



HAL
open science

Bogomolov on tori revisited.

Francesco Amoroso

► **To cite this version:**

| Francesco Amoroso. Bogomolov on tori revisited.. 2007. hal-00132119

HAL Id: hal-00132119

<https://hal.science/hal-00132119>

Preprint submitted on 20 Feb 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Bogomolov on tori revisited.

Francesco AMOROSO

*Laboratoire de mathématiques Nicolas Oresme, CNRS UMR 6139
Université de Caen, Campus II, BP 5186
14032 Caen Cédex, France*

1 Introduction.

Let $V \subseteq \mathbb{G}_m^n \subseteq \mathbb{P}^n$ be a geometrically irreducible variety which is not torsion (*i. e.* not a translate of a subtorus by a torsion point). For $\theta > 0$ let $V(\theta)$ be the set of $\alpha \in V(\overline{\mathbb{Q}})$ of Weil's height $h(\alpha) \leq \theta$. By the toric case of Bogomolov conjecture (which is a theorem of Zhang),

$$\hat{\mu}^{\text{ess}}(V) = \inf\{\theta > 0, \overline{V(\theta)} = V\} > 0.$$

If we assume moreover that V is not a translate of a subtorus by a point (eventually of infinite order) we can give a lower bound for $\hat{\mu}^{\text{ess}}(V)$ depending only on $\deg(V)$ (see [Bom-Zan 1995], [Dav-Phi 1999], [Sch 1996]).

Let us define the obstruction index $\omega(V)$ as the minimum degree of an hypersurface containing V . We remark that $\omega(V) \leq n \deg(V)^{1/\text{codim}(V)}$ ([Cha]). Assume that V is not transverse (*i. e.* is not contained in a translate of a subtorus). In [Amo-Dav 2003] we conjecture

$$\hat{\mu}^{\text{ess}}(V) \geq c(n)\omega(V)^{-1}$$

for some $c(n) > 0$ and we prove

$$\hat{\mu}^{\text{ess}}(V) \geq c(n)\omega(V)^{-1}(\log(3\omega(V)))^{-\lambda(\text{codim}(V))}$$

where $\lambda(k) = (9(3k)^{k+1})^k$.

The aim of this paper is to give a more simple proof of a slightly improved (and explicit) version of this result (theorem 4.1), based on a very simple determinant argument (see section 2). More precisely the proof presented here

- avoid the use of the absolute Siegel's lemma of Zhang (see [Dav-Phi 1999], lemme 4.7)
- don't need any variant of zero's lemma and the subsequent combinatorial arguments (section 4 of [Amo-Dav 2003])

- don't use the weighted obstruction index $\omega(T; V)$ defined in [Amo-Dav 2003], definition 2.3.

Let

$$V^0 = V \setminus \bigcup_{B \subseteq V} B.$$

where the union is on the set of translates B of subgroups of positive dimension contained in V . In [Amo-Dav 2006], theorem 1.5 we deduce from a lower bound for the essential minimum of V , a lower bound for height for all but finitely points of V^0 . Here we prove (theorem 5.1) an again slightly improved (and explicit) version of that result. We also correct a mistake which appears in that paper: in *op. cit.*, theorem 1.5, $\delta(V)$ must be defined as the minimum degree δ such that V is, as a set, intersection of hypersurface of degree $\leq \delta$ (see remark 5.2 for details).

The determinant argument allow us to prove also very precise results concerning the normalized height $\hat{h}(V)$ of an hypersurface V (see section 3 for the definition). In this special case we conjecture :

Conjecture 1.1 *Assume one of the following:*

- i) V is geometrically irreducible and it is not a translate of a subtorus.*
- ii) V is defined and irreducible over the rationals and is not torsion.*

Then, there exists an absolute constant $c > 0$ such that $\hat{h}(V) \geq c$.

We remark that Lehmer's conjecture implies conjecture ii), *via* an argument of Lawton. We shall prove

Theorem 1.2 *Let $V \subseteq \mathbb{G}_m^n$ be an hypersurface of multi-degrees (D_1, \dots, D_n) with discrete stabilizer. Then, if $n \geq 9$ and*

$$\max D_j \leq 3^{2^n}$$

we have

$$\hat{h}(V) \geq \frac{1}{23}.$$

This result shows that an eventual example contradicting conjecture i) in n variable must be realized by polynomials of very big degree (or comes from an hypersurface of less variables). This could suggests an even more optimistic conjecture:

Let V be a geometric irreducible hypersurface of \mathbb{G}_m^n with discrete stabilizer. Then $\hat{h}(V) \geq f(n)$, where $f(n) \rightarrow +\infty$ for $n \rightarrow \infty$.

In section 3 we also provide a counterexample to this last statement.

2 A determinant argument.

The following proposition is the key argument for the proof of the main theorems.

Let $S \subseteq \mathbb{P}_n$ and let $I \subset \mathbb{C}[\mathbf{x}]$ be the ideal defining its Zariski closure. For $\nu \in \mathbb{N}$ we denote by $H(S; \nu)$ the Hilbert function $\dim[\mathbb{C}[\mathbf{x}]/I]_\nu$. Let T be a positive integer and let $I^{(T)}$ be the T -symbolic power of I , *i. e.* the ideal of polynomials vanishing on S with multiplicity $\geq T$. We put $H(S, T; \nu) = \dim[\mathbb{C}[\mathbf{x}]/I^{(T)}]_\nu$.

Similarly, if $S \subseteq (\mathbb{P}_1)^n$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n) \in \mathbb{N}^n$ we denote its multi-homogeneous Hilbert function by

$$H(S; \boldsymbol{\nu}) = \dim([\overline{\mathbb{Q}}[x_1, \dots, x_n]/I]_{\nu_1, \dots, \nu_n})$$

where $I \subset \mathbb{C}[\mathbf{x}]$ is the ideal defining \overline{S} . More generally, if T is a positive integer we put $H(S, T; \boldsymbol{\nu}) = \dim([\overline{\mathbb{Q}}[x_1, \dots, x_n]/I^{(T)}]_{\nu_1, \dots, \nu_n})$.

Proposition 2.1 *Let ν, T be positive integers and let p be a prime number. Let also h be a positive real number and S be a subset (eventually infinite) of \mathbb{G}_m^n of points of height $\leq h$. Then*

$$h \geq \left(1 - \frac{H(S, T; \nu)}{H(\ker[p] \cdot S; \nu)}\right) \frac{T \log p}{p\nu} - \frac{n}{2\nu} \log(\nu + 1). \quad (2.1)$$

In particular, if

$$H(S, T; \nu) \leq \frac{1}{2} H(\ker[p] \cdot S; \nu) \quad (2.2)$$

and

$$T \log p \geq 2np \log(\nu + 1), \quad (2.3)$$

then

$$h \geq \frac{T \log p}{4p\nu} \geq \frac{n \log(\nu + 1)}{2\nu}.$$

Proof. Let for brevity $S' = \ker[p]S$. We consider the (eventually infinite) matrix

$$(\boldsymbol{\beta}^\lambda)_{\substack{\boldsymbol{\beta} \in S' \\ |\lambda| \leq \nu}}$$

of rang $L = H(\ker[p] \cdot S; \nu)$. We select $\boldsymbol{\beta}_1, \dots, \boldsymbol{\beta}_L \in S'$ and $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_L$ with $|\boldsymbol{\lambda}_j| \leq \nu$ such that the determinant

$$\Delta = \left| \det(\boldsymbol{\beta}_i^{\boldsymbol{\lambda}_j})_{i,j=1, \dots, L} \right|$$

is non-zero. Let $L_0 = H(\ker[p] \cdot S; \nu) - H(S, T; \nu)$. Then, by definition, there exist linearly independent polynomials $G_k = \sum_{j=1}^{L_0} g_{kj} \mathbf{x}^{\boldsymbol{\lambda}_j}$ ($k = 1, \dots, L_0$) vanishing on S with multiplicity $\geq T$. Let K be a sufficiently large field and let v be a non archimedean place of K dividing p . After renumbering the multi-indexes $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_L$ and after making some linear combinations, we can assume

