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Trees, valuations and the Green-Lazarsfeld set.

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1 Introduction.

The aim of this paper is the study of the relationship between two objects, the Green-Lazarsfeld set and the Bieri Neumann Strebel invariant, which appear simultaneously in 1987 ([GL], [BNS]). Let us recall some basic definitions.

Let Γ be a finitely generated group, and K be a field. A 1-character χ is an homomorphism from Γ to K^* ; in this article we will only consider 1-characters, and call them characters. A character χ is called exceptional if $H^1(\Gamma, \chi) \neq 0$, or more geometrically if χ can be realized as the linear part of a fixed point free affine action of Γ on a K -line.

The set of exceptional characters, $E^1(\Gamma, K)$ is a subset of the abelian group $\text{Hom}(\Gamma, K^*)$, and our aim is to understand its geometry, in particular if Γ is the fundamental group of a Kähler manifold.

Motivated by the pioneering work of M. Green and R. Lazarsfeld, algebraic geometers studied the case where $K = \mathbb{C}$ is the field of complex numbers, and $\Gamma = \pi_1(X)$ is the fundamental group of a projective or more generally a Kähler manifold. In this case, the geometry of $\text{Hom}(\Gamma, K^*)$ is well understood : it is the union of a finite set, made up with torsion characters, and a finite set of translates of subtori. This result has been conjectured by A. Beauville and F. Catanese, ([Be]) proved by C. Simpson [Si 2]) for projective manifolds, and extended by F. Campana to the Kähler case (see [Ca] for a detailed introduction). The main tools used by C. Simpson were the flat hyper-Kähler structure of $\text{Hom}(\Gamma, \mathbb{C}^*)$ and the Schneider-Lang theorem in transcendence theory. Another proof, model theoretic, has been proposed by R. Pink and D. Roessler [PR].

The definition of a exceptional class in the sense of Bieri Neumann Strebel is easier to explain in the case of an integral cohomology class (an element of $H^1(\Gamma, \mathbb{Z})$). Such a class is *exceptional* if it can be realized as the translation class of a parabolic, non loxodromic action of Γ in some tree.

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The link between these two notions, explained in the next paragraph, can be sketched as follows. Let χ be a Green-Lazarsfeld character of Γ . Suppose $\chi(\Gamma)$ is not contained in the ring of algebraic integers of K . There exists a discrete non archimedean valuation on K such that $v \circ \chi$ is a non trivial homomorphism to \mathbb{Z} . It appears that $v \circ \chi$ is an exceptional class in the sense of Bieri Neumann Strebel. More precisely, one can find a parabolic action of Γ on the Bruhat Tits tree of K_v , with translation length $v \circ \chi$.

Due to the work of C. Simpson [Si 3], M. Gromov and R. Shoen [GS], exceptional cohomology classes on Kähler manifold are well understood (see also [De] for a detailed study of the BNS invariant of a Kähler group). Let X be a Kähler manifold, and ω an exceptional class ; there exists a holomorphic map F from X to a hyperbolic Riemann orbifold Σ such that ω belongs to $F^*H^1(\Sigma, \mathbb{Z})$. Recall that a complex 2-orbifold Σ is a Riemann surface S marked by a finite set of marked points $\{(q_1, m_1) \dots, (q_n, m_n)\}$, where the m_i 's are integers ≥ 2 . A map $F : X \rightarrow \Sigma$ is called holomorphic if it is holomorphic in the usual sense, and for every q_i the multiplicity of the fiber $F^{-1}(q_i)$ is divisible by m_i . The main result of this paper is a description of the (generalized) Green Lazarsfeld set of $\pi_1(X)$ in terms of the finite list of its fibrations on hyperbolic 2-orbifolds.

Theorem *Let Γ be the fundamental group of a Kähler manifold X , $(F_i, \Sigma_i)_{1 \leq i \leq n}$ the family of fibration of X over hyperbolic 2-orbifolds. Let K be a field of characteristic p (if $p = 0$, $K = \mathbb{C}$), $\bar{F}_p \subset K$ the algebraic closure of F_p in K . Then $E^1(\Gamma, K)$ is the union of a finite set of torsion characters (contained in $E^1(\Gamma, \bar{F}_p)$ if $p > 0$) and the union $\bigcup_{1 \leq i \leq n} F_i^* E^1(\pi_1^{\text{orb}}(\Sigma_i), K^*)$.*

Remarks a) Let $\Sigma = (S; (q_i, m_i)_{1 \leq i \leq n})$ a hyperbolic 2-orbifold, and $\Gamma = \pi_1^{\text{orb}}(\Sigma)$ its fundamental group. Then, by a simple computation (see prop.22), one checks that $E^1(\pi_1^{\text{orb}}(\Sigma), K^*) = \text{Hom}(\pi_1^{\text{orb}}(\Sigma), K^*) = (K^*)^{2g} \times \Phi$, where Φ is a finite abelian group, unless $g = 1$ and for all i , $m_i \not\equiv 0 \pmod{\text{char } K}$. If $g = 1$ and for all i $m_i \equiv 0 \pmod{\text{char } K}$, $E^1(\pi_1^{\text{orb}}(\Sigma), K^*)$ is finite, made of torsion characters.

