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► **To cite this version:**

| Patrick Le Meur. Topological invariants of piecewise hereditary algebras. 2008. hal-00131235v3

HAL Id: hal-00131235

<https://hal.science/hal-00131235v3>

Preprint submitted on 8 Aug 2008 (v3), last revised 26 Feb 2009 (v4)

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Topological invariants of piecewise hereditary algebras

Patrick Le Meur

8th August 2008

Abstract

We investigate the Galois coverings of piecewise hereditary algebras. In particular, we study their behaviour under derived equivalences. For a piecewise hereditary algebra, we prove that there exists a universal Galois covering whose group of automorphisms is free and depends only on the derived category of the algebra. As a corollary, we prove that the algebra is simply connected if and only if its first Hochschild cohomology vanishes.

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Introduction

Let k be an algebraically closed field and A be a basic finite dimensional k -algebra (or, simply, an algebra). The representation theory of algebras deals with the description of the category $\text{mod } A$ of finite dimensional (right) A -modules. Also, one wishes to classify these algebras up to Morita and derived equivalence. From this point of view, some classes of algebras play an important rôle: The *hereditary* algebras, that is, path algebras kQ of finite quivers Q with no oriented cycle; the *tilted* algebras, that is, of the form $\text{End}_{kQ}(T)$, where T is a tilting kQ -module; and, more generally, the *piecewise hereditary* algebras, that is, whose bounded derived category $\mathcal{D}^b(\text{mod } A)$ is equivalent to $\mathcal{D}^b(\mathcal{H})$ where \mathcal{H} is a Hom-finite hereditary abelian category with split idempotents (if $\mathcal{H} = \text{mod } kQ$, then A is called *piecewise hereditary of type Q*). These algebras are particularly well understood (see [2, 14, 26, 27], for instance).

The piecewise hereditary algebras arise in the study of other fields of representation theory. For example, the cluster category C_A of a piecewise hereditary algebra has been introduced in [5, 7] as a tool to study conjectures arising in the theory of cluster algebras (see [12]). Another example is the study of self-injective algebras, that is, algebras A such that $A \simeq DA$ as right A -modules. Indeed, to any algebra A is associated the repetitive category \hat{A} , which is a Galois covering with group \mathbb{Z} of the trivial extension $A \ltimes DA$ (see [18]). Assume that some group G acts freely on \hat{A} , thus defining a Galois covering $\hat{A} \rightarrow \hat{A}/G$ with group G . If \hat{A}/G is a finite dimensional algebra, that is, if it has finitely many objects as a category, then it is self-injective. It is proved in [28] that any self-injective algebra of polynomial growth and admitting a Galois covering by a strongly simply connected category is of the form \hat{A}/G for some tilted algebra A , and some infinite cyclic group G . The class of self-injective algebras of type Q has been the object of many studies recently (see [30, 31, 32]).

In this text we investigate the Galois coverings of piecewise hereditary algebras. The Galois covering of algebras and, more generally, of k -categories, were introduced in [24] and [13] for the classification of representation-finite algebras. Consider A as a locally bounded k -category: If $1 = e_1 + \dots + e_n$ is a decomposition of the unity into primitive orthogonal idempotents, then $\text{ob}(A) = \{e_1, \dots, e_n\}$, and the space of morphisms from e_i to e_j is $e_j A e_i$. Then a *Galois covering* of the k -category A is a k -linear functor $F: \mathcal{C} \rightarrow A$ where \mathcal{C} is a k -category endowed with a free action of G , that is, G acts freely on $\text{ob}(\mathcal{C})$, such that $F \circ g = F$ for every $g \in G$ and the induced functor $\mathcal{C}/G \rightarrow A$ is an isomorphism (see [13]). In such a situation, $\text{mod } \mathcal{C}$ and $\text{mod } A$ are related by the so-called push-down functor $F_\lambda: \text{mod } \mathcal{C} \rightarrow \text{mod } A$, that is, the extension-of-scalars functor. Often, F_λ allows nice comparisons between $\text{mod } \mathcal{C}$ and $\text{mod } A$. For example: The action of G on \mathcal{C} naturally defines an action $(g, X) \mapsto {}^g X$ of G on \mathcal{C} -modules.

When this action is free on indecomposable \mathcal{C} -modules, F_λ defines an isomorphism of translation quivers between $\Gamma(\text{mod } \mathcal{C})/G$ and a union of components of the Auslander-Reiten quiver $\Gamma(\text{mod } A)$ of A (see [13, 11]).

The comparisons allowed by the covering techniques raise naturally the following questions: Given an algebra A , is it possible to describe all the Galois coverings of A (in particular, does A admit a universal Galois covering, as happens in topology)? Is it possible to characterise the simple connectedness of A (that is, the fact that A has no proper Galois covering by a connected and locally bounded k -category)? In view of the above discussion on self-injective algebras, these questions are particularly relevant when A is piecewise hereditary of type Q . In case $A = kQ$, the answers are well-known: the Galois coverings of kQ correspond to the ones of the underlying graph of Q ; and kQ is simply connected if and only if Q is a tree, which is also equivalent to the vanishing of the first Hochschild cohomology group $\text{HH}^1(kQ)$ (see [8]). Keeping in mind the general objective of representation theory, one can wonder if the data of the Galois coverings of A , and the fact that A is simply connected or not, does depend only on the bounded derived category $\mathcal{D}^b(\text{mod } A)$. Again, it is natural to first treat this problem for piecewise hereditary algebras. Up to now, there are no general solutions to the above problems. The question of the description of the Galois coverings, and the one of the characterisation of simple connectedness have found a satisfactory answer in the case of standard representation-finite algebras (see [13, 6]). This is mainly due to the fact that the Auslander-Reiten quiver is connected and completely describes the module category in this case. However, the infinite-representation case seems to be more complicated. As an example, there exist string algebras which admit no universal Galois covering (see [22]). In the present text, we study the above problems when A is piecewise hereditary. As a main result, we prove the following theorem.

Theorem 1. *Let A be a connected and piecewise hereditary algebra. Then A admits a universal Galois covering $\widehat{\mathcal{C}} \rightarrow A$ with group a free group $\pi_1(A)$ whose rank depends only on $\mathcal{D}^b(\text{mod } A)$. This means that $\widehat{\mathcal{C}}$ is connected and locally bounded and for any Galois covering $\mathcal{C} \rightarrow A$ with group G where \mathcal{C} connected and locally bounded there exists a commutative diagram:*

$$\begin{array}{ccc} \widehat{\mathcal{C}} & & \mathcal{C} \\ \downarrow & \searrow & \downarrow \\ A & \xrightarrow{\sim} & A \end{array}$$

where the bottom horizontal arrow is an isomorphism extending the identity map on $\text{ob}(A)$. Moreover, $\widehat{\mathcal{C}} \rightarrow \mathcal{C}$ is Galois with group N such that there is an exact sequence of groups $1 \rightarrow N \rightarrow \pi_1(Q) \rightarrow G \rightarrow 1$. If A is piecewise hereditary of type Q , then $\pi_1(A) \simeq \pi_1(Q)$.

The above theorem implies that the Galois coverings of a piecewise hereditary algebra are determined by the factor groups of $\pi_1(A)$. Also it shows that the data of the Galois coverings is an invariant of the derived category. Therefore, so does the simple connectedness. Using the fact that the Hochschild cohomology is invariant under derived equivalences (see [19]), we deduce the following corollary of our main result.

Corollary 2. *Let A be a connected and piecewise hereditary algebra. The following are equivalent:*

- (a) A is simply connected.
- (b) $\text{HH}^1(A) = 0$.

If, moreover, A is piecewise hereditary of type Q , then these conditions are equivalent to the following one:

- (c) Q is a tree.

This corollary generalises some of the results of [3, 1] which studied the same characterisation for tilted algebras of tame type, and for tame tilted algebras. Also, this corollary gives a new class of algebras for which the following question of A. Skowroński (see [29, Pb. 1]) has a positive answer: *Is A simply connected if and only if $\text{HH}^1(A) = 0$?* Originally, this question was asked for tame triangular algebras.

