2$ are well defined on $U_1 \cup U_2$. Indeed the support of the form $d\Phi_1$ is contained in $U_1 \cap U_2$, as $d\Phi_1 = -d\Phi_2$. So $a_1 \diamond_\Phi a_2 \in \mathcal{A}^{k_1+k_2}(N, N \setminus (F_1 \cap F_2))$.

The following relation is immediate to verify:

$$d_{\mathrm{rel}}(a_1 \diamond_\Phi a_2) = (d_{\mathrm{rel}}a_1) \diamond_\Phi a_2 + (-1)^{k_1} a_1 \diamond_\Phi (d_{\mathrm{rel}}a_2).$$

Thus $\diamond_\Phi$ defines a map $\mathcal{H}^*(N, N \setminus F_1) \times \mathcal{H}^*(N, N \setminus F_2) \to \mathcal{H}^*(N, N \setminus (F_1 \cap F_2))$.

Let us see that this product do not depend of the choice of the partition of unity. If we have another partition $\Phi' = (\Phi'_1, \Phi'_2)$, then $\Phi_1 - \Phi'_1 = -(\Phi_2 - \Phi'_2)$. It is immediate to verify that, if $d_{\rm rel}(a_1) = 0$ and $d_{\rm rel}(a_2) = 0$, one has

$$a_1 \diamond_\Phi a_2 - a_1 \diamond_{\Phi'} a_2 = d_{\rm rel}\Big(0, (-1)^{k_1}(\Phi_1 - \Phi'_1)\beta_1 \wedge \beta_2\Big).$$

So the product on the relative cohomology spaces will be denoted by $\diamond$.

2.2. **Inverse limit of cohomology with support.** Let F be a closed subset of N. We consider the set $\mathcal{F}_F$ of all open neighborhoods U of F which is ordered by the relation $U \leq V$ if and only if $V \subset U$. For any $U \in \mathcal{F}_F$, we consider the algebra $\mathcal{A}^*_U(N)$ of differential forms on N with support contained in U (that is vanishing on a neighborhood of $N \setminus U$): this algebra is stable under the de Rham differential d, and we denote by $\mathcal{H}^*_U(N)$ the corresponding cohomology algebra. If $U \leq V$, we have then an inclusion map $\mathcal{A}^*_V(N) \hookrightarrow \mathcal{A}^*_U(N)$ which gives rise to a map $f_{U,V} : \mathcal{H}^*_V(N) \to \mathcal{H}^*_U(N)$.

Definition 2.2. *We denote by $\mathcal{H}^*_F(N)$ the inverse limit of the inverse system $(\mathcal{H}^*_U(N), f_{U,V}; U, V \in \mathcal{F}_F)$.*

When F is compact, we have another system. For $U \in \mathcal{F}_F$, we take the vector space $\mathcal{H}^*_c(U)$ and for $U \leq V$, we take the canonical map $h_{U,V} : \mathcal{H}^*_c(V) \to \mathcal{H}^*_c(U)$. We denote by $\widetilde{\mathcal{H}}^*_F(N)$ the inverse limit of the inverse system $(\mathcal{H}^*_c(U), h_{U,V}; U, V \in \mathcal{F}_F)$. Let us recall the following basic facts :

- When F is compact, $\widetilde{\mathcal{H}}^*_F(N) \simeq \mathcal{H}^*_F(N)$.
- $\mathcal{H}^*_F(N) = \{0\}$ if $F = \emptyset$.
- There is a natural map $\mathcal{H}^*_F(N) \to \mathcal{H}^*(N)$. If F is compact, this map factors through $\mathcal{H}^*_F(N) \to \mathcal{H}^*_c(N)$.
- If $F \subset F'$ are two closed subsets of N, there is a natural map $\mathcal{H}^*_F(N) \to \mathcal{H}^*_{F'}(N)$,
- If F and R are two closed subsets of N, there is a natural product

$$\mathcal{H}^*_F(N) \times \mathcal{H}^*_R(N) \xrightarrow{\wedge} \mathcal{H}^*_{F\cap R}(N). \tag{6}$$

The operation (6) is defined via the wedge product on forms. It is well defined since for any neighborhood W of $F \cap R$, we can find neighborhoods U of F and V of R such that $U \cap V \subset W$.

Now we define a natural map from $\mathcal{H}^*(N, N \setminus F)$ into $\mathcal{H}^*_F(N)$.

Let $\beta \in \mathcal{A}^*(N \setminus F)$. If χ is a function on N which is identically 1 on a neighborhood of F, note that $d\chi\beta$ defines a differential form on N, since $d\chi$ is equal to 0 in a neighborhood of F.

Proposition 2.3. *For any open neighborhood U of F, we choose $\chi \in \mathcal{C}^\infty(N)$ with support in U and equal to 1 in a neighborhood of F.*

- *The map*

$$\mathrm{p}^\chi_U(\alpha, \beta) = \chi\alpha + d\chi\beta \tag{7}$$

defines a homomorphism of complexes $\mathrm{p}^\chi_U : \mathcal{A}^*(N, N \setminus F) \to \mathcal{A}^*_U(N)$.

In consequence, let $\alpha \in \mathcal{A}^*(N)$ be a closed *form and $\beta \in \mathcal{A}^*(N \setminus F)$ such that $\alpha|_{N\setminus F} = d\beta$, then $\mathrm{p}^\chi_U(\alpha, \beta)$ is a* closed *differential form supported in U.*

- *The cohomology class of $\mathrm{p}^\chi_U(\alpha, \beta)$ in $\mathcal{H}^*_U(N)$ does not depend of χ. We denote this class by $\mathrm{p}_U(\alpha, \beta) \in \mathcal{H}^*_U(N)$.*

• *For any neighborhoods $V \subset U$ of F, we have $f_{U,V} \circ \mathrm{p}_V = \mathrm{p}_U$.*

Proof. The equation $\mathrm{p}_U^\chi \circ d_{\mathrm{rel}} = d \circ \mathrm{p}_U^\chi$ is immediate to check. In particular $\mathrm{p}_U^\chi(\alpha, \beta)$ is closed, if $d_{\mathrm{rel}}(\alpha, \beta) = 0$. Directly: $d(\chi\alpha + d\chi\beta) = d\chi\alpha - d\chi d\beta = 0$. This proves the first point. For two different choices χ and χ', we have

$$\begin{aligned} \mathrm{p}_U^\chi(\alpha,\beta) - \mathrm{p}_U^{\chi'}(\alpha,\beta) &= (\chi - \chi')\alpha + d(\chi - \chi')\beta \\ &= d\left((\chi - \chi')\beta\right). \end{aligned}$$

Since $\chi - \chi' = 0$ in a neighborhood of F, the right hand side of the last equation is well defined, and is an element of $\mathcal{A}_U^*(N)$. This proves the second point. Finally, the last point is immediate, since $\mathrm{p}_U^\chi(\alpha, \beta) = \mathrm{p}_V^\chi(\alpha, \beta)$ for $\chi \in \mathcal{C}^\infty(N)$ with support in $V \subset U$. □

Definition 2.4. *Let $\alpha \in \mathcal{A}^*(N)$ be a* closed *form and $\beta \in \mathcal{A}^*(N \setminus F)$ such that $\alpha|_{N\setminus F} = d\beta$. We denote by $\mathrm{p}_F(\alpha, \beta) \in \mathcal{H}_F^*(N)$ the element defined by the sequence $\mathrm{p}_U(\alpha, \beta) \in \mathcal{H}_U^*(N)$, $U \in \mathcal{F}_F$. We have then a morphism*

$$\mathrm{p}_F : \mathcal{H}^*(N, N \setminus F) \to \mathcal{H}_F^*(N). \tag{8}$$

The following proposition summarizes the functorial properties of p.

Proposition 2.5. • *If $F \subset F'$ are closed subsets of N, then the diagram*

$$\begin{array}{ccc} \mathcal{H}^*(N, N \setminus F) & \xrightarrow{\mathrm{p}_F} & \mathcal{H}_F^*(N) \\ \downarrow & & \downarrow \\ \mathcal{H}^*(N, N \setminus F') & \xrightarrow{\mathrm{p}_{F'}} & \mathcal{H}_{F'}^*(N) \end{array} \tag{9}$$

is commutative.

• *If F_1, F_2 are closed subsets of N, then the diagram*

$$\begin{array}{ccccc} \mathcal{H}^*(N, N \setminus F_1) & \times & \mathcal{H}^*(N, N \setminus F_2) & \xrightarrow{\Diamond} & \mathcal{H}^*(N, N \setminus (F_1 \cap F_2)) \\ \downarrow{\scriptstyle \mathrm{p}_{F_1}} & & \downarrow{\scriptstyle \mathrm{p}_{F_2}} & & \downarrow{\scriptstyle \mathrm{p}_{F_1\cap F_2}} \\ \mathcal{H}_{F_1}^*(N) & \times & \mathcal{H}_{F_2}^*(N) & \xrightarrow{\wedge} & \mathcal{H}_{F_1\cap F_2}^*(N) \end{array} \tag{10}$$

is commutative.

Proof. The proof of the first point is left to the reader. Let us prove the second point. Let W be a neighborhood of $F_1 \cap F_2$. Let V_1, V_2 be respectively neighborhoods of F_1 and F_2 such that $V_1 \cap V_2 \subset W$. Let $\chi_i \in \mathcal{C}^\infty(N)$ supported in V_i and equal to 1 in a neighborhood of F_i. Then $\chi_1\chi_2$ is supported in W and equal to 1 in a neighborhood of $F_1 \cap F_2$. Let $\Phi_1 + \Phi_2 = 1_{N\setminus(F_1\cap F_2)}$ be a partition of unity relative to the decomposition $N \setminus (F_1 \cap F_2) = N \setminus F_1 \cup N \setminus F_2$.

Then one checks easily that

$$\mathrm{p}_{V_1}^{\chi_1}(a_1) \wedge \ \mathrm{p}_{V_2}^{\chi_2}(a_2) - \mathrm{p}_W^{\chi_1\chi_2}(a_1 \diamond_\Phi a_2)$$

is equal to

$$d\Big((-1)^{k_1+1}\chi_1 d\chi_2(\Phi_1\beta_1\beta_2) + (-1)^{k_1}\chi_2 d\chi_1(\beta_1\Phi_2\beta_2)\Big)$$

for $d_{\rm rel}$-closed forms $a_i = (\alpha_i, \beta_i) \in \mathcal{A}^{k_i}(N, N \setminus F_i)$. Remark that $\Phi_1\beta_1\beta_2$ is defined on $N \setminus F_2$, so that $d\chi_2(\Phi_1\beta_1\beta_2)$ is well defined on N and supported in V_2. Thus the form $(-1)^{k_1+1}\chi_1 d\chi_2(\Phi_1\beta_1\beta_2) + (-1)^{k_1}\chi_2 d\chi_1(\beta_1\Phi_2\beta_2)$ is well defined on N and supported in $V_1 \cap V_2 \subset W$. Since $\mathrm{p}_{V_1}^{\chi_1}(a_1) \wedge \mathrm{p}_{V_2}^{\chi_2}(a_2)$ and $\mathrm{p}_W^{\chi_1\chi_2}(a_1 \diamond_\Phi a_2)$ represent respectively the W-component of $\mathrm{p}_{F_1}(a_1) \wedge \mathrm{p}_{F_2}(a_2)$ and $\mathrm{p}_{F_1\cap F_2}(a_1 \diamond a_2)$, this proves that $\mathrm{p}_{F_1}(a_1) \wedge \mathrm{p}_{F_2}(a_2) = \mathrm{p}_{F_1\cap F_2}(a_1 \diamond a_2)$.

□

If we take $F' = N$ in (9), we see that the map $\mathrm{p}_F : \mathcal{H}^*(N, N \setminus F) \to \mathcal{H}_F^*(N)$ factors the natural map $\mathcal{H}^*(N, N \setminus F) \to \mathcal{H}^*(N)$.