In every cases, the Green Lazarsfeld set is the union of a finite set of torsion characters and a finite set of abelian groups which are translates of tori ; this is our generalization of Simpson's theorem.

b) The main tool used by Simpson to prove his theorem [Si 2] was the study of algebraic triple tori ; if $\text{char } K \neq 0$ no such a structure is available. Our proof furnishes a geometric (i.e. non arithmetic) alternative to Simpson's proof in the case of characteristic 0. In fact, in this case ($\text{char } K = 0$) our method proves that $E^1(\Gamma, K)$ is made with a finite set of *integral* characters (in the sense of Bass [Ba]), and the union $\bigcup_{1 \leq i \leq n} F_i^* \text{Hom}(\pi_1^{\text{orb}}(\Sigma_i), K^*)$; the conclusion follows from the study of the absolute value $|\chi|$ of exceptional characters, which was already done by A. Beauville[Be].

In a recent preprint, [CS], C. Simpson and K. Corlette study the variety of characters of a Kähler group Γ , $\text{Hom}^{\text{ss}}(\Gamma, \text{PSL}(2, \mathbb{C})/\text{PSL}(2, \mathbb{C}))$ from a very similar point of view ; they prove in particular that a Zariski dense representation

of a Kähler group which is not *integral* in the sense of Bass factorizes through a fibration over a hyperbolic 2-orbifold. Their proof is based on the same idea as ours : if a representation ρ is not integral, there exists a valuation on the field generated by $\rho(\Gamma)$ such that the action of Γ on the Bruhat Tits building is non elementary. The conclusion follows by applying the theory of Gromov Shoen on harmonic maps with value in a tree. Using Simpson's work on Higgs bundles they prove further a rigid representation come from a complex variation of Hodge structure.

In paragraph 2, we explain the relationship between the Green-Lazarsfeld and Bieri-Neumann-Strebel invariants ; in paragraph 3 we study the Green-Lazarsfeld set of a metabelian group : a finiteness result on this set is established. These two paragraphs are purely group theoretic, and no Kähler structure is mentioned. In the paragraph 4 we prove the main result.

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2 From an affine action on a line to a parabolic action on a tree.

2.1 Affine action on the line : the Green-Lazarsfeld set

Let K be a field. The affine group of transformation of a K -line, $\text{Aff}_1(K)$, is isomorphic to $K^* \ltimes K$. We identify this group with the set of upper triangular $(2, 2)$ matrices $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ with values in K .

Let Γ be a finitely generated group. An affine action of Γ on the line is a morphism $\rho : \Gamma \rightarrow \text{Aff}_1(K)$. One can write $\rho(g) = \begin{pmatrix} \chi(g) & \theta(g) \\ 0 & 1 \end{pmatrix}$. The linear part of ρ is an homomorphism $\chi : \Gamma \rightarrow K^*$. Its translation part $\theta : \Gamma \rightarrow K$ is a 1-cocycle of Γ with value in χ , i.e. a function which satisfies $\theta(gh) = \theta(g) + \chi(g)\theta(h)$. The representation ρ is conjugate to a diagonal representation if and only if $\rho(\Gamma)$ fixes a point $\mu \in K$, or equivalently if and only if there exists a $\mu \in K$ such that $\theta(g) = \mu(-1 + \chi(g))$ is a coboundary.

Definition 1 *A character $\chi \in \text{Hom}(\Gamma, K^*)$ is exceptional if it can be realized as the linear part of a fixed point free affine action of Γ on the line, i.e if $H^1(\Gamma, \chi) \neq 0$. The set of exceptional characters $E^1(\Gamma, K)$ is called the Green-Lazarsfeld set of Γ .*

2.2 Parabolic action on a tree : the Bieri Neumann Strebel invariant.

Let T be a simplicial tree. We endow T with its natural simplicial metric, and think of T as a complete geodesic space. Let us recall the definitions of

the boundary of T , and of the Busemann cocycle associated to a point in this boundary.

A ray in T is an isometric map $r : [a, +\infty[\rightarrow T$. Two rays $r : [a, +\infty[\rightarrow T$, $s : [b, +\infty[\rightarrow T$ are equivalent (or asymptotic) if they coincide after a certain time : there exists a', b' s.t. for all $t \geq 0$ $r(a' + t) = s(b' + t)$. The boundary of T , denoted ∂T , is the set of equivalence classes of rays. If $\alpha \in \partial T$ and $r : [a, +\infty[\rightarrow T$ represents α , for every point x , the function $t \rightarrow d(x, r(t)) - t$ is eventually constant. Its limit $b_r(x)$ is called the Busemann function of r . If s is equivalent to r , the difference $b_r - b_s$ is a constant.