$$G_k = \sum_{j=1}^{L-k+1} g_{kj} \mathbf{x}^{\boldsymbol{\lambda}_j}$$

and moreover

$$|g_{k,j}|_v \begin{cases} \leq 1, & \text{if } j = 1, \dots, L - k; \\ = 1, & \text{if } j = L - k + 1; \end{cases}$$

for $k = 1, \dots, L_0$. By elementary operations on columns we replace the last L_0 columns of Δ by the columns

$${}^\tau(G_k(\beta_1), \dots, G_k(\beta_L)), \quad k = 1, \dots, L_0.$$

Let Δ' the new determinant; then $|\Delta'|_v = |\Delta|_v$. Since G_k vanish on S with multiplicity $\geq T$ and since its coefficients are v -integers, we also have

$$|G_k(\beta_i)|_v \leq p^{-T/(p-1)} \max\{1, |\beta_{i,1}|_v, \dots, |\beta_{i,n}|_v\}^\nu \quad (i = 1, \dots, L; k = 1, \dots, L_0).$$

By developing Δ' with respect to the last L_0 columns we obtain

$$|\Delta'|_v = |\Delta|_v \leq p^{-L_0 T/(p-1)} \prod_{i=1}^L \max\{1, |\beta_{i,1}|_v, \dots, |\beta_{i,n}|_v\}^{\nu L}.$$

By the product's formula (using a trivial lower bound for $v \nmid p$)

$$1 \leq p^{-L_0 T/(p-1)} L^{L/2} e^{\nu h L}$$

and, using $L \leq \binom{\nu+1}{n} \leq (\nu+1)^n$,

$$\log h \geq \frac{L_0}{L} \times \frac{T \log p}{p\nu} - \frac{n}{2\nu} \log(\nu+1)$$

and the statement of proposition 2.1 follows. □

The following is a multihomogeneous version of proposition 2.1.

Proposition 2.2 *Let ν_1, \dots, ν_n, T be positive integers and let p be a prime number. Let also h_1, \dots, h_n be a positive real number and S be a subset (eventually infinite) of \mathbb{G}_m^n of points α satisfying $h(\alpha_j) \leq h_j$ for $j = 1, \dots, n$. Then*

$$\nu_1 h_1 + \dots + \nu_n h_n \geq \left(1 - \frac{H(S, T; \nu)}{H(\ker[p] \cdot S; \nu)}\right) \frac{T \log p}{p} - \frac{n}{2} \log(\nu_{\max} + 1) \quad (2.4)$$

where $\nu_{\max} = \max\{\nu_1, \dots, \nu_n\}$.

Proof. Let for brevity $S' = \ker[p]S$. We consider the matrix

$$(\beta^\lambda)_{\substack{\beta \in S' \\ |\lambda_1| \leq \nu_1, \dots, |\lambda_n| \leq \nu_1}}$$

of rang $L = H(\ker[p] \cdot S; \nu)$. We select $\beta_1, \dots, \beta_L \in S'$ and $\lambda_1, \dots, \lambda_L$ with $|\lambda_{j,l}| \leq \nu_l$ such that the determinant

$$\Delta = |\det(\beta_i^{\lambda_j})_{i,j=1,\dots,L}|$$

is non-zero. Let $L_0 = H(\ker[p] \cdot S; \nu) - H(S, T; \nu)$. Then, by definition, there exists linearly independent polynomials $G_k = \sum_{j=1}^L g_{kj} \mathbf{x}^{\lambda_j}$ ($k = 1, \dots, L_0$) vanishing on S with multiplicity $\geq T$. Let K be a sufficiently large field and let v be a non archimedean place of K dividing p . After renumbering the multi-index $\lambda_1, \dots, \lambda_L$ and after making some linear combinations, we can assume

$$G_k = \sum_{j=1}^{L-k+1} g_{kj} \mathbf{x}^{\lambda_j}$$

and moreover

$$|g_{k,j}|_v \begin{cases} \leq 1, & \text{if } j = 1, \dots, L-k; \\ = 1, & \text{if } j = L-k+1; \end{cases}$$

for $k = 1, \dots, L_0$. By elementary operations on columns we replace the last L_0 columns of Δ by the columns

$$\tau(G_k(\beta_1), \dots, G_k(\beta_L)), \quad k = 1, \dots, L_0.$$

Let Δ' the new determinant; then $|\Delta'|_v = |\Delta|_v$. Since G_k vanish on S with multiplicity $\geq T$ and since its coefficients are v -integers, we also have

$$|G_k(\beta_i)|_v \leq p^{-T/(p-1)} \prod_{j=1}^n \max\{1, |\beta_{i,j}|_v\}^{\nu_j} \quad (i = 1, \dots, L; k = 1, \dots, L_0).$$

By developing Δ' with respect to the last L_0 columns we obtain

$$|\Delta'|_v = |\Delta|_v \leq p^{-L_0 T/(p-1)} \prod_{i=1}^L \prod_{j=1}^n \max\{1, |\beta_{i,j}|_v\}^{\nu_j L}.$$

By the product's formula (using a trivial lower bound for $v \nmid p$)

$$1 \leq p^{-L_0 T/(p-1)} L^{L/2} e^{(\nu_1 h_1 + \dots + \nu_n h_n) L}$$

and, using $L \leq (\nu_{\max} + 1)^n$,

$$\nu_1 h_1 + \dots + \nu_n h_n \geq \frac{L_0}{L} \times \frac{T \log p}{p} - \frac{n}{2} \log(\nu_{\max} + 1)$$

and the statement of proposition 2.2 follows. □

3 Hypersurfaces.

In this section we are interested in the case of a hypersurface V . For these varieties we have a “natural” definition of height (which coincide with the previous one) since we can extend the Mahler measure to polynomials in several variables. Let $f \in \mathbb{C}[x_1, \dots, x_n]$; we define its Mahler measure as:

$$M(P) = \exp \int_0^1 \cdots \int_0^1 \log |f(e^{2\pi i t_1}, \dots, e^{2\pi i t_n})| dt_1 \dots dt_n.$$

Let now K be a number field and let V be an hypersurface in \mathbb{G}_m^n defined over K :

$$V = \{\alpha \in \mathbb{G}_m^n \text{ such that } f(\alpha) = 0\}$$

for some polynomial $f \in K[\mathbf{x}]$ (irreducible over $\overline{\mathbb{Q}}[\mathbf{x}]$). We define:

$$\hat{h}(V) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log M_v(f),$$

where $M_v(f)$ is the maximum of the v -adic absolute values of the coefficients of f if v is non archimedean, and $M_v(f)$ is the Mahler measure of σf if v is an archimedean place associated with the embedding $\sigma : K \hookrightarrow \overline{\mathbb{Q}}$.

We prove:

Proposition 3.1 *Let $V \subseteq \mathbb{G}_m^n$ be an hypersurface of multi-degrees D_1, \dots, D_n and assume that V is not a translated of a torus. Let $D_{\max} = \max\{D_1, \dots, D_n\}$. Then, for any prime number $p \geq 5$,*

$$\hat{h}(V) \geq \frac{\log p}{7p} - \frac{nk' \log p}{p^{k'}} - \frac{n \log(n^2 D_{\max})}{2p^{k'}}. \quad (3.5)$$

where k' is the codimension of the stabilizer of V .

Proof. Since V is not a translated of a torus, $k' \geq 2$. This implies $n \geq 2$ and $p^{k'} \geq 9$.