The methods we use to prove Theorem 1 allow us to prove the last main result of this text. It shows that the Galois coverings have a nice behaviour for piecewise hereditary algebras.

Theorem 3. *Let A be piecewise hereditary of type Q . Let $F: \mathcal{C} \rightarrow A$ be a Galois covering with group G where \mathcal{C} is connected and locally bounded. Then \mathcal{C} is piecewise hereditary of type a quiver Q' , that is, $\mathcal{D}^b(\text{mod } kQ')$ and $\mathcal{D}^b(\text{mod } \mathcal{C})$ are triangle equivalent, such that there exists a Galois covering of quivers $Q' \rightarrow Q$ with group G .*

We now give some explanations on the proof of Theorem 1. For unexplained notions, we refer the reader to the next section. Assume that A is piecewise hereditary. It is known from [16, Thm. 2.6] that there exists an algebra B such that $A \simeq \text{End}_{\mathcal{D}^b(\text{mod } B)}(X)$ for some tilting complex $X \in \mathcal{D}^b(\text{mod } B)$, and such that B has one of the following forms:

1. $B = kQ$, with Q a finite quiver with no oriented cycle.
2. B is a squid algebra (see Section 6 for a reminder on squid algebras).

It is easy to check that Theorem 1 holds true for path algebras of quivers and for squid algebras. Therefore, we are reduced to proving that Theorem 1 holds for A and only if it holds for $\text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$ for any tilting complex $T \in \mathcal{D}^b(\text{mod } A)$. Roughly speaking, we need a correspondence between the Galois coverings of A and those of $\text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$. Therefore, we use a construction introduced in [21] for tilting modules. This construction is as follows. Given a Galois covering $F: \mathcal{C} \rightarrow A$ with group G , the push-down functor $F_\lambda: \text{mod } \mathcal{C} \rightarrow \text{mod } A$ is exact and therefore induces an exact functor $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$. Also, the G -action on modules extends to a G -action on $\mathcal{D}^b(\text{mod } \mathcal{C})$ by triangle automorphisms. Now, let $T \in \mathcal{D}^b(\text{mod } A)$ be a tilting complex and $T = T_1 \oplus \dots \oplus T_n$ be an indecomposable decomposition. Assume that the following conditions hold true for every $i \in \{1, \dots, n\}$:

(H₁) There exists an indecomposable \mathcal{C} -module \tilde{T}_i such that $F_\lambda \tilde{T}_i = T_i$.

(H₂) The stabiliser $\{g \in G \mid {}^g \tilde{T}_i \simeq \tilde{T}_i\}$ is the trivial group.

Under these assumptions, the complexes ${}^g \tilde{T}_i$ (for $g \in G$ and $i \in \{1, \dots, n\}$) form a full subcategory of $\mathcal{D}^b(\text{mod } \mathcal{C})$ which we denote by $\text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C})}(\tilde{T})$. Then $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$ induces a Galois covering with group G :

$$\begin{array}{ccc} \text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C})}(\tilde{T}) & \rightarrow & \text{End}_{\mathcal{D}^b(\text{mod } A)}(T) \\ {}^g \tilde{T}_i & \mapsto & T_i \\ {}^g \tilde{T}_i \xrightarrow{u} {}^h \tilde{T}_j & \mapsto & T_i \xrightarrow{F_\lambda(u)} T_j \end{array} .$$

Hence, (H₁) and (H₂) are technical conditions which allow one to associate a Galois covering of $\text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$ to a Galois covering of A . In particular, if A admits a universal Galois covering, then the associated Galois covering of $\text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$ is a good candidate for being a universal Galois covering. We prove that this is indeed the case provided that the following technical condition is satisfied:

(H₃) If $\psi: A \xrightarrow{\sim} A$ is an automorphism such that $\psi(x) = x$ for every $x \in \text{ob}(A)$, then $\psi_\lambda T_i \simeq T_i$, for every i .

We therefore need to prove the assertions (H₁), (H₂) and (H₃) for every Galois covering $F: \mathcal{C} \rightarrow A$ and every tilting complex $T \in \mathcal{D}^b(\text{mod } A)$.

The text is therefore organised as follows. In Section 1, we recall some useful definitions and fix some notations. In Section 2, we define the exact functor $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$ associated to a Galois covering $F: \mathcal{C} \rightarrow A$. In Section 3, we introduce elementary transformations on tilting complexes using approximations. The main result of the section asserts that for any tilting complexes T, T' , there exists a sequence of elementary transformations relating T and T' . We prove the assertions (H₁), (H₂) and (H₃) in Section 4 using the elementary transformations. We prove Theorem 3 as an application of these results. Then, in Section 5, we establish a correspondence between the Galois coverings of A and those of $\text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$ for every tilting complex T . Finally, we prove Theorem 1 and Corollary 2 in Section 6.

1 Definitions and notations

Modules over k -categories

We refer the reader to [4] for the definition of k -categories and locally bounded k -categories. All locally bounded k -categories are assumed to be small and all functors between k -categories are assumed to be k -linear (our module categories and derived categories will be skeletally small).

Let \mathcal{C} be a k -category. Following [4], a (right) \mathcal{C} -module is a k -linear functor $M: \mathcal{C}^{op} \rightarrow \text{MOD } k$ where $\text{MOD } k$ is the category of k -vector spaces. The category of \mathcal{C} -modules is denoted by $\text{MOD } \mathcal{C}$. A module $M \in \text{MOD } \mathcal{C}$ is called *finite dimensional* if $\sum_{x \in \text{ob}(\mathcal{C})} \dim_k M(x) < \infty$. The category of finite dimensional \mathcal{C} -modules is denoted by $\text{mod } \mathcal{C}$. Note

that the indecomposable projective \mathcal{C} -module associated to $x \in \text{ob}(\mathcal{C})$ is equal to $\mathcal{C}(-, x)$. The projective dimension of a \mathcal{C} -module X is denoted by $\text{pd}_{\mathcal{C}}(X)$. If $X \in \text{mod } \mathcal{C}$, then $\text{add}(X)$ denotes the smallest full subcategory of $\text{mod } \mathcal{C}$ closed under direct summands and direct sums. Finally, $\mathcal{D}^b(\text{MOD } \mathcal{C})$ (or $\mathcal{D}^b(\text{mod } \mathcal{C})$) denotes the derived category of complexes of \mathcal{C} -modules (or of bounded complexes of finite dimensional \mathcal{C} -modules, respectively) with shift functor denoted by [1].

We refer the reader to [2] for notions on tilting theory. If A is an algebra, an A -module T is called *tilting* if: (a) T is multiplicity-free; (b) $\text{pd}_A(T) \leq 1$; (c) $\text{Ext}_A^1(T, T) = 0$; (d) for every indecomposable projective A -module P there is an exact sequence $0 \rightarrow P \rightarrow X \rightarrow Y \rightarrow 0$ in $\text{mod } A$ where $X, Y \in \text{add}(T)$. Let \mathcal{H} be a hereditary abelian category. An object $T \in \mathcal{H}$ is called *tilting* (see [16]) if: (a) T is multiplicity-free; (b) $\text{Ext}_{\mathcal{H}}^1(T, T) = 0$; (c) whenever $\text{Hom}_{\mathcal{H}}(T, X) = \text{Ext}_{\mathcal{H}}^1(T, X) = 0$ for $X \in \mathcal{H}$, then $X = 0$.

If \mathcal{A} is an additive category, then $\text{ind } \mathcal{A}$ denotes the full subcategory of all indecomposable objects of \mathcal{A} .

Galois coverings of k -categories

Let $F: \mathcal{E} \rightarrow \mathcal{B}$ be a Galois covering with group G between k -categories (see the introduction). It is called *connected* if both \mathcal{C} and \mathcal{B} are connected and locally bounded.