Definition 2 (*Busemann cocycle*). Let Γ be a group acting on T , and $\alpha \in \partial T$. If Γ fixes α , one defines a homomorphism, the Busemann cocycle, by the formula :

$$\omega_\alpha : \Gamma \rightarrow \mathbb{Z}$$

$$\omega_\alpha(g) = b_r \circ g - b_r$$

Definition 3 (*Exceptional classes*) The action of Γ is called parabolic if it fixes some point at infinity. It is called exceptional if it fixes a unique point at infinity, and if the associated Busemann cocycle is not trivial. A class $\omega \in H^1(\Gamma, \mathbb{Z})$ is exceptional if it can be realized as the Busemann cocycle of an exceptional action of Γ in some tree. The set of exceptional classes is denoted $\mathcal{E}^1(\Gamma, \mathbb{Z})$.

Remark 1 A topological definition of an exceptional class can also be given, in the case where Γ is finitely presented. Let $\Gamma = \pi_1(X)$, where X is a compact manifold, and let ω be some class in $H^1(\Gamma, \mathbb{Z})$. One represents ω by a closed 1-form w on X and considers a primitive $F : \tilde{X} \rightarrow \mathbb{R}$ of the lift of w to the universal cover of X . Then ω is exceptional iff $F \geq 0$ has several components on which F is unbounded (see [Bi], [Le], [Bro]).

Remark 2 The notion of an exceptional class, defined by Bieri Neumann Strebel and studied by several authors, in particular [Bro], [Le], is more general : it concerns homomorphism with value in \mathbb{R} and can be defined along the same lines, using \mathbb{R} -trees instead of combinatorial trees. Our point of view is that of Brown ; it is interesting to remark that [Bro], [BNS] and [GL] are published in the same issue of the same journal, but apparently nobody remarked that [Bro] and [GL] studied the same object from a different point of view. This remark justifies the choice of our title.

2.3 Discrete valuations and Bruhat-Tits trees.

In this paragraph we fix a field K . Let $v : K^* \rightarrow \mathbb{Z}$ be a discrete non archimedean valuation on K . Bruhat and Tits [BT] constructed a tree T_v with an action of $\mathrm{PGL}(2, K)$. One should think of the action of $\mathrm{PGL}(2, K)$ of T_v as an analogue

of the action of $\mathrm{PGL}(2, \mathbb{C})$ on the hyperbolic space of dimension 3 ; we recall below some basic facts about this action (see [Se] for a detailed study).

Let $O_v \subset K$ denote the valuation ring $v \geq 0$. The *vertices* of T_v are the homothety classes of O_v -lattices, i.e. free O_v -modules of rank 2, in K^2 . The *boundary* of this tree is the projective line $P^1(\bar{K}_v)$ over the v -completion of K .

By the general theory of lattices, if Λ, Λ' are two lattices, one can find a O_v -base of Λ such that, in this base, Λ' is generated by $(t^a, 0)$ and $(0, t^b)$ for some t with $v(t) = 1$; hence up to homothety by $(1, 0)$ and $(0, t^n)$, for $n = b - a$. Then the distance between Λ , and Λ' is $|n|$, and the segment between Λ and Λ' is the set of lattices generated by $(1, 0)$ and $(0, t^k)$, $k = 1, n$. More generally if l, l' are two different lines in K^2 , considered as points in ∂T_v , the geodesic from l to l' is the set of product of lattices in l and l' .

The matrix $g_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ fixes the lattice Λ_n generated by $(1, 0)$ and $(0, t^n)$ for $n \leq v(u)$. The matrix $g_u = \begin{pmatrix} t^n & u \\ 0 & 1 \end{pmatrix}$ transforms Λ_m to Λ_{m+n} if $m + n \leq v(u)$.

Acting on T_v the Borel sub-group $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ is *parabolic* : it fixes an end of T_v (namely the line generated by the first basis vector), but neither a point of T_v nor a pair of points of ∂T_v .

The Busemann cocycle of this parabolic subgroup is $b \left(\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \right) = v(\alpha)$.

The relation between the Green-Lazarsfeld set and the Bieri-Neumann-Strebel invariant is now simple to explain.

Proposition 1 *Let $\chi \in H^1(\Gamma, K^*)$. Suppose that $\chi \in E^1(\Gamma, K^*)$ and let $\theta \in H^1(\Gamma, \chi) \neq 0$. Let $\rho : \Gamma \rightarrow \mathrm{GL}(2, K)$ be defined by $\rho(g) = \begin{pmatrix} \chi(g) & \theta(g) \\ 0 & 1 \end{pmatrix}$. If $v \circ \chi \in H^1(\Gamma, \mathbb{Z})$ is not 0, ρ is an exceptional action on T_v .*

Proof. By construction the action of Γ on T_v fixes a point at infinity. It contains an hyperbolic element as $v \circ \chi \neq 0$, but the action cannot fix a line : the other point in the boundary $P^1(\bar{K}_v)$ would be fixed by the group Γ , and ρ would be conjugate to diagonalizable action. The orbit of any point of Γ is therefore a minimal tree which is not a line. \square

3 Metabelian groups

If Γ is a group, let $\Gamma' = [\Gamma, \Gamma]$ its derived group. Recall that a group is *metabelian* if Γ' is abelian, or $\Gamma^2 = (\Gamma)'$ is trivial. If Γ is a f.g. group, Γ/Γ^2 is metabelian.

