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# Bogomolov on tori revisited.

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## 1 Introduction.

Let $V \subseteq \mathbb{G}_m^n \subseteq \mathbb{P}^n$ be a geometrically irreducible variety which is not torsion (*i. e.* not a translate of a subtorus by a torsion point). For $\theta > 0$ let $V(\theta)$ be the set of $\boldsymbol{\alpha} \in V(\overline{\mathbb{Q}})$ of Weil's height $h(\boldsymbol{\alpha}) \leq \theta$. By the toric case of Bogomolov conjecture (which is a theorem of Zhang),

$$\hat{\mu}^{\mathsf{ess}}(V) = \inf\{\theta > 0, \quad \overline{V(\theta)} = V\} > 0 \, .$$

If we assume moreover that $V$ is not a translate of a subtorus by a point (eventually of infinite order) we can give a lower bound for $\hat{\mu}^{\mathsf{ess}}(V)$ depending only on $\deg(V)$ (see [Bom-Zan 1995], [Dav-Phi 1999], [Sch 1996]).

Let us define the obstruction index $\omega(V)$ as the minimum degree of an hypersurface containing $V$. We remark that $\omega(V) \leq n \deg(V)^{1/\operatorname{codim}(V)}$ ([Cha]). Assume that $V$ is not transverse (*i. e.* is not contained in a translate of a subtorus). In [Amo-Dav 2003] we conjecture

$$\hat{\mu}^{\mathsf{ess}}(V) \geq c(n)\omega(V)^{-1}$$

for some $c(n) > 0$ and we prove

$$\hat{\mu}^{\mathsf{ess}}(V) \geq c(n)\omega(V)^{-1}(\log(3\omega(V))^{-\lambda(\operatorname{codim}(V))}$$

where $\lambda(k) = \big(9(3k)^{k+1}\big)^k$.

The aim of this paper is to give a more simple proof of a slightly improved (and explicit) version of this result (theorem 4.1), based on a very simple determinant argument (see section 2). More precisely the proof presented here

- avoid the use of the absolute Siegel's lemma of Zhang (see [Dav-Phi 1999], lemme 4.7)
- don't need any variant of zero's lemma and the subsequent combinatorial arguments (section 4 of [Amo-Dav 2003])

- don't use the weighted obstruction index $\omega(T;V)$ defined in [Amo-Dav 2003], definition 2.3.

Let

$$V^0 = V \setminus \bigcup_{B \subseteq V} B.$$

where the union is on the set of translates $B$ of subgroups of positive dimension contained in $V$. In [Amo-Dav 2006], theorem 1.5 we deduce from a lower bound for the essential minimum of $V$, a lower bound for height for all but finitely points of $V^0$. Here we prove (theorem 5.1) an again slightly improved (and explicit) version of that result. We also correct a mistake which appears in that paper: in *op. cit.*, theorem 1.5, $\delta(V)$ must be defined as the minimum degree $\delta$ such that $V$ is, as a set, intersection of hypersurface of degree $\leq \delta$ (see remark 5.2 for details).

The determinant argument allow us to prove also very precise results concerning the normalized height $\hat{h}(V)$ of an hypersurface $V$ (see section 3 for the definition). In this special case we conjecture :

**Conjecture 1.1** *Assume one of the following:*

*i) $V$ is geometrically irreducible and it is not a translate of a subtorus.*

*ii) $V$ is defined and irreducible over the rationals and is not torsion.*

*Then, there exists an absolute constant $c > 0$ such that $\hat{h}(V) \geq c$.*

We remark that Lehmer's conjecture implies conjecture ii), *via* an argument of Lawton. We shall prove

**Theorem 1.2** *Let $V \subseteq \mathbb{G}_m^n$ be an hypersurface of multi-degrees $(D_1, \ldots, D_n)$ with discrete stabilizer. Then, if $n \geq 9$ and*

$$\max D_j \leq 3^{2^n}$$

*we have*

$$\hat{h}(V) \geq \frac{1}{23}\,.$$

This result shows that an eventual example contradicting conjecture i) in $n$ variable must be realized by polynomials of very big degree (or comes from an hypersurface of less variables). This could suggests an even more optimistic conjecture:

*Let $V$ be a geometric irreducible hypersurface of $\mathbb{G}_m^n$ with discrete stabilizer. Then $\hat{h}(V) \geq f(n)$, where $f(n) \to +\infty$ for $n \to \infty$.*

In section 3 we also provide a counterexample to this last statement.

## 2 A determinant argument.

The following proposition is the key argument for the proof of the main theorems.

Let $S \subseteq \mathbb{P}_n$ and let $I \subset \mathbb{C}[\mathbf{x}]$ be the ideal defining its Zariski closure. For $\nu \in \mathbb{N}$ we denote by $H(S;\nu)$ the Hilbert function $\dim[\mathbb{C}[\mathbf{x}]/I]_\nu$. Let $T$ be a positive integer and let $I^{(T)}$ be the $T$-symbolic power of $I$, *i. e.* the ideal of polynomials vanishing on $S$ with multiplicity $\geq T$. We put $H(S,T;\nu) = \dim[\mathbb{C}[\mathbf{x}]/I^{(T)}]_\nu$.

Similarly, if $S \subseteq (\mathbb{P}_1)^n$ and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$ we denote its multi-homogeneous Hilbert function by

$$H(S;\boldsymbol{\nu}) = \dim([\overline{\mathbb{Q}}[x_1, \ldots, x_n]/I]_{\nu_1,\ldots,\nu_n})$$

where $I \subset \mathbb{C}[\mathbf{x}]$ is the ideal defining $\overline{S}$. More generally, if $T$ is a positive integer we put $H(S,T;\boldsymbol{\nu}) = \dim([\overline{\mathbb{Q}}[x_1, \ldots, x_n]/I^{(T)}]_{\nu_1,\ldots,\nu_n})$.

**Proposition 2.1** *Let $\nu$, $T$ be positive integers and let $p$ be a prime number. Let also $h$ be a positive real number and $S$ be a subset (eventually infinite) of $\mathbb{G}_m^n$ of points of height $\leq h$. Then*

$$h \geq \left(1 - \frac{H(S,T;\nu)}{H(\ker[p] \cdot S;\nu)}\right) \frac{T \log p}{p\nu} - \frac{n}{2\nu} \log(\nu+1) \ . \tag{2.1}$$

*In particular, if*

$$H(S,T;\nu) \leq \frac{1}{2} H(\ker[p] \cdot S;\nu) \tag{2.2}$$

*and*

$$T \log p \geq 2np \log(\nu+1) \ , \tag{2.3}$$

*then*

$$h \geq \frac{T \log p}{4p\nu} \geq \frac{n \log(\nu+1)}{2\nu} \ .$$

**Proof.** Let for brevity $S' = \ker[p]S$. We consider the (eventually infinite) matrix

$$(\boldsymbol{\beta}^{\boldsymbol{\lambda}})_{\substack{\boldsymbol{\beta} \in S' \\ |\boldsymbol{\lambda}| \leq \nu}}$$

of rang $L = H(\ker[p] \cdot S;\nu)$. We select $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_L \in S'$ and $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_L$ with $|\boldsymbol{\lambda}_j| \leq \nu$ such that the determinant

$$\Delta = \left| \det(\boldsymbol{\beta}_i^{\boldsymbol{\lambda}_j})_{i,j=1,\ldots,L} \right|$$

is non-zero. Let $L_0 = H(\ker[p] \cdot S;\nu) - H(S,T;\nu)$. Then, by definition, there exist linearly independent polynomials $G_k = \sum_{j=1}^{L} g_{kj} \mathbf{x}^{\boldsymbol{\lambda}_j}$ $(k = 1, \ldots, L_0)$ vanishing on $S$ with multiplicity $\geq T$. Let $K$ be a sufficiently large field and let $v$ be a non archimedean place of $K$ dividing $p$. After renumbering the multi-indexes $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_L$ and after making some linear combinations, we can assume

$$G_k = \sum_{j=1}^{L-k+1} g_{kj} \mathbf{x}^{\boldsymbol{\lambda}_j}$$

and moreover

$$|g_{k,j}|_v \begin{cases} \leq 1, & \text{if } j = 1, \dots, L-k; \\ = 1, & \text{if } j = L-k+1; \end{cases}$$

for $k = 1, \dots, L_0$. By elementary operations on columns we replace the last $L_0$ columns of $\Delta$ by the columns

$$^{\tau}\big(G_k(\boldsymbol{\beta}_1), \dots, G_k(\boldsymbol{\beta}_L)\big) \ , \qquad k = 1, \dots, L_0 \ .$$