Let A be a connected and locally bounded k -category and $x_0 \in \text{ob}(A)$. A *pointed Galois covering* $F: (\mathcal{C}, x) \rightarrow (A, x_0)$ is a connected Galois covering $F: \mathcal{C} \rightarrow A$ endowed with $x \in \text{ob}(\mathcal{C})$ such that $F(x) = x_0$. A *morphism of*

pointed Galois coverings $F \xrightarrow{u} F'$ from $F: (\mathcal{C}, x) \rightarrow (A, x_0)$ to $F': (\mathcal{C}', x') \rightarrow (A, x_0)$ is a functor $u: \mathcal{C} \rightarrow \mathcal{C}'$ such that $F' \circ u = F$ and $u(x) = x'$. Note that, given F and F' , there is at most one such morphism (see [20, Lem. 3.1]). This defines the category $\text{Gal}(A, x_0)$ of pointed Galois coverings. If $F \in \text{Gal}(A, x_0)$, then we let F^\rightarrow be the full subcategory of $\text{Gal}(A, x_0)$ with objects those F' such that there exists a morphism $F \rightarrow F'$.

Covering properties on module categories (see [4, 24])

Let $F: \mathcal{E} \rightarrow \mathcal{B}$ be a Galois covering with group G . The G -action on \mathcal{E} defines a G -action on $\text{MOD } \mathcal{E}$: If $M \in \text{MOD } \mathcal{E}$ and $g \in G$, then ${}^g M := F \circ g^{-1} \in \text{MOD } \mathcal{E}$. If $X \in \text{MOD } \mathcal{E}$, the stabiliser of X is the subgroup $G_X := \{g \in G \mid {}^g X \simeq X\}$ of G . The Galois covering F defines two exact functors: The extension-of-scalars functor $F_\lambda: \text{MOD } \mathcal{E} \rightarrow \text{MOD } \mathcal{B}$ which is called the *push-down functor* and the restriction-of-scalars functor $F: \text{MOD } \mathcal{B} \rightarrow \text{MOD } \mathcal{E}$ which is called the *pull-up functor*. They form an adjoint pair (F_λ, F) and F_λ is G -invariant, that is, $F_\lambda \circ g = F_\lambda$ for every $g \in G$. We refer the reader to [4] for details on F_λ and F . For any $M, N \in \text{mod } \mathcal{E}$, the following maps induced by F_λ are bijective:

$$\bigoplus_{g \in G} \text{Hom}_{\mathcal{E}}({}^g M, N) \rightarrow \text{Hom}_{\mathcal{B}}(F_\lambda M, F_\lambda N) \quad \text{and} \quad \bigoplus_{g \in G} \text{Hom}_{\mathcal{E}}(M, {}^g N) \rightarrow \text{Hom}_{\mathcal{B}}(F_\lambda M, F_\lambda N).$$

An indecomposable module $X \in \text{mod } \mathcal{B}$ is called *of the first kind with respect to F* if and only if $F_\lambda \tilde{X} \simeq X$ for some $\tilde{X} \in \text{mod } \mathcal{E}$ (necessarily indecomposable). In such a case, one may choose \tilde{X} such that $F_\lambda \tilde{X} = X$. More generally, $X \in \text{mod } \mathcal{B}$ is called of the first kind with respect to F if and only if it is the direct sum of indecomposable \mathcal{B} -modules of the first kind with respect to F .

2 Covering techniques on the bounded derived category

Let $F: \mathcal{C} \rightarrow A$ be a Galois covering with group G and with \mathcal{C} and A locally bounded categories of finite global dimension. The G -action on $\text{mod } \mathcal{C}$ naturally defines a G -action on $\mathcal{D}^b(\text{mod } \mathcal{C})$, still denoted by $(g, M) \mapsto {}^g M$, by triangle automorphisms. We introduce an exact functor $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$ induced by $F_\lambda: \text{mod } \mathcal{C} \rightarrow \text{mod } A$.

Proposition 2.1. *There exists an exact functor $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \text{mod } \mathcal{C} & \xrightarrow{\quad} & \mathcal{D}^b(\text{mod } \mathcal{C}) \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ \text{mod } A & \xrightarrow{\quad} & \mathcal{D}^b(\text{mod } A) \end{array} .$$

The functor $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$ has the covering property, that is, it is G -invariant, and the two following maps are linear bijections for every $M, N \in \mathcal{D}^b(\text{mod } \mathcal{C})$:

$$\bigoplus_{g \in G} \mathcal{D}^b(\text{mod } \mathcal{C})({}^g M, N) \xrightarrow{F_\lambda} \mathcal{D}^b(\text{mod } A)(F_\lambda M, F_\lambda N),$$

and

$$\bigoplus_{g \in G} \mathcal{D}^b(\text{mod } \mathcal{C})(M, {}^g N) \xrightarrow{F_\lambda} \mathcal{D}^b(\text{mod } A)(F_\lambda M, F_\lambda N).$$

Proof: The existence and exactness of $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$ follows from the exactness of $F_\lambda: \text{mod } \mathcal{C} \rightarrow \text{mod } A$. On the other hand, F_λ induces an additive functor $F_\lambda: \mathcal{K}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{K}^b(\text{mod } A)$ between bounded homotopy categories of complexes. It easily checked that it has the covering property in the sense of the proposition. Since A and \mathcal{C} have finite global dimension, we deduce that $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$ has the covering property. \square

Remark 2.2. It follows from the preceding proposition that $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$ is faithful.

We are mainly interested in indecomposable objects $X \in \mathcal{D}^b(\text{mod } A)$ which are of the form $F_\lambda \tilde{X}$ for some $\tilde{X} \in \mathcal{D}^b(\text{mod } \mathcal{C})$. The following shows that the possible objects \tilde{X} lie in the same G -orbit for a given X .

Lemma 2.3. *Let $X, Y \in \mathcal{D}^b(\text{mod } \mathcal{C})$ be such that $F_\lambda X$ and $F_\lambda Y$ are indecomposable and isomorphic in $\mathcal{D}^b(\text{mod } A)$. Then $X \simeq {}^g Y$ for some $g \in G$.*

Proof: Let $u: F_\lambda X \rightarrow F_\lambda Y$ be an isomorphism in $\mathcal{D}^b(\text{mod } A)$. By Proposition 2.1, there exists $(u_g)_g \in \bigoplus_{g \in G} \mathcal{D}^b(\text{mod } \mathcal{C})(X, {}^g Y)$ such that $u = \sum_{g \in G} F_\lambda u_g$. Since $F_\lambda X$ and $F_\lambda Y$ are indecomposable, there exists $g_0 \in G$ such that $F_\lambda(u_{g_0}): F_\lambda X \rightarrow F_\lambda Y$ is an isomorphism. Since $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$ is exact and faithful, $u_{g_0}: X \rightarrow {}^{g_0} Y$ is an isomorphism in $\mathcal{D}^b(\text{mod } \mathcal{C})$. \square

3 Transforming tilting complexes into tilting modules

Let \mathcal{H} be a hereditary abelian category with tilting objects. Let n be the rank of its Grothendieck group. For short, we set $\text{Hom} := \text{Hom}_{\mathcal{D}^b(\mathcal{H})}$ and $\text{Ext}^i(X, Y) := \text{Hom}_{\mathcal{D}^b(\mathcal{H})}(X, Y[i])$. We write \mathcal{T} for the class of objects $T \in \mathcal{D}^b(\mathcal{H})$ such that:

- (a) T is multiplicity-free and has n indecomposable summands.
- (b) $\text{Ext}^i(T, T) = 0$ for every $i \geq 1$.