We assume first that $p \nmid [\text{Stab}(V) : \text{Stab}(V)^0]$, so that $V' = \ker[p]V$ is a union of $p^{k'}$ translate of V , and we prove

$$\hat{h}(V) \geq \frac{\log p}{7p} - \frac{nk' \log p}{p^{k'}} - \frac{n \log(n D_{\max})}{2p^{k'}}, \quad (3.6)$$

Let $\varepsilon > 0$ and assume $D_{\max} = D_n$. The proposition 2.7 of [Amo-Dav 2000] shows that the set

$$S = \{(\zeta_1, \dots, \zeta_{n-1}, \alpha) \in V(\overline{\mathbb{Q}}), \zeta_1, \dots, \zeta_{n-1} \text{ roots of unity, } h(\alpha) \leq \hat{h}(V)/D_n + \varepsilon\}$$

is Zariski dense in V . We apply proposition 2.2 with $h_1 = \dots = h_{n-1} = 0$ and $h_n = \hat{h}(V)/D_n + \varepsilon$. We choose, for $j = 1, \dots, n-1$,

$$\nu_j = np^{k'} D_j - 1$$

and $\nu_n = p^{k'} D_n - 1$. We remark that $\nu_{\max} = \max\{\nu_1, \dots, \nu_n\} \leq np^{k'} D_{\max} - 1$. We also choose $T = \lceil p^{k'}/2 \rceil$. Then

$$\begin{aligned} H(V, T; \boldsymbol{\nu}) &= (\nu_1 + 1) \cdots (\nu_n + 1) - (\nu_1 - TD_1 + 1) \cdots (\nu_n - D_n + 1) \\ &= n^{n-1} p^{k'n} - \frac{1}{2} \left(n - \frac{1}{2} \right)^{n-1} p^{k'n} \end{aligned}$$

and

$$\begin{aligned} H(V'; \boldsymbol{\nu}) &= (\nu_1 + 1) \cdots (\nu_n + 1) - (\nu_1 - p^{k'} D_1 + 1) \cdots (\nu_n - p^{k'} D_n + 1) \\ &= n^{n-1} p^{k'n} \end{aligned}$$

so that

$$1 - \frac{H(V, T; \boldsymbol{\nu})}{H(V'; \boldsymbol{\nu})} \geq \frac{1}{2} \left(1 - \frac{1}{2n} \right)^{n-1} \geq \frac{1}{2\sqrt{e}}.$$

Inequality (2.4) now gives

$$\begin{aligned} \nu_n h_n &= (p^{k'} D_n - 1) \left(\frac{\hat{h}(V)}{D_n} + \varepsilon \right) \\ &\geq \frac{T \log p}{2\sqrt{ep}} - \frac{n}{2} \log(\nu_{\max} + 1) \\ &\geq \frac{p^{k'} \log p}{4\sqrt{ep}} - \frac{\log p}{2\sqrt{ep}} - \frac{n}{2} \log(np^{k'} D_{\max}) \\ &\geq \frac{p^{k'} \log p}{7p} - nk' \log p - \frac{n}{2} \log(nD_{\max}). \end{aligned}$$

By letting $\varepsilon \mapsto 0$ we obtain the lower bound (3.6).

If $\text{Stab}(V)$ is not connected, by inspection of the proof of proposition 2.4 of [Amo-Dav 2000] we obtain an hypersurface W with connected stabilizer of the same codimension k' , multi-degree (D'_1, \dots, D'_n) with $D'_j \leq nD_j$ and normalized height $\hat{h}(W) \leq \hat{h}(V)$. Therefore, by (3.6),

$$\hat{h}(V) \geq \hat{h}(W) \geq \frac{\log p}{7p} - \frac{nk' \log p}{p^{k'}} - \frac{n \log(n^2 D_{\max})}{2p^{k'}}.$$

□

Let now assume $k' = n$, i. e. $\text{Stab}(V)$ discrete. Choosing $p = 5$ we obtain:

Theorem 3.2 *Let $V \subseteq \mathbb{G}_m^n$ be an hypersurface of multi-degrees (D_1, \dots, D_n) with discrete stabilizer. Then, if $n \geq 9$ and*

$$\max D_j \leq 3^{2^n}$$

we have

$$\hat{h}(V) \geq \frac{1}{23} .$$

Proof. We apply the proposition above with $p = 5$, assuming $D_{\max} \leq 3^{2^n}$ and $k' = n$. We obtain

$$\begin{aligned} \hat{h}(V) &\geq \frac{\log 5}{35} - \frac{n^2 \log 5}{5^n} - \frac{n \log(n^2 D_{\max})}{2 \times 5^n} \\ &\geq \frac{\log 5}{35} - \frac{n^2 \log 5}{5^n} - \frac{2n \log n}{2 \times 5^n} - \frac{n 2^n \log 3}{2 \times 5^n} =: f(n) . \end{aligned}$$

An easy computation shows that f is an increasing function and $f(9) > 1/23$. □

As stated in the introduction, we could conjecture that for any geometric irreducible hypersurface $V \subseteq \mathbb{G}_m^n$ with discrete stabilizer we had $\hat{h}(V) \geq f(n)$ for some function $f(n) \rightarrow +\infty$ for $n \rightarrow \infty$. This is false, as the the following example prove. Let $F(x_1) = x_1^3 - x_1 - 1$ and define inductively

$$F_n(x_1, \dots, x_n) = F^*(x_1, \dots, x_{n-1})x_n - F(x_1, \dots, x_{n-1})$$

where F^* indicated the reciprocal polynomial. Since the rational function

$$R(x_1, \dots, x_{n-1}) = \frac{F(x_1, \dots, x_{n-1})}{F^*(x_1, \dots, x_{n-1})}$$

satisfy $|R(z_1, \dots, z_{n-1})| = 1$ for $|z_1| = \dots = |z_{n-1}| = 1$, we have for any integer n $M(F_n) = \theta_0$ where θ_0 is the root > 1 of F_1 . Moreover, it is easy to see that F_n is irreducible (over $\overline{\mathbb{Q}}$ if $n \geq 2$) and that $V_n = \{F_n = 0\}$ has trivial stabilizer.

We conclude this section with a more a general (and technical) lower bound for the normalized height of an hypersurface:

Theorem 3.3 *Let $V \subseteq \mathbb{G}_m^n$ be an hypersurface of multi-degrees (D_1, \dots, D_n) and assume that V is not a translated of a torus. Then,*

$$\hat{h}(V) \geq \frac{1}{56} \times \max \left(\frac{\log(n \log(n^2 D_{\max}))}{k'}, 1 \right) \times \left(\frac{\log(n \log(n^2 D_{\max}))}{28nk' \log(n^2 D_{\max})} \right)^{1/(k'-1)}$$

where k' is the codimension of the stabilizer of V and $D_{\max} = \max D_j$. In particular,

$$\hat{h}(V) \geq \frac{\log(n \log(n^2 D_{\max}))^2}{6272n \log(n^2 D_{\max})} .$$

Proof. Let

$$N = \left(\frac{28nk' \log(n^2 D_{\max})}{\log(n \log(n^2 D_{\max}))} \right)^{1/(k'-1)} \quad (3.7)$$

and choose a prime number p such that $N \leq p \leq 2N$. By

$$\log x \leq x^{1/2} \quad (x > 0) \quad (3.8)$$

we have $\log(n \log(n^2 D_{\max})) \leq \log(n(n^2 D_{\max})^{1/2}) \leq \log(n^2 D_{\max})$; hence

$$p^{k'-1} \geq 28nk' .$$

We also remark that, again by (3.8),

$$\log p \geq \frac{\log(28n^{1/2}k' \log(n^2 D_{\max})^{1/2})}{k'-1} \geq \frac{\log(n \log(n^2 D_{\max}))}{2k'} \quad (3.9)$$

Therefore,

$$p^{k'-1} \log p \geq 14n \log(n^2 D_{\max}) .$$

Thus, by proposition 3.1 we have

$$\begin{aligned} \hat{h}(V) &\geq \frac{\log p}{7p} - \frac{nk' \log p}{p^{k'}} - \frac{n \log(n^2 D_{\max})}{2p^{k'}} \\ &\geq \frac{\log p}{7p} - \frac{\log p}{28p} - \frac{\log p}{28p} \\ &= \frac{\log p}{14p} . \end{aligned}$$