Let $\Delta'$ the new determinant; then $|\Delta'|_v = |\Delta|_v$. Since $G_k$ vanish on $S$ with multiplicity $\geq T$ and since its coefficients are $v$-integers, we also have

$$|G_k(\boldsymbol{\beta}_i)|_v \leq p^{-T/(p-1)} \max\{1, |\beta_{i,1}|_v, \dots, |\beta_{i,n}|_v\}^{\nu} \qquad (i = 1, \dots, L;\ k = 1, \dots, L_0) \ .$$

By developping $\Delta'$ with respect to the last $L_0$ columns we obtain

$$|\Delta'|_v = |\Delta|_v \leq p^{-L_0 T/(p-1)} \prod_{i=1}^{L} \max\{1, |\beta_{i,1}|_v, \dots, |\beta_{i,n}|_v\}^{\nu L} \ .$$

By the product's formula (using a trivial lower bound for $v \nmid p$)

$$1 \leq p^{-L_0 T/(p-1)} L^{L/2} e^{\nu h L}$$

and, using $L \leq \binom{\nu+1}{n} \leq (\nu+1)^n$,

$$\log h \geq \frac{L_0}{L} \times \frac{T \log p}{p\nu} - \frac{n}{2\nu} \log(\nu+1)$$

and the statement of proposition 2.1 follows.

□

The following is a multihomogeneous version of proposition 2.1.

**Proposition 2.2** *Let $\nu_1, \dots, \nu_n$, $T$ be positive integers and let $p$ be a prime number. Let also $h_1, \dots, h_n$ be a positive real number and $S$ be a subset (eventually infinite) of $\mathbb{G}_m^n$ of points $\boldsymbol{\alpha}$ satisfying $h(\alpha_j) \leq h_j$ for $j = 1, \dots, n$. Then*

$$\nu_1 h_1 + \dots + \nu_n h_n \geq \left(1 - \frac{H(S, T; \boldsymbol{\nu})}{H(\ker[p] \cdot S; \boldsymbol{\nu})}\right) \frac{T \log p}{p} - \frac{n}{2} \log(\nu_{\max} + 1) \qquad (2.4)$$

*where $\nu_{\max} = \max\{\nu_1, \dots, \nu_n\}$.*

**Proof.** Let for brevity $S' = \ker[p]S$. We consider the matrix

$$(\boldsymbol{\beta}^{\boldsymbol{\lambda}})_{\substack{\boldsymbol{\beta} \in S' \\ |\lambda_1| \leq \nu_1, \dots, |\lambda_n| \leq \nu_1}}$$

of rang $L = H(\ker[p] \cdot S; \boldsymbol{\nu})$. We select $\boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_L \in S'$ and $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_L$ with $|\lambda_{j,l}| \leq \nu_l$ such that the determinant

$$\Delta = \left| \det(\boldsymbol{\beta}_i^{\boldsymbol{\lambda}_j})_{i,j=1,\ldots,L} \right|$$

is non-zero. Let $L_0 = H(\ker[p] \cdot S; \boldsymbol{\nu}) - H(S, T; \boldsymbol{\nu})$. Then, by definition, there exists linearly independent polynomials $G_k = \sum_{j=1}^{L} g_{kj} \mathbf{x}^{\boldsymbol{\lambda}_j}$ $(k = 1, \ldots, L_0)$ vanishing on $S$ with multiplicity $\geq T$. Let $K$ be a sufficiently large field and let $v$ be a non archimedean place of $K$ dividing $p$. After renumbering the multi-index $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_L$ and after making some linear combinations, we can assume

$$G_k = \sum_{j=1}^{L-k+1} g_{kj} \mathbf{x}^{\boldsymbol{\lambda}_j}$$

and moreover

$$|g_{k,j}|_v \begin{cases} \leq 1, & \text{if } j = 1, \ldots, L-k; \\ = 1, & \text{if } j = L-k+1; \end{cases}$$

for $k = 1, \ldots, L_0$. By elementary operations on columns we replace the last $L_0$ columns of $\Delta$ by the columns

$${}^{\tau}\big(G_k(\boldsymbol{\beta}_1), \ldots, G_k(\boldsymbol{\beta}_L)\big) , \qquad k = 1, \ldots, L_0 .$$

Let $\Delta'$ the new determinant; then $|\Delta'|_v = |\Delta|_v$. Since $G_k$ vanish on $S$ with multiplicity $\geq T$ and since its coefficients are $v$-integers, we also have

$$|G_k(\boldsymbol{\beta}_i)|_v \leq p^{-T/(p-1)} \prod_{j=1}^{n} \max\{1, |\beta_{i,j}|_v\}^{\nu_j} \qquad (i = 1, \ldots, L;\ k = 1, \ldots, L_0) .$$

By developping $\Delta'$ with respect to the last $L_0$ columns we obtain

$$|\Delta'|_v = |\Delta|_v \leq p^{-L_0 T/(p-1)} \prod_{i=1}^{L} \prod_{j=1}^{n} \max\{1, |\beta_{i,j}|_v\}^{\nu_j L} .$$

By the product's formula (using a trivial lower bound for $v \nmid p$)

$$1 \leq p^{-L_0 T/(p-1)} L^{L/2} e^{(\nu_1 h_1 + \cdots + \nu_n h_n) L}$$

and, using $L \leq (\nu_{\max} + 1)^n$,

$$\nu_1 h_1 + \cdots + \nu_n h_n \geq \frac{L_0}{L} \times \frac{T \log p}{p} - \frac{n}{2} \log(\nu_{\max} + 1)$$

and the statement of proposition 2.2 follows.

□

## 3 Hypersurfaces.

In this section we are interested in the case of a hypersurface $V$. For these varieties we have a "natural" definition of height (which coincide with the previous one) since we can extend the Mahler measure to polynomials in several variables. Let $f \in \mathbb{C}[x_1, \dots, x_n]$; we define its Mahler measure as:

$$M(P) = \exp \int_0^1 \cdots \int_0^1 \log |f\left(e^{2\pi i t_1}, \dots, e^{2\pi i t_n}\right)| dt_1 \dots dt_n.$$

Let now $K$ be a number field and let $V$ be an hypersurface in $\mathbb{G}_m^n$ defined over $K$:

$$V = \{\boldsymbol{\alpha} \in \mathbb{G}_m^n \text{ such that } f(\boldsymbol{\alpha}) = 0\}$$

for some polynomial $f \in K[\mathbf{x}]$ (irreducible over $\overline{\mathbb{Q}}[\mathbf{x}]$). We define:

$$\hat{h}(V) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in \mathcal{M}_K} [K_v : \mathbb{Q}_v] \log M_v(f),$$

where $M_v(f)$ is the maximum of the $v$-adic absolute values of the coefficients of $f$ if $v$ is non archimedean, and $M_v(f)$ is the Mahler measure of $\sigma f$ if $v$ is an archimedean place associated with the embedding $\sigma \colon K \hookrightarrow \overline{\mathbb{Q}}$.