We identify an object in \mathcal{T} with its isomorphism class. A complex T lies in \mathcal{T} if and only if $T[1] \in \mathcal{T}$. Also, all tilting complexes in $\mathcal{D}^b(\mathcal{H})$, and, therefore, all tilting object in \mathcal{H} , lie in \mathcal{T} . Given $T \in \mathcal{D}^b(\mathcal{H})$, we let $\langle T \rangle$ be the smallest full subcategory of $\mathcal{D}^b(\mathcal{H})$ containing T and stable under direct sums, direct summands and shifts in either direction. The aim of this section is to define elementary transformations on objects in \mathcal{T} which, by repetition, allow one to relate any two objects in \mathcal{T} . For this purpose, we introduce some notation. Given $T \in \mathcal{T}$, we have a unique decomposition $T = Z_0[i_0] \oplus Z_1[i_0 + 1] \oplus \dots \oplus Z_l[i_0 + l]$ where $Z_0, \dots, Z_l \in \mathcal{H}$, and $Z_0, Z_l \neq 0$. Here, Z_i needs not be indecomposable. We let $r(T)$ be the number of indecomposable summands of $Z_1[i_0 + 1] \oplus \dots \oplus Z_l[i_0 + l]$. Note that: $r(T) \in \{0, \dots, n - 1\}$; $r(T) = 0$ if and only if $T[-i_0]$ is a tilting object in \mathcal{H} ; and $r(T) = r(T[1])$. We are interested in transformations which map an object $T \in \mathcal{T}$ to T' such that $r(T') < r(T)$. Hence, by repeating the process, we may end up with a tilting object in \mathcal{H} (up to a shift).

Transformations of the first kind

Our first elementary transformation is given by the following lemma.

Lemma 3.1. *Let $T \in \mathcal{T}$. There exists $T' \in \mathcal{T}$ such that $T' \in \langle T \rangle$, such that $r(T') \leq r(T)$, and such that $T' = Z'_0 \oplus Z'_1[1] \oplus \dots \oplus Z'_{l'}[l']$, where:*

- (a) $Z'_0, \dots, Z'_{l'} \in \mathcal{H}$ and $Z'_0, Z'_{l'} \neq 0$.
- (b) $\text{Hom}(Z'_0, Z'_1[1]) \neq 0$ if $l' \neq 0$.

Proof: Given $T' \in \langle T \rangle$, we have the unique decomposition $T' = Z'_0[i'_0] \oplus Z'_1[i'_0 + 1] \oplus \dots \oplus Z'_{l'}[i'_0 + l']$ as explained at the beginning of the section. We choose $T' \in \langle T \rangle \cap \mathcal{T}$ such that $r(T') \leq r(T)$ and such that the pair $(l', r(T'))$ is minimal for the lexicographical order. We may assume that $i_0 = 0$. We prove that T' satisfies (a) and (b). If $l' = 0$, there is nothing to prove. So we assume that $l' > 0$. Assume first that $Z'_1 = 0$. Then we let T'' be as follows:

$$T'' := Z'_0 \oplus Z'_2[1] \oplus Z'_3[2] \oplus \dots \oplus Z'_{l'}[l' - 1].$$

Then $T'' \in \langle T' \rangle = \langle T \rangle$. Also, $\text{Ext}^i(T'', T'') = 0$ for every $i \geq 1$ because $T' \in \mathcal{T}$ and \mathcal{H} is hereditary. Finally, T'' is the direct sum of n pairwise non isomorphic indecomposable objects. Thus, $T'' \in \langle T \rangle \cap \mathcal{T}$ and $(l' - 1, r(T'')) < (l', r(T'))$ which contradicts the minimality of $(l', r(T'))$. So $Z'_1 \neq 0$. Now, assume that $\text{Hom}(Z'_0, Z'_1[1]) = 0$. We let T'' be the following object:

$$T'' := (Z'_0 \oplus Z'_1) \oplus Z'_2[2] \oplus Z'_3[3] \oplus \dots \oplus Z'_{l'}[l'].$$

As above, we have $T'' \in \langle T \rangle \cap \mathcal{T}$ and $(l', r(T'')) < (l', r(T'))$ which contradicts the minimality of $(l', r(T'))$. So $\text{Hom}(Z'_0, Z'_1[1]) \neq 0$. \square

With the notations of Lemma 3.1, we say that T and T' are related by a *transformation of the first kind*.

Transformations of the second kind

We now turn to the second elementary transformation. It is inspired by the characterisation of the quiver of tilting objects in hereditary categories (see [17] and also [5] for the corresponding construction in cluster categories). Let $T, T' \in \mathcal{T}$ be such that $T = X \oplus \overline{T}$ with X indecomposable, $T' = Y \oplus \overline{T}$ with Y indecomposable, and there exists a triangle $X \xrightarrow{u} M \xrightarrow{v} Y \rightarrow X[1]$ such that u is a left minimal $\text{add}(\overline{T})$ -approximation or v is a right minimal $\text{add}(\overline{T})$ -approximation. In such a situation, we say that T and T' are related by a *transformation of the second kind*. Note that both u and v are minimal $\text{add}(\overline{T})$ -approximations, as shows the following lemma.

Lemma 3.2. *Let $T \in \mathcal{T}$. Assume that $T = X \oplus \overline{T}$ with X indecomposable.*

- (a) *Let $X \xrightarrow{u} M \xrightarrow{v} Y \rightarrow X[1]$ be a triangle where u is a left minimal $\text{add}(\overline{T})$ -approximation. Then v is a right minimal $\text{add}(\overline{T})$ -approximation.*
- (b) *Let $Y \xrightarrow{u} M \xrightarrow{v} X \rightarrow Y[1]$ be a triangle where v is a right minimal $\text{add}(\overline{T})$ -approximation. Then u is a left minimal $\text{add}(\overline{T})$ -approximation.*

Proof: We only (a), because the proof of (b) is similar. Every morphism $\overline{T} \rightarrow Y$ factorises through v because $\text{Hom}(\overline{T}, X[1]) = 0$. So v is a right $\text{add}(\overline{T})$ -approximation. Let $\alpha: M \rightarrow M$ be a morphism such that $v\alpha = v$. So there exists $\lambda: M \rightarrow X$ such that $u\lambda = \alpha - \text{Id}_M$. Note that u is not a section because T is multiplicity-free. So λu , and therefore $u\lambda$, is nilpotent. We deduce that $\alpha = \text{Id}_M + u\lambda$ is an isomorphism. Therefore, v is right minimal. \square

It is not true that any two objects $T, T' \in \mathcal{T}$ can be related by a sequence of transformations of the first or second kind. However, we have the following result from [17].

Proposition 3.3. *see [17] Assume that \mathcal{H} has no non-zero projective object or $\mathcal{H} = \text{mod } kQ$ with Q a finite quiver of Dynkin type. Let $T, T' \in \mathcal{H}$ be tilting objects. There exists a sequence $T = T_0, \dots, T_i = T'$ of tilting objects in \mathcal{H} such that T_i and T_{i+1} are related by a transformation of the second kind for every i .*

We are going to prove that any $T \in \mathcal{T}$ can be related to some tilting object in \mathcal{H} by a sequence of transformations of the first or of the second kind. Let $T \in \mathcal{T}$. With the notations established at the beginning of the section, assume that $\text{Hom}(Z_0, Z_1[1]) \neq 0$. Since the ordinary quiver of $\text{End}(T)$ has no oriented cycle, there exists $M \in \text{add}(Z_1[i_0 + 1])$ indecomposable such that:

1. $\text{Hom}(Z_0[i_0], M) \neq 0$.
2. $\text{Hom}(Z, M) = 0$ for any indecomposable direct summand Z of $\bigoplus_{t=1}^i Z_t[i_0 + t]$ not isomorphic to M .