3.1 The Green-Lazarsfeld set of a metabelian group.

If K is a field, the Green-Lazarsfeld set $E^1(\Gamma, K)$ of the group Γ only depends on its metabelianized Γ/Γ^2 as it only depends of the set of representation of Γ

in the metabelian group $\text{Aff}_1(K) = K^* \rtimes K$.

Let Γ be a metabelian group. We write $1 \rightarrow [\Gamma, \Gamma] \rightarrow \Gamma \rightarrow Q \rightarrow 1$, where $Q = \Gamma/[\Gamma, \Gamma]$ is the abelianized group, and $[\Gamma, \Gamma]$ is abelian. As an abelian group, $M = [\Gamma, \Gamma]$ is not necessary f.g, however we can let Q acts on $[\Gamma, \Gamma]$ by conjugation, so that M can be promoted as a $\mathbb{Z}Q$ module. The following fact is basic and well-known.

Lemma 1 *The module M is finitely generated as a $\mathbb{Z}Q$ module.*

If g_1, \dots, g_r are generators of Γ , the commutators $h_{ij} = [g_i, g_j]$ generate $[\Gamma, \Gamma]$ as a $\mathbb{Z}Q$ module : if $[g, h]$ if $h = ab$ we have $[g, h] = [g, ab] = gag^{-1}a^{-1}agbg^{-1}b^{-1}a^{-1} = [g, a]a[g, b]a^{-1} = [g, a]a_*[g, b]$, and the result follows by induction. \square

Theorem 1 *Let Γ be a finitely generated group. Given a prime number p (p might be 0), there exists a finite number of fields K_ν of characteristic p and of finite transcendence degree over F_p (if $p = 0$, set $F_p = \mathbb{Q}$) and characters $\xi_\nu : \Gamma \rightarrow K_\nu^*$ such that :*

1. $H^1(\Gamma, \xi_\nu) \neq 0$, i.e. $\xi_\nu \in E^1(\Gamma, K_\nu)$
2. If K is a field of characteristic p and $\chi \in E^1(\Gamma, K)$ a Green-Lazarsfeld character, then there exists an index ν s.t. $\ker \chi \supset \ker \xi_\nu$.

Proof Let F_p be the field with p elements and $F_p[Q]$ the group ring of Q with F_p coefficients. Let $M_p = [\Gamma, \Gamma] \otimes F_p, \mathcal{J} \subset F_p[Q]$ the annihilator of M_p , and $A = F_p[Q]/\mathcal{J}$. As Q is a finitely generated abelian group, isomorphic to $\mathbb{Z}^r \times \Phi$, with Φ finite abelian, A is a noetherian ring. Thus A admits a finite number of minimal prime ideals $(\mathfrak{p}_\nu)_{1 \leq \nu \leq \nu_0}$. Let k_i be the field of fraction of A/\mathfrak{p}_i , and ξ_i be the natural character $\Gamma \rightarrow Q \rightarrow A/\mathfrak{p}_i \rightarrow k_i$. Up to re-ordering the list of these ideals, we may assume that for $1 \leq i \leq \nu_1, H^1(\Gamma, \xi_i) \neq 0$.

The theorem 8 is a consequence of the following :

Lemma 2 *Let $\chi \in E^1(\Gamma, K)$ be an exceptional character, $\chi \neq 1$, and let \mathfrak{p} be a minimal prime ideal contained in $\ker \chi$. Then, the character $\xi_{\mathfrak{p}}$ belongs to $E^1(\Gamma, k_{\mathfrak{p}}^*)$, i.e. $H^1(\Gamma, \xi) \neq 0$.*

Let $M_{\mathfrak{p}} = M \otimes A_{\mathfrak{p}}$, and $M_0 = M \otimes_A K = M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$. Note that M_0 is a finitely generated $k_{\mathfrak{p}}$ vector space, on which Γ acts by homotheties : the action of g is the homothety of ratio $\xi(g)$. Let $\pi : [\Gamma, \Gamma] \rightarrow M_0$ the canonical map. We shall prove that $H^1(\Gamma, M) \neq 0$.

For some $g_0 \in \Gamma, \xi(g_0)$ is not 1 (as an element of $k_{\mathfrak{p}}$) : if not $\Gamma = \ker \xi_{\mathfrak{p}}$ so $\chi = 1$.

The map $\Gamma \rightarrow M_0$ defined by $c(g) = \pi(g_0 g g_0^{-1} g^{-1})$ satisfies $c(gh) = \pi(g_0 g h g_0^{-1} h^{-1} g^{-1}) = \pi(g_0 g g_0^{-1} g^{-1}) + \pi(g g_0 h g_0^{-1} h^{-1} g^{-1}) = c(g) + \xi(g)\pi(g_0 h g_0^{-1} h^{-1}) = c(g) + \xi(g)c(h)$. Therefore c is a 1-cocycle of Γ with value in M .