We prove:

**Proposition 3.1** *Let $V \subseteq \mathbb{G}_m^n$ be an hypersurface of multi-degrees $D_1, \dots, D_n$ and assume that $V$ is not a translated of a torus. Let $D_{\max} = \max\{D_1, \dots, D_n\}$. Then, for any prime number $p \geq 5$,*

$$\hat{h}(V) \geq \frac{\log p}{7p} - \frac{nk' \log p}{p^{k'}} - \frac{n \log(n^2 D_{max})}{2p^{k'}} \ . \tag{3.5}$$

*where $k'$ is the codimension of the stabilizer of $V$.*

**Proof.** Since $V$ is not a translated of a torus, $k' \geq 2$. This implies $n \geq 2$ and $p^{k'} \geq 9$.

We assume first that $p \nmid [\mathrm{Stab}(V) : \mathrm{Stab}(V)^0]$, so that $V' = \ker[p]V$ is a union of $p^{k'}$ translate of $V$, and we prove

$$\hat{h}(V) \geq \frac{\log p}{7p} - \frac{nk' \log p}{p^{k'}} - \frac{n \log(nD_{max})}{2p^{k'}} \ , \tag{3.6}$$

Let $\varepsilon > 0$ and assume $D_{\max} = D_n$. The proposition 2.7 of [Amo-Dav 2000] shows that the set

$$S = \{(\zeta_1, \dots, \zeta_{n-1}, \alpha) \in V(\overline{\mathbb{Q}}),\ \zeta_1, \dots, \zeta_{n-1} \text{ roots of unity},\ h(\alpha) \leq \hat{h}(V)/D_n + \varepsilon\}$$

is Zariski dense in $V$. We apply proposition 2.2 with $h_1 = \cdots = h_{n-1} = 0$ and $h_n = \hat{h}(V)/D_n + \varepsilon$. We choose, for $j = 1, \dots, n-1$,

$$\nu_j = np^{k'} D_j - 1$$

and $\nu_n = p^{k'} D_n - 1$. We remark that $\nu_{\max} = \max\{\nu_1, \ldots, \nu_n\} \leq n p^{k'} D_{\max} - 1$. We also choose $T = [p^{k'}/2]$. Then

$$H(V, T; \boldsymbol{\nu}) = (\nu_1 + 1) \cdots (\nu_n + 1) - (\nu_1 - T D_1 + 1) \cdots (\nu_n - D_n + 1)$$

$$= n^{n-1} p^{k'n} - \frac{1}{2} \left( n - \frac{1}{2} \right)^{n-1} p^{k'n}$$

and

$$H(V'; \boldsymbol{\nu}) = (\nu_1 + 1) \cdots (\nu_n + 1) - (\nu_1 - p^{k'} D_1 + 1) \cdots (\nu_n - p^{k'} D_n + 1)$$

$$= n^{n-1} p^{k'n}$$

so that

$$1 - \frac{H(V, T; \boldsymbol{\nu})}{H(V'; \boldsymbol{\nu})} \geq \frac{1}{2} \left( 1 - \frac{1}{2n} \right)^{n-1} \geq \frac{1}{2\sqrt{e}} \, .$$

Inequality (2.4) now gives

$$\nu_n h_n = (p^{k'} D_n - 1) \left( \frac{\hat{h}(V)}{D_n} + \varepsilon \right)$$

$$\geq \frac{T \log p}{2\sqrt{e} p} - \frac{n}{2} \log(\nu_{\max} + 1)$$

$$\geq \frac{p^{k'} \log p}{4\sqrt{e} p} - \frac{\log p}{2\sqrt{e} p} - \frac{n}{2} \log(n p^{k'} D_{\max})$$

$$\geq \frac{p^{k'} \log p}{7p} - n k' \log p - \frac{n}{2} \log(n D_{\max}) \, .$$

By letting $\varepsilon \mapsto 0$ we obtain the lower bound (3.6).

If $\mathrm{Stab}(V)$ is not connected, by inspection of the proof of proposition 2.4 of [Amo-Dav 2000] we obtain an hypersurface $W$ with connected stabilizer of the same codimension $k'$, multi-degree $(D'_1, \ldots, D'_n)$ with $D'_j \leq n D_j$ and normalized height $\hat{h}(W) \leq \hat{h}(V)$. Therefore, by (3.6),

$$\hat{h}(V) \geq \hat{h}(W) \geq \frac{\log p}{7p} - \frac{n k' \log p}{p^{k'}} - \frac{n \log(n^2 D_{max})}{2 p^{k'}} \, .$$

□

Let now assume $k' = n$, *i. e.* $\mathrm{Stab}(V)$ discrete. Choosing $p = 5$ we obtain:

**Theorem 3.2** *Let $V \subseteq \mathbb{G}_m^n$ be an hypersurface of multi-degrees $(D_1, \ldots, D_n)$ with discrete stabilizer. Then, if $n \geq 9$ and*

$$\max D_j \leq 3^{2^n}$$

*we have*

$$\hat{h}(V) \geq \frac{1}{23} \ .$$

**Proof.** We apply the proposition above with $p = 5$, assuming $D_{\max} \leq 3^{2^n}$ and $k' = n$. We obtain

$$\hat{h}(V) \geq \frac{\log 5}{35} - \frac{n^2 \log 5}{5^n} - \frac{n \log(n^2 D_{max})}{2 \times 5^n}$$

$$\geq \frac{\log 5}{35} - \frac{n^2 \log 5}{5^n} - \frac{2n \log n}{2 \times 5^n} - \frac{n 2^n \log 3}{2 \times 5^n} =: f(n) \ .$$

An easy computation shows that $f$ is an increasing function and $f(9) > 1/23$.

□

As stated in the introduction, we could conjecture that for any geometric irreducible hypersurface $V \subseteq \mathbb{G}_m^n$ with discrete stabilizer we had $\hat{h}(V) \geq f(n)$ for some function $f(n) \to +\infty$ for $n \to \infty$. This is false, as the the following example prove. Let $F(x_1) = x_1^3 - x_1 - 1$ and define inductively

$$F_n(x_1, \ldots, x_n) = F^*(x_1, \ldots, x_{n-1}) x_n - F(x_1, \ldots, x_{n-1})$$

where $F^*$ indicated the reciprocal polynomial. Since the rational function

$$R(x_1, \ldots, x_{n-1}) = \frac{F(x_1, \ldots, x_{n-1})}{F^*(x_1, \ldots, x_{n-1})}$$

satisfy $|R(z_1, \ldots, z_{n-1})| = 1$ for $|z_1| = \cdots = |z_{n-1}| = 1$, we have for any integer $n$ $M(F_n) = \theta_0$ where $\theta_0$ is the root $> 1$ of $F_1$. Moreover, it is easy to see that $F_n$ is irreducible (over $\overline{\mathbb{Q}}$ if $n \geq 2$) and that $V_n = \{F_n = 0\}$ has trivial stabilizer.