Let \bar{T} be such that $T = \bar{T} \oplus M$. Let $B \rightarrow M$ be a right minimal $\text{add}(\bar{T})$ -approximation of M . Complete it into a triangle in $\mathcal{D}^b(\mathcal{H})$:

$$M^* \rightarrow B \rightarrow M \rightarrow M^*[1]. \quad (\Delta)$$

Lemma 3.4. *With the above setting, let $T' = \bar{T} \oplus M^*$. Then $T' \in \mathcal{T}$ and T, T' are related by a transformation of the second kind. Moreover, $r(T') < r(T)$.*

Proof: We only need to prove that $T' \in \mathcal{C}$. We may assume that $i_0 = 0$. By hypothesis on M , we have $B \in \text{add}(Z_0) \subseteq \mathcal{H}$. Since $M \in \mathcal{H}[1]$, the triangle Δ does not split. We now list some properties on T' . In most cases, the proof is due to arguments taken from [5, § 6]. Although these arguments were originally given in the setting of cluster categories (that is, triangulated categories which are Calabi–Yau of dimension 2), it is easily verified that they still work in our situation (that is, the Calabi–Yau property is unnecessary):

1. $\text{Ext}^1(\bar{T}, M^*) = 0$ (see [5, Lem. 6.3]).
2. $\text{Ext}^i(\bar{T}, M^*) = 0$ for every $i \geq 2$. Indeed, applying $\text{Hom}(\bar{T}, -)$ to Δ gives the exact sequence

$$0 = \text{Ext}^{i-1}(\bar{T}, M) \rightarrow \text{Ext}^i(\bar{T}, M^*) \rightarrow \text{Ext}^i(\bar{T}, B) = 0.$$

3. $\text{Ext}^i(M^*, \bar{T}) = 0$ for every $i \geq 1$. Indeed, applying $\text{Hom}(-, \bar{T})$ to Δ gives the exact sequence

$$0 = \text{Ext}^i(B, \bar{T}) \rightarrow \text{Ext}^i(M^*, \bar{T}) \rightarrow \text{Ext}^{i+1}(M, \bar{T}) = 0.$$

4. The map $M^* \rightarrow B$ is a left minimal $\text{add}(\bar{T})$ -approximation (see [5, Lem. 6.4]).
5. M^* is indecomposable and does not lie in $\text{add}(\bar{T})$ (see [5, Lems. 6.5, 6.6]). Therefore, T' is the direct sum of n pairwise indecomposable objects.
6. $M^* \in \mathcal{H}$. Indeed, M is indecomposable, and there are two non-zero maps $M[-1] \rightarrow M^*$ and $M^* \rightarrow B$ with $M[-1], B \in \mathcal{H}$.
7. $\text{Ext}^1(M^*, M^*) = 0$ (see [5, Lem. 6.7]).
8. $\text{Ext}^i(M^*, M^*) = 0$ for every $i \geq 2$ because M^* is indecomposable and \mathcal{H} is hereditary.

The facts 1. – 8. express that $T' \in \mathcal{T}$. Moreover, $r(T') < r(T)$ because $M^* \in \mathcal{H}$ and $M \in \mathcal{H}[1]$. \square

Lemma 3.5. *Let $T \in \mathcal{T}$. Let \mathcal{A} be the smallest subclass of \mathcal{T} containing T and stable under transformations of the first or of the second kind. Then \mathcal{A} contains a tilting object in \mathcal{H} .*

Proof: Let $T' \in \mathcal{A}$ be such that $r(T')$ is minimal for this property. Assume that $r(T') > 0$. Then by applying Lemma 3.1 (if necessary) and then Lemma 3.4, we find $T'' \in \mathcal{A}$ such that $r(T'') < r(T')$. This contradicts the minimality of $r(T')$. Hence, $r(T') = 0$, and there exists an integer i_0 such that $T'[-i_0]$ is a tilting object in \mathcal{H} and lies in \mathcal{A} . \square

The following result is a direct consequence of Proposition 3.3 and Lemma 3.5.

Proposition 3.6. *Assume that \mathcal{H} has no non-zero projective object or $\mathcal{H} = \text{mod } kQ$ with Q a finite quiver of Dynkin type. Let $T \in \mathcal{T}$. Then \mathcal{T} is the smallest subset of \mathcal{T} containing T and stable under transformations of the first or the second kind.*

Remark 3.7. 1. A tilting object in \mathcal{H} generates $\mathcal{D}^b(\mathcal{H})$. By definition of the two kinds of transformations, Proposition 3.6 implies, under the same hypotheses, that any $T \in \mathcal{T}$ generates $\mathcal{D}^b(\mathcal{H})$.

2. A piecewise hereditary algebra A is derived equivalent to \mathcal{H} where \mathcal{H} has no non-zero projective object or is equivalent to $\text{mod } kQ$ with a finite quiver Q of Dynkin type (see [16]). Therefore, the conclusion of Proposition 3.6 holds true if one replaces \mathcal{H} by $\text{mod } A$.

4 Tilting complexes of the first kind

Throughout this section, we assume that A is a connected and piecewise hereditary algebra. We fix a Galois covering $F: \mathcal{C} \rightarrow A$ with group G and with \mathcal{C} locally bounded. We use Proposition 2.1 without reference. The aim of this section is to prove that the following facts hold true for any tilting complex $T \in \mathcal{D}^b(\text{mod } A)$:

- (H₁) For every indecomposable direct summand X of T , there exists $\tilde{X} \in \mathcal{D}^b(\text{mod } \mathcal{C})$ such that $F_\lambda \tilde{X} \simeq X$ in $\mathcal{D}^b(\text{mod } A)$.
- (H₂) $\tilde{X} \not\cong {}^g \tilde{X}$ for every indecomposable direct summand X of T and $g \in G \setminus \{1\}$.
- (H₃) If $\psi: A \xrightarrow{\sim} A$ is an automorphism such that $\psi(x) = x$ for every $x \in \text{ob}(A)$, then $\psi_\lambda X \simeq X$ in $\mathcal{D}^b(\text{mod } A)$ for every indecomposable direct summand X of T .

For this purpose, we fix a triangle equivalence $\Theta: \mathcal{D}^b(\mathcal{H}) \rightarrow \mathcal{D}^b(\text{mod } A)$, where \mathcal{H} is a hereditary abelian category with tilting objects of one of the two following forms (see [16]):

1. $\mathcal{H} = \text{mod } kQ$ with Q a finite quiver of Dynkin type.
2. \mathcal{H} has no non-zero projective and no non-zero injective object.

Some results presented in this section have been proved in [21, § 3] in the case of tilting modules.

Proof of assertion (H₁)

In this paragraph, we prove the following.

Proposition 4.1. *Let $T \in \mathcal{D}^b(\text{mod } A)$ be a tilting complex. Then:*

- (a) *For every indecomposable direct summand X of T , there exists $\tilde{X} \in \mathcal{D}^b(\text{mod } \mathcal{C})$ (necessarily indecomposable) such that $F_\lambda \tilde{X} \simeq X$.*

Moreover, the class $\{\tilde{X} \in \mathcal{D}^b(\text{mod } \mathcal{C}) \mid F_\lambda \tilde{X} \text{ is an indecomposable direct summand of } T\}$ satisfies the following:

- (b) *It generates the triangulated category $\mathcal{D}^b(\text{mod } \mathcal{C})$.*
- (c) *It is stable under the action of G .*
- (d) *$\mathcal{D}^b(\text{mod } \mathcal{C})(\tilde{X}, {}^g \tilde{Y}[i]) = 0$ for every \tilde{X}, \tilde{Y} in this class, $i \neq 0$ and $g \in G$.*

We need the two following dual lemmas.

Lemma 4.2. *Let $\Delta: X \xrightarrow{u} M \rightarrow Y \rightarrow X[1]$ be triangle in $\mathcal{D}^b(\text{mod } A)$ such that:*

- (a) *There exists $\tilde{X} \in \mathcal{D}^b(\text{mod } \mathcal{C})$ satisfying $X = F_\lambda \tilde{X}$.*
- (b) *$M = M_1 \oplus \dots \oplus M_t$ where M_1, \dots, M_t are indecomposables such that there exist indecomposable objects $\tilde{M}_1, \dots, \tilde{M}_t$ satisfying $F_\lambda \tilde{M}_i = M_i$ for every i .*
- (c) *$\mathcal{D}^b(\text{mod } A)(Y, M[1]) = 0$.*

Then Δ is isomorphic to a triangle in $\mathcal{D}^b(\text{mod } A)$:

$$X \xrightarrow{\begin{bmatrix} F_\lambda u'_1 \\ \vdots \\ F_\lambda u'_t \end{bmatrix}} M_1 \oplus \dots \oplus M_t \rightarrow Y \rightarrow X[1],$$

where $u'_i \in \mathcal{D}^b(\text{mod } \mathcal{C})(\tilde{X}, {}^{g_i} \tilde{M}_i)$ for some $g_i \in G$ for every i .