2.3. **Integration.** We consider now the case where F is a *compact* subset of N. We have then a map

$$\mathrm{p}_c : \mathcal{H}^*(N, N \setminus F) \longrightarrow \mathcal{H}_c^*(N) \tag{11}$$

which is equal to the composition of p_F with the natural map $\mathcal{H}_F^*(N) \to \mathcal{H}_c^*(N)$. If $a \in \mathcal{H}^*(N, N \setminus F)$ is represented by the $d_{\rm rel}$-closed differential form $(\alpha, \beta) \in \mathcal{A}^*(N, N \setminus F)$, the class $\mathrm{p}_c(a) \in \mathcal{H}_c^*(N)$ is represented by the differential form $\mathrm{p}_U^\chi(\alpha, \beta) = \chi\alpha + d\chi\beta$ where χ is a function with *compact* support.

Definition 2.6. *Suppose that N is oriented. If $a \in \mathcal{H}^*(N, N \setminus F)$, the integral of a is defined by*

$$\int_N a := \int_N \mathrm{p}_c(a).$$

Suppose that N is oriented and **compact**. The elements α and $\mathrm{p}_c(a)$ coincide in $\mathcal{H}^*(N)$, hence

$$\int_N a = \int_N \alpha$$

if N is compact.

Suppose now that N is non-compact. Let $a = [\alpha, \beta]$, and assume that the form α is **integrable**. The two sides of the equality above are defined. However, it is usually not true that $\int_N a = \int_N \alpha$. An interesting case is the relative Thom form t_V of a vector space V: then $t_V = [0, \beta]$, with β a particular closed form on $V \setminus \{0\}$. Here the integral of $\alpha = 0$ is equal to 0, while $\int_V t_V = 1$. See Example 5.3.2.

In some important cases studied in Subsection 5.2.2, we will however prove that the integral of $\mathrm{p}_c(a)$ is indeed the same than the integral than α. As we have

$$\begin{aligned} \mathrm{p}_U^\chi(\alpha, \beta) - \alpha &= (\chi - 1)\alpha + d\chi\beta \\ &= d\Big((\chi - 1)\beta\Big), \end{aligned} \tag{12}$$

the comparison between the integral of $\mathrm{p}_c(a)$ and the one of α will follow from the careful study of the behavior on N of the form $(\chi - 1)\beta$.

3. Quillen's relative Chern Character

3.1. **Chern form of a super-connection.** For an introduction to the Quillen's notion of super-connection, see [7]. If $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded space, we set $|v| = 0$, if $v \in E^+$ and $|v| = 1$ if $v \in E^-$.

If $\mathcal{E}$ is a complex vector bundle on a manifold N, we denote by $\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))$ the algebra of $\mathrm{End}(\mathcal{E})$-valued differential forms on N.

Let ∇ be a connection on $\mathcal{E}$. The curvature ∇^2 of ∇ is a $\mathrm{End}(\mathcal{E})$-valued two-form on N. Recall that the Chern character of $\mathcal{E}$ is the de Rham cohomology class of the closed differential form $\mathrm{Chern}(\mathcal{E}) := \mathrm{Tr}(\exp(\frac{-\nabla^2}{2i\pi}))$. Here we simply denote by $\mathrm{Ch}(\mathcal{E}) \in \mathcal{H}^*(N)$ the de Rham cohomology class of $\mathrm{Tr}(\exp(\nabla^2))$. We will call it the (non normalized) Chern character of $\mathcal{E}$.

More generally, let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ be a $\mathbb{Z}_2$-graded complex vector bundle on a manifold N. Taking in account the $\mathbb{Z}_2$-grading of $\mathrm{End}(\mathcal{E})$, the algebra $\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))$ is a $\mathbb{Z}_2$-graded algebra: $\mathcal{A}^*(N, \mathrm{End}(\mathcal{E})) = [\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))]^+ \oplus [\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))]^-$, where for example $[\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))]^+ = \mathcal{A}^+(N, \mathrm{End}(\mathcal{E})^+) \oplus \mathcal{A}^-(N, \mathrm{End}(\mathcal{E})^-)$. The super-trace on $\mathrm{End}(\mathcal{E})$ extends to a map $\mathrm{Str} : \mathcal{A}^*(N, \mathrm{End}(\mathcal{E})) \to \mathcal{A}^*(N)$.

Let $\mathbb{A}$ be a super-connection on $\mathcal{E}$ and $\mathbf{F} = \mathbb{A}^2$ its curvature, an element of $[\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))]^+$. *The Chern form* of $(\mathcal{E}, \mathbb{A})$ is the closed differential form

$$\mathrm{Ch}(\mathbb{A}) := \mathrm{Str}(e^{\mathbf{F}}).$$

More precisely, in local coordinates, let $\mathcal{E} = N \times E$ be a trivial $\mathbb{Z}_2$-graded complex vector bundle with fiber a finite dimensional $\mathbb{Z}_2$-graded complex vector space E. A super-connection on $\mathcal{E}$ is an operator $\mathbb{A}$ on $\mathcal{A}^*(N) \otimes E$ of the form $d + A$ where $A \in [\mathcal{A}^*(N) \otimes \mathrm{End}(E)]^-$. Then the operator $\mathbb{A}^2$ is given by the multiplication by the element $\mathbf{F} = dA + A^2$ of $[\mathcal{A}^*(N) \otimes \mathrm{End}(E)]^+$ (that we may sometimes denote also by $\mathbb{A}^2$). More explicitly, if $A = \sum_a A_a \omega_a$ where A_a are endomorphisms of E and ω_a forms on N, then

$$\mathbf{F} = \sum_a (-1)^{|A_a|} A_a d\omega_a + \sum_{a,b} (-1)^{|\omega_a||A_b|} A_a A_b \omega_a \omega_b. \tag{13}$$

We will use the following transgression formulaes.

Proposition 3.1. • *Let $\mathbb{A}_t$, for $t \in \mathbb{R}$, be a one parameter family of super-connections on E, and let $\frac{d}{dt}\mathbb{A}_t \in [\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))]^-$. Let $\mathbf{F}_t$ be the curvature of $\mathbb{A}_t$. Then one has*

$$\frac{d}{dt}\,\mathrm{Ch}(\mathbb{A}_t) = d\left(\mathrm{Str}\left((\frac{d}{dt}\mathbb{A}_t)e^{\mathbf{F}_t}\right)\right). \tag{14}$$

• *Let $\mathbb{A}(s,t)$ be a two-parameter family of super-connections. Here $s, t \in \mathbb{R}$. We denote by $\mathbf{F}(s,t)$ the curvature of $\mathbb{A}(s,t)$. Then:*

$$\begin{aligned}\frac{d}{ds}\,\mathrm{Str}\left((\frac{d}{dt}\mathbb{A}(s,t))\,e^{\mathbf{F}(s,t)}\right) - \frac{d}{dt}\,\mathrm{Str}\left((\frac{d}{ds}\mathbb{A}(s,t))\,e^{\mathbf{F}(s,t)}\right)\\ = d\left(\int_0^1 \mathrm{Str}\left((\frac{d}{ds}\mathbb{A}(s,t))e^{u\mathbf{F}(s,t)}(\frac{d}{dt}\mathbb{A}(s,t))\,e^{(1-u)\mathbf{F}(s,t)}\right)du\right).\end{aligned}$$

Proof. These formulae are well known, and are derived easily from the two identities: $\mathbf{F} = \mathbb{A}^2$, and $d\,\mathrm{Str}(\alpha) = \mathrm{Str}[\mathbb{A}, \alpha]$ for any $\alpha \in \mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))$.

We give for example the proof of the second point. We write $\mathbb{A}$ instead of $\mathbb{A}(s,t)$, and $\mathbf{F} = \mathbb{A}(s,t)^2$ instead of $\mathbf{F}(s,t)$. By Volterra's formula (38),

$$\frac{d}{ds}((\frac{d}{dt}\mathbb{A})\,e^{\mathbf{F}}) - \frac{d}{dt}((\frac{d}{ds}\mathbb{A})\,e^{\mathbf{F}}) =$$

$$\int_0^1 (\frac{d}{dt}\mathbb{A})\,e^{u\mathbf{F}}[\mathbb{A}, \frac{d}{ds}\mathbb{A}]e^{(1-u)\mathbf{F}}du - \int_0^1 (\frac{d}{ds}\mathbb{A})\,e^{u\mathbf{F}}[\mathbb{A}, \frac{d}{dt}\mathbb{A}]e^{(1-u)\mathbf{F}}du.$$

Now

$$d\left(\mathrm{Str}\left((\frac{d}{ds}\mathbb{A})e^{u\mathbf{F}}(\frac{d}{dt}\mathbb{A})\,e^{(1-u)\mathbf{F}}\right)\right)=\mathrm{Str}\left([\mathbb{A},(\frac{d}{ds}\mathbb{A})e^{u\mathbf{F}}(\frac{d}{dt}\mathbb{A})\,e^{(1-u)\mathbf{F}}]\right)$$

$$=\mathrm{Str}\left([\mathbb{A},\frac{d}{ds}\mathbb{A}]e^{u\mathbf{F}}(\frac{d}{dt}\mathbb{A})\,e^{(1-u)\mathbf{F}}\right)-\mathrm{Str}\left((\frac{d}{ds}\mathbb{A})\,e^{u\mathbf{F}}[\mathbb{A},\frac{d}{dt}\mathbb{A}]e^{(1-u)\mathbf{F}}\right).$$

We can commute the even element $[\mathbb{A},\frac{d}{ds}\mathbb{A}]e^{u\mathbf{F}}$ of $\mathcal{A}^*(N,\mathrm{End}(\mathcal{E}))$ with $(\frac{d}{dt}\mathbb{A})e^{(1-u)\mathbf{F}}$ under the super-trace. Integrating from 0 to 1, we obtain our lemma. □

In particular, the cohomology class defined by $\mathrm{Ch}(\mathbb{A})$ in $\mathcal{H}^*(N)$ is independent of the choice of the super-connection $\mathbb{A}$ on $\mathcal{E}$. By definition, this is the Chern character $\mathrm{Ch}(\mathcal{E})$ of $\mathcal{E}$. By choosing $\mathbb{A}=\begin{pmatrix}\nabla^+ & 0\\ 0 & \nabla^-\end{pmatrix}$ where $\nabla^\pm$ are connections on $\mathcal{E}^\pm$, this class is just $\mathrm{Ch}(\mathcal{E}^+)-\mathrm{Ch}(\mathcal{E}^-)$. However, different choices of $\mathbb{A}$ define very different looking representatives of $\mathrm{Ch}(\mathcal{E})$.

3.2. **Quillen's relative Chern character of a morphism.** Let $\mathcal{E}=\mathcal{E}^+\oplus\mathcal{E}^-$ be a $\mathbb{Z}_2$-graded complex vector bundle on a manifold N and $\sigma:\mathcal{E}^+\to\mathcal{E}^-$ be a smooth morphism. At each point $n\in N$, $\sigma(n):\mathcal{E}^+_n\to\mathcal{E}^-_n$ is a linear map. The *support* of σ is the closed subset of N

$$\mathrm{Supp}(\sigma)=\{n\in N\mid \sigma(n)\text{ is not invertible}\}.$$

Remark 3.2. *It may look strange to define the support of σ at the closed set of points where $\det(\sigma(n))=0$, the support of a function (differential form, etc) being the closure of the set of points where this function is* **not zero**. *But this is indeed very natural: the fact that $\det(\sigma(n))=0$ is equivalent to the fact that the vector space $\mathrm{CoKer}(\sigma(n))\oplus\mathrm{Ker}(\sigma(n))$ is* **not zero**.

Definition 3.3. *The morphism σ is elliptic if $\mathrm{Supp}(\sigma)$ is compact.*

Recall that the data $(\mathcal{E}^+,\mathcal{E}^-,\sigma)$ defines an element of the K-theory $\mathbf{K}^0(N)$ of N when σ is elliptic.