We conclude this section with a more a general (and technical) lower bound for the normalized height of an hypersurface:

**Theorem 3.3** *Let $V \subseteq \mathbb{G}_m^n$ be an hypersurface of multi-degrees $(D_1, \ldots, D_n)$ and assume that $V$ is not a translated of a torus. Then,*

$$\hat{h}(V) \geq \frac{1}{56} \times \max\left(\frac{\log(n \log(n^2 D_{\max}))}{k'}, 1\right) \times \left(\frac{\log(n \log(n^2 D_{\max}))}{28 n k' \log(n^2 D_{\max})}\right)^{1/(k'-1)}$$

*where $k'$ is the codimension of the stabilizer of $V$ and $D_{\max} = \max D_j$. In particular,*

$$\hat{h}(V) \geq \frac{\log(n \log(n^2 D_{\max}))^2}{6272 n \log(n^2 D_{\max})} \ .$$

**Proof.** Let

$$N = \left( \frac{28nk' \log(n^2 D_{\max})}{\log(n \log(n^2 D_{\max}))} \right)^{1/(k'-1)} \tag{3.7}$$

and choose a prime number $p$ such that $N \leq p \leq 2N$. By

$$\log x \leq x^{1/2} \qquad (x > 0) \tag{3.8}$$

we have $\log(n \log(n^2 D_{\max})) \leq \log(n(n^2 D_{\max})^{1/2}) \leq \log(n^2 D_{\max})$; hence

$$p^{k'-1} \geq 28nk' \ .$$

We also remark that, again by (3.8),

$$\log p \geq \frac{\log(28n^{1/2}k' \log(n^2 D_{\max})^{1/2})}{k'-1} \geq \frac{\log(n \log(n^2 D_{\max}))}{2k'} \tag{3.9}$$

Therefore,

$$p^{k'-1} \log p \geq 14n \log(n^2 D_{\max}) \ .$$

Thus, by proposition 3.1 we have

$$\begin{aligned} \hat{h}(V) &\geq \frac{\log p}{7p} - \frac{nk' \log p}{p^{k'}} - \frac{n \log(n^2 D_{max})}{2p^{k'}} \\ &\geq \frac{\log p}{7p} - \frac{\log p}{28p} - \frac{\log p}{28p} \\ &= \frac{\log p}{14p} \ . \end{aligned}$$

By (3.9) we obtain:

$$\begin{aligned} \hat{h}(V) &\geq \frac{1}{14} \times \max\left( \frac{\log(n \log(n^2 D_{\max}))}{2k'}, \log 2 \right) \times \frac{1}{2N} \\ &\geq \frac{1}{56} \times \max\left( \frac{\log(n \log(n^2 D_{\max}))}{k'}, 1 \right) \times \left( \frac{\log(n \log(n^2 D_{\max}))}{28nk' \log(n^2 D_{\max})} \right)^{1/(k'-1)} \ . \end{aligned}$$

This prove the first inequality of theorem 3.3. For the second one, we remark that $k' \geq 2$ and $k'(nk')^{1/(k-1)} \leq 4n$. So

$$\begin{aligned} \hat{h}(V) &\geq \frac{1}{56} \times \max\left( \frac{\log(n \log(n^2 D_{\max}))}{k'}, 1 \right) \times \left( \frac{\log(n \log(n^2 D_{\max}))}{28nk' \log(n^2 D_{\max})} \right)^{1/(k'-1)} \\ &\geq \frac{\log(n \log(n^2 D_{\max}))^2}{56 \times 28 \times 4n \log(n^2 D_{\max})} \\ &= \frac{\log(n \log(n^2 D_{\max}))^2}{6272n \log(n^2 D_{\max})} \ . \end{aligned}$$

□

# 4 Essential minimum.

In this section we prove the following theorem, which slightly umprove theorem 1.4 of [Amo-Dav 2003]:

**Theorem 4.1** *Let $V$ be a subvariety of $\mathbb{G}_m^n$ of codimension $k < n$. Then either there exists a translate $B$ of a subgroup such that $V \subseteq B \subsetneq \mathbb{G}_m^n$ and*

$$\deg(B)^{1/\operatorname{codim}(B)} \leq \big(250n^3 \log(2n\omega(V))\big)^{\lambda(k)+1}\omega(V)$$

*or*

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \big(2400n^4 \log(2n\omega(V))\big)^{-\lambda(k)}\omega(V)^{-1}$$

*where $\lambda(k) = \frac{k+1}{k}\big((k+1)^k - 1\big) - 1 \leq n^n - 3$.*

Proposition 2.1 gives the following result:

**Proposition 4.2** *Let $V$ be a subvariety of $\mathbb{G}_m^n$ et let $\omega = \omega(V)$. Let also $p$ be a prime, $3 \leq p \leq \omega$ and assume :*

$$\hat{\mu}^{\mathsf{ess}}(V) < \frac{\log p}{10np\omega} \ .$$

*Then,*

$$\omega([p]V) \leq \frac{18n^2\omega \log(5n\omega)}{\log p} \ .$$

**Proof.** Let $h$ such that $\hat{\mu}^{\mathsf{ess}}(V) < h < \frac{\log p}{10np\omega}$ and let

$$S = \{\boldsymbol{\alpha} \in V, \quad h(\boldsymbol{\alpha}) < h\} \ .$$

Thus $H(S, T; \nu) = H(V, T; \nu)$ and $H(\ker[p] \cdot S; \nu) = H(\ker[p] \cdot V; \nu)$. Let us define

$$T = \left[\frac{7np \log(5n\omega)}{\log p}\right]$$

and $\nu = (2n+1)\omega T$. We first show that there exists a a non zero polynomial $F \in \overline{\mathbb{Q}}[x_1, \ldots, x_n]$ of total degree $\leq \nu$, vanishing on $\ker[p]V$. Since $3 \leq p \leq \omega$, we have

$$\nu + 1 \leq 3n\omega \cdot 7np \cdot 5n\omega + 1 \leq (5n\omega)^3$$

and $T \log p \geq 6np \log(5n\omega)$. Thus inequality (2.3) of proposition 2.1, *i. e.* $T \log p \geq 2np \log(\nu + 1)$, is satisfied. We also have

$$\frac{T \log p}{4p\nu} = \frac{\log p}{4p(2n+1)\omega} > h \ .$$

By proposition 2.1, we must have

$$H(\ker[p] \cdot V; \nu) < 2H(V, T; \nu) \leq 2\left(\binom{\nu+n}{n} - \binom{\nu - \omega T + n}{n}\right) \ .$$

We remark that

$$\binom{\nu+n}{n}\binom{\nu-\omega T+n}{n}^{-1} = \prod_{j=1}^{n} \frac{\nu+j}{\nu-\omega T+j} \leq \left(1+\frac{\omega T}{\nu-\omega T}\right)^n$$

$$= \left(1+\frac{1}{2n}\right)^n \leq \sqrt{e} < 2 \ .$$

Thus

$$H(\ker[p]\cdot V;\nu) < \binom{\nu+n}{n} \ ,$$

*i. e.* there exists a non zero polynomial $F \in \overline{\mathbb{Q}}[x_1,\ldots,x_n]$ vanishing on $\ker[p]V$ of total degree $\leq \nu$. By the zero's lemma of P. Philippon (see [Phi 1986]), there exists a variety $Z$ containing $V$ such that

$$\deg(\ker[p]Z) \leq \nu^{\operatorname{codim}(Z)} \ .$$

Indeed, let $W$ be the algebraic set defined by the equations $F(\boldsymbol{\zeta}\mathbf{x}) = 0$ for $\boldsymbol{\zeta} \in \ker[p]$. Since $F$ vanishes on $\ker[p]V$, there exists a geometrically irriducible component $Z$ of $W$ containing $V$. Since $W$ is stable by translation by $p$-torsion points, all $\boldsymbol{\zeta}V$ are components of $W$ for $\boldsymbol{\zeta} \in \ker[p]$. Proposition 3.3 of [Phi 1986] (with $p = 1$, $N_1 = n$ and $D_1 = \nu$) then gives the desired upper bound for $\deg(\ker[p]Z)$.
Since

$$\deg(\ker[p]Z) = \deg([p]^{-1}[p]Z) = p^{\operatorname{codim}(Z)}\deg([p]Z)$$

we obtain

$$\omega([p]V) \leq \deg([p]Z)^{1/\operatorname{codim}(Z)} \leq p^{-1}\nu \ .$$

We finally remark that

$$\frac{1}{p}\nu \leq \frac{1}{p}\cdot\frac{5}{2}n\omega\cdot\frac{7np\log(5n\omega)}{\log p} < \frac{18n^2\omega\log(5n\omega)}{\log p} \ .$$

□

In order to prove theorem 4.1 we need, as in [Amo-Dav 2003], a descent argument. In what follows we fix a geometrically irreducible subvariety $V \subsetneq \mathbb{G}_m^n$ of dimension $k < n$ (thus $n \geq 2$) and we let $\omega = \omega(V)$. For $j = 1,\ldots,k$ let $\rho_j = (k+1)^{k-j+1} - 1$ and $P_j = (2\Delta)^{\rho_j}$ where $\Delta = Cn^3\log(2n\omega)$ and $C = 120$.