By (3.9) we obtain:

$$\begin{aligned} \hat{h}(V) &\geq \frac{1}{14} \times \max \left(\frac{\log(n \log(n^2 D_{\max}))}{2k'}, \log 2 \right) \times \frac{1}{2N} \\ &\geq \frac{1}{56} \times \max \left(\frac{\log(n \log(n^2 D_{\max}))}{k'}, 1 \right) \times \left(\frac{\log(n \log(n^2 D_{\max}))}{28nk' \log(n^2 D_{\max})} \right)^{1/(k'-1)} . \end{aligned}$$

This prove the first inequality of theorem 3.3. For the second one, we remark that $k' \geq 2$ and $k'(nk')^{1/(k-1)} \leq 4n$. So

$$\begin{aligned} \hat{h}(V) &\geq \frac{1}{56} \times \max \left(\frac{\log(n \log(n^2 D_{\max}))}{k'}, 1 \right) \times \left(\frac{\log(n \log(n^2 D_{\max}))}{28nk' \log(n^2 D_{\max})} \right)^{1/(k'-1)} \\ &\geq \frac{\log(n \log(n^2 D_{\max}))^2}{56 \times 28 \times 4n \log(n^2 D_{\max})} \\ &= \frac{\log(n \log(n^2 D_{\max}))^2}{6272n \log(n^2 D_{\max})} . \end{aligned}$$

□

4 Essential minimum.

In this section we prove the following theorem, which slightly unproves theorem 1.4 of [Amo-Dav 2003]:

Theorem 4.1 *Let V be a subvariety of \mathbb{G}_m^n of codimension $k < n$. Then either there exists a translate B of a subgroup such that $V \subseteq B \subsetneq \mathbb{G}_m^n$ and*

$$\deg(B)^{1/\text{codim}(B)} \leq (250n^3 \log(2n\omega(V)))^{\lambda(k)+1} \omega(V)$$

or

$$\hat{\mu}^{\text{ess}}(V) \geq (2400n^4 \log(2n\omega(V)))^{-\lambda(k)} \omega(V)^{-1}$$

where $\lambda(k) = \frac{k+1}{k}((k+1)^k - 1) - 1 \leq n^n - 3$.

Proposition 2.1 gives the following result:

Proposition 4.2 *Let V be a subvariety of \mathbb{G}_m^n and let $\omega = \omega(V)$. Let also p be a prime, $3 \leq p \leq \omega$ and assume :*

$$\hat{\mu}^{\text{ess}}(V) < \frac{\log p}{10np\omega}.$$

Then,

$$\omega([p]V) \leq \frac{18n^2\omega \log(5n\omega)}{\log p}.$$

Proof. Let h such that $\hat{\mu}^{\text{ess}}(V) < h < \frac{\log p}{10np\omega}$ and let

$$S = \{\alpha \in V, \quad h(\alpha) < h\}.$$

Thus $H(S, T; \nu) = H(V, T; \nu)$ and $H(\ker[p] \cdot S; \nu) = H(\ker[p] \cdot V; \nu)$. Let us define

$$T = \left\lceil \frac{7np \log(5n\omega)}{\log p} \right\rceil$$

and $\nu = (2n+1)\omega T$. We first show that there exists a non zero polynomial $F \in \overline{\mathbb{Q}}[x_1, \dots, x_n]$ of total degree $\leq \nu$, vanishing on $\ker[p]V$. Since $3 \leq p \leq \omega$, we have

$$\nu + 1 \leq 3n\omega \cdot 7np \cdot 5n\omega + 1 \leq (5n\omega)^3$$

and $T \log p \geq 6np \log(5n\omega)$. Thus inequality (2.3) of proposition 2.1, i. e. $T \log p \geq 2np \log(\nu + 1)$, is satisfied. We also have

$$\frac{T \log p}{4p\nu} = \frac{\log p}{4p(2n+1)\omega} > h.$$

By proposition 2.1, we must have

$$H(\ker[p] \cdot V; \nu) < 2H(V, T; \nu) \leq 2 \left(\binom{\nu+n}{n} - \binom{\nu-\omega T+n}{n} \right).$$

We remark that

$$\begin{aligned} \binom{\nu+n}{n} \binom{\nu-\omega T+n}{n}^{-1} &= \prod_{j=1}^n \frac{\nu+j}{\nu-\omega T+j} \leq \left(1 + \frac{\omega T}{\nu-\omega T}\right)^n \\ &= \left(1 + \frac{1}{2n}\right)^n \leq \sqrt{e} < 2. \end{aligned}$$

Thus

$$H(\ker[p] \cdot V; \nu) < \binom{\nu+n}{n},$$

i. e. there exists a non zero polynomial $F \in \overline{\mathbb{Q}}[x_1, \dots, x_n]$ vanishing on $\ker[p]V$ of total degree $\leq \nu$. By the zero's lemma of P. Philippon (see [Phi 1986]), there exists a variety Z containing V such that

$$\deg(\ker[p]Z) \leq \nu^{\text{codim}(Z)}.$$

Indeed, let W be the algebraic set defined by the equations $F(\zeta \mathbf{x}) = 0$ for $\zeta \in \ker[p]$. Since F vanishes on $\ker[p]V$, there exists a geometrically irreducible component Z of W containing V . Since W is stable by translation by p -torsion points, all ζV are components of W for $\zeta \in \ker[p]$. Proposition 3.3 of [Phi 1986] (with $p = 1$, $N_1 = n$ and $D_1 = \nu$) then gives the desired upper bound for $\deg(\ker[p]Z)$.

Since

$$\deg(\ker[p]Z) = \deg([p]^{-1}[p]Z) = p^{\text{codim}(Z)} \deg([p]Z)$$

we obtain

$$\omega([p]V) \leq \deg([p]Z)^{1/\text{codim}(Z)} \leq p^{-1}\nu.$$

We finally remark that

$$\frac{1}{p}\nu \leq \frac{1}{p} \cdot \frac{5}{2}n\omega \cdot \frac{7np \log(5n\omega)}{\log p} < \frac{18n^2\omega \log(5n\omega)}{\log p}.$$

□

In order to prove theorem 4.1 we need, as in [Amo-Dav 2003], a descent argument. In what follows we fix a geometrically irreducible subvariety $V \subsetneq \mathbb{G}_m^n$ of dimension $k < n$ (thus $n \geq 2$) and we let $\omega = \omega(V)$. For $j = 1, \dots, k$ let $\rho_j = (k+1)^{k-j+1} - 1$ and $P_j = (2\Delta)^{\rho_j}$ where $\Delta = Cn^3 \log(2n\omega)$ and $C = 120$.