In the following, **we do not** assume σ elliptic. We recall Quillen's construction [18] of a cohomology class $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ in $\mathcal{H}^*(N,N\setminus\mathrm{Supp}(\sigma))$. The definition will involve several choices. We choose Hermitian structures on $\mathcal{E}^\pm$ and a super-connection $\mathbb{A}$ on $\mathcal{E}$ **without** 0 **exterior degree term**.

We associate to the morphism σ the odd Hermitian endomorphism of $\mathcal{E}$ defined by

$$v_\sigma=\begin{pmatrix}0 & \sigma^*\\ \sigma & 0\end{pmatrix}. \tag{15}$$

Then $v_\sigma^2=\begin{pmatrix}\sigma^*\sigma & 0\\ 0 & \sigma\sigma^*\end{pmatrix}$ is a non negative even Hermitian endomorphism of $\mathcal{E}$. The support of σ coincides with the set of elements $n\in N$ where the spectrum of $v_\sigma^2(n)$ contains 0.

Definition 3.4. *If H is an Hermitian endomorphism of V and $h\in\mathbb{R}$, we write $H\geq h$ when $(Hw,w)\geq h\|w\|^2$ for any $w\in V$. Then $H\geq h$ if and only if the smallest eigenvalue of H is larger than h.*

Consider the family of super-connections

$$\mathbb{A}^{\sigma}(t) = \mathbb{A} + it\, v_{\sigma}, \quad t \in \mathbb{R}. \tag{16}$$

The curvature of $\mathbb{A}^{\sigma}(t)$ is the even element $\mathbf{F}(\sigma, \mathbb{A}, t) \in [\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}))]^+$ defined by :

$$\mathbf{F}(\sigma, \mathbb{A}, t) = (itv_{\sigma} + \mathbb{A})^2 = -t^2 v_{\sigma}^2 + it[\mathbb{A}, v_{\sigma}] + \mathbb{A}^2. \tag{17}$$

Here $-t^2 v_{\sigma}^2$ is the term of exterior degree 0. As the super-connection $\mathbb{A}$ do not have 0 exterior degree term, both elements $it[\mathbb{A}, v_{\sigma}]$ and $\mathbb{A}^2$ are sums of terms with strictly positive exterior degrees. For example, if $\mathbb{A} = \nabla^+ \oplus \nabla^-$ is a direct sum of connections, then $it[\mathbb{A}, v_{\sigma}] \in \mathcal{A}^1(N, \mathrm{End}(\mathcal{E})^-)$ and $\mathbb{A}^2 \in \mathcal{A}^2(N, \mathrm{End}(\mathcal{E})^+)$.

Definition 3.5. *We denote* $\mathrm{Ch}(\sigma, \mathbb{A}, t)$ *the Chern form of* $(\mathcal{E}, \mathbb{A}^{\sigma}(t))$*, that is*

$$\mathrm{Ch}(\sigma, \mathbb{A}, t) := \mathrm{Str}\left(e^{\mathbf{F}(\sigma, \mathbb{A}, t)}\right).$$

We define $\eta(\sigma, \mathbb{A}, t) := -\mathrm{Str}\left(iv_{\sigma}\, e^{\mathbf{F}(\sigma, \mathbb{A}, t)}\right)$*, and we call it the transgression form.*

As $iv_{\sigma} = \frac{d}{dt}\mathbb{A}^{\sigma}(t)$, we have $\frac{d}{dt}\mathrm{Ch}(\sigma, \mathbb{A}, t) = -d(\eta(\sigma, \mathbb{A}, t))$. After integration, it gives the following equality of differential forms on N

$$\mathrm{Ch}(\mathbb{A}) - \mathrm{Ch}(\sigma, \mathbb{A}, t) = d\left(\int_0^t \eta(\sigma, \mathbb{A}, s)ds\right), \tag{18}$$

since $\mathrm{Ch}(\mathbb{A}) = \mathrm{Ch}(\sigma, \mathbb{A}, 0)$.

Proposition 3.6. *Let* $\mathcal{K}$ *be a compact subset of* N *and let* $h \geq 0$ *such that* $v_{\sigma}^2(n) \geq h$ *when* $n \in \mathcal{K}$*. There exists a polynomial* $\mathcal{P}_{\mathcal{K}}$ *of degree* $\dim N$ *such that, on* $\mathcal{K}$*,*

$$\left\| e^{\mathbf{F}(\sigma, \mathbb{A}, t)} \right\| \leq \mathcal{P}_{\mathcal{K}}(t) e^{-ht^2} \quad \text{for all } t \geq 0. \tag{19}$$

In particular, when $\mathcal{K}$ *is contained in* $N \setminus \mathrm{Supp}(\sigma)$*, then* $\mathrm{Ch}(\sigma, \mathbb{A}, t)$ *and* $\eta(\sigma, \mathbb{A}, t)$ *tends to* 0 *exponentially fast when* t *tends to infinity.*

Proof. We work on $\mathrm{End}(E) \otimes \mathcal{A}$, where $\mathcal{A} = \oplus_{k=0}^{\dim N} \mathcal{A}^k(N)$. To estimate $\|e^{\mathbf{F}(\sigma, \mathbb{A}, t)}\|$, we employ Lemma 6.1 of the Appendix, with $H = t^2 v_{\sigma}^2$, and $R = -it[\mathbb{A}, v_{\sigma}] - \mathbb{A}^2$. Here R is a sum of $\mathrm{End}(E)$-valued differential forms on N with strictly positive exterior degrees. Remark that R is a polynomial in t of degree 1. Lemma 6.1 gives us the estimate $\|e^{\mathbf{F}(\sigma, \mathbb{A}, t)}\| \leq \mathcal{P}(\|R\|)e^{-ht^2}$ with $\mathcal{P}$ an explicit polynomial with positive coefficients of degree $\dim N$. Using the fact that $\|R\| \leq at + b$ on $\mathcal{K}$, we obtain the estimate (19) on N.

If $\mathcal{K}$ is contained in $N \setminus \mathrm{Supp}(\sigma)$, we can find $h > 0$ such that $v_{\sigma}^2(n) \geq h$ when $n \in \mathcal{K}$. Thus we see that $\|e^{\mathbf{F}(\sigma, \mathbb{A}, t)}\|$ decreases exponentially fast, when t tends to infinity.

□

The former estimates allows us to take the limit $t \to \infty$ in (18) on the open subset $N \setminus \mathrm{Supp}(\sigma)$. There, the differential form $\mathrm{Ch}(\sigma, \mathbb{A}, t) = \mathrm{Str}\left(e^{\mathbf{F}(\sigma, \mathbb{A}, t)}\right)$ tends to 0 as t goes to ∞, and we get the following important lemma due to Quillen.

Lemma 3.7. [18] *We can define on* $N \setminus \mathrm{Supp}(\sigma)$ *the differential form*

$$\beta(\sigma, \mathbb{A}) = \int_0^{\infty} \eta(\sigma, \mathbb{A}, t)dt \tag{20}$$

and we have $\mathrm{Ch}(\mathbb{A})|_{N \setminus \mathrm{Supp}(\sigma)} = d\left(\beta(\sigma, \mathbb{A})\right)$.

We are in the situation of Definition 2.1. The closed form $\mathrm{Ch}(\mathbb{A})$ on N and the form $\beta(\sigma,\mathbb{A})$ on $N\backslash\mathrm{Supp}(\sigma)$ define an even relative cohomology class $[\mathrm{Ch}(\mathbb{A}),\beta(\sigma,\mathbb{A})]$ in $\mathcal{H}^*(N,N\setminus\mathrm{Supp}(\sigma))$.

Proposition 3.8. • *The class* $[\mathrm{Ch}(\mathbb{A}),\beta(\sigma,\mathbb{A})]\in\mathcal{H}^*(N,N\setminus\mathrm{Supp}(\sigma))$ *does not depend of the choice of* $\mathbb{A}$*, nor on the Hermitian structure on* $\mathcal{E}$*. We denote it by* $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ *and we call it the Quillen Chern character.*

• *Let* F *be a closed subset of* N*. For* $s\in[0,1]$*, let* $\sigma_s:\mathcal{E}^+\to\mathcal{E}^-$ *be a family of smooth morphisms such that* $\mathrm{Supp}(\sigma_s)\subset F$*. Then all classes* $\mathrm{Ch}_{\mathrm{rel}}(\sigma_s)$ *coincide in* $\mathcal{H}^*(N,N\setminus F)$.

Proof. Although the proofs are standard, we give the details, since we will use them in a forthcoming article in a more general context. Let us prove the first point. Let $\mathbb{A}_s$, $s\in[0,1]$, be a smooth one parameter family of super-connections on $\mathcal{E}$ without 0 exterior degree terms. Let $\mathbb{A}(s,t)=\mathbb{A}_s+itv_\sigma$. Thus $\frac{d}{ds}\mathbb{A}(s,t)=\frac{d}{ds}\mathbb{A}_s$ and $\frac{d}{dt}\mathbb{A}(s,t)=iv_\sigma$. Let $\mathbf{F}(s,t)$ be the curvature of $\mathbb{A}(s,t)$. We have

$$\frac{d}{ds}\mathrm{Ch}(\mathbb{A}_s)=d\gamma_s,\quad\text{with}\quad\gamma_s=\mathrm{Str}\Big((\frac{d}{ds}\mathbb{A}_s)e^{\mathbf{F}(s,0)}\Big),$$

and $\eta(\sigma,\mathbb{A}_s,t)=-\mathrm{Str}\big((\frac{d}{dt}\mathbb{A}(s,t))e^{\mathbf{F}(s,t)}\big)$. We apply the double transgression formula of Proposition 3.1, and we obtain

$$\frac{d}{ds}\eta(\sigma,\mathbb{A}_s,t)=-\frac{d}{dt}\mathrm{Str}\Big((\frac{d}{ds}\mathbb{A}(s,t))e^{\mathbf{F}(s,t)}\Big)-d\nu(s,t) \tag{21}$$

with

$$\begin{aligned}\nu(s,t)&=\int_0^1\mathrm{Str}\Big((\frac{d}{ds}\mathbb{A}(s,t))e^{u\mathbf{F}(s,t)}(\frac{d}{dt}\mathbb{A}(s,t))e^{(1-u)\mathbf{F}(s,t)}\Big)du\\&=\int_0^1\mathrm{Str}\Big((\frac{d}{ds}\mathbb{A}_s)e^{u\mathbf{F}(s,t)}(iv_\sigma)e^{(1-u)\mathbf{F}(s,t)}\Big)du.\end{aligned}$$

For $u,s\in[0,1]$ and $t\geq0$, we consider the element of $\mathcal{A}^*(N,\mathrm{End}(\mathcal{E}))$ defined by

$$I(u,s,t)=(\frac{d}{ds}\mathbb{A}_s)e^{u\mathbf{F}(s,t)}(iv_\sigma)e^{(1-u)\mathbf{F}(s,t)}.$$

On a compact subset $\mathcal{K}$ of $N\setminus F$, σ is invertible and we can find $h>0$ such that $v_\sigma^2(n)\geq h$ when $n\in\mathcal{K}$. We have $u\mathbf{F}(s,t)=-t^2uv_\sigma^2-uR_{t,s}$, with $R_{t,s}=-it[\mathbb{A}_s,v_\sigma]-\mathbb{A}_s^2$ which is a sum of terms with strictly positive exterior degrees. Remark that $R_{t,s}$ is a polynomial of degree 1 in t. By the estimate of Lemma 6.1 of the Appendix, we obtain

$$\begin{aligned}\Big\|I(u,s,t)\Big\|&\leq\Big\|\frac{d}{ds}\mathbb{A}_s\Big\|\Big\|v_\sigma\Big\|\Big\|e^{u\mathbf{F}(s,t)}\Big\|\Big\|e^{(1-u)\mathbf{F}(s,t)}\Big\|\\&\leq\Big\|\frac{d}{ds}\mathbb{A}_s\Big\|\Big\|v_\sigma\Big\|\,e^{-uht^2}\mathcal{P}(u\|R_{t,s}\|)\,e^{-(1-u)ht^2}\mathcal{P}((1-u)\|R_{t,s}\|)\\&\leq\Big\|\frac{d}{ds}\mathbb{A}_s\Big\|\Big\|v_\sigma\Big\|\,e^{-ht^2}\mathcal{P}(u\|R_{t,s}\|)\mathcal{P}((1-u)\|R_{t,s}\|)\end{aligned}$$

where $\mathcal{P}$ is a polynomial of degree less or equal to $\dim N$. So, we can find a constant C such that $\|I(u,s,t)\|\leq C(1+t^2)^{\dim N}e^{-ht^2}$ for all $u,s\in[0,1]$ and $t\geq0$. Thus we can integrate Equation (21) in t, from 0 to ∞. Since $-\int_0^\infty\frac{d}{dt}[\mathrm{Str}((\frac{d}{ds}\mathbb{A}(s,t))e^{\mathbf{F}(s,t)})]dt=\gamma_s$, it follows that

$$\frac{d}{ds}\mathrm{Ch}(\mathbb{A}_s)=d\gamma_s,\qquad\frac{d}{ds}\beta(\sigma,\mathbb{A}_s)=\gamma_s-d\epsilon_s \tag{22}$$

where

$$\epsilon_s = \int_0^\infty \nu(s,t)dt, \quad \text{and} \quad \nu(s,t) = \int_0^1 I(u,s,t)du.$$