The following elementary relations will be used several time

**Lemma 4.3** *We have:*

*i)* $\log(2n\omega) > 1$ *and* $\Delta > 960$.

*ii) For $j \in \{0, \dots, k\}$ we have*

$$\sum_{l=j+1}^{k} \rho_l = (k+1)\frac{(k+1)^{k-j}-1}{k} - (k-j) \ .$$

**Definition 4.4** *Let $\mathcal{W}$ be the set of triples $(s, \mathbf{p}, \mathbf{W})$, where $s \in [0, k]$ is an integer, $\mathbf{p} = (p_1, \dots, p_s)$ is a $s$-tuple of prime numbers with $P_i/2 \leq p_i \leq P_i$, and where $\mathbf{W} = (W_0, \dots, W_s)$ is a $(s+1)$-tuple of strict geometrically irreducible subvarieties $\subsetneq \mathbb{G}_m^n$, satisfying:*

*i) $V \subseteq W_0$. Moreover, for $i = 1, \dots, s$,*

$$[p_i]W_{i-1} \subseteq W_i \quad \textit{and} \quad p_i \nmid [\mathrm{Stab}(W_{i-1}) : \mathrm{Stab}(W_{i-1})^0] \ ;$$

*ii) For $i = 0, \dots, s$*

$$\deg(W_i)^{1/\operatorname{codim}(W_i)} \leq \Delta^{k-i} p_{i+1} \cdots p_k \omega([p_1 \dots p_i]V) \ ;$$

*iii) For $i = 1, \dots, s$*

$$\omega([p_1 \dots p_i]V) \leq \Delta \omega([p_1 \dots p_{i-1}]V) \ .$$

**Remark 4.5** Let $(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W}$ and assume $0 \leq i \leq j \leq s$. Then

$$\omega([p_1 \dots p_j]V) \leq \Delta^{j-i} \omega([p_1 \dots p_i]V) \ .$$

We want to prove that there exists $(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W}$, such that $\dim(W_{i-1}) = \dim(W_i)$ for at least one index $i$. Let

$$\mathcal{W}_0 = \{(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W}, \ \text{such that} \ \dim(W_0) < \dim(W_1) < \cdots < \dim(W_s)\} \ .$$

**Proposition 4.6** *Assume*

$$\hat{\mu}^{\mathsf{ess}}(V) < \Big(10n\Delta^{k-1} P_1 \cdots P_k \omega\Big)^{-1} \ . \tag{4.10}$$

*Then $\mathcal{W}_0 \neq \mathcal{W}$.*

In order to prove proposition 4.6, we endow the set of finite sequences of integers with the following (total) order $\preccurlyeq$. Let $(v) = (v_i)_{0 \leq i \leq s}$ and $(v') = (v'_j)_{0 \leq j \leq s'}$ two such sequences. Then $(v) \preccurlyeq (v')$ if

$$(v_i)_{0 \leq i \leq \min\{s,s'\}} < (v'_i)_{0 \leq i \leq \min\{s,s'\}}$$

for the lexicographical order, or if $(v_i)_{0 \leq i \leq \min\{s,s'\}} = (v'_i)_{0 \leq i \leq \min\{s,s'\}}$ and $s \geq s'$.

We also need the following technical lemma:

**Lemma 4.7** *Let $s \in \mathbb{N}$, $p_1, \ldots, p_s, p_{s+1}$ positive integers, $W_0, \ldots, W_s \subsetneq \mathbb{G}_m^n$ geometrically irreducible subvarieties. Let us assume $V \subseteq W_0$ and $[p_i]W_{i-1} \subseteq W_i$ for $i = 1, \ldots, s$. Then, there exists an integer $s' \in [0, s+1]$ and a geometrically irreducible subvariety $Z_{s'}$ of degree*

$$\deg(Z_{s'}) \leq p_{s'+1} \ldots p_{s+1} \omega([p_1 \ldots p_{s+1}]V) \deg(W_{s'}) \ , \tag{4.11}$$

*such that $[p_{s'}]W_{s'-1} \subseteq Z_{s'}$, $\operatorname{codim}(Z_{s'}) = \operatorname{codim}(W_{s'}) + 1$ (with the following convention: $\operatorname{codim}(W_{s+1}) = 0$, $\deg(W_{s+1}) = 1$, $W_{-1} = V$ and $p_0 = 1$) and:*

$$(\dim(W_0), \ldots, \dim(W_{s'-1}), \dim(Z_{s'})) \prec (\dim(W_0), \ldots, \dim(W_s)) \ . \tag{4.12}$$

**Proof.** Let $Z_{s+1}$ be an hypersurface containing $[p_1 \ldots p_{s+1}]V$ of minimal degree $\omega([p_1 \ldots p_{s+1}]V)$. Thus if $s' = s+1$ (4.11) is satisfied. We construct by induction subvarieties $Z_0, \ldots, Z_s$ such that, for $i = 0, \ldots, s$,

i) $Z_i \subseteq W_i$ and $Z_i \neq W_i \Rightarrow \operatorname{codim}(Z_i) = \operatorname{codim}(W_i) + 1$.

ii) $[p_{i+1} \ldots p_{s+1}]Z_i \subseteq Z_{s+1}$.

iii) $[p_{i+1}]Z_i \subseteq Z_{i+1}$.

iv) $\deg(Z_i) \leq p_{i+1} \ldots p_{s+1} \omega([p_1 \ldots p_{s+1}]V) \deg(W_i)$.

We start by the construction of $Z_0$. If $[p_1 \ldots p_{s+1}]W_0 \subseteq Z_{s+1}$, we set $Z_0 = W_0$. Otherwise we choose for $Z_0$ a geometrically irreducible component of maximal dimension of $W_0 \cap [p_1 \ldots p_{s+1}]^{-1} Z_{s+1}$ containing $V$. By Bézout's inequality we have:

$$\deg(Z_0) \leq \deg(W_0) \deg([p_1 \ldots p_{s+1}]^{-1} Z_{s+1}) \leq p_1 \ldots p_{s+1} \omega([p_1 \ldots p_{s+1}]V) \deg(W_0) \ .$$

Let now $i \in [0, s-1]$ be an integer and assume that $Z_0, \ldots, Z_i$ satisfy conditions i)–iv). If

$$[p_{i+2} \ldots p_{s+1}]W_{i+1} \subseteq Z_{s+1} \ ,$$

we set $Z_{i+1} = W_{i+1}$. Otherwise we choose for $Z_{i+1}$ a geometrically irreducible component of maximal dimension of $[p_{i+2} \ldots p_{s+1}]^{-1} Z_{s+1} \cap W_{i+1}$ containing $[p_{i+1}]Z_i$. We can do this, since $[p_{i+1}]W_i \subseteq W_{i+1}$ (by assumption) $Z_i \subseteq W_i$ (by induction i)) and since

$$[p_{i+1} \ldots p_{s+1}]Z_i \subseteq Z_{s+1}$$

(by induction i)). The variety $Z_{i+1}$ verify conditions i)–iii). As before, by Bézout's inequality we have:

$$\deg(Z_{i+1}) \leq p_{i+2} \ldots p_{s+1} \omega([p_1 \ldots p_{s+1}]V) \deg(W_{i+1}) \ .$$

and the variety $Z_{i+1}$ also verify condition iv).