Lemma 4.3. *Let $\Delta: X \rightarrow M \xrightarrow{v} Y \rightarrow X[1]$ be triangle in $\mathcal{D}^b(\text{mod } A)$ such that:*

- (a) *There exists $\tilde{Y} \in \mathcal{D}^b(\text{mod } \mathcal{C})$ satisfying $Y = F_\lambda \tilde{Y}$.*
- (b) *$M = M_1 \oplus \dots \oplus M_t$ where M_1, \dots, M_t are indecomposables such that there exist indecomposable objects $\tilde{M}_1, \dots, \tilde{M}_t$ satisfying $F_\lambda \tilde{M}_i = M_i$ for every i .*
- (c) *$\mathcal{D}^b(\text{mod } A)(M, X[1]) = 0$.*

Then Δ is isomorphic to a triangle in $\mathcal{D}^b(\text{mod } A)$:

$$X \rightarrow M_1 \oplus \dots \oplus M_t \xrightarrow{[F_\lambda v'_1 \quad \dots \quad F_\lambda v'_t]} Y \rightarrow X[1],$$

where $v'_i \in \mathcal{D}^b(\text{mod } \mathcal{C})({}^{g_i} \tilde{M}_i, \tilde{Y})$ for some $g_i \in G$ for every i .

Proof of Lemma 4.2: We say that a morphism $u \in \mathcal{D}^b(\text{mod } A)(X, M_i)$ is homogeneous of degree $g \in G$ if and only if there exists $u' \in \mathcal{D}^b(\text{mod } \mathcal{C})(\tilde{X}, {}^g\tilde{M}_i)$ such that $u = F_\lambda(u')$. Since $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$ has the covering property, any morphism $X \rightarrow M_i$ is (uniquely) the sum of d non zero homogeneous morphisms of pairwise

different degrees (with $d \geq 0$). Let $u = \begin{bmatrix} u_1 \\ \vdots \\ u_t \end{bmatrix}$ with $u_i: X \rightarrow M_i$ for each i . We may assume that $u_1: X \rightarrow M_1$

is not homogeneous. Thus $u_1 = h_1 + \dots + h_d$, where $d \geq 2$ and $h_1, \dots, h_d: X \rightarrow M_1$ are non-zero homogeneous morphisms of pairwise different degree. In order to prove the lemma, it suffices to prove that Δ is isomorphic to

a triangle $X \xrightarrow{u'} M \rightarrow Y \rightarrow X[1]$ with $u' = \begin{bmatrix} u'_1 \\ u_2 \\ \vdots \\ u_t \end{bmatrix}$ such that u'_1 is equal to the sum of at most $d - 1$ non-zero

homogeneous morphisms $X \rightarrow M_1$ of pairwise different degrees. For simplicity we adopt the following notations:

1. $\overline{M} = M_2 \oplus \dots \oplus M_t$ (so $M = M_1 \oplus \overline{M}$),
2. $\overline{u} = \begin{bmatrix} u_2 \\ \vdots \\ u_t \end{bmatrix}: X \rightarrow \overline{M}$ (so $u = \begin{bmatrix} u_1 \\ \overline{u} \end{bmatrix}: X \rightarrow M_1 \oplus \overline{M}$),
3. $\overline{h} = h_2 + \dots + h_d: X \rightarrow M_1$ (so $u_1 = h_1 + \overline{h}$).

Applying the functor $\mathcal{D}^b(\text{mod } A)(-, M_1)$ to Δ gives the exact sequence:

$$\mathcal{D}^b(\text{mod } A)(M_1 \oplus \overline{M}, M_1) \xrightarrow{\text{Hom}(u, -)} \mathcal{D}^b(\text{mod } A)(X, M_1) \rightarrow \mathcal{D}^b(\text{mod } A)(Y, M_1[1]) = 0.$$

So there exists $[\lambda, \mu]: M_1 \oplus \overline{M} \rightarrow M_1$ such that $h_1 = [\lambda, \mu]u$. Hence:

$$h_1 = \lambda u_1 + \mu \overline{u} = \lambda h_1 + \lambda \overline{h} + \mu \overline{u}. \quad (i)$$

We distinguish two cases according to whether $\lambda \in \text{End}_{\mathcal{D}^b(\text{mod } A)}(M_1, M_1)$ is invertible or nilpotent:

- If λ is invertible, then the following is an isomorphism in $\mathcal{D}^b(\text{mod } A)$:

$$\theta := \begin{bmatrix} \lambda & \mu \\ 0 & \text{Id}_{\overline{M}} \end{bmatrix}: M_1 \oplus \overline{M} \rightarrow M_1 \oplus \overline{M}.$$

Using (i) we deduce an isomorphism of triangles:

$$\begin{array}{ccccccc} X & \xrightarrow{\begin{bmatrix} u_1 \\ \overline{u} \end{bmatrix}} & M_1 \oplus \overline{M} & \longrightarrow & Y & \longrightarrow & X[1] & \Delta \\ \parallel & & \downarrow \theta & & \downarrow \sim & & & \\ X & \xrightarrow{\begin{bmatrix} h_1 \\ \overline{u} \end{bmatrix}} & M_1 \oplus \overline{M} & \longrightarrow & Y & \longrightarrow & X[1] & \Delta' \end{array}$$

Since $h_1: X \rightarrow M_1$ is homogeneous, Δ' satisfies the our requirements.

- If λ is nilpotent, let $p \geq 0$ be such that $\lambda^p = 0$. Using (i) we get the following equalities:

$$\begin{aligned} h_1 &= \lambda^2 h_1 + (\lambda^2 + \lambda) \overline{h} + (\lambda + \text{Id}_{M_1}) \mu \overline{u} \\ &\vdots \\ &\vdots \\ h_1 &= \lambda^t h_1 + (\lambda^t + \lambda^{t-1} + \dots + \lambda) \overline{h} + (\lambda^{t-1} + \dots + \lambda + \text{Id}_{M_1}) \mu \overline{u} \\ &\vdots \\ &\vdots \\ h_1 &= \lambda^p h_1 + (\lambda^p + \lambda^{p-1} + \dots + \lambda) \overline{h} + (\lambda^{p-1} + \dots + \lambda + \text{Id}_{M_1}) \mu \overline{u}. \end{aligned}$$

Since $\lambda^p = 0$ and $u_1 = h_1 + \overline{h}$ we infer that:

$$u_1 = \lambda' \overline{h} + \lambda' \mu \overline{u},$$

where λ' is the invertible morphism $\text{Id}_{M_1} + \lambda + \dots + \lambda^{p-1} \in \text{End}_{\mathcal{D}^b(\text{mod } A)}(M_1, M_1)$. So we have an isomorphism in $\mathcal{D}^b(\text{mod } A)$:

$$\theta := \begin{bmatrix} \lambda' & \lambda' \mu \\ 0 & \text{Id}_{\overline{M}} \end{bmatrix}: M_1 \oplus \overline{M} \rightarrow M_1 \oplus \overline{M}.$$