The first equality in Equations (22) holds on N, and the second on $N \setminus \mathrm{Supp}(\sigma)$. These equations (22) exactly mean that

$$\frac{d}{ds}\left(\mathrm{Ch}(\mathbb{A}_s), \beta(\sigma, \mathbb{A}_s)\right) = d_{\mathrm{rel}}\left(\gamma_s, \epsilon_s\right).$$

So the cohomology class $[\mathrm{Ch}(\mathbb{A}), \beta(\sigma, \mathbb{A})]$ in $\mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma))$ does not depend of the choice of $\mathbb{A}$.

With a similar proof, we see that the cohomology class of $[\mathrm{Ch}(\mathbb{A}), \beta(\sigma, \mathbb{A})] \in \mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma))$ does not depend on the choice of Hermitian structure on $\mathcal{E}$.

We now prove the second point. We consider the super-connection $\mathbb{A}(s,t) = itv_{\sigma_s} + \mathbb{A}$. Thus $\frac{d}{ds}\mathbb{A}(s,t) = it\frac{d}{ds}v_{\sigma_s}$ and $\frac{d}{dt}\mathbb{A}(s,t) = iv_{\sigma_s}$. Let $\mathbf{F}(s,t)$ be the curvature of $\mathbb{A}(s,t)$. Let $\eta(\sigma_s, \mathbb{A}, t) = -\mathrm{Str}((\frac{d}{dt}\mathbb{A}(s,t))e^{\mathbf{F}(s,t)})$. By the double transgression formula,

$$\frac{d}{ds}\eta(\sigma_s, \mathbb{A}, t) = -\frac{d}{dt}\,\mathrm{Str}\left(it(\frac{d}{ds}v_{\sigma_s})e^{\mathbf{F}(s,t)}\right) - d\nu(s,t) \tag{23}$$

with

$$\begin{aligned}\nu(s,t) &= \int_0^1 \mathrm{Str}\left((\frac{d}{ds}\mathbb{A}(s,t))e^{u\mathbf{F}(s,t)}(\frac{d}{dt}\mathbb{A}(s,t))e^{(1-u)\mathbf{F}(s,t)}\right)du \\ &= \int_0^1 \mathrm{Str}\left((it\frac{d}{ds}v_{\sigma_s})e^{u\mathbf{F}(s,t)}(iv_{\sigma_s})e^{(1-u)\mathbf{F}(s,t)}\right)du.\end{aligned}$$

On a compact subset of $N \setminus F$, the Hermitian endomorphisms $v_{\sigma_s}^2$ are strictly positive and we can find $h > 0$, uniformly in $s \in [0,1]$, such that $v_{\sigma_s}^2 \geq h$. The estimate of Lemma 6.1 of the Appendix shows that we can integrate Equation (23) in t from 0 to ∞. Here the first term of the right hand side integrates to 0. This gives the relation $\frac{d}{ds}\beta(\sigma_s, \mathbb{A}) = d\epsilon_s$ on $N \setminus \mathrm{Supp}(\sigma)$ with $\epsilon_s = \int_0^\infty \nu(s,t)dt$. Thus we obtain

$$\frac{d}{ds}\left(\mathrm{Ch}(\mathbb{A}), \beta(\sigma_s, \mathbb{A})\right) = d_{\mathrm{rel}}\left(0, \epsilon_s\right),$$

so that the class of $(\mathrm{Ch}(\mathbb{A}), \beta(\sigma_s, \mathbb{A}))$ does not depend of s. □

We have defined a representative of the relative Chern class $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ using the one-parameter family $\mathbb{A}^\sigma(t)$ of super-connections, for t varying between 0 and ∞. We can define another representative as follows. We have $\mathrm{Ch}(\sigma, \mathbb{A}, t) = d(\beta(\sigma, \mathbb{A}, t))$ with

$$\beta(\sigma, \mathbb{A}, t) = \int_t^\infty \eta(\sigma, \mathbb{A}, s)ds \tag{24}$$

Lemma 3.9. *For any $t \in \mathbb{R}$, the relative Chern character $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ satifies*

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma) = \Big[\,\mathrm{Ch}(\sigma, \mathbb{A}, t), \beta(\sigma, \mathbb{A}, t)\Big] \quad \text{in} \quad \mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma)).$$

Proof. It is easy to check that

$$\Big(\,\mathrm{Ch}(\mathbb{A}), \beta(\sigma, \mathbb{A})\Big) - \Big(\,\mathrm{Ch}(\sigma, \mathbb{A}, t), \beta(\sigma, \mathbb{A}, t)\Big) = d_{\mathrm{rel}}(\delta(t), 0), \tag{25}$$

with $\delta(t) = \int_0^t \eta(\sigma, \mathbb{A}, s)ds$. □

Remark 3.10. *Quillen relative Chern character seems to be very related to the "multiplicative K-theory" defined by Connes-Karoubi (see* [15], [16]*). For example, even if $\mathcal{E}^+, \mathcal{E}^-$ are flat bundles, the Quillen Chern character is usually non zero, as it also encodes the odd closed differential form $\omega = \beta(\sigma, \mathbb{A})$.*

4. Multiplicative property of $\mathrm{Ch}_{\mathrm{rel}}$

Let $\mathcal{E}_1, \mathcal{E}_2$ be two $\mathbb{Z}_2$-graded complex vector bundles on a manifold N. The space $\mathcal{E}_1 \otimes \mathcal{E}_2$ is a $\mathbb{Z}_2$-graded complex vector bundle with even part $\mathcal{E}_1^+ \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^- \otimes \mathcal{E}_2^-$ and odd part $\mathcal{E}_1^- \otimes \mathcal{E}_2^+ \oplus \mathcal{E}_1^+ \otimes \mathcal{E}_2^-$.

Remark 4.1. *If E_1 and E_2 are super vector spaces, the super-algebra* $\mathrm{End}(E_1) \otimes \mathrm{End}(E_2)$ *is identified with the super-algebra* $\mathrm{End}(E_1 \otimes E_2)$ *via the following rule. For $v_1 \in E_1, v_2 \in E_2, A \in \mathrm{End}(E_1), B \in \mathrm{End}(E_2)$ homogeneous*

$$(A \otimes B)(v_1 \otimes v_2) = (-1)^{|B||v_1|} A v_1 \otimes B v_2.$$

The super-algebra $\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}_1 \otimes \mathcal{E}_2))$ can be identified with $\mathcal{A}^*(N, \mathrm{End}(\mathcal{E}_1)) \otimes \mathcal{A}^*(N, \mathrm{End}(\mathcal{E}_2))$ where the tensor is taken in the sense of super-algebras. Then, if $A \in \mathcal{A}^0(N, \mathrm{End}(\mathcal{E}_1)^-)$ and $B \in \mathcal{A}^0(N, \mathrm{End}(\mathcal{E}_2)^-)$ are odd endomorphisms, we have $(A \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes B)^2 = A^2 \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes B^2$.

Let $\sigma_1 : \mathcal{E}_1^+ \to \mathcal{E}_1^-$ and $\sigma_2 : \mathcal{E}_2^+ \to \mathcal{E}_2^-$ be two smooth morphisms. With the help of Hermitian structures, we define the morphism

$$\sigma_1 \odot \sigma_2 : (\mathcal{E}_1 \otimes \mathcal{E}_2)^+ \longrightarrow (\mathcal{E}_1 \otimes \mathcal{E}_2)^-$$

by $\sigma_1 \odot \sigma_2 := \sigma_1 \otimes \mathrm{Id}_{\mathcal{E}_2^+} + \mathrm{Id}_{\mathcal{E}_1^+} \otimes \sigma_2 + \mathrm{Id}_{\mathcal{E}_1^-} \otimes \sigma_2^* + \sigma_1^* \otimes \mathrm{Id}_{\mathcal{E}_2^-}$.

Let v_{σ_1} and v_{σ_2} be the odd Hermitian endomorphisms of $\mathcal{E}_1, \mathcal{E}_2$ associated to σ_1 and σ_2 (see (15)). Then $v_{\sigma_1 \odot \sigma_2} = v_{\sigma_1} \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes v_{\sigma_2}$ and $v^2_{\sigma_1 \odot \sigma_2} = v^2_{\sigma_1} \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes v^2_{\sigma_2}$. Thus the square $v^2_{\sigma_1 \odot \sigma_2}$ is the sum of two commuting non negative Hermitian endomorphisms $v^2_{\sigma_1} \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes v^2_{\sigma_2}$. It follows that

$$\mathrm{Supp}(\sigma_1 \odot \sigma_2) = \mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2).$$

Remark 4.2. Exterior product

When N_1 and N_2 are two distinct manifolds and $\sigma_1 : \mathcal{E}_1^+ \to \mathcal{E}_1^-$, $\sigma_2 : \mathcal{E}_2^+ \to \mathcal{E}_2^-$ two smooth morphisms respectively over N_1, N_2, the exterior product $\sigma_1 \odot_{ext} \sigma_2$ is the product of $\sigma_1 \otimes \mathrm{Id}_{\mathcal{E}_2}$ and of $\mathrm{Id}_{\mathcal{E}_2} \otimes \sigma_2$ over $N_1 \times N_2$. In this case, which will be especially important in our forthcoming article on index theory of transversally elliptic operators,

$$\mathrm{Supp}(\sigma_1 \odot_{ext} \sigma_2) = \mathrm{Supp}(\sigma_1) \times \mathrm{Supp}(\sigma_2).$$

We can now state the main result of this paper.

Theorem 4.3. (Quillen's Chern character is multiplicative) *Let σ_1, σ_2 be two morphisms over N. The relative cohomology classes*

- $\mathrm{Ch}_{\mathrm{rel}}(\sigma_k) \in \mathcal{H}^*(N, N \setminus \mathrm{Supp}(\sigma_k))$, $k = 1, 2$,
- $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) \in \mathcal{H}^*(N, N \setminus (\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)))$,

satisfy the following equality

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) = \mathrm{Ch}_{\mathrm{rel}}(\sigma_1) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2)$$

in $\mathcal{H}^*(N, N \setminus (\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)))$. *Here* $\diamond$ *is the product of relative classes (see (5)).*

Proof. For $k = 1, 2$, we choose super-connections $\mathbb{A}_k$, without 0 exterior degree terms. We consider the closed forms $c_k(t) := \mathrm{Ch}(\sigma_k, \mathbb{A}_k, t)$ and the transgression forms $\eta_k(t) := \eta(\sigma_k, \mathbb{A}_k, t)$ so that $\frac{d}{dt}(c_k(t)) = -d(\eta_k(t))$. Let $\beta_k = \int_0^\infty \eta_k(t)dt$. A representative of $\mathrm{Ch}_{\mathrm{rel}}(\sigma_k)$ is $(c_k(0), \beta_k)$.