We now choose the integer $s'$. We define $s'$ as the least integer $i$ such that $Z_i \subsetneq W_i$, if such an integer exists. Otherwise we set $s' = s+1$. We remark that in both cases (4.12) holds.

□

**Proof of proposition 4.6.** The set $\mathcal{W}_0$ is a finite non-empty set (indeed, let $W_0$ be an hypersurface of $\mathbb{G}_m^n$ containing $V$ of degree $\omega$; then $(0, \emptyset, (W_0)) \in \mathcal{W}_0$). Thus, there exists a minimal element $(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W}_0$, *i. e.*

$$(\dim W_i)_{0 \le i \le s} \preccurlyeq (\dim W_i')_{0 \le i \le s'}.$$

for all $(s', \mathbf{p}', \mathbf{W}') \in \mathcal{W}_0$. We remark that $s \le k-1$, since

$$n - k = \dim(V) \le \dim(W_0) < \dim(W_1) < \cdots < \dim(W_s) \le n-1 \ .$$

We need the following computation:

**Lemma 4.8** *There exists a prime $p_{s+1}$ such that $P_{s+1}/2 \le p_{s+1} \le P_{s+1}$ and*

$$p_{s+1} \nmid [\mathrm{Stab}(W_s) : \mathrm{Stab}(W_s)^0] \ .$$

**Proof.** By Theorems 9 and 10 of [Ros-Sch 1962], $\sum_{p \le x} \log p \le 1.02x$ for $x \ge 1$ and $\sum_{p \le x} \log p \ge 0.84x$ for $x \ge 101$. Thus

$$\begin{aligned} \sum_{P_{s+1}/2 \le p \le P_{s+1}} \log p &\ge \big(0.84 - 1.02/2\big) P_{s+1} \\ &> P_{s+1}/4 \ . \end{aligned}$$

If for any prime $p$ with $P_{s+1}/2 \le p \le P_{s+1}$ we had $p \mid [\mathrm{Stab}(W_s) : \mathrm{Stab}(W_s)^0]$, then

$$2 \log \deg(W_s) \ge P_{s+1}/4 \ ,$$

since $\deg(\mathrm{Stab}(W_s)) \le \deg(W_s)^2$. By assertion ii) of definition 4.4 and by remark 4.5, we have :

$$\begin{aligned} \log \deg(W_s) &\le \mathrm{codim}(W_s)\big(k \log(\Delta + \sum_{j=s+1}^{k} \log P_j + \log(\omega)\big) \\ &\le k\Big(\big(k + \sum_{j=s+1}^{k} \log \rho_j\big) \log(2\Delta) + \log \omega\Big) \ . \end{aligned}$$

Using the inequality $\log x < x^{1/3}$ $(x > 100)$ with $x = 2\Delta$ (see lemma 4.3 i)) we obtain

$$\log \deg(W_s) \le k\big(k + 1 + \sum_{j=s+1}^{k} \log \rho_j\big)(2Cn^3)^{1/3} \log(2n\omega) \ .$$

Since $s \le k-1$, we have, using lemma 4.3 ii),

$$\begin{aligned} k\big(k + 1 + \sum_{j=s+1}^{k} \log \rho_j\big) &= k(k+1) + (k+1)^{k-s+1} - (k+1) - k(k-s) \\ &= (k+1)^{k-s+1} + ks - 1 \\ &\le 2(k+1)^{2(k-s)} \ . \end{aligned}$$

Thus, by setting $a = (k+1)^{(k-s)} \geq 2$,

$$2 \log \deg(W_s) \leq 4a^2 (2Cn^3)^{1/3} \log(2n\omega)$$

and

$$\frac{P_{s+1}/4}{2 \log \deg(W_s)} \geq \frac{\big(2Cn^3 \log(2n\omega)\big)^{a-1}}{16a^2 (2Cn^3)^{1/3} \log(2n\omega)}$$

$$\geq \frac{\big(16C\big)^{a-4/3}}{16a^2} =: f(a) \ .$$

An easy computation shows that $f(a) \geq f(2) > 1$. Contradiction.

□

By the previous lemma, there exists a prime number $p_{s+1} \in [P_{s+1}/2, P_{s+1}]$ such that $p_{s+1} \nmid [\mathrm{Stab}(W_s) : \mathrm{Stab}(W_s)^0]$. We want to apply proposition 4.2 to the variety $V' = [p_1 \dots p_s]V$ choosing $p = p_{s+1}$. We have

$$\hat{\mu}^{\mathsf{ess}}(V') \leq p_1 \dots p_s \hat{\mu}^{\mathsf{ess}}(V)$$

and, by iii) of definition 4.4

$$\omega(V') \leq \Delta^s \omega(V) \ .$$

Thus, by assumption (4.10),

$$\omega(V')\hat{\mu}^{\mathsf{ess}}(V') \leq \Delta^s p_1 \cdots p_s \omega \hat{\mu}^{\mathsf{ess}}(V)$$

$$< (10nP_{s+1})^{-1}$$

$$\leq \frac{\log p_{s+1}}{10np_{s+1}} \ .$$

Proposition 4.2 shows that:

$$\omega([p_{s+1}]V') \leq \frac{18n^2 \log(5n\omega(V'))}{\log p_{s+1}} \omega([p_1 \dots p_s]V)$$

$$\leq 18n^2 \log(5n\omega(V'))\omega(V') \ .$$

Since $s \leq k-1 \leq n$, we have, using remark 4.5,

$$5n\omega(V') \leq 5n\Delta^s \omega \leq \big((C\sqrt{5}/32)(2n\omega)^5\big)^n \ .$$

Thus

$$\Delta - 18n^2 \log(5n\omega(V')) \geq Cn^3 \log(2n\omega) - 18n^3 \log\big((C\sqrt{5}/32)(2n\omega)^5\big)$$

$$\geq n^3\big((C - 18 \times 5)\log(4) - 18 \log(C\sqrt{5}/32)\big) > 0$$

and

$$\omega([p_1 \dots p_{s+1}]V) = \omega([p_{s+1}]V') \leq \Delta\omega(V') = \Delta\omega([p_1 \dots p_s]V) \ .$$

We apply now lemme 4.7. We obtain an integer $s'$ such that $0 \leq s' \leq s+1 \leq k$ and a subvariety $Z_{s'}$ satisfying the properties described in this lemma. We want to show that

$$(s', (p_1, \dots, p_{s'}), (W_0, \dots, W_{s'-1}, Z_{s'})) \in \mathcal{W} \ .$$

All conditions i)–iii) of definition 4.4 are trivially verified, except eventually for the upper bound of $\deg(Z_{s'})$. Using inequality (4.11) of lemma 4.7, the upper bound for the degree of $W_{s'}$ (point ii) of definition 4.4), remark 4.5 and the relation $\operatorname{codim}(Z_{s'}) = \operatorname{codim}(W_{s'+1}) + 1$, we get:

$$\begin{aligned}
\deg(Z_{s'}) &\leq p_{s'+1} \dots p_{s+1} \omega([p_1 \dots p_{s+1}]V) \deg(W_{s'}) \\
&\leq p_{s'+1} \dots p_{s+1} \Delta^{s-s'+1} \omega([p_1 \dots p_{s'}]V) \deg(W_{s'}) \\
&\leq \Delta^{k-s'} p_{s'+1} \cdots p_k \omega([p_1 \dots p_{s'}]V) \deg(W_{s'}) \\
&\leq \Big(\Delta^{k-s'} p_{s'+1} \cdots p_k \omega([p_1 \dots p_{s'}]V)\Big)^{1+\operatorname{codim}(W_{s'+1})} \\
&\leq \Big(\Delta^{k-s'} p_{s'+1} \cdots p_k \omega([p_1 \dots p_{s'}]V)\Big)^{\operatorname{codim}(Z_{s'})} \ .
\end{aligned}$$