Let us prove, by contradiction, that the cohomology class of c is not 0.

For every $m \in M_0$, $c(m) = (\xi(g_0)m - m) = (\xi(g_0) - 1)m$. If $c = 0$, as $\xi(g_0) \neq 1$, then $M_0 = 0$. But if $M_0 = 0$, $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} = 0$, i.e. $\mathfrak{p}M_{\mathfrak{p}} = M_{\mathfrak{p}}$, and $M_{\mathfrak{p}} = 0$ by the Nakayama lemma (\mathfrak{p} is the unique maximal ideal of $A_{\mathfrak{p}}$), i.e. $M = \mathfrak{p}M$. But $\mathfrak{p} \subset \ker \chi$, so this would imply that $M \otimes_A K = 0$ and $H^1(\Gamma, \chi) = 0$.

If this cocycle is a coboundary we could find some $m \in M_0$ s.t. $c(g) = (1 - \xi(g))m$, but $c(g_0) = 0$, and $\xi(g_0) \neq 1$, so c would be 0.

In order to prove lemma 9, we see that, for every linear map $l = M_0 \rightarrow K$, $l \circ c$ is a non trivial 1 - cocycle.

This proves theorem 8. □

Remark 3 The previous proof is a combination of arguments by [BG] and [Bre]. In their remarkable paper R. Bieri and J. Groves describe the BNS invariant of a metabelian group in terms of the finite set of field k_{ν} and characters ξ_{ν} for a finite set of primes p (the primes p for which $[\Gamma, \Gamma]$ has p -torsion). For every such a field and every valuation on it, $v \circ \xi_{\nu}$ is exceptional. This provides a map from the cone of valuations on k_{ν} to the BNS set. This set turns out to be the union of the images of these cones. In [Bre], Breuillard proves along the same lines, that a metabelian not virtually nilpotent group admits a non trivial affine action.

4 Fundamental groups of Kähler manifolds.

4.1 Fiberings a Kähler manifold.

For the general study of orbifolds and their fundamental groups, we refer to W. Thurston [Th] chap. 13. Complex 2-orbifolds are 2-orbifolds with singularities modeled on the quotient of the unit disk by the action of $\mathbb{Z}/n\mathbb{Z}$. The usefulness of this notion in our context of (fiberings complex manifolds to Riemann surfaces) has been pointed out by C. Simpson [Si 1].

Definition 4 *Complex 2-orbifold, and holomorphic maps.* A complex 2-orbifold Σ is a Riemann surface S marked by a finite set of marked points $\{(q_1, m_1) \dots, (q_n, m_n)\}$, where the m_i 's are integers ≥ 2 .

Let X be a complex manifold, $f : X \rightarrow \Sigma$ a map. Let $x \in X$, $q = f(x)$. Let $m \in \mathbb{N}^*$ be the multiplicity of q , so that there exists an holomorphic map $u : D(0, r) \subset \mathbb{C} \rightarrow (\Sigma, q)$ which is a ramified cover of order m of a neighborhood of q . Then, f is called holomorphic at x , if there exists a neighborhood U of q and a lift $\tilde{f} : U \rightarrow D$, holomorphic at x such that $f = u \circ \tilde{f}$.

Definition 5 *Fundamental group.* Let $\Sigma = (S; \{(q_1, m_1) \dots, (q_n, m_n)\})$ be a 2-orbifold. Let $q \in S \setminus \{(q_1, m_1) \dots, (q_n, m_n)\}$. The fundamental group -in the sense of orbifolds- of Σ at the point p is the quotient $\pi_1^{\text{orb}}(\Sigma, p) = \pi_1(S \setminus \{q_1, \dots, q_n\}) / \ll$

$\gamma_i^{m_i} \gg$, where γ_i is the class of homotopy (well defined up to conjugacy) of a small circle turning once around q_i , and $\ll \gamma_i^{m_i} \gg$ is the normal subgroup generated by all the conjugates of $\gamma_i^{m_i}$.

Example 1 (This is the main example, see [Th] chap. 13) Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a uniform (discrete co-compact) lattice. The quotient $S = D/\Gamma$ of the unit disk by the action of Γ is a Riemann surface. If $p \in D$, its stabilizer is a finite hence cyclic subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Modulo the action of Γ there are only a finite set of points $\{q_1, \dots, q_n\}$ with non trivial stabilizers of order m_i . The quotient orbifold is $\Sigma = (S; \{(q_1, m_1) \dots, (q_n, m_n)\})$. One proves that $\Gamma = \pi_1^{\mathrm{orb}}(\Sigma)$. An orbifold is called hyperbolic if it is obtained in this way ; an orbifold is hyperbolic if and only if its Euler characteristic $\chi^{\mathrm{orb}}(\Sigma) = \chi(S) - \sum_{1 \leq i \leq n} (1 - \frac{1}{m_i})$ is non positive.

The following definition is useful to understand the structure of Kähler groups (see [ABCKT]).