For the symbol $\sigma_1 \odot \sigma_2$, we consider $\mathbb{A}(t) = \mathbb{A} + itv_{\sigma_1 \odot \sigma_2}$ where $\mathbb{A} = \mathbb{A}_1 \otimes \mathrm{Id}_{\mathcal{E}_2} + \mathrm{Id}_{\mathcal{E}_1} \otimes \mathbb{A}_2$. Then $\mathrm{Ch}(\mathbb{A}) = c_1(0)c_2(0)$. Furthermore, it is easy to see that the transgression form for the family $\mathbb{A}(t)$ is $\eta(t) = \eta_1(t)c_2(t) + c_1(t)\eta_2(t)$. Let $\beta_{12} = \int_0^\infty \eta(t)dt$. A representative of $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2)$ is $(c_1(0)c_2(0), \beta_{12})$.

We consider the open subsets $U = N \setminus (\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2))$, and $U_k = N \setminus \mathrm{Supp}(\sigma_k)$. Let $\Phi_1 + \Phi_2 = 1_U$ be a partition of unity subordinate to the decomposition $U = U_1 \cup U_2$. We need the following lemma.

Lemma 4.4. *The integrals*

$$\begin{aligned} B_1 &= \int_{0 \le t \le s} \Phi_1 \eta_1(s) \wedge \eta_2(t) ds\, dt, \\ B_2 &= \int_{0 \le s \le t} \Phi_2 \eta_1(s) \wedge \eta_2(t) ds\, dt \end{aligned}$$

are well defined differential forms supported on U.

Proof. The function $(s, t) \mapsto \Phi_1 \eta_1(s) \wedge \eta_2(t)$ is a function on $\mathbb{R}^2$ with values in $\mathcal{A}^*(U)$. We have to see that the integral B_1 is convergent on the domain $0 \le t \le s$. This fact follows directly from the estimates of Proposition 3.6. Indeed, let $\mathcal{K}$ be a compact subset of U. Since Φ_1 is supported on $U_1 = N \setminus \mathrm{Supp}(\sigma_1)$, there exists $h > 0$, and a polynomial $\mathcal{P}_1$ in s such that, on $\mathcal{K}$,

$$\|\Phi_1 \eta_1(s)\| \le \mathcal{P}_1(s) e^{-hs^2} \quad \text{for} \quad s \ge 0.$$

On the other hand, there exists a polynomial $\mathcal{P}_2$ in t such that, on $\mathcal{K}$,

$$\|\eta_2(t)\| \le \mathcal{P}_2(t) \quad \text{for} \quad t \ge 0.$$

Then, when $0 \le t \le s$, we have, on $\mathcal{K}$: $\|\Phi_1 \eta_1(s) \wedge \eta_2(t)\| \le \mathcal{P}_1(s)\mathcal{P}_2(t)\, e^{-hs^2}$ and the integral B_1 is absolutely convergent on $0 \le t < s$. Reversing the role $1 \leftrightarrow 2$, we prove in the same way that B_2 is well defined.

We now prove

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_1, \mathbb{A}_1) \diamond_\Phi \mathrm{Ch}_{\mathrm{rel}}(\sigma_2, \mathbb{A}_2) - \mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2, \mathbb{A}) = d_{\mathrm{rel}}\Big(0, B_1 - B_2\Big). \tag{26}$$

Indeed

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_1, \mathbb{A}_1) \diamond_\Phi \mathrm{Ch}_{\mathrm{rel}}(\sigma_2, \mathbb{A}_2) = \Big(c_1(0)c_2(0), \Phi_1\beta_1 c_2(0) + c_1(0)\Phi_2\beta_2 - d\Phi_1\beta_1\beta_2\Big),$$

so that the first member of Equality (26) is

$$\Big(0, \Phi_1\beta_1 c_2(0) + c_1(0)\Phi_2\beta_2 - d\Phi_1\beta_1\beta_2 - \beta_{12}\Big).$$

Thus we need to check that $dB_2 - dB_1 = \Phi_1\beta_1 c_2(0) + c_1(0)\Phi_2\beta_2 - d\Phi_1\beta_1\beta_2 - \beta_{12}$. We have $dB_1 = R_1 + S_1$ with

$$\begin{aligned} R_1 &= d\Phi_1 \wedge \int_{0\leq t\leq s} \eta_1(s)\eta_2(t)\, ds\, dt, \\ S_1 &= \Phi_1 \int_{0\leq t\leq s} \Big(d\eta_1(s)\eta_2(t) - \eta_1(s)d\eta_2(t)\Big) ds\, dt. \end{aligned}$$

Now we use $d\eta_j(s) = -\frac{d}{ds}c_j(s)$, so that we obtain

$$\begin{aligned} S_1 &= \Phi_1 \int_{0\leq t\leq s} \left((-\frac{d}{ds}c_1(s))\eta_2(t) + \eta_1(s)(\frac{d}{dt}c_2(t))\right) ds\, dt \\ &= \Phi_1\Big(\int_0^\infty c_1(t)\eta_2(t)dt + \int_0^\infty \eta_1(s)c_2(s)ds\Big) - c_2(0)\Phi_1\beta_1, \end{aligned}$$

that is $S_1 = \Phi_1\beta_{12} - c_2(0)\Phi_1\beta_1$. Similarly, we compute $dB_2 = R_2 + S_2$, with

$$\begin{aligned} R_2 &= d\Phi_2 \wedge \int_{0\leq s\leq t} \eta_1(s)\eta_2(t)\, ds\, dt, \\ S_2 &= -\Phi_2\beta_{12} + c_1(0)\Phi_2\beta_2 \end{aligned}$$

So finally as $d\Phi_1 = -d\Phi_2$

$$dB_2 - dB_1 = -d\Phi_1 \int_0^\infty\int_0^\infty \eta_1(s)\eta_2(t)dsdt - \beta_{12} + c_2(0)\Phi_1\beta_1 + c_1(0)\Phi_2\beta_2$$

which was the equation to prove.

□

□

5. Chern character of a morphism

We employ notations of Section 3.2

5.1. **The Chern Character.** Let $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$ be a morphism on N. Following Subsection 2.2, we consider the image of $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ through the map $\mathcal{H}^*(N, N\setminus \mathrm{Supp}(\sigma)) \to \mathcal{H}^*_{\mathrm{Supp}(\sigma)}(N)$. Applying Propositions 2.3 and 3.8, we obtain the following theorem.

Theorem 5.1. • *For any neighborhood U of* $\mathrm{Supp}(\sigma)$*, take $\chi \in \mathcal{C}^\infty(N)$ which is equal to 1 in a neighborhood of* $\mathrm{Supp}(\sigma)$ *and with support contained in U. The differential form*

$$c(\sigma, \mathbb{A}, \chi) = \chi\ \mathrm{Ch}(\mathbb{A}) + d\chi\, \beta(\sigma, \mathbb{A}) \tag{27}$$

is closed *and supported in U. Its cohomology class $c_U(\sigma) \in \mathcal{H}^*_U(N)$ does not depend of the choice of $\mathbb{A}, \chi$ and the Hermitian structures on $\mathcal{E}^\pm$. Furthermore, the inverse family $c_U(\sigma)$ when U runs over the neighborhoods of* $\mathrm{Supp}(\sigma)$ *defines a class*

$$\mathrm{Ch}_{\mathrm{good}}(\sigma) \in \mathcal{H}^*_{\mathrm{Supp}(\sigma)}(N).$$

The image of this class in $\mathcal{H}^*(N)$ is the Chern character $\mathrm{Ch}(\mathcal{E})$ *of $\mathcal{E}$.*

• *Let F be a closed subset of N. For $s \in [0,1]$, let $\sigma_s : \mathcal{E}^+ \to \mathcal{E}^-$ be a family of smooth morphisms such that* $\mathrm{Supp}(\sigma_s) \subset F$*. Then all classes* $\mathrm{Ch}_{\mathrm{good}}(\sigma_s)$ *coincide in $\mathcal{H}^*_F(N)$.*

Using Lemma 3.9 we get

Lemma 5.2. *For any $t \geq 0$, the class $\mathrm{Ch}_{\mathrm{good}}(\sigma)$ can be defined with the forms $c(\sigma, \mathbb{A}, \chi, t) = \chi\ \mathrm{Ch}(\sigma, \mathbb{A}, t) + d\chi\, \beta(\sigma, \mathbb{A}, t)$.*

Proof. It is due to the following transgression

$$c(\sigma, \mathbb{A}, \chi) - c(\sigma, \mathbb{A}, \chi, t) = d(\chi\delta(t)), \tag{28}$$

which follows from (25). □

In some situations the Chern form $\mathrm{Ch}(\sigma, \mathbb{A}, 1)$ enjoys good properties relative to the integration. So it is natural to compare the differential form $c(\sigma, \mathbb{A}, \chi)$ and $\mathrm{Ch}(\sigma, \mathbb{A}, 1)$.

Lemma 5.3. *We have*

$$c(\sigma, \mathbb{A}, \chi) - \mathrm{Ch}(\sigma, \mathbb{A}, 1) = d\left(\chi \int_0^1 \eta(\sigma, \mathbb{A}, s)ds\right) + d\Big((\chi - 1)\beta(\sigma, \mathbb{A}, 1)\Big).$$

Proof. This follows immediately from the transgressions (12) and (28).

□

Definition 5.4. *When σ is elliptic, we denote by*

$$\mathrm{Ch}_c(\sigma) \in \mathcal{H}_c^*(N) \tag{29}$$

the cohomology class with compact support which is the image of $\mathrm{Ch}_{\mathrm{good}}(\sigma) \in \mathcal{H}^*_{\mathrm{Supp}(\sigma)}(N)$ through the canonical map $\mathcal{H}^*_{\mathrm{Supp}(\sigma)}(N) \to \mathcal{H}_c^*(N)$.

A representative of $\mathrm{Ch}_c(\sigma)$ is given by $c(\sigma, \mathbb{A}, \chi)$, where $\chi \in \mathcal{C}^\infty(N)$ is chosen with a compact support, and equal to 1 in a neighborhood of $\mathrm{Supp}(\sigma)$ and $c(\sigma, \mathbb{A}, \chi)$ is given by Formula (27).

We will now rewrite Theorem 4.3 for the Chern classes $\mathrm{Ch}_{\mathrm{good}}$ and Ch_c. Let $\sigma_1 : \mathcal{E}_1^+ \to \mathcal{E}_1^-$ and $\sigma_2 : \mathcal{E}_2^+ \to \mathcal{E}_2^-$ be two smooth morphisms. Let $\sigma_1 \odot \sigma_2 : (\mathcal{E}_1 \otimes \mathcal{E}_2)^+ \to (\mathcal{E}_1 \otimes \mathcal{E}_2)^-$ their product.

Following (6), the product of the elements $\mathrm{Ch}_{\mathrm{good}}(\sigma_k) \in \mathcal{H}^*_{\mathrm{Supp}(\sigma_k)}(N)$ for $k = 1, 2$ belongs to $\mathcal{H}^*_{\mathrm{Supp}(\sigma_1)\cap\mathrm{Supp}(\sigma_2)}(N) = \mathcal{H}^*_{\mathrm{Supp}(\sigma_1\odot\sigma_2)}(N)$.