The following elementary relations will be used several time

Lemma 4.3 *We have:*

i) $\log(2n\omega) > 1$ and $\Delta > 960$.

ii) For $j \in \{0, \dots, k\}$ we have

$$\sum_{l=j+1}^k \rho_l = (k+1) \frac{(k+1)^{k-j} - 1}{k} - (k-j).$$

Definition 4.4 Let \mathcal{W} be the set of triples $(s, \mathbf{p}, \mathbf{W})$, where $s \in [0, k]$ is an integer, $\mathbf{p} = (p_1, \dots, p_s)$ is a s -tuple of prime numbers with $P_i/2 \leq p_i \leq P_i$, and where $\mathbf{W} = (W_0, \dots, W_s)$ is a $(s+1)$ -tuple of strict geometrically irreducible subvarieties $\subsetneq \mathbb{G}_m^n$, satisfying:

i) $V \subseteq W_0$. Moreover, for $i = 1, \dots, s$,

$$[p_i]W_{i-1} \subseteq W_i \quad \text{and} \quad p_i \nmid [\text{Stab}(W_{i-1}) : \text{Stab}(W_{i-1})^0];$$

ii) For $i = 0, \dots, s$

$$\deg(W_i)^{1/\text{codim}(W_i)} \leq \Delta^{k-i} p_{i+1} \cdots p_k \omega([p_1 \dots p_i]V);$$

iii) For $i = 1, \dots, s$

$$\omega([p_1 \dots p_i]V) \leq \Delta \omega([p_1 \dots p_{i-1}]V).$$

Remark 4.5 Let $(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W}$ and assume $0 \leq i \leq j \leq s$. Then

$$\omega([p_1 \dots p_j]V) \leq \Delta^{j-i} \omega([p_1 \dots p_i]V).$$

We want to prove that there exists $(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W}$, such that $\dim(W_{i-1}) = \dim(W_i)$ for at least one index i . Let

$$\mathcal{W}_0 = \{(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W}, \text{ such that } \dim(W_0) < \dim(W_1) < \dots < \dim(W_s)\}.$$

Proposition 4.6 Assume

$$\hat{\mu}^{\text{ess}}(V) < \left(10n\Delta^{k-1}P_1 \cdots P_k \omega\right)^{-1}. \quad (4.10)$$

Then $\mathcal{W}_0 \neq \mathcal{W}$.

In order to prove proposition 4.6, we endow the set of finite sequences of integers with the following (total) order \preceq . Let $(v) = (v_i)_{0 \leq i \leq s}$ and $(v') = (v'_j)_{0 \leq j \leq s'}$ two such sequences. Then $(v) \preceq (v')$ if

$$(v_i)_{0 \leq i \leq \min\{s, s'\}} < (v'_i)_{0 \leq i \leq \min\{s, s'\}}$$

for the lexicographical order, or if $(v_i)_{0 \leq i \leq \min\{s, s'\}} = (v'_i)_{0 \leq i \leq \min\{s, s'\}}$ and $s \geq s'$.

We also need the following technical lemma:

Lemma 4.7 *Let $s \in \mathbb{N}$, p_1, \dots, p_s, p_{s+1} positive integers, $W_0, \dots, W_s \subsetneq \mathbb{G}_m^n$ geometrically irreducible subvarieties. Let us assume $V \subseteq W_0$ and $[p_i]W_{i-1} \subseteq W_i$ for $i = 1, \dots, s$. Then, there exists an integer $s' \in [0, s+1]$ and a geometrically irreducible subvariety $Z_{s'}$ of degree*

$$\deg(Z_{s'}) \leq p_{s'+1} \dots p_{s+1} \omega([p_1 \dots p_{s+1}]V) \deg(W_{s'}) , \quad (4.11)$$

such that $[p_{s'}]W_{s'-1} \subseteq Z_{s'}$, $\text{codim}(Z_{s'}) = \text{codim}(W_{s'}) + 1$ (with the following convention: $\text{codim}(W_{s+1}) = 0$, $\deg(W_{s+1}) = 1$, $W_{-1} = V$ and $p_0 = 1$) and:

$$(\dim(W_0), \dots, \dim(W_{s'-1}), \dim(Z_{s'})) \prec (\dim(W_0), \dots, \dim(W_s)) . \quad (4.12)$$

Proof. Let Z_{s+1} be an hypersurface containing $[p_1 \dots p_{s+1}]V$ of minimal degree $\omega([p_1 \dots p_{s+1}]V)$. Thus if $s' = s+1$ (4.11) is satisfied. We construct by induction subvarieties Z_0, \dots, Z_s such that, for $i = 0, \dots, s$,

- i) $Z_i \subseteq W_i$ and $Z_i \neq W_i \Rightarrow \text{codim}(Z_i) = \text{codim}(W_i) + 1$.
- ii) $[p_{i+1} \dots p_{s+1}]Z_i \subseteq Z_{s+1}$.
- iii) $[p_{i+1}]Z_i \subseteq Z_{i+1}$.
- iv) $\deg(Z_i) \leq p_{i+1} \dots p_{s+1} \omega([p_1 \dots p_{s+1}]V) \deg(W_i)$.

We start by the construction of Z_0 . If $[p_1 \dots p_{s+1}]W_0 \subseteq Z_{s+1}$, we set $Z_0 = W_0$. Otherwise we choose for Z_0 a geometrically irreducible component of maximal dimension of $W_0 \cap [p_1 \dots p_{s+1}]^{-1}Z_{s+1}$ containing V . By Bézout's inequality we have:

$$\deg(Z_0) \leq \deg(W_0) \deg([p_1 \dots p_{s+1}]^{-1}Z_{s+1}) \leq p_1 \dots p_{s+1} \omega([p_1 \dots p_{s+1}]V) \deg(W_0) .$$

Let now $i \in [0, s-1]$ be an integer and assume that Z_0, \dots, Z_i satisfy conditions i)–iv). If

$$[p_{i+2} \dots p_{s+1}]W_{i+1} \subseteq Z_{s+1} ,$$

we set $Z_{i+1} = W_{i+1}$. Otherwise we choose for Z_{i+1} a geometrically irreducible component of maximal dimension of $[p_{i+2} \dots p_{s+1}]^{-1}Z_{s+1} \cap W_{i+1}$ containing $[p_{i+1}]Z_i$. We can do this, since $[p_{i+1}]W_i \subseteq W_{i+1}$ (by assumption) $Z_i \subseteq W_i$ (by induction i)) and since

$$[p_{i+1} \dots p_{s+1}]Z_i \subseteq Z_{s+1}$$

(by induction i)). The variety Z_{i+1} verify conditions i)–iii). As before, by Bézout's inequality we have:

$$\deg(Z_{i+1}) \leq p_{i+2} \dots p_{s+1} \omega([p_1 \dots p_{s+1}]V) \deg(W_{i+1}) .$$

and the variety Z_{i+1} also verify condition iv).

We now choose the integer s' . We define s' as the least integer i such that $Z_i \subsetneq W_i$, if such an integer exists. Otherwise we set $s' = s+1$. We remark that in both cases (4.12) holds.

□

Proof of proposition 4.6. The set \mathcal{W}_0 is a finite non-empty set (indeed, let W_0 be an hypersurface of \mathbb{G}_m^n containing V of degree ω ; then $(0, \emptyset, (W_0)) \in \mathcal{W}_0$). Thus, there exists a minimal element $(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W}_0$, *i. e.*

$$(\dim W_i)_{0 \leq i \leq s} \preccurlyeq (\dim W'_i)_{0 \leq i \leq s'}.$$

for all $(s', \mathbf{p}', \mathbf{W}') \in \mathcal{W}_0$. We remark that $s \leq k - 1$, since

$$n - k = \dim(V) \leq \dim(W_0) < \dim(W_1) < \dots < \dim(W_s) \leq n - 1.$$

We need the following computation:

Lemma 4.8 *There exists a prime p_{s+1} such that $P_{s+1}/2 \leq p_{s+1} \leq P_{s+1}$ and*

$$p_{s+1} \nmid [\text{Stab}(W_s) : \text{Stab}(W_s)^0].$$

Proof. By Theorems 9 and 10 of [Ros-Sch 1962], $\sum_{p \leq x} \log p \leq 1.02x$ for $x \geq 1$ and $\sum_{p \leq x} \log p \geq 0.84x$ for $x \geq 101$. Thus

$$\begin{aligned} \sum_{P_{s+1}/2 \leq p \leq P_{s+1}} \log p &\geq (0.84 - 1.02/2)P_{s+1} \\ &> P_{s+1}/4. \end{aligned}$$