Definition 6 A Kähler manifold X fibers if there exists a pair (Σ, F) where $\Sigma = (S; \{(q_1, m_1) \dots, (q_n, m_n)\})$ is a hyperbolic 2-orbifold, and $F : X \rightarrow \Sigma$ an holomorphic map with connected fibers. Two such maps $F : X \rightarrow \Sigma, F' : X' \rightarrow \Sigma'$ are equivalent if the fibers of F and F' are the same and images in Σ and Σ' of singular fibers have same order. In this case there exists an holomorphic isomorphism from S to S' which maps singular points of S to singular points of S' preserving the multiplicity.

Let $\pi : X \rightarrow S$ be an holomorphic map from a compact complex surface to a curve. If $q \in S$ is a singular value of π , the analytic set $\pi^{-1}(q)$ can be decomposed in a finite union of irreducible sets, (D_i) . Away from a set of complex dimension $n - 2$ in D_i , hence of complex codimension 2 in X , the map p can be written $\pi(z_1, \dots, z_n) = z_1^{d_i}$, where d_i is the multiplicity of D_i . The multiplicity of the fiber $\pi^{-1}(q)$ is by definition $m = \mathrm{pgcd}(d_i)$. Let Σ be the orbifold whose underlying space is S , singular points are singular values of π with corresponding multiplicity.

Lemma 3 $\pi : X \rightarrow \Sigma$ is holomorphic.

By construction, locally in the neighborhood of a point of $\pi^{-1}(q), \pi(x) = f_1^{d_1} \dots f_k^{d_k} + \mathrm{cte}$, with $m | \mathrm{pgcd} d_i$ \square

The following finiteness theorem is well-known in the smooth case, and implicit in the litterature at several places ; we give below a short proof based on the hyperbolic geometry of hyperbolic orbifolds.

Theorem 2 Let X be a compact complex manifold. There exists, up to equivalence, a finite set of pair (Σ_i, F_i) where Σ_i is a complex hyperbolic 2-orbifold, $F_i : X \rightarrow \Sigma_i$ is holomorphic with connected fibers \square .

Let us give a proof of this (well known) fact based on the Kobayashi-hyperbolicity of a hyperbolic 2 orbifold :there exist no holomorphic map from \mathbb{C} to an hyperbolic 2-orbifold as there exists no holomorphic map from \mathbb{C} to the unit disk. Thus, by the Bloch principle, as X is compact there exists a uniform bound on the differential of an holomorphic map $F : X \rightarrow \Sigma$. Therefore the set of pairs (F, Σ) is *compact* (two such orbifold are ε -close if they are close for the Gromov-Hausdorff topology, i.e. there exists a map between them which is isometric up to an error of ε). But this compact space has only isolated points : if $F_1 : X \rightarrow \Sigma_1$ is given, and, and the (Gromov-Hausdorff) distance of F to F_1 is smaller than the diameter of Σ_1 (for instance $\leq 1/2 \text{diam}(X)$ where X is endowed the Kobayashi pseudo-metric) all the fibers of F_1 are send by F inside a disk (or an annulus in the case of the Margulis constant)therefore to a constant by the maximum principle ; in other words F factorizes through F_1 and induces an isomorphism between Σ and Σ_1 . \square

Remark 4 This proof shows that the number of pairs (F, Σ) for a given complex manifold X can be bounded by the Kobayashi diameter of X .

The following is well known (see [Si 1] [CKO]) .

Theorem 3 *Let $F : X \rightarrow S$ by an holomorphic map with connected fibers from the complex manifold X to a complex curve S . Let Σ be the orbifold whose singular points are the singular values of p and multiplicity the multiplicity of the corresponding fiber. Let $Y = F^{-1}(b)$ be the fiber of a non singular point of S . Let $\pi'_1(Y)$ the image in $\pi_1(X)$ of $\pi_1(Y)$. One has the exact sequence*

$$1 \rightarrow \pi'_1(Y) \rightarrow \pi_1(X) \rightarrow \pi_1^{\text{orb}}(\Sigma) \rightarrow 1$$

in particular the kernel of $\pi_ : \pi_1(X) \rightarrow \pi_1^{\text{orb}}(\Sigma)$ is finitely generated.* \square

4.2 Valuations.

The next result is a reformulation of a fibration theorem of Gromov-Shoen [GS]and Simpson [Si 3]in terms of the exceptional set in the sense of Bieri Neumann Strebel ; see also [De] for a more general study of the BNS invariant of a Kähler group, where $\omega \in H^1(\Gamma, \mathbb{R})$ rather than $H^1(\Gamma, \mathbb{Z})$.