Theorem 5.5. • *We have the equality*

$$\mathrm{Ch}_{\mathrm{good}}(\sigma_1) \wedge \mathrm{Ch}_{\mathrm{good}}(\sigma_2) = \mathrm{Ch}_{\mathrm{good}}(\sigma_1 \odot \sigma_2) \quad \text{in} \quad \mathcal{H}^*_{\mathrm{Supp}(\sigma_1\odot\sigma_2)}(N).$$

• *If the morphisms σ_1, σ_2 are elliptic, we have*

$$\mathrm{Ch}_c(\sigma_1) \wedge \mathrm{Ch}_c(\sigma_2) = \mathrm{Ch}_c(\sigma_1 \odot \sigma_2) \quad \text{in} \quad \mathcal{H}_c^*(N).$$

Proof. The second point is a consequence of the first point. Theorem 4.3 tells us that $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) = \mathrm{Ch}_{\mathrm{rel}}(\sigma_1) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2)$ holds in $\mathcal{H}^*(N, N \setminus (\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)))$. We apply now the maps "p" and use the diagram (10):

$$\begin{aligned}
\mathrm{Ch}_{\mathrm{good}}(\sigma_1 \odot \sigma_2) &= \mathrm{p}_{12}\Big(\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2)\Big) \\
&= \mathrm{p}_{12}\Big(\mathrm{Ch}_{\mathrm{rel}}(\sigma_1) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2)\Big) \\
&= \mathrm{p}_1(\mathrm{Ch}_{\mathrm{rel}}(\sigma_1)) \wedge \mathrm{p}_2(\mathrm{Ch}_{\mathrm{rel}}(\sigma_2)) \\
&= \mathrm{Ch}_{\mathrm{good}}(\sigma_1) \wedge \mathrm{Ch}_{\mathrm{good}}(\sigma_2),
\end{aligned}$$

where $\mathrm{p}_k := \mathrm{p}_{F_k}$ with $F_k = \mathrm{Supp}(\sigma_k)$ and $\mathrm{p}_{12} := \mathrm{p}_{F_1\cap F_2}$.

□

The second point of Theorem 5.5 has the following interesting refinement. Let σ_1, σ_2 be two morphisms on N which are **not elliptic**, and assume that the product $\sigma_1 \odot \sigma_2$ is **elliptic**. Since $\mathrm{Supp}(\sigma_1) \cap \mathrm{Supp}(\sigma_2)$ is compact, we consider neighborhoods U_k of $\mathrm{Supp}(\sigma_k)$ such that $\overline{U_1} \cap \overline{U_2}$ is compact. Let $\chi_k \in \mathcal{C}^\infty(N)$ supported on U_k and equal to 1 in a neighborhood of $\mathrm{Supp}(\sigma_k)$. Then, the differential form $c(\sigma_1, \mathbb{A}_1, \chi_1) \wedge c(\sigma_2, \mathbb{A}_2, \chi_2)$ is compactly supported on N, and we have

$$\mathrm{Ch}_c(\sigma_1 \odot \sigma_2) = \Big[c(\sigma_1, \mathbb{A}_1, \chi_1) \wedge c(\sigma_2, \mathbb{A}_2, \chi_2)\Big] \quad \text{in} \quad \mathcal{H}_c^*(N).$$

Note that the differential forms $c(\sigma_k, \mathbb{A}_k, \chi_k)$ are not compactly supported.

5.2. **Comparison with other constructions.**

5.2.1. *Trivialization outside* $\mathrm{Supp}(\sigma)$. Outside the support of σ, the complex vector bundles $\mathcal{E}^+$ and $\mathcal{E}^-$ are "the same", so that it is natural to construct representatives of $\mathrm{Ch}(\mathcal{E}) = \mathrm{Ch}(\mathcal{E}^+) - \mathrm{Ch}(\mathcal{E}^-)$ which are zero "outside" the support of σ by the following identifications of bundle with connections. For simplicity, we assume in this section that σ is elliptic.

A connection $\nabla = \nabla^+ \oplus \nabla^-$ is said "adapted" to the morphism σ when the following holds

$$\begin{aligned} \nabla^- \circ \sigma + \sigma \circ \nabla^+ &= 0, \\ \nabla^+ \circ \sigma^* + \sigma^* \circ \nabla^- &= 0, \end{aligned} \tag{30}$$

outside a compact neighborhood of $\mathrm{Supp}(\sigma)$. An adapted connection is denoted by ∇^{adap}.

A construction of an adapted connection can be done as follows. First we choose a Hermitian structure on $\mathcal{E}^+$ and a Hermitian connection ∇^+ on $\mathcal{E}^+ \to N$. Let U, V be compact neighborhoods of $\mathrm{Supp}(\sigma)$ such that $\overline{V} \subset U$. Since σ is invertible outside $\overline{V}$, there exists a Hermitian structure on $\mathcal{E}^-$ such that σ is a unitary isomorphism between $\mathcal{E}^+$ and $\mathcal{E}^-$ outside $\overline{V}$, so that $\sigma^* = \sigma^{-1}$. Define the connection $\nabla^-_{out} = -\sigma \circ \nabla^+ \circ \sigma^{-1}$ on $\mathcal{E}^-$. Then the connection $\nabla_{out} = \nabla^+ \oplus \nabla^-_{out}$ satisfies Relations (30) on $N \setminus \overline{V}$. We consider now any connection $\nabla_{in} = \nabla^+_{in} \oplus \nabla^-_{in}$ defined on U. Let $\psi \in \mathcal{C}^\infty(N)$, supported in U and equal to 1 in a neighborhood of $\overline{V}$. Finally the following connection is adapted to σ:

$$\nabla^{\mathrm{adap}} = \psi \nabla_{in} + (1 - \psi)\nabla_{out}. \tag{31}$$

Proposition 5.6. *Let* ∇^{adap} *be a connection* adapted *to* $\sigma : \mathcal{E}^+ \to \mathcal{E}^-$. *Then the differential form* $\mathrm{Ch}(\nabla^{adap})$ *is compactly supported and its cohomology class coincides with* $\mathrm{Ch}_c(\sigma)$ *in* $\mathcal{H}_c^*(N)$.

Proof. Suppose that ∇^{adap} satisfies (30) outside a compact neighborhood C of $\mathrm{Supp}(\sigma)$. We verify that the forms $\mathrm{Ch}(\nabla^{\mathrm{adap}})$ as well as the form $\beta(\sigma, \nabla^{\mathrm{adap}})$ are supported on C. Thus if $\chi \in \mathcal{C}^\infty(N)$ is equal to 1 on C, we see that the differential forms $c(\sigma, \nabla^{\mathrm{adap}}, \chi)$ and $\mathrm{Ch}(\nabla^{\mathrm{adap}})$ coincide. □

Remark 5.7. *If* F *is* **compact**, *the closed differential form* $\mathrm{Ch}(\nabla^{adap})$ *represents the Chern character of a difference bundle* $[\mathcal{D}^+] - [\mathcal{D}^-]$, *where* $[\mathcal{D}^+]$ *and* $[\mathcal{D}^-]$ *are complex vector bundles (isomorphic outside* F*) on a compactification* N_F *of* N *(see*

for example [8]*). Thus* $\mathrm{Ch}(\nabla^{adap})$ *is a representative of the Chern character as defined by Atiyah and al. in* [2, 5]*. In this case, Theorem 5.5 is just the multiplicativity property of the Chern character in absolute theories.*

5.2.2. *Gaussian look.* In [17], Mathai-Quillen gives an explicit representative with "Gaussian look" of the Bott class of a complex vector bundle $N \to B$. The purpose of this paragraph is to compare the Mathai-Quillen construction of Chern characters with "Gaussian look" and the Quillen relative construction.

Let N be a real vector bundle over a manifold B. We denote by $\pi : N \to B$ the projection. We denote by (x, ξ) a point of N with $x \in B$ and $\xi \in N_x$. Let $\mathcal{E}^{\pm} \to B$ be two Hermitian vector bundles. We consider a morphism $\sigma : \pi^*\mathcal{E}^+ \to \pi^*\mathcal{E}^-$.

We choose a metric on the fibers of the fibration $N \to B$. We work under the following assumption on σ.

Assumption 5.8. *The morphism* $\sigma : \pi^*\mathcal{E}^+ \to \pi^*\mathcal{E}^-$ *and all its partial derivatives have at most a polynomial growth along the fibers of* $N \to B$*. Moreover we assume that, for any compact subset* $\mathcal{K}_B$ *of* B*, there exist* $R \geq 0$ *and* $c > 0$ *such that*[1] $v_\sigma^2(x, \xi) \geq c\|\xi\|^2$ *when* $\|\xi\| \geq R$ *and* $x \in \mathcal{K}_B$*.*

We may define the sub-algebra $\mathcal{A}^*_{\text{dec-rap}}(N)$ of forms on N such that all partial derivatives are rapidly decreasing along the fibers. Let $\mathcal{H}^*_{\text{dec-rap}}(N)$ be the corresponding cohomology algebra. Under Assumption 5.8, the support of σ intersects the fibers of π in compact sets. We have then a canonical map from $\mathcal{H}^*_{\text{Supp}(\sigma)}(N)$ into $\mathcal{H}^*_{\text{dec-rap}}(N)$. We will now compute the image of $\mathrm{Ch}_{\text{good}}(\sigma)$ under this map.

Let $\nabla = \nabla^+ \oplus \nabla^-$ be a connection on $\mathcal{E} \to B$, and consider the super-connection $\mathbb{A} = \pi^*\nabla$ so that $\mathbb{A}^\sigma(t) = \pi^*\nabla + itv_\sigma$. Then, the form $\mathrm{Ch}(\sigma, \mathbb{A}, 1)$ has a "Gaussian" look.

Lemma 5.9. *The differential forms* $\mathrm{Ch}(\sigma, \mathbb{A}, 1)$ *and* $\beta(\sigma, \mathbb{A}, 1)$ *are rapidly decreasing along the fibers.*

Proof. The curvature of $\mathbb{A}^\sigma(t)$ is

$$\mathbf{F}(t) = \pi^*\mathbf{F} - t^2v_\sigma^2 + it[\pi^*\nabla, v_\sigma].$$

Here $\mathbf{F} \in \mathcal{A}^2(B, \mathrm{End}(\mathcal{E}))$ is the curvature of ∇. Assumption 5.8 implies that $[\pi^*\nabla, v_\sigma] \in \mathcal{A}^1(N, \mathrm{End}(\pi^*\mathcal{E}))$ has at most a polynomial growth along the fibers. Furthermore, for any compact subset $\mathcal{K}_B$ of the basis, there exists $R \geq 0$ and $c > 0$ such that $v_\sigma^2(x, \xi) \geq c\|\xi\|^2$ when $\|\xi\| \geq R$ and $x \in \mathcal{K}_B$.

To estimate $e^{\mathbf{F}(t)}$, we apply Lemma 6.1 of the Appendix, with $H = t^2v_\sigma^2$, and $R = -\pi^*\mathbf{F} - it[\pi^*\nabla, v_\sigma]$. The smallest eigenvalue of H is greater or equal to $t^2c\|\xi\|^2$, when $\|\xi\| \geq R$, and R is a sum of terms with strictly positive exterior degrees. Remark that R is a polynomial in t of degree 1 and is bounded in norm by a polynomial in $\|\xi\|$ along the fibers. It follows that from Lemma 6.1 that, for $t \geq 0$, we have

$$\left\| e^{\mathbf{F}(t)} \right\| (x, \xi) \leq \mathcal{P}(\|R\|)e^{-t^2c\|\xi\|^2}.$$

Our estimates on the polynomial growth of R in t and $\|\xi\|$ implies that there exists a polynomial $\mathcal{Q}$ such that, for $t \geq 0$,

$$\left\| e^{\mathbf{F}(t)} \right\| (x, \xi) \leq \mathcal{Q}(t\|\xi\|)e^{-t^2c\|\xi\|^2}, \tag{32}$$

[1]This inequality means that $\|\sigma(x, \xi)w\|^2 \geq c\|\xi\|^2\|w\|^2$ for any $w \in \mathcal{E}_x$.

for $(x,\xi)\in N$, $x\in\mathcal{K}_B$, $\|\xi\|\geq R$.