If for any prime p with $P_{s+1}/2 \leq p \leq P_{s+1}$ we had $p \mid [\text{Stab}(W_s) : \text{Stab}(W_s)^0]$, then

$$2 \log \deg(W_s) \geq P_{s+1}/4,$$

since $\deg(\text{Stab}(W_s)) \leq \deg(W_s)^2$. By assertion ii) of definition 4.4 and by remark 4.5, we have :

$$\begin{aligned} \log \deg(W_s) &\leq \text{codim}(W_s) \left(k \log(\Delta) + \sum_{j=s+1}^k \log P_j + \log(\omega) \right) \\ &\leq k \left((k + \sum_{j=s+1}^k \log \rho_j) \log(2\Delta) + \log \omega \right). \end{aligned}$$

Using the inequality $\log x < x^{1/3}$ ($x > 100$) with $x = 2\Delta$ (see lemma 4.3 i)) we obtain

$$\log \deg(W_s) \leq k(k+1 + \sum_{j=s+1}^k \log \rho_j) (2Cn^3)^{1/3} \log(2n\omega).$$

Since $s \leq k - 1$, we have, using lemma 4.3 ii),

$$\begin{aligned} k(k+1 + \sum_{j=s+1}^k \log \rho_j) &= k(k+1) + (k+1)^{k-s+1} - (k+1) - k(k-s) \\ &= (k+1)^{k-s+1} + ks - 1 \\ &\leq 2(k+1)^{2(k-s)}. \end{aligned}$$

Thus, by setting $a = (k + 1)^{(k-s)} \geq 2$,

$$2 \log \deg(W_s) \leq 4a^2(2Cn^3)^{1/3} \log(2n\omega)$$

and

$$\begin{aligned} \frac{P_{s+1}/4}{2 \log \deg(W_s)} &\geq \frac{(2Cn^3 \log(2n\omega))^{a-1}}{16a^2(2Cn^3)^{1/3} \log(2n\omega)} \\ &\geq \frac{(16C)^{a-4/3}}{16a^2} =: f(a). \end{aligned}$$

An easy computation shows that $f(a) \geq f(2) > 1$. Contradiction. □

By the previous lemma, there exists a prime number $p_{s+1} \in [P_{s+1}/2, P_{s+1}]$ such that $p_{s+1} \nmid [\text{Stab}(W_s) : \text{Stab}(W_s)^0]$. We want to apply proposition 4.2 to the variety $V' = [p_1 \dots p_s]V$ choosing $p = p_{s+1}$. We have

$$\hat{\mu}^{\text{ess}}(V') \leq p_1 \dots p_s \hat{\mu}^{\text{ess}}(V)$$

and, by iii) of definition 4.4

$$\omega(V') \leq \Delta^s \omega(V).$$

Thus, by assumption (4.10),

$$\begin{aligned} \omega(V') \hat{\mu}^{\text{ess}}(V') &\leq \Delta^s p_1 \dots p_s \omega \hat{\mu}^{\text{ess}}(V) \\ &< (10nP_{s+1})^{-1} \\ &\leq \frac{\log p_{s+1}}{10np_{s+1}}. \end{aligned}$$

Proposition 4.2 shows that:

$$\begin{aligned} \omega([p_{s+1}]V') &\leq \frac{18n^2 \log(5n\omega(V'))}{\log p_{s+1}} \omega([p_1 \dots p_s]V) \\ &\leq 18n^2 \log(5n\omega(V')) \omega(V'). \end{aligned}$$

Since $s \leq k - 1 \leq n$, we have, using remark 4.5,

$$5n\omega(V') \leq 5n\Delta^s \omega \leq ((C\sqrt{5}/32)(2n\omega)^5)^n.$$

Thus

$$\begin{aligned} \Delta - 18n^2 \log(5n\omega(V')) &\geq Cn^3 \log(2n\omega) - 18n^3 \log((C\sqrt{5}/32)(2n\omega)^5) \\ &\geq n^3((C - 18 \times 5) \log(4) - 18 \log(C\sqrt{5}/32)) > 0 \end{aligned}$$

and

$$\omega([p_1 \dots p_{s+1}]V) = \omega([p_{s+1}]V') \leq \Delta\omega(V') = \Delta\omega([p_1 \dots p_s]V) .$$

We apply now lemme 4.7. We obtain an integer s' such that $0 \leq s' \leq s+1 \leq k$ and a subvariety $Z_{s'}$ satisfying the properties described in this lemma. We want to show that

$$(s', (p_1, \dots, p_{s'}), (W_0, \dots, W_{s'-1}, Z_{s'})) \in \mathcal{W} .$$

All conditions i)–iii) of definition 4.4 are trivially verified, except eventually for the upper bound of $\deg(Z_{s'})$. Using inequality (4.11) of lemma 4.7, the upper bound for the degree of $W_{s'}$ (point ii) of definition 4.4), remark 4.5 and the relation $\text{codim}(Z_{s'}) = \text{codim}(W_{s'+1}) + 1$, we get:

$$\begin{aligned} \deg(Z_{s'}) &\leq p_{s'+1} \dots p_{s+1} \omega([p_1 \dots p_{s+1}]V) \deg(W_{s'}) \\ &\leq p_{s'+1} \dots p_{s+1} \Delta^{s-s'+1} \omega([p_1 \dots p_{s'}]V) \deg(W_{s'}) \\ &\leq \Delta^{k-s'} p_{s'+1} \dots p_k \omega([p_1 \dots p_{s'}]V) \deg(W_{s'}) \\ &\leq \left(\Delta^{k-s'} p_{s'+1} \dots p_k \omega([p_1 \dots p_{s'}]V) \right)^{1+\text{codim}(W_{s'+1})} \\ &\leq \left(\Delta^{k-s'} p_{s'+1} \dots p_k \omega([p_1 \dots p_{s'}]V) \right)^{\text{codim}(Z_{s'})} . \end{aligned}$$

Thus $(s', (p_1, \dots, p_{s'}), (W_0, \dots, W_{s'-1}, Z_{s'})) \in \mathcal{W}$. Since

$$(\dim(W_0), \dots, \dim(W_{s'-1}), \dim(Z_{s'})) \prec (\dim(W_0), \dots, \dim(W_s))$$

by relation (4.12) of lemma 4.7 and since $(s, \mathbf{p}, \mathbf{W})$ is a minimal element of \mathcal{W}_0 , we deduce that:

$$(s', (p_1, \dots, p_{s'}), (W_0, \dots, W_{s'-1}, Z_{s'})) \notin \mathcal{W}_0 .$$

□

4.1 Proof of theorem 4.1

Let V be a geometrically irreducible subvariety of \mathbb{G}_m^n of codimension $k < n$ which satisfy the assumption of proposition 4.6. By this proposition, there exists $(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W} \setminus \mathcal{W}_0$. Thus there exists an index i such that

$$\text{codim}(W_{i-1}) = \text{codim}(W_i) = r, \quad [p_i]W_{i-1} \subseteq W_i, \quad [p_1 \dots p_{i-1}]V \subseteq W_i ;$$

and $p_i \nmid [\text{Stab}(W_{i-1}) : \text{Stab}(W_{i-1})^0]$.