Theorem 4 *Let $\omega \in H^1(\Gamma, \mathbb{Z})$. Then ω is exceptional iff there exist a hyperbolic orbifold Σ , an holomorphic map $F : X \rightarrow \Sigma$ such that $\omega \in F^* H^1(\Sigma, \mathbb{Z})$.*

Let η be a closed holomorphic $(1, 0)$ form whose real part is the harmonic representative of ω . Let \tilde{X} the universal cover of X , and $F : \tilde{X} \rightarrow \mathbb{R}$ a primitive of $\text{Re } \eta$. From the definition (Remark 5) of \mathcal{E}^1 we know that $F \geq 0$ is not connected ; [Si 3]applies. One can also apply the proof of corollary 9.2 of [GS] to the foliation defined by the complex valued closed $(1, 0)$ form whose real part is the harmonic representative of ω .

To prove the converse (which will not be used), one remarks that for every $w \in H^1(\Sigma, \mathbb{Z})$, its pull back to $H^1(\Sigma, \mathbb{Z})$ is exceptional, as $\pi_1^{\text{orb}}(\Sigma)$ is hyperbolic, and the kernel of $\pi_1^{\text{orb}}(\Sigma) \rightarrow \mathbb{Z}$ cannot be finitely generated. \square

4.3 The Green-Lazarsfeld set of a Kähler group.

Let K be a field. Recall that a character $\chi : \Gamma \rightarrow K^*$ is called integral in the sense of Bass [Ba] if $\chi(\Gamma) \subset O$, the ring of algebraic integers of K .

Proposition 2 *Let X be a Kähler manifold, $\chi \in E^1(\Gamma, K^*)$ be a character. If χ is not integral, X fibers over a 2-orbifold Σ such that $\chi \in F^*E^1(\pi_1^{\text{orb}}(\Sigma), K^*)$.*

Proof. Let v be some valuation such that $\omega = v \circ \chi \neq 0$. Let Γ acts on T_v . By prop.6 this action is exceptional. Applying Thm. 18 we get a pair F, Σ such that $\omega \in F^*H^1(\Sigma, \mathbb{Z})$. From the exact sequence of Theorem 17, we see that $\pi_1'(Y)$ is a finitely generated normal subgroup of Γ made up with elliptic elements. As $\pi_1'(Y)$ is *finitely generated*, the subtree of T_v made up with fixed points of $\pi_1'(F)$ is not empty. As $\pi_1'(Y)$ is normal, it is invariant by the action of Γ . Therefore the boundary of this tree contains at least 3 distinct elements. Thus acting on $P^1(K)$ $\pi_1'(Y)$ fixes three different points and is the identity : $\pi_1'(Y) \subset \ker \rho$, and ρ descends to some character on $\pi_1^{\text{orb}}(\Sigma)$. \square

The following proposition is a reformulation of a result by Beauville [Be] (Cor 3.6), it will be used to study the cohomology class of $v \circ \chi$, for the archimedean valuation $v(z) = \ln |z|$ an $\chi : \Gamma \rightarrow \mathbb{C}^*$ a character.

Proposition 3 *Let X be a Kähler manifold, $\chi \in E^1(\Gamma, \mathbb{C}^*)$ be character. If $|\chi| \neq 1$, there exist an holomorphic map $F : X \rightarrow \Sigma$ from X to a 2-orbifold Σ such that $\chi \in F^*E^1(\pi_1^{\text{orb}}(\Sigma), K^*)$.*

Combining propositions 19 and 20, we get the description of the GL set of a Kähler manifold in terms of its fibering over hyperbolic 2-orbifolds. It generalizes results by M. Green R. Lazarsfeld [GL], A. Beauville [Be], C. Simpson [Si 2], F. Campana [Ca], R. Pink D. Roessler [PR], who studied the case where the field K is the field of complex numbers.

Theorem 5 *Let Γ be the fundamental group of a Kähler manifold X , $(F_i, \Sigma_i)_{1 \leq i \leq n}$ the family of fibration of X over hyperbolic 2-orbifolds. Let K be a field of characteristic p (if $p = 0$, $K = \mathbb{C}$), $\bar{F}_p \subset K$ the algebraic closure of F_p in K . Then $E^1(\Gamma, K)$ is made with a finite set of torsion characters (contained in $E^1(\Gamma, \bar{F}_p)$ if $p > 0$) and the union of $F_i^* \text{Hom}(\pi_1^{\text{orb}}(\Sigma_i), K^*)$.*

Proof We shall prove that a character χ which is not in the union $\bigcup F_i^* \text{Hom}(\pi_1^{\text{orb}}(\Sigma_i), K^*)$ must be a torsion character of bounded order. Let us fix such a character χ .

From theorem 8, we know that there exists a finite number of fields K_ν and characters ξ_ν such that $H^1(\Gamma, \xi_\nu) \neq 0$, and for every $\chi \in E^1(\Gamma, K)$ there exists an index ν for which $\ker \xi_\nu \subset \ker \chi$. If ξ_ν is not integral, there exists a 2-orbifold Σ and a holomorphic map $F : X \rightarrow \Sigma$ such that $\ker F_* \supset \ker \xi_\nu$: therefore $\ker F_* \supset \ker \chi$ and $\chi \in F^*E^1(\pi_1^{\text{orb}}(\Sigma))$.

Thus, as $\chi \notin \bigcup F_i^* \text{Hom}(\pi_1^{\text{orb}}(\Sigma_i), K^*)$ χ is integral.