This implies that $\mathrm{Ch}(\sigma,\mathbb{A},1)=\mathrm{Str}(e^{\mathbf{F}(1)})$ is rapidly decreasing along the fibers. Consider now $\beta(\sigma,\mathbb{A},1)=-i\int_1^\infty \mathrm{Str}(v_\sigma e^{\mathbf{F}(t)})dt$ which is defined (at least) for $\|\xi\|\geq R$. The estimate (32) shows also that $\beta(\sigma,\mathbb{A},1)$ is rapidly decreasing along the fibers. We can prove in the same way that all partial derivatives of $\mathrm{Ch}(\sigma,\mathbb{A},1)$ and $\beta(\sigma,\mathbb{A},1)$ are rapidly decreasing along the fibers: hence $\mathrm{Ch}(\sigma,\mathbb{A},1)\in\mathcal{A}^*_{\text{dec-rap}}(N)$ and $\beta(\sigma,\mathbb{A},1)\in\mathcal{A}^*_{\text{dec-rap}}(N\setminus\mathrm{Supp}(\sigma))$. □

Proposition 5.10. *The differential form* $\mathrm{Ch}(\sigma,\mathbb{A},1)\in\mathcal{A}^*_{\text{dec-rap}}(N)$ *represents the image of the class* $\mathrm{Ch}_{\text{good}}(\sigma)\in\mathcal{H}^*_{\mathrm{Supp}(\sigma)}(N)$ *in* $\mathcal{H}^*_{\text{dec-rap}}(N)$.

Proof. Choosing χ supported on $\|\xi\|\leq R+1$ and equal to 1 in a neighborhood of $\|\xi\|\leq R$, the transgression formula of Lemma 5.3: $c(\sigma,\mathbb{A},\chi)-\mathrm{Ch}(\sigma,\mathbb{A},1)=d(\chi\int_0^1\eta(\sigma,\mathbb{A},s)ds)+d\left((\chi-1)\beta(\sigma,\mathbb{A},1)\right)$ implies our proposition, since the form $c(\sigma,\mathbb{A},\chi)$ represents $\mathrm{Ch}_{\text{good}}(\sigma)$ in $\mathcal{H}^*_{\text{dec-rap}}(N)$.

□

When the fibers of $\pi:N\to B$ are oriented, we have an integration morphism $\pi_*:\mathcal{H}^*_{\text{dec-rap}}(N)\to\mathcal{H}^*(B)$.

Corollary 5.11. *We have* $\pi_*\,\mathrm{Ch}(\sigma,\mathbb{A},1)=\pi_*\,\mathrm{Ch}_{\text{good}}(\sigma)$ *in* $\mathcal{H}^*(B)$.

Remark 5.12. *Very simple examples of Mathai-Quillen classes with "Gaussian look" as well as relative Quillen classes are given in Examples 5.3.2 and 5.3.3.*

5.3. **Examples.** If $\mathcal{E}$ is a trivial bundle $N\times V$ on a manifold N, an endomorphism of $\mathrm{End}(\mathcal{E})$ is determined by a map from N to $\mathrm{End}(V)$. We employ the same notation for both objects, so that if σ is a map from N to $\mathrm{End}(V)$, we also denote by σ the bundle map $\sigma[n,v]=[n,\sigma(n)v]$, for $n\in N$ and $v\in V$. We sometimes denote by $[V]$ the trivial vector bundle with fiber V.

We will use the following convention. Let $V=V^+\oplus V^-$ be a $\mathbb{Z}_2$-graded finite dimensional complex vector space with $V^\pm=\mathbb{C}$. Let $\mathcal{A}$ be a super-commutative algebra (the ring of differential forms on a manifold for example). The elements of the super-algebra $\mathcal{A}\otimes\mathrm{End}(V)$ will be represented by 2×2 matrices with coefficients in $\mathcal{A}$. This algebra operates on the space $\mathcal{A}\otimes V$. We take the following convention: the forms are always considered as operating first: for example, if $V^+=\mathbb{C}$ and $V^-=\mathbb{C}$, the matrix $\begin{pmatrix}0&\alpha\\ \beta&0\end{pmatrix}$ represents the operator

$$\begin{pmatrix}0&\alpha\\ \beta&0\end{pmatrix}:=\begin{pmatrix}0&1\\ 0&0\end{pmatrix}\alpha+\begin{pmatrix}0&0\\ 1&0\end{pmatrix}\beta \tag{33}$$

on $\mathcal{A}\otimes V$.

5.3.1. *The cotangent bundle* $\mathbf{T}^*S^1$. We consider $\mathbf{T}^*S^1:=S^1\times\mathbb{R}$ the cotangent bundle to the circle S^1. The group $\mathbf{K}^0(\mathbf{T}^*S^1)$ of K-theory is generated, as a $\mathbb{Z}$-module, by the class $[\sigma]$ of the following elliptic symbol.

Take $\mathcal{E}^+=\mathcal{E}^-$ the trivial bundles $\mathbf{T}^*S^1\times\mathbb{C}$ over $\mathbf{T}^*S^1$. Let u be a function on $\mathbb{R}$ satisfying $u(\xi)=1$ if $|\xi|>1$ and $u(\xi)=0$ if $|\xi|<1/2$. The symbol $\sigma:\mathcal{E}^+\to\mathcal{E}^-$

is defined by a smooth map $\sigma : \mathbf{T}^*S^1 \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}) = \mathbb{C}$ satisfying

$$\sigma([e^{i\theta},\xi]) = \begin{cases} u(\xi)e^{i\theta}, & \text{if} \quad \xi \geq 0; \\ u(\xi), & \text{if} \quad \xi \leq 0. \end{cases}$$

Here $\mathrm{Supp}(\sigma)$ is the set $\{[e^{i\theta},\xi];\, u(\xi) = 0\}$ and is compact. Note that the class $[\sigma] \in \mathbf{K}^0(\mathbf{T}^*S^1)$ does not depend on the choice of the function u.

We choose on $\mathcal{E}^{\pm}$ the trivial connections $\nabla^+ = \nabla^- = d$ and we let $\mathbb{A} = \nabla^+ \oplus \nabla^-$ on $\mathcal{E}^+ \oplus \mathcal{E}^-$. Then $\mathrm{Ch}(\mathbb{A}) = 0$.

For $\xi \geq 0$, the curvature $\mathbf{F}(\sigma, \mathbb{A}, t)$ of the super-connection

$$\mathbb{A}^{\sigma}(t) = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} + \begin{pmatrix} 0 & itu(\xi)e^{-i\theta} \\ itu(\xi)e^{i\theta} & 0 \end{pmatrix}$$

is represented by the following matrix (see Equation 13 and convention (33)) $\mathbf{F}(\sigma, \mathbb{A}, t)$):

$$\begin{pmatrix} -t^2u(\xi)^2 & 0 \\ 0 & -t^2u(\xi)^2 \end{pmatrix} - it \begin{pmatrix} 0 & e^{-i\theta}(u'(\xi)d\xi - iu(\xi)d\theta) \\ e^{i\theta}(u'(\xi)d\xi + iu(\xi)d\theta) & 0 \end{pmatrix}.$$

For $\xi \leq 0$, we have

$$\mathbf{F}(\sigma, \mathbb{A}, t) = \begin{pmatrix} -t^2u(\xi)^2 & 0 \\ 0 & -t^2u(\xi)^2 \end{pmatrix} - it \begin{pmatrix} 0 & u'(\xi)d\xi \\ u'(\xi)d\xi & 0 \end{pmatrix}.$$

For $\xi \geq 0$, $e^{\mathbf{F}(\sigma,\mathbb{A},t)}$ is represented by the matrix

$$e^{-t^2u(\xi)^2} \begin{pmatrix} 1 + it^2u(\xi)u'(\xi)d\xi\, d\theta & -ite^{-i\theta}(u'(\xi)d\xi - iu(\xi)d\theta) \\ -ite^{i\theta}(u'(\xi)d\xi + iu(\xi)d\theta) & 1 - it^2u(\xi)u'(\xi)d\xi\, d\theta \end{pmatrix}, \tag{34}$$

while for $\xi \leq 0$

$$e^{\mathbf{F}(\sigma,\mathbb{A},t)} = e^{-t^2u(\xi)^2} \begin{pmatrix} 1 & -itu'(\xi)d\xi \\ -itu'(\xi)d\xi & 1 \end{pmatrix}. \tag{35}$$

Thus $\eta(\sigma, \mathbb{A}, t) = -\,\mathrm{Str}\left(iv_\sigma\, e^{\mathbf{F}(\sigma,\mathbb{A},t)}\right)$ is given by

$$\eta(\sigma, \mathbb{A}, t)([e^{i\theta},\xi]) = \begin{cases} -2ite^{-t^2u(\xi)^2}u(\xi)^2 d\theta, & \text{if} \quad \xi \leq 0; \\ 0, & \text{if} \quad \xi \geq 0. \end{cases}$$

Finally, integrating $\eta(\sigma, \mathbb{A}, t)$ in t from 0 to ∞, we find that $\beta(\sigma, \mathbb{A})([e^{i\theta},\xi]) = -id\theta$ when $\xi \geq 0$ and $u(\xi) \neq 0$, and $\beta(\sigma, \mathbb{A})([e^{i\theta},\xi]) = 0$ in the other cases.

Proposition 5.13. *The relative Chern class* $\mathrm{Ch}_{\mathrm{rel}}(\sigma)$ *is the class* $[0, \beta(\sigma, \mathbb{A})]$.

For a function $\chi \in \mathcal{C}^\infty(\mathbb{R})$ compactly supported and equal to 1 on $[-1, 1]$, we have then

$$c(\sigma, \mathbb{A}, \chi) = -i\, \mathbf{1}_{\geq 0}\, d\chi \wedge d\theta$$

where $\mathbf{1}_{\geq 0}$ is the characteristic function of the interval $[0, \infty[$.

We take the orientation $d\theta \wedge d\xi$ on T^*S^1. The differential form $c(\sigma, \mathbb{A}, \chi)$ is compactly supported and of integral equal to $-2i\pi$ on $\mathbf{T}^*S^1$.

Remark 5.14. *From Formulae (34) and (35), we see that the Chern character of the super-connection* $\mathbb{A}^{\sigma}(t)$ *is thus the differential form supported on* $\xi \geq 0$ *and equal for* $\xi \geq 0$ *to*

$$\mathrm{Ch}(\mathbb{A}, \sigma, t) = 2it^2e^{-t^2u(\xi)^2}u(\xi)u'(\xi)d\xi \wedge d\theta.$$

For any $t > 0$, *the integral of* $\mathrm{Ch}(\mathbb{A}, \sigma, t)$ *on* $\mathbf{T}^*S^1$ *is equal to* $-2i\pi$.

Assume that the function u is identically equal to 0 on $[a,b]$ with $a<0$ and $b>0$ and strictly positive when $\xi>b$, so that the support of σ is the interval $[a,b]$. It is easy to see then that the limit when t tends to ∞ of $\mathrm{Ch}(\mathbb{A},\sigma,t)$ is the current on $S^1\times\mathbb{R}$ given $\phi\mapsto -i\int_0^{2\pi}\phi(\theta,b)d\theta$.

5.3.2. *The space $\mathbb{R}^2$.* Now we consider the case where $N=\mathbb{R}^2\simeq\mathbb{C}$. Take $\mathcal{E}^+=\mathcal{E}^-$ the trivial bundles $N\times\mathbb{C}$ over N. We consider Bott's symbol $\sigma_b:\mathcal{E}^+\to\mathcal{E}^-$ which is given by the map $\sigma_b(z)=z$ for $z\in N\simeq\mathbb{C}$. The support of σ_b is reduced to the origin $\{0\}$, thus σ_b defines an element of $\mathbf{K}^0(\mathbb{R}^2)$. Recall that the Bott isomorphism tells us that $\mathbf{K}^0(\mathbb{R}^2)$ is a free $\mathbb{Z}$-module with base σ_b.