Consequently we have an isomorphism of triangles:

$$\begin{array}{ccccccc}
X & \xrightarrow{\begin{bmatrix} \bar{h} \\ \bar{u} \end{bmatrix}} & M_1 \oplus \bar{M} & \longrightarrow & Y & \longrightarrow & X[1] & \Delta' \\
\parallel & & \downarrow \theta & & \downarrow \sim & & & \\
X & \xrightarrow{\begin{bmatrix} u_1 \\ \bar{u} \end{bmatrix}} & M_1 \oplus \bar{M} & \longrightarrow & Y & \longrightarrow & X[1] & \Delta
\end{array}$$

where $\bar{h} = h_2 + \dots + h_p$ is the sum of $p - 1$ non zero homogeneous morphisms of pairwise different degrees. So Δ' satisfies our requirements. \square

The proof of Lemma 4.3 is the dual of the one of Lemma 4.2 so we omit it. Now, we can prove Proposition 4.1. **Proof of Proposition 4.1:** If (a) holds true, then so does (c) because $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$ is G -invariant. Recall that $\Theta: \mathcal{D}^b(\mathcal{H}) \rightarrow \mathcal{D}^b(\text{mod } A)$ is a triangle equivalence. As in Section 2, we write \mathcal{T} (or \mathcal{T}') for the set of isomorphism classes of objects $T \in \mathcal{D}^b(\mathcal{H})$ (or $T \in \mathcal{D}^b(\text{mod } A)$) such that:

1. T is the direct sum of n pairwise non isomorphic indecomposable objects.
2. $\mathcal{D}^b(\mathcal{H})(T, T[i]) = 0$ (or $\mathcal{D}^b(\text{mod } A)(T, T[i]) = 0$, respectively) for every $i \geq 1$.

Therefore:

- (i) Θ defines a bijection $\Theta: \mathcal{T} \rightarrow \mathcal{T}'$. Under this bijection, tilting complexes in $\mathcal{D}^b(\mathcal{H})$ correspond to tilting complexes in $\mathcal{D}^b(\text{mod } A)$.

We prove that (a) and (b) hold true for any $T \in \mathcal{T}'$ (and therefore for any tilting object in $\mathcal{D}^b(\text{mod } A)$). For this purpose, we use the results of Section 3. First of all, remark that:

- (ii) The assertions (a) and (b) hold true for $T = A$. In this case, $F_\lambda \tilde{X}$ is an indecomposable summand of A if and only if \tilde{X} is an indecomposable projective \mathcal{C} -module.

If $\tilde{X} \in \mathcal{D}^b(\mathcal{H})$, then $F_\lambda(\tilde{X}[1]) = (F_\lambda \tilde{X})[1]$. Therefore:

- (iii) Let $T \in \mathcal{T}$. Let $T' \in \mathcal{T}$ be obtained from T by a transformation of the first kind. Then (a) and (b) hold true for $\Theta(T)$ if and only if they do so for $\Theta(T')$.

Now, assume that $T, T' \in \mathcal{T}'$ are such that $\Theta^{-1}(T')$ is obtained from $\Theta^{-1}(T)$ by a transformation of the second third kind. We prove that (a) and (b) hold for T if and only if they do so for T' . In such a situation there exist $X, Y \in \mathcal{D}^b(\text{mod } A)$ indecomposable and $\bar{T} \in \mathcal{D}^b(\text{mod } A)$ such that $T = X \oplus \bar{T}$ and $T' = Y \oplus \bar{T}$. Also, there exists a triangle in $\mathcal{D}^b(\text{mod } A)$ of one the two following forms:

1. $X \rightarrow M \rightarrow Y \rightarrow X[1]$ where $M \in \text{add}(\bar{T})$.
2. $Y \rightarrow M \rightarrow X \rightarrow Y[1]$ where $M \in \text{add}(\bar{T})$.

Assume that (a) and (b) hold true for T and that there is a triangle $X \rightarrow M \rightarrow Y \rightarrow X[1]$ (the other cases are dealt with using similar arguments). In order to prove that (a) and (b) hold true for T' , we prove that $Y \simeq F_\lambda \tilde{Y}$ for some $\tilde{Y} \in \mathcal{D}^b(\text{mod } \mathcal{C})$. Fix an indecomposable decomposition $M = M_1 \oplus \dots \oplus M_t$. By assumption on T , there exist indecomposable objects $\tilde{X}, \tilde{M}_1, \dots, \tilde{M}_t \in \mathcal{D}^b(\text{mod } \mathcal{C})$ such that $F_\lambda \tilde{X} \simeq X, F_\lambda \tilde{M}_1 \simeq M_1, \dots, F_\lambda \tilde{M}_t \simeq M_t$. Using these isomorphisms, we identify $F_\lambda \tilde{X}$ and $F_\lambda \tilde{M}_i$ to X and M_i , respectively. By Lemma 4.2, there exist $g_1, \dots, g_t \in G$ and morphisms $u_i \in \mathcal{D}^b(\text{mod } \mathcal{C})(\tilde{X}, {}^{g_i} \tilde{M}_i)$ (for $i \in \{1, \dots, t\}$) such that the triangle $X \rightarrow M \rightarrow Y \rightarrow X[1]$ is isomorphic to a triangle of the form:

$$X \xrightarrow{\begin{bmatrix} F_\lambda u_1 \\ \vdots \\ F_\lambda u_t \end{bmatrix}} M \rightarrow Y \rightarrow X[1].$$

Set $u = \begin{bmatrix} u_1 \\ \vdots \\ u_t \end{bmatrix}: \tilde{X} \rightarrow \tilde{M}_1 \oplus \dots \oplus \tilde{M}_t$. We complete u into a triangle $\tilde{X} \xrightarrow{u} \tilde{M}_1 \oplus \dots \oplus \tilde{M}_t \xrightarrow{v} \tilde{X}[1]$ in $\mathcal{D}^b(\text{mod } \mathcal{C})$.

So, we have a triangle $X \xrightarrow{F_\lambda(u)} M \xrightarrow{F_\lambda(v)} F_\lambda \tilde{Y} \rightarrow X[1]$ in $\mathcal{D}^b(\text{mod } A)$. Therefore, $Y \simeq F_\lambda \tilde{Y}$. So, (a) holds for T' and the class $\{\tilde{Z} \mid F_\lambda \tilde{Z} \text{ is an indecomposable direct summand of } T'\}$ coincides with the class $\{{}^g \tilde{Y} \mid g \in G\} \cup \{\tilde{Z} \mid F_\lambda \tilde{Z} \text{ is an indecomposable direct summand of } \bar{T}\}$ (see Lemma 2.3). Because (b) holds true for T and because of the triangle $\tilde{X} \rightarrow \tilde{M}_1 \oplus \dots \oplus \tilde{M}_t \rightarrow \tilde{Y} \rightarrow \tilde{X}[1]$, we deduce that (b) holds for T' . So we have proved the following:

(iv) Let $T, T' \in \mathcal{T}'$ be such that $\Theta^{-1}(T')$ is obtained from $\Theta^{-1}(T)$ by a transformation of the second kind. Then (a) and (b) hold true for T if and only if they do so for T' .

By Proposition 3.6 and (i – iv), the assertions (a), (b) and (c) are satisfied for any $T \in \mathcal{T}$. Finally, if T is a tilting complex, then (d) follows from the fact that $\mathcal{D}^b(\text{mod } A)(T, T[i]) = 0$ for every $i \neq 0$ and from Proposition 2.1. \square

It is interesting to note that the transformations of the second kind in $\mathcal{D}^b(\text{mod } A)$ give rise to transformations of the second kind in $\mathcal{D}^b(\text{mod } \mathcal{C})$. Indeed, let T, T' be in \mathcal{T}' where T' is as in the proof of Proposition 4.1. Assume that $T = M \oplus \overline{T}$ with M indecomposable, $T = M^* \oplus \overline{T}$ with M^* indecomposable and there exists a triangle $\Delta: M \xrightarrow{\sim} B \xrightarrow{\sim} M^* \rightarrow M[1]$ in $\mathcal{D}^b(\text{mod } A)$ where $M \rightarrow B$ (or $B \rightarrow M^*$) is a left minimal $\text{add}(\overline{T})$ -approximation of M (or a right minimal $\text{add}(\overline{T})$ -approximation of M^* , respectively). Then the following holds.