Thus $(s', (p_1, \dots, p_{s'}), (W_0, \dots, W_{s'-1}, Z_{s'})) \in \mathcal{W}$. Since

$$(\dim(W_0), \dots, \dim(W_{s'-1}), \dim(Z_{s'})) \prec (\dim(W_0), \dots, \dim(W_s))$$

by relation (4.12) of lemma 4.7 and since $(s, \mathbf{p}, \mathbf{W})$ is a minimal element of $\mathcal{W}_0$, we deduce that:

$$(s', (p_1, \dots, p_{s'}), (W_0, \dots, W_{s'-1}, Z_{s'})) \notin \mathcal{W}_0 \ .$$

□

## 4.1 Proof of theorem 4.1

Let $V$ be a geometrically irreducible subvariety of $\mathbb{G}_m^n$ of codimension $k < n$ which satisfy the assumption of proposition 4.6. By this proposition, there exists $(s, \mathbf{p}, \mathbf{W}) \in \mathcal{W} \setminus \mathcal{W}_0$. Thus there exists an index $i$ such that

$$\operatorname{codim}(W_{i-1}) = \operatorname{codim}(W_i) = r, \qquad [p_i]W_{i-1} \subseteq W_i, \qquad [p_1 \dots p_{i-1}]V \subseteq W_i \ ;$$

and $p_i \nmid [\operatorname{Stab}(W_{i-1}) : \operatorname{Stab}(W_{i-1})^0]$.

Assume first that $W_i$ is a translate of a subtorus. Then the same is true for the connected component $B$ of $[p_1 \dots p_i]^{-1} W_i$ containing $V$ and we have, using ii) of definition 4.4 and remark 4.5,

$$\begin{aligned}(\deg B)^{1/\operatorname{codim}(B)} &\leq (p_1 \cdots p_i)^{1/r} \Delta^k p_{i+1} \cdots p_k \\ &\leq \Delta^k P_1 \cdots P_k \\ &\leq (2\Delta)^{\lambda(k)+1}\end{aligned}$$

where

$$\lambda(k) + 1 = k + \sum_{j=1}^{k} \rho_j = \frac{k+1}{k}\big((k+1)^k - 1\big) \ .$$

Assume now that $W_i$ is not a translate of a subtorus. Thus

$$p_i \deg(W_{i-1}) \leq \deg(W_i) \ \ .$$

Since $W_{i-1} \supseteq [p_1 \dots p_{i-1}]V$, we have, using ii) and iii) of definition 4.4,

$$\begin{aligned}\omega([p_1 \dots p_{i-1}]V) &\leq \big(\deg(W_{i-1})\big)^{1/r} \\ &\leq p_i^{-1/r} \big(\deg(W_i)\big)^{1/r} \\ &\leq p_i^{-1/r} \Delta^{k-i} p_{i+1} \cdots p_k \omega([p_1 \dots p_i]V) \\ &\leq p_i^{-1/r} \Delta^{k-i} p_{i+1} \cdots p_k \times \Delta\omega([p_1 \dots p_{i-1}]V) \ .\end{aligned}$$

Since $r \leq k$ and $P_i/2 \leq p_i \leq P_i$, we get :

$$\begin{aligned}p_i^{-1/r} \Delta^{k-i} p_{i+1} \cdots p_k \Delta &\leq P_i^{-1/k} 2^{1/k} \Delta^{k-i+1} P_{i+1} \cdots P_k \\ &< P_i^{-1/k} (2\Delta)^{k-i+1} P_{i+1} \cdots P_k \\ &= (2\Delta)^b\end{aligned}$$

where (see lemma 4.3 ii))

$$\begin{aligned}b &= -\frac{\rho_i}{k} + k - i + 1 + \sum_{j=i+1}^{k} \rho_j \\ &= -\frac{(k+1)^{k-i+1} - 1}{k} + (k - i + 1) + (k+1)\frac{(k+1)^{k-i} - 1}{k} - (k - i) \\ &= 0 \ .\end{aligned}$$

This is a contradiction. Hence

$$\hat{\mu}^{\mathsf{ess}}(V) \geq \Big(10n\Delta^{k-1}P_1\cdots P_k\omega(V)\Big)^{-1} .$$

We finally remark that

$$10n\Delta^{k-1}P_1\cdots P_k \leq \big(20n\Delta\big)^{\lambda(k)} .$$

Theorem 4.1 is proved.

□

# 5 Petit points.

Given an algebraic set $V \subseteq \mathbb{G}_m^n$ we define, following [Bom-Zan 1995] and [Sch 1996],

$$V^0 = V \setminus \bigcup_{B \subseteq V} B.$$

where the union is on the set of translates $B$ of subgroups of positive dimension contained in $V$. In this section we prove a slightly improved version of theorem 1.5 of [Amo-Dav 2006]:

**Theorem 5.1** *Let $V \subsetneq \mathbb{G}_m^n$ be an algebraic set defined by equations of degree $\leq \delta$. Then, for all but finitely many $\boldsymbol{\alpha} \in V^0$ we have*

$$\hat{h}(\boldsymbol{\alpha}) \geq \theta := \big(2400n^3\log(2n\delta)\big)^{-n^n+3}\delta^{-1} .$$

*More precisely, the set of $\alpha \in V$ of height $< \theta$ is contained in a finite union $B_1 \cup \cdots \cup B_m$ of translate of subtori such that*

$$\deg(B_j) \leq \big(250n^3\log(2n\delta)\big)^{(2n)^n}\delta^{2^{\mathrm{codim}(B_j)}-1}$$

**Proof.**

It is enough to prove the following statement:

*Let $V \subsetneq \mathbb{G}_m^n$ be an algebraic set defined by equations of degree $\leq \delta$ and let $Z$ be a geometrically irreducible subvariety of $V$ of positive dimension, satifying*

$$\hat{\mu}^{\mathsf{ess}}(Z) \leq \big(2400n^3\log(2n\delta)\big)^{-n^n+3}\delta^{-1} . \tag{5.13}$$

*Then, there exists a translate $B$ of a subtorus of codimension $r$ such that $Z \subseteq B \subseteq V$ and*

$$\deg(B) \leq \big(250n^3\log(2n\delta)\big)^{(2n)^n}\delta^{2^r-1} .$$

We prove this last statement by induction on $n$. If $n = 2$ it is easily implied by theorem 4.1. Assume $n \geq 3$ and that the conclusion holds for all algebraic set defined by equations of degree $\leq \delta'$ in $\mathbb{G}_m^{n-1}$. Assume further that there exists a positive integer $\delta$, an algebraic set $V \subsetneq \mathbb{G}_m^n$ defined by equations of degree $\leq \delta$ and a geometrically irreducible subvariety $Z$ of $V$ which satisfies (5.13). Let $k = \operatorname{codim}(Z)$. In particular, since $\omega(Z) \leq \delta$ and $\lambda(k) \leq n^n - 3$, theorem 4.1 gives a translate $B = \boldsymbol{\alpha} H$ of codimension $k'$ containing $Z$, and such that

$$\big(\deg(B)\big)^{1/k'} \leq \big(250 n^3 \log(2n\delta)\big)^{n^n-2}\delta \ . \tag{5.14}$$

We can assume $\boldsymbol{\alpha} \in Z$ and $\hat{h}(\boldsymbol{\alpha}) \leq 2\hat{\mu}^{\mathsf{ess}}(Z)$; thus we have :

$$\hat{\mu}^{\mathsf{ess}}(\boldsymbol{\alpha}^{-1}Z) \leq \hat{h}(\boldsymbol{\alpha}^{-1}) + \hat{\mu}^{\mathsf{ess}}(Z) \leq n\hat{h}(\boldsymbol{\alpha}) + \hat{\mu}^{\mathsf{ess}}(Z) \leq 3n\hat{\mu}^{\mathsf{ess}}(Z) \ . \tag{5.15}$$