Assume first that W_i is a translate of a subtorus. Then the same is true for the connected component B of $[p_1 \dots p_i]^{-1}W_i$ containing V and we have, using ii) of definition 4.4 and remark 4.5,

$$\begin{aligned} (\deg B)^{1/\text{codim}(B)} &\leq (p_1 \dots p_i)^{1/r} \Delta^k p_{i+1} \dots p_k \\ &\leq \Delta^k P_1 \dots P_k \\ &\leq (2\Delta)^{\lambda(k)+1} \end{aligned}$$

where

$$\lambda(k) + 1 = k + \sum_{j=1}^k \rho_j = \frac{k+1}{k} ((k+1)^k - 1) .$$

Assume now that W_i is not a translate of a subtorus. Thus

$$p_i \deg(W_{i-1}) \leq \deg(W_i) .$$

Since $W_{i-1} \supseteq [p_1 \dots p_{i-1}]V$, we have, using ii) and iii) of definition 4.4,

$$\begin{aligned} \omega([p_1 \dots p_{i-1}]V) &\leq (\deg(W_{i-1}))^{1/r} \\ &\leq p_i^{-1/r} (\deg(W_i))^{1/r} \\ &\leq p_i^{-1/r} \Delta^{k-i} p_{i+1} \dots p_k \omega([p_1 \dots p_i]V) \\ &\leq p_i^{-1/r} \Delta^{k-i} p_{i+1} \dots p_k \times \Delta \omega([p_1 \dots p_{i-1}]V) . \end{aligned}$$

Since $r \leq k$ and $P_i/2 \leq p_i \leq P_i$, we get :

$$\begin{aligned} p_i^{-1/r} \Delta^{k-i} p_{i+1} \dots p_k \Delta &\leq P_i^{-1/k} 2^{1/k} \Delta^{k-i+1} P_{i+1} \dots P_k \\ &< P_i^{-1/k} (2\Delta)^{k-i+1} P_{i+1} \dots P_k \\ &= (2\Delta)^b \end{aligned}$$

where (see lemma 4.3 ii))

$$\begin{aligned} b &= -\frac{\rho_i}{k} + k - i + 1 + \sum_{j=i+1}^k \rho_j \\ &= -\frac{(k+1)^{k-i+1} - 1}{k} + (k - i + 1) + (k+1) \frac{(k+1)^{k-i} - 1}{k} - (k - i) \\ &= 0 . \end{aligned}$$

This is a contradiction. Hence

$$\hat{\mu}^{\text{ess}}(V) \geq \left(10n\Delta^{k-1}P_1 \cdots P_k \omega(V)\right)^{-1}.$$

We finally remark that

$$10n\Delta^{k-1}P_1 \cdots P_k \leq (20n\Delta)^{\lambda(k)}.$$

Theorem 4.1 is proved. □

5 Petit points.

Given an algebraic set $V \subseteq \mathbb{G}_m^n$ we define, following [Bom-Zan 1995] and [Sch 1996],

$$V^0 = V \setminus \bigcup_{B \subseteq V} B.$$

where the union is on the set of translates B of subgroups of positive dimension contained in V . In this section we prove a slightly improved version of theorem 1.5 of [Amo-Dav 2006]:

Theorem 5.1 *Let $V \subsetneq \mathbb{G}_m^n$ be an algebraic set defined by equations of degree $\leq \delta$. Then, for all but finitely many $\alpha \in V^0$ we have*

$$\hat{h}(\alpha) \geq \theta := (2400n^3 \log(2n\delta))^{-n^2+3} \delta^{-1}.$$

More precisely, the set of $\alpha \in V$ of height $< \theta$ is contained in a finite union $B_1 \cup \cdots \cup B_m$ of translate of subtori such that

$$\deg(B_j) \leq (250n^3 \log(2n\delta))^{(2n)^n} \delta^{2^{\text{codim}(B_j)-1}}$$

Proof.

It is enough to prove the following statement:

Let $V \subsetneq \mathbb{G}_m^n$ be an algebraic set defined by equations of degree $\leq \delta$ and let Z be a geometrically irreducible subvariety of V of positive dimension, satisfying

$$\hat{\mu}^{\text{ess}}(Z) \leq (2400n^3 \log(2n\delta))^{-n^2+3} \delta^{-1}. \quad (5.13)$$

Then, there exists a translate B of a subtorus of codimension r such that $Z \subseteq B \subseteq V$ and

$$\deg(B) \leq (250n^3 \log(2n\delta))^{(2n)^n} \delta^{2^r-1}.$$

We prove this last statement by induction on n . If $n = 2$ it is easily implied by theorem 4.1. Assume $n \geq 3$ and that the conclusion holds for all algebraic set defined by equations of degree $\leq \delta'$ in \mathbb{G}_m^{n-1} . Assume further that there exists a positive integer δ , an algebraic set $V \subsetneq \mathbb{G}_m^n$ defined by equations of degree $\leq \delta$ and a geometrically irreducible subvariety Z of V which satisfies (5.13). Let $k = \text{codim}(Z)$. In particular, since $\omega(Z) \leq \delta$ and $\lambda(k) \leq n^n - 3$, theorem 4.1 gives a translate $B = \alpha H$ of codimension k' containing Z , and such that

$$(\deg(B))^{1/k'} \leq (250n^3 \log(2n\delta))^{n^n - 2} \delta. \quad (5.14)$$

We can assume $\alpha \in Z$ and $\hat{h}(\alpha) \leq 2\hat{\mu}^{\text{ess}}(Z)$; thus we have :

$$\hat{\mu}^{\text{ess}}(\alpha^{-1}Z) \leq \hat{h}(\alpha^{-1}) + \hat{\mu}^{\text{ess}}(Z) \leq n\hat{h}(\alpha) + \hat{\mu}^{\text{ess}}(Z) \leq 3n\hat{\mu}^{\text{ess}}(Z). \quad (5.15)$$

We now fix a \mathbb{Z} -base $\mathbf{a}_1, \dots, \mathbf{a}_{k'}$ of the \mathbb{Z} -module

$$\Lambda := \left\{ \boldsymbol{\lambda} \in \mathbb{Z}^n, \text{ t.q. } \forall \mathbf{x} \in H, \mathbf{x}^{\boldsymbol{\lambda}} = 1 \right\} \subseteq \mathbb{Z}^n$$

and we consider the $n \times k'$ matrix $A = (a_{i,j})$. Let $E = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Then (see for instance [Ber-Phi 1988]) the degree of H is the maximum of the absolute values of the $k' \times k'$ subdeterminants of A , and $\text{Vol}(E/\Lambda)$ is their quadratic mean. Thus

$$\text{Vol}(E/\Lambda) \leq \binom{n}{k'}^{1/2} \deg(B) \leq n^{k'} \deg(B).$$

Let us consider the cube $[-1/2, 1/2]^n \subset \mathbb{R}^n$; by a theorem of Vaaler (see [Vaaler 1979])

$$\text{Vol}(C \cap E) \geq 1.$$

Thus, by Minkowski's theorem on convex bodies, there exists a non-zero $\boldsymbol{\lambda} \in \Lambda$ such that:

$$\max_{1 \leq i \leq n} \{|\lambda_i|\} \leq n \deg(B)^{1/k'}.$$

Since H is connected, we can assume $\lambda_1, \dots, \lambda_n$ coprime and also $\lambda_n = D$. Then the equation

$$\mathbf{x}^{\boldsymbol{\lambda}} = 1$$

defines a subtorus $H' \supseteq H$ of codimension 1 and degree

$$D \leq n \deg(B)^{1/k'} \leq (2n)^{-2} (250n^3 \log(2n\delta))^{n^n} \delta. \quad (5.16)$$

If $\alpha H' \subseteq V$ we are done. Assume the contrary. We consider the isogeny $\mathbb{G}_m^{n-1} \rightarrow H'$ defined by

$$\varphi(\mathbf{x}) = \left(x_1^{\lambda_n}, \dots, x_{n-1}^{\lambda_n}, x_1^{-\lambda_1} \dots x_{n-1}^{-\lambda_{n-1}} \right).$$

We remark that, for any $\beta \in \mathbb{G}_m^{n-1}$,

$$h(\varphi(\beta)) \geq h(\beta^{\lambda_n}) = \lambda_n h(\beta) = Dh(\beta). \quad (5.17)$$

Let

$$V' = \varphi^{-1}(\alpha^{-1}V \cap H) \subseteq \mathbb{G}_m^{n-1}$$

Since $\alpha H' \not\subseteq V$ we have $V' \subsetneq \mathbb{G}_m^{n-1}$. Moreover, let $F_j(\mathbf{x})$ ($j = 1, \dots, N$) be equations defining V ; then V' is defined by the equations

$$F_j(x_1^{\lambda_n}, \dots, x_{n-1}^{\lambda_n}, x_1^{-\lambda_1} \dots x_{n-1}^{-\lambda_{n-1}}) = 0$$

of degree

$$\leq \delta' = \max\{\lambda_n, |\lambda_1 + \dots + \lambda_{n-1}|\}\delta \leq nD\delta.$$