Let us first discuss the case of positive characteristic. If ξ_ν is integral, then $\xi_\nu(\Gamma)$ is made with roots of unity of K_ν . But K_ν is of finite transcendence

degree over F_p so admits only a finite number of roots of unity of degree d_ν (see [Ba] for instance). Therefore, χ is a torsion character of order d dividing d_ν .

Suppose now that $\text{char } K = 0$, and ξ_ν is integral. Thus K_ν is a number field, and $\xi_\nu(\Gamma)$ is contained in the ring O_ν of integers of ξ_ν . If $|\xi_\nu| \neq 1$, or if one of its conjugates $\sigma(\xi_\nu)$ has $|\sigma(\xi_\nu)| \neq 1$, as $H^1(\Gamma, \xi_\nu) \neq 0$ we know (prop. 20) that there exists a 2-orbifold Σ and a holomorphic map $F : X \rightarrow \Sigma$ such that $\ker F_* \supset \ker \xi_\nu$; the previous argument apply and proves that $\chi \in F^* E^1(\pi_1^{\text{orb}}(\Sigma))$. Therefore, χ must be a root of unity, by a theorem of Kronecker, of bounded degree d , as the degree of the n -th cyclotomic polynomial goes to infinity with n , and as d divides the degree of K_ν . The rest of the argument is unchanged. \square

Thus, the theorem 21 reduces the computation of $E^1(\Gamma, K^*)$ to the case where Γ is the fundamental group of a 2-orbifold.

Proposition 4 *Let $\Gamma = \pi_1^{\text{orb}}(\Sigma)$, for $\Sigma = (S; (q_i, m_i)_{1 \leq i \leq n})$ a hyperbolic 2-orbifold then,*

$E^1(\pi_1^{\text{orb}}(\Sigma), K^*) = \text{Hom}(\pi_1^{\text{orb}}(\Sigma), K^*)$ unless $g = 1$ and for all i , $m_i \not\equiv 0 \pmod{\text{char } K}$.

If $g = 1$ and for all i $m_i \equiv 0 \pmod{\text{char } K}$, $E^1(\pi_1^{\text{orb}}(\Sigma), K^)$ is finite, made of torsion characters.*

Let $\chi : \pi_1^{\text{orb}}(\Sigma) \rightarrow K^*$ be a representation. If $\chi = 1$, $H^1(\pi_1^{\text{orb}}(\Sigma), K^*) = \text{Hom}(\pi_1^{\text{orb}}(\Sigma), K^*) \neq 0$. If $g > 1$, consider a simple closed curves on S such that c are homologous to 0, which separated S in two compact surface of positive genus S_1, S_2 , with common boundary c and such that all singular points are in S_2 ; if $g = 1$ consider a curve c , which bounds a disk \bar{D} on S containing all singular points q_i , and let $S_1 = S \setminus \text{int}(D)$ be the other component. One consider a representation $\chi : \pi_1^{\text{orb}}(\Sigma) \rightarrow K^*$, and note that $\chi(c) = 1$ as c is homologous to 0. We think of χ as a local system on Σ and we will use a Mayer Vietoris exact sequence.

First note that if $\chi|_{\pi_1(S_1)}$ and $\chi|_{\pi_1(S_2)}$ are not 1, then $H^1(\pi_1^{\text{orb}}(\Sigma), K^*) \neq 0$: let $x_0 \in K$, there exists a unique 1-cocycle c such that $c(g) = x_0(1 - \chi(g))$ is $g \in \pi_1(S_1)$, $c(g) = 0$ if $g \in S_2$.

If $\chi|_{S_1} = 1$, as $H^1(S_1, \partial S_1, K) = K^2$, one can find a 1-cocycle c whose restriction on S_2 or D is 0, and restriction on S^1 is not trivial.

We are left to the case $\chi|_{S_2}$ or $\chi|_D = 1$. If $g(S_2) > 0$ $H^1(\pi_1^{\text{orb}}(\Sigma_2), C, K) \rightarrow K^{2g}$ and the previous argument apply.

The remaining case is $g = 1$, $\chi|_{\pi_1^{\text{orb}}(D)} = 1$, $\chi|_{\pi_1(S_1)} \neq 1$. Note that in this case χ is a torsion character. Furthermore, $H^1(\pi_1^{\text{orb}}(\Sigma_2), K) = \{(z_1, \dots, z_n) \in K / m_i z_i = 0\}$. This space is 0 unless $m_i \equiv 0 \pmod{\text{char } K}$ for some i . On the other hand, if $\rho|_{\pi_1(S_1)} \neq 0$ the homomorphism $H^1(\pi_1(S_1), \rho) \rightarrow K$ which sends θ to $\theta(c)$ is an isomorphism. Using the exact sequence of Mayer Vietoris, we see that $H^1(\pi_1^{\text{orb}}(\Sigma), \chi) \neq 0$ if $g > 1$ or $g = 1$ and for some i , m_i divides the characteristic of K . \square

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