We choose on $\mathcal{E}^\pm$ the trivial connections $\nabla^+=\nabla^-=d$. Let $\mathbb{A}=\nabla^+\oplus\nabla^-$ on $\mathcal{E}^+\oplus\mathcal{E}^-$. The curvature $\mathbf{F}(\sigma_b,\mathbb{A},t)$ of the super-connection $\mathbb{A}^{\sigma_b}(t)=\begin{pmatrix} d & 0\\ 0 & d\end{pmatrix}+\begin{pmatrix} 0 & it\overline{z}\\ itz & 0\end{pmatrix}$ has the matrix form (see (33))

$$\mathbf{F}(\sigma_b,\mathbb{A},t)=\begin{pmatrix} -t^2|z|^2 & 0\\ 0 & -t^2|z|^2\end{pmatrix}-it\begin{pmatrix} 0 & d\overline{z}\\ dz & 0\end{pmatrix}.$$

Thus

$$e^{\mathbf{F}(\sigma_b,\mathbb{A},t)}=e^{-t^2|z|^2}\begin{pmatrix} 1-\frac{t^2}{2}dzd\overline{z} & -itd\overline{z}\\ -itdz & 1+\frac{t^2}{2}dzd\overline{z}\end{pmatrix}.$$

Thus $\eta(t)=-\operatorname{Str}\left(iv_{\sigma_b}\,e^{\mathbf{F}(\sigma_b,\mathbb{A},t)}\right)$ is equal to

$$-e^{-t^2|z|^2}\operatorname{Str}\left(\begin{pmatrix} 0 & i\overline{z}\\ iz & 0\end{pmatrix}\begin{pmatrix} 1-\frac{t^2}{2}dzd\overline{z} & itd\overline{z}\\ itdz & 1+\frac{t^2}{2}dzd\overline{z}\end{pmatrix}\right).$$

Thus

$$\eta(t)=-t(\overline{z}dz-zd\overline{z})e^{-t^2|z|^2}. \tag{36}$$

When $z\neq 0$, we obtain

$$\begin{aligned}\beta(\sigma_b,\mathbb{A})(z) &= \int_0^\infty\eta(t)dt\\ &= \frac{1}{2|z|^2}(zd\overline{z}-\overline{z}dz)=-i\,d(\arg z).\end{aligned}$$

Thus we have

$$\mathrm{Ch}_{\mathrm{rel}}(\sigma_b)=[0,-i\,d(\arg z)]. \tag{37}$$

It is easy that $\mathcal{H}^*(\mathbb{C},\mathbb{C}-\{0\})=\mathbb{C}\,\mathrm{Ch}_{\mathrm{rel}}(\sigma_b)$.

Take $f\in\mathcal{C}^\infty(\mathbb{R})$ with compact support and equal to 1 in a neighborhood of 0. Let $\chi(z):=f(\|z\|^2)$. Then the class $\mathrm{Ch}_c(\sigma_b)\in\mathcal{H}_c^*(\mathbb{R}^2)$ is represented by the differential form $c(\sigma_b,\mathbb{A},\chi)=\chi\,\mathrm{Ch}(\mathbb{A})+d\chi\beta(\sigma_b,\mathbb{A})$. Here the differential form $\mathrm{Ch}(\mathbb{A})$ is identically equal to 0. We obtain

$$\begin{aligned}c(\sigma_b,\mathbb{A},\chi) &= d(f(|z|^2))\wedge\beta(\sigma_b,\mathbb{A})\\ &= -f'(|z|^2)d\overline{z}\wedge dz.\end{aligned}$$

Remark that $c(\sigma_b,\mathbb{A},\chi)$ is compactly supported and of integral equal to $2i\pi$ on $\mathbb{R}^2$ (with orientation $dx\wedge dy$). Thus $\frac{1}{2i\pi}c(\sigma_b,\mathbb{A},\chi)$ is a representative of the Thom form of $\mathbb{R}^2$.

Remark 5.15. *For $t > 0$, the Chern character of the super-connection $\mathbb{A}^{\sigma_b}(t)$ is the degree 2 differential form with "Gaussian look"*

$$\mathrm{Ch}(\sigma_b, \mathbb{A}, t) = -e^{-t^2|z|^2} t^2 dz d\overline{z}.$$

For any $t > 0$, $\mathrm{Ch}(\sigma_b, \mathbb{A}, t)$ and $c(\sigma_b, \mathbb{A}, \chi)$ coincide in the cohomology $\mathcal{H}^*_{\text{dec-rap}}(\mathbb{R}^2)$, as follows from Proposition 5.10. In particular they have the same integral.

We also see that the limit, when t tends to ∞ of $\mathrm{Ch}(\sigma_b, \mathbb{A}, t)$ is the Dirac distribution $(2i\pi)\delta_0$.

5.3.3. *The multiplicativity property on $\mathbb{C}^2$.* Following the notations of preceding example, we consider $\mathbb{C}^2$ with coordinates $z = (z_1, z_2)$ and morphisms $\sigma_1 = z_1$ and $\sigma_2 = z_2$. Then the tensor product morphism is

$$\sigma_1 \odot \sigma_2 = \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix}.$$

The morphism $\sigma_1 \odot \sigma_2$ has support $z_1 = z_2 = 0$. A calculation similar to the calculation done in the preceding section gives the following

Proposition 5.16. *The relative chern class $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2) \in \mathcal{H}^*(\mathbb{C}^2, \mathbb{C}^2 \setminus (0,0))$ is represented by $(0, \beta_{12})$, where*

$$\beta_{12} = \frac{-1}{2|z|^4}\Big((\overline{z_1}dz_1 - z_1 d\overline{z_1}) \wedge d\overline{z_2} \wedge dz_2 + (\overline{z_2}dz_2 - z_2 d\overline{z_2}) \wedge d\overline{z_1} \wedge dz_1\Big)$$

is a closed form on $\mathbb{C}^2 \setminus (0,0)$.

Remark that β_{12} is invariant under the symmetry group $U(2)$ of $\mathbb{C}^2$.

Recall that $\mathrm{Ch}_{\mathrm{rel}}(\sigma_k) = [0, \beta_k]$, with $\beta_k = -\frac{\overline{z_k}dz_k - z_k d\overline{z_k}}{2|z_k|^2}$. The wedge product $\beta_1 \wedge \beta_2$ is not defined on $\mathbb{C}^2 \setminus (0,0)$. Introduce a partition of unity Φ_1, Φ_2 with respect to the covering $U_1 \cup U_2$ of $\mathbb{C}^2 \setminus (0,0)$, with $U_k = \{z, z_k \neq 0\}$. Then the relative product $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1) \diamond \mathrm{Ch}_{\mathrm{rel}}(\sigma_2)$ has representative $(0, \beta)$, with

$$\beta = -d\Phi_1 \wedge \beta_1 \wedge \beta_2.$$

We now compute the forms B_1, B_2 of the equation (5). The form $\eta_k(t)$ have been computed (Equation 36). From this it is easy to compute $B_1 = \Phi_1 \int_{0 \leq s \leq t} \eta_1(t)\eta_2(s) ds\, dt$ and B_2. We obtain

$$\begin{aligned} B_1 &= \Phi_1(z_1, z_2) \frac{(\overline{z_1}dz_1 - z_1 d\overline{z_1}) \wedge (\overline{z_2}dz_2 - z_2 d\overline{z_2})}{4|z_1|^2(|z_1|^2 + |z_2|^2)}, \\ B_2 &= \Phi_2(z_1, z_2) \frac{(\overline{z_1}dz_1 - z_1 d\overline{z_1}) \wedge (\overline{z_2}dz_2 - z_2 d\overline{z_2})}{4|z_2|^2(|z_1|^2 + |z_2|^2)}. \end{aligned}$$

Here $B_1 - B_2$ is a two form which is well defined on $\mathbb{C}^2 \setminus (0,0)$ and the relation $\Phi_1 + \Phi_2 = 1$ imply

$$\beta_{12} - \beta = d(B_1 - B_2).$$

This shows that the class $\mathrm{Ch}_{\mathrm{rel}}(\sigma_1 \odot \sigma_2)$ is the product $[0, \beta_1] \diamond [0, \beta_2]$.

We can now look at the different representatives of the Chern class with compact support $\mathrm{Ch}_c(\sigma_1 \odot \sigma_2) \in \mathcal{H}^*_c(\mathbb{C}^2)$.

Let $f \in \mathcal{C}^\infty(\mathbb{R})$ with compact support and equal to 1 in a neighborhood of 0. We consider the functions $\chi(z) = f(|z_1|^2 + |z_2|^2)$ and $\chi_k(z_k) = f(|z_k|^2)$. Let $\Omega = d\overline{z_1} \wedge dz_1 \wedge d\overline{z_2} \wedge dz_2$.

Proposition 5.17. *The Chern class* $\mathrm{Ch}_c(\sigma_1 \odot \sigma_2) \in \mathcal{H}_c^*(\mathbb{C}^2)$ *is represented either by*

$$c(\sigma_1 \odot \sigma_2, \mathbb{A}, \chi) = -\frac{f'\left(|z_1|^2 + |z_2|^2\right)}{|z_1|^2 + |z_2|^2}\,\Omega$$

or by

$$c(\sigma_1, \mathbb{A}_1, \chi_1) \wedge c(\sigma_2, \mathbb{A}_2, \chi_2) = f'\left(|z_1|^2\right) f'\left(|z_2|^2\right)\,\Omega.$$

Clearly the first representative is "better", as it is invariant by the full symmetry group $SO(4)$ on $\mathbb{C}^2 = \mathbb{R}^4$.

In a forthcoming article, for any oriented Euclidean vector bundle $N \to B$, we will give "canonical" explicit representatives of the relative Bott class and of the Thom class in $\mathcal{H}^*(N, N \setminus B)$, similar to the one given in Proposition 5.16.

6. Appendix

We give a proof of the estimate used in this article. It is based on Volterra's expansion formula: if H and R are elements in a finite dimensional associative algebra, then $e^{(H+R)} = e^H + \sum_{k=1}^{\infty} I_k(H,R)$ where

$$I_k(H,R) = \int_{\Delta_k} e^{s_1 H} R e^{s_2 H} R \cdots R e^{s_k H} R e^{s_{k+1} H} ds_1 \cdots ds_k \tag{38}$$

Here Δ_k is the simplex $\{s_i \geq 0\,;\, s_1 + s_2 + \cdots + s_k + s_{k+1} = 1\}$ which has the volume $\frac{1}{k!}$ for the measure $ds_1 \cdots ds_k$.

Now, let $A = \oplus_{i=0}^{R} A_i$ be a complex finite dimensional graded commutative algebra with a norm $\|\cdot\|$ such that $\|ab\| \leq \|a\|\|b\|$. We denote by $A_+ = \oplus_{i=1}^{R} A_i$. Thus $\omega^{R+1} = 0$ for any $\omega \in A_+$. Let V be a finite dimensional Hermitian vector space. Then $\mathrm{End}(V) \otimes A$ is an algebra with a norm still denoted by $\|\cdot\|$. If $H \in \mathrm{End}(V)$, we denote also by H the element $H \otimes 1$ in $\mathrm{End}(V) \otimes A$.

We denote $\mathrm{Herm}(V) \subset \mathrm{End}(V)$ the subspace formed by the Hermitian endomorphisms. When $H \in \mathrm{Herm}(V)$, we denote $\mathrm{sm(H)} \in \mathbb{R}$ the smallest eigenvalue of H : we have

$$\left\| e^{-H} \right\| = e^{-\mathrm{sm(H)}}, \quad \text{for all} \quad H \in \mathrm{Herm}(V).$$

Lemma 6.1. *Let* $\mathcal{P}(t) = \sum_{k=0}^{R} \frac{t^k}{k!}$. *Then, for any* $R \in \mathrm{End}(V) \otimes A_+$, *and* $H \in \mathrm{Herm}(V)$, *we have*

$$\left\| e^{-(H+R)} \right\| \leq e^{-\mathrm{sm(H)}} \mathcal{P}(\|R\|).$$

Proof. Let $c = \mathrm{sm(H)}$. Then $\|e^{-uH}\| = e^{-uc}$ for all $u \geq 0$. The term $I_k(H,R)$ of the Volterra expansion vanishes for $k > R$ since the term $e^{s_1 H} R \cdots R e^{s_{k+1} H}$ belongs to $\mathrm{End}(V) \otimes A_k$. The norm of the term $I_k(H,R)$ is bounded by $\frac{1}{k!} e^{-c} \|R\|^k$. Summing up in k, we obtain our estimate. □

The preceding estimates hold if we work in the algebra $\mathrm{End}(E) \otimes A$, where E is a super-vector space and A a super-commutative algebra.

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