We now fix a $\mathbb{Z}$-base $\mathbf{a}_1, \dots \mathbf{a}_{k'}$ of the $\mathbb{Z}$-module

$$\Lambda := \Big\{ \boldsymbol{\lambda} \in \mathbb{Z}^n, \text{ t.q. } \forall\, \mathbf{x} \in H,\ \mathbf{x}^{\boldsymbol{\lambda}} = 1 \Big\} \subseteq \mathbb{Z}^n$$

and we consider the $n \times k'$matrix $A = (a_{i,j})$. Let $E = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Then (see for instance [Ber-Phi 1988]) the degree of $H$ is the maximum of the absolute values of the $k' \times k'$ subdeterminants of $A$, and $\operatorname{Vol}(E/\Lambda)$ is their quadratic mean. Thus

$$\operatorname{Vol}(E/\Lambda) \leq \binom{n}{k'}^{1/2} \deg(B) \leq n^{k'} \deg(B) \ .$$

Let us consider the cube $[-1/2, 1/2]^n \subset \mathbb{R}^n$ ; by a theorem of Vaaler (see [Vaaler 1979])

$$\operatorname{Vol}(C \cap E) \geq 1 \ .$$

Thus, by Minkowski's theorem on convex bodies, there exists a non-zero $\boldsymbol{\lambda} \in \Lambda$ such that:

$$\max_{1 \leq i \leq n} \{|\lambda_i|\} \leq n \deg(B)^{1/k'} \ .$$

Since $H$ is connected, we can assume $\lambda_1, \dots, \lambda_n$ coprime and also $\lambda_n = D$. Then the equation

$$\boldsymbol{x}^{\boldsymbol{\lambda}} = 1$$

defines a subtorus $H' \supseteq H$ of codimenion 1 and degree

$$D \leq n \deg(B)^{1/k'} \leq (2n)^{-2}\big(250 n^3 \log(2n\delta)\big)^{n^n} \delta \ . \tag{5.16}$$

If $\boldsymbol{\alpha} H' \subseteq V$ we are done. Asume the contrary. We consider the isogeny $\mathbb{G}_m^{n-1} \twoheadrightarrow H'$ defined by

$$\varphi(\mathbf{x}) = \Big( x_1^{\lambda_n}, \dots, x_{n-1}^{\lambda_n}, x_1^{-\lambda_1} \cdots x_{n-1}^{-\lambda_{n-1}} \Big) \ .$$

We remark that, for any $\boldsymbol{\beta} \in \mathbb{G}_m^{n-1}$,

$$h\big(\varphi(\boldsymbol{\beta})\big) \geq h\big(\boldsymbol{\beta}^{\lambda_n}\big) = \lambda_n h\big(\boldsymbol{\beta}\big) = Dh(\boldsymbol{\beta}) \ . \tag{5.17}$$

Let

$$V' = \varphi^{-1}\left(\boldsymbol{\alpha}^{-1}V \cap H\right) \subseteq \mathbb{G}_m^{n-1}$$

Since $\boldsymbol{\alpha}H' \not\subseteq V$ we have $V' \subsetneq \mathbb{G}_m^{n-1}$. Moreover, let $F_j(\mathbf{x})$ $(j = 1, \ldots, N)$ be equations defining $V$; then $V'$ is defined by the equations

$$F_j\left(x_1^{\lambda_n}, \ldots, x_{n-1}^{\lambda_n}, x_1^{-\lambda_1} \ldots x_{n-1}^{-\lambda_{n-1}}\right) = 0$$

of degree

$$\leq \delta' = \max\{\lambda_n, |\lambda_1 + \cdots + \lambda_{n-1}|\}\delta \leq nD\delta \ .$$

Let $Z'$ be a geometrically irreducible component of $\varphi^{-1}\left(\boldsymbol{\alpha}^{-1}Z \cap H\right) \subseteq V'$. We have, by (5.17) and (5.15),

$$D\hat{\mu}^{\mathsf{ess}}(Z') \leq \hat{\mu}^{\mathsf{ess}}(\varphi(Z')) = \hat{\mu}^{\mathsf{ess}}(\boldsymbol{\alpha}^{-1}Z) \leq 3n\hat{\mu}^{\mathsf{ess}}(Z) \ .$$

Using the upper bound for $\hat{\mu}^{\mathsf{ess}}(Z)$ and the inequality $\delta' \leq nD\delta$, we deduce

$$\begin{aligned}\hat{\mu}^{\mathsf{ess}}(Z') &\leq 3nD^{-1}\big(2400n^3\log(2n\delta)\big)^{-n^n+3}\delta^{-1} \\ &\leq 3n^2\big(2400n^3\log(2n\delta)\big)^{-n^n+3}\delta'^{-1}\end{aligned}$$

Using the inequalities $\delta' \leq nD\delta$, (5.16) and $\log x < x$ we get

$$2n\delta' \leq 2n^2D\delta \leq (250n^3 \cdot 2n\delta)^{n^n}\delta \leq (2n\delta)^{(250n^3)^{n-1}} \ . \tag{5.18}$$

Thus

$$\big(2400(n-1)^3\log(2(n-1)\delta')\big)^{(n-1)^{n-1}-3} \leq (3n^2)^{-1}(2400n^3)^a\log(2n\delta)^{n^n-3}$$

where

$$a = 1 + n\big((n-1)^{n-1} - 3\big) \leq n^n - 3 \ .$$

Therefore

$$\hat{\mu}^{\mathsf{ess}}(Z') \leq \big(2400(n-1)^3\log(2(n-1)\delta')\big)^{-(n-1)^{n-1}+3}\delta'^{-1}$$

By induction there exists a translate $B' \subseteq V'$ of a subtorus of codimension $r'$ such that $Z' \subseteq B'$ and

$$\deg(B') \leq \big(250n^3\log(2n\delta')\big)^{(2n)^{n-1}}\delta'^{2^{r'}-1} \ .$$

Let for brevity $K = 250n^3\log(2n\delta)$. From the inequalities (5.18) and $\delta' \leq nD\delta$ we get

$$\deg(B') \leq K^{2^{n-1}n^n}(nD\delta)^{2^{r'}-1} \ .$$

Then $Z = \boldsymbol{\alpha}\varphi(Z') \subseteq \varphi(B') \subseteq V$, $r = \operatorname{codim} \varphi(B') = r' + 1$ and

$$\begin{aligned}
\deg \varphi(B') &\leq D \deg(B') \\
&\leq K^{2^{n-1}n^n} n^{2^{r'}-1} D^{2^{r'}} \delta^{2^{r'}-1} \\
&\leq K^{2^{n-1}n^n + 2^{r'}n^n} \delta^{2^{r'+1}-1} \\
&\leq K^{(2n)^n} \delta^{2^r-1}
\end{aligned}$$

where we have used the upper bound (5.16) for $D$.

□

**Remark 5.2** In [Amo-Dav 2006], theorem 1.5 we assume that $V$ is geometrically irreducible (which is not necessary) and that $V$ is incompletely defined by forms of degree $\leq \delta$, *i. e.* it is a component of a complete intersection of hypersurfaces of degree $\leq \delta$. Unfortunately, there is a mistake in the proof: at page 561, point (a), we cannot ensure that $V'$ is incompletely defined by forms of degree $\leq nD\delta$. The problem is the following: if $V$ is incompletely defined by forms of degree $\leq \delta$, $Z$ is an hpersurface of degree $\leq \delta$ which not contains $V$, then an irreducible component of $V \cap Z$ is not a *priori* incompletely defined by forms of degree $\leq \delta$.

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