Let Z' be a geometrically irreducible component of $\varphi^{-1}(\alpha^{-1}Z \cap H) \subseteq V'$. We have, by (5.17) and (5.15),

$$D\hat{\mu}^{\text{ess}}(Z') \leq \hat{\mu}^{\text{ess}}(\varphi(Z')) = \hat{\mu}^{\text{ess}}(\alpha^{-1}Z) \leq 3n\hat{\mu}^{\text{ess}}(Z).$$

Using the upper bound for $\hat{\mu}^{\text{ess}}(Z)$ and the inequality $\delta' \leq nD\delta$, we deduce

$$\begin{aligned} \hat{\mu}^{\text{ess}}(Z') &\leq 3nD^{-1}(2400n^3 \log(2n\delta))^{-n^2+3}\delta^{-1} \\ &\leq 3n^2(2400n^3 \log(2n\delta))^{-n^2+3}\delta'^{-1} \end{aligned}$$

Using the inequalities $\delta' \leq nD\delta$, (5.16) and $\log x < x$ we get

$$2n\delta' \leq 2n^2D\delta \leq (250n^3 \cdot 2n\delta)^{n^2} \delta \leq (2n\delta)^{(250n^3)^{n-1}}. \quad (5.18)$$

Thus

$$(2400(n-1)^3 \log(2(n-1)\delta'))^{(n-1)^{n-1}-3} \leq (3n^2)^{-1}(2400n^3)^a \log(2n\delta)^{n^2-3}$$

where

$$a = 1 + n((n-1)^{n-1} - 3) \leq n^n - 3.$$

Therefore

$$\hat{\mu}^{\text{ess}}(Z') \leq (2400(n-1)^3 \log(2(n-1)\delta'))^{-(n-1)^{n-1}+3}\delta'^{-1}$$

By induction there exists a translate $B' \subseteq V'$ of a subtorus of codimension r' such that $Z' \subseteq B'$ and

$$\deg(B') \leq (250n^3 \log(2n\delta'))^{(2n)^{n-1}} \delta'^{2^{r'}-1}.$$

Let for brevity $K = 250n^3 \log(2n\delta)$. From the inequalities (5.18) and $\delta' \leq nD\delta$ we get

$$\deg(B') \leq K^{2^{n-1}n^n} (nD\delta)^{2^{r'}-1}.$$

Then $Z = \alpha\varphi(Z') \subseteq \varphi(B') \subseteq V$, $r = \text{codim } \varphi(B') = r' + 1$ and

$$\begin{aligned} \deg \varphi(B') &\leq D \deg(B') \\ &\leq K^{2^{n-1}n^n} n^{2^{r'}-1} D^{2^{r'}} \delta^{2^{r'}-1} \\ &\leq K^{2^{n-1}n^n + 2^{r'}n^n} \delta^{2^{r'}+1-1} \\ &\leq K^{(2n)^n} \delta^{2^r-1} \end{aligned}$$

where we have used the upper bound (5.16) for D .

□

Remark 5.2 In [Amo-Dav 2006], theorem 1.5 we assume that V is geometrically irreducible (which is not necessary) and that V is incompletely defined by forms of degree $\leq \delta$, *i. e.* it is a component of a complete intersection of hypersurfaces of degree $\leq \delta$. Unfortunately, there is a mistake in the proof: at page 561, point (a), we cannot ensure that V' is incompletely defined by forms of degree $\leq nD\delta$. The problem is the following: if V is incompletely defined by forms of degree $\leq \delta$, Z is an hypersurface of degree $\leq \delta$ which not contains V , then an irreducible component of $V \cap Z$ is not *a priori* incompletely defined by forms of degree $\leq \delta$.

References

- [Amo-Dav 1999] F. Amoroso et S. David. “Le problème de Lehmer en dimension supérieure”, *J. Reine Angew. Math.* **513**, 145-179 (1999).
- [Amo-Dav 2000] F. Amoroso et S. David. “Minoration de la hauteur normalisée des hypersurfaces”, *Acta Arith.* **92** (2000), **4**, 340-366.
- [Amo-Dav 2001] F. Amoroso et S. David. “Densité des points à coordonnées multiplicativement indépendantes”, *Ramanujan J.* **5**, 237-246 (2001).
- [Amo-Dav 2003] F. Amoroso et S. David. “Minoration de la hauteur normalisée dans un tore”, *Journal de l’Institut de Mathématiques de Jussieu*, **2** (2003), no. 3, 335-381.
- [Amo-Dav 2006] F. Amoroso et S. David. “Points de petite hauteur sur une sous-varité d’un tore”, *Compos. Math.* **142** (2006), 551-562.
- [Amo-Zan 2000] F. Amoroso and U. Zannier. “A relative Dobrowolski’s lower bound over abelian extensions”, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **29** (2000), no 3, 711-727.

- [Ber-Phi 1988] D. BERTRAND et P. PHILIPPON. “Sous-groupes algébriques de groupes algébriques commutatifs. III”. *J. Math.*, t. **32**, pages 263–280, 1988.
- [Bom-Zan 1995] E. Bombieri and U. Zannier. “Algebraic points on subvarieties of \mathbb{G}_m^n ”. *Internat. Math. Res. Notices*, **7** (1995), 333-347.
- [Cha] M. Chardin – “Une majoration de la fonction de Hilbert et ses conséquences pour l’interpolation algébrique”, *Bulletin de la Société Mathématique de France*, **117**, 305–318, (1988).
- [Cha-Phi] M. Chardin et P. Philippon – “Régularité et interpolation”, *J. Algebr. Geom.*, **8**, no. 3, 471–481, (1999) ; *erratum*, *ibidem*, **11**, 599–600, (2002).
- [Dav-Phi 1999] S. David et P. Philippon. “Minorations des hauteurs normalisées des sous-variétés des tores”. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, **xxviii** (1999), no. 3, 489-543; Errata, *ibidem* **xxix** (2000), no 3, 729-731.
- [Leh 1933] D. H. Lehmer. “Factorization of certain cyclotomic functions”, *Ann. of Math.* **34** (1933), 461-479.
- [Lau 1984] M. Laurent. “Equations diophantiennes exponentielles”. *Invent. Math.* **78** (1984), 299-327.
- [Law 1983] W. Lawton. “A Problem of Boyd concerning Geometric Means of Polynomial”. *J. of Number Theory*, **16** (1983), 356-362.
- [Phi 1986] P. Philippon. “Lemmes de zéros dans les groupes algébriques commutatifs”. *Bull. Soc. Math. France*, **114** (1986), 353-383.
- [Pon 2005] C. Pontreau “Geometric lower bounds for the normalized height of hypersurfaces”. *International Journal of Number Theory*, to appear.
- [Ros-Sch 1962] J. B. Rosser and L. Schoenfeld. “Approximate formulas for some functions of prime numbers”. *Ill. J. Math.*, **6** (1962), 64–94.
- [Sch 1991] W. M. Schmidt. *Diophantine approximation and Diophantine equations*. Springer Lecture Notes in Mathematics, t. **1467**, Springer-Verlag, Berlin–Heidelberg–New-York, viii & 217 pages, 1991.
- [Sch 1996] W. M. Schmidt. “Heights of points on subvarieties of \mathbb{G}_m^n ”. In “Number Theory 93-94”, S. David editor, London Math. Soc. Ser., volume 235, Cambridge University Press, 1996.
- [Vaaler 1979] J. D. Vaaler. “A geometric inequality with applications to linear forms”. *Pacific J. Math.*, t. **83** n°2, pages 543–553, 1979.
- [Zha 1995] S. Zhang. “Positive line bundles on arithmetic varieties”. *J. Amer. Math. Soc.*, **8** (1995), no. 1, 187-221.