Lemma 4.4. *Keep the above setting. Let $B = \bigoplus_{i=1}^t B_i$ be an indecomposable decomposition (maybe with multiplicities). Then there exists a triangle $\tilde{\Delta}: \tilde{M} \xrightarrow{u} \bigoplus_{i=1}^t {}^g \tilde{B}_i \xrightarrow{v} {}^{g_0} \tilde{M}^* \rightarrow M[1]$ in $\mathcal{D}^b(\text{mod } \mathcal{C})$ whose image under F_λ is isomorphic to Δ . Moreover, if \mathcal{X} (or \mathcal{X}') denoted the additive full subcategory of $\mathcal{D}^b(\text{mod } \mathcal{C})$ generated by the indecomposables $X \in \mathcal{D}^b(\text{mod } \mathcal{C})$ not isomorphic to \tilde{M} (or to \tilde{M}^*) and such that $F_\lambda X$ is an indecomposable summand of T (or of T' , respectively), then:*

(a) u is a left minimal \mathcal{X} -approximation.

(b) v is a right minimal \mathcal{X}' -approximation.

Proof: The existence of $\tilde{\Delta}$ follows from the proof of Proposition 4.1. So $F_\lambda(u)$ is a left minimal $\text{add}(\overline{T})$ -approximation. This and the exactness of F_λ imply that u is left minimal. Let $f: \tilde{M} \rightarrow {}^g \tilde{Y}$ be a non-zero morphism where ${}^g \tilde{Y} \in \mathcal{X}$. The linear map $\bigoplus_{h \in G} \mathcal{D}^b(\text{mod } \mathcal{C})(\tilde{M}, {}^g \tilde{M}) \rightarrow \text{End}_{\mathcal{D}^b(\text{mod } A)}(M, M)$ induced by F_λ is bijective. Also $\dim_k \text{End}_{\mathcal{D}^b(\text{mod } A)}(M, M) = 1$ because M is an indecomposable and $\mathcal{D}^b(\text{mod } A)(M, M[i]) = 0$ for every $i > 0$. So ${}^g \tilde{Y} \not\cong {}^h \tilde{M}$ for every $h \neq 1$. Therefore, $Y \in \text{add}(\overline{T})$ and $F_\lambda(f)$ factorises through $F_\lambda(u)$:

$$\begin{array}{ccc} M & \xrightarrow{F_\lambda(u)} & \bigoplus_{i=1}^t B_i \\ & \searrow F_\lambda(f) & \downarrow f' \\ & & Y \end{array} .$$

There exists $(f'_h)_h \in \bigoplus_{h \in G} \mathcal{D}^b(\text{mod } \mathcal{C})(\bigoplus_{i=1}^t {}^g \tilde{B}_i, {}^h \tilde{Y})$ such that $f' = \sum_{h \in G} F_\lambda(f'_h)$ because of the covering property of F_λ . So $F_\lambda(f) = \sum_{h \in G} F_\lambda(f'_h u)$ and $f = f'_g u$ for the same reason. Thus u is a left minimal \mathcal{X} -approximation. Similarly, v is a right minimal \mathcal{X}' -approximation. \square

Since tilting A -modules are particular cases of tilting complexes, we get the following result.

Corollary 4.5. *Let A be a connected piecewise hereditary algebra. Let $F: \mathcal{C} \rightarrow A$ be a Galois covering with group G where \mathcal{C} is locally bounded. Let T be a tilting A -module and $X \in \text{mod } A$ be an indecomposable summand of T . Then there exists $\tilde{X} \in \text{mod } \mathcal{C}$ such that $F_\lambda \tilde{X} \simeq X$.*

Proof: By Proposition 4.1, such an \tilde{X} exists in $\mathcal{D}^b(\text{mod } \mathcal{C})$. We prove that \tilde{X} is isomorphic to a \mathcal{C} -module. Let $P \in \text{mod } \mathcal{C}$ be projective, and let $i \in \mathbb{Z} \setminus \{0\}$. Then $F_\lambda P \in \text{mod } A$ is projective and $\mathcal{D}^b(\text{mod } A)(F_\lambda P, X[i]) = 0$ because X is an A -module. On the other hand, the spaces $\mathcal{D}^b(\text{mod } A)(F_\lambda P, X[i])$ and $\bigoplus_{g \in G} \mathcal{D}^b(\text{mod } \mathcal{C})({}^g P, \tilde{X}[i])$ are isomorphic. So $\mathcal{D}^b(\text{mod } \mathcal{C})(P, \tilde{X}[i]) = 0$ for every $i \neq 0$. Thus, $\tilde{X} \simeq H^0(\tilde{X}) \in \text{mod } \mathcal{C}$. \square

Proof of assertion (H₂)

Proposition 4.6. *Let A be a piecewise hereditary algebra. Let $F: \mathcal{C} \rightarrow A$ be a Galois covering with group G , where \mathcal{C} is locally bounded. Let $X \in \text{ind } A$ be a direct summand of a tilting complex in $\mathcal{D}^b(\text{mod } A)$. Assume that $F_\lambda \tilde{X} \simeq X$ for some $\tilde{X} \in \mathcal{D}^b(\text{mod } \mathcal{C})$. Then ${}^g \tilde{X} \not\cong \tilde{X}$ for every $g \in G \setminus \{1\}$.*

Proof: We have $\dim_k \text{End}_{\mathcal{D}^b(\text{mod } A)}(X, X) = 1$ because X is indecomposable and $\mathcal{D}^b(\text{mod } A)(X, X[i]) = 0$ for every $i \neq 0$. On the other hand, the spaces $\bigoplus_{g \in G} \mathcal{D}^b(\text{mod } \mathcal{C})({}^g \tilde{X}, \tilde{X})$ and $\text{End}_{\mathcal{D}^b(\text{mod } A)}(X, X)$ are isomorphic. So $\mathcal{D}^b(\text{mod } \mathcal{C})({}^g \tilde{X}, \tilde{X}) = 0$ and, therefore, ${}^g \tilde{X} \not\cong \tilde{X}$ if $g \neq 1$. \square

Proof of assertion (H_3)

If $\psi: A \rightarrow A$ is an automorphism (and therefore a Galois covering with trivial group), then $\psi_\lambda: \text{mod } A \rightarrow \text{mod } A$ is an equivalence. It thus induces a triangle equivalence $\psi_\lambda: \mathcal{D}^b(\text{mod } A) \rightarrow \mathcal{D}^b(\text{mod } A)$ (see Proposition 2.1).

Proposition 4.7. *Let A be a connected and piecewise hereditary algebra and $\psi: A \xrightarrow{\sim} A$ be an automorphism such that $\psi(x) = x$, for every $x \in \text{ob}(A)$. Let $T \in \mathcal{D}^b(\text{mod } A)$ be a tilting complex. Then $\psi_\lambda X \simeq X$ in $\mathcal{D}^b(\text{mod } A)$ for every indecomposable summand X of T .*

Proof: Since $\psi(x) = x$ for every $x \in \text{ob}(A)$, we have:

(i) The conclusion of the proposition holds true if X is an indecomposable projective A -module.

Recall that $\Theta: \mathcal{D}^b(\mathcal{H}) \rightarrow \mathcal{D}^b(\text{mod } A)$ is a triangle equivalence. We keep the notations \mathcal{T} and \mathcal{T}' introduced in the proof of Proposition 4.1. We prove the proposition for any $T \in \mathcal{T}'$. By construction of Θ , we have:

(ii) Θ induces a bijection $\Theta: \mathcal{T} \rightarrow \mathcal{T}'$. Under this bijection, tilting complexes in $\mathcal{D}^b(\mathcal{H})$ correspond to tilting complexes in $\mathcal{D}^b(\text{mod } A)$.

Since $\psi_\lambda: \mathcal{D}^b(\text{mod } A) \rightarrow \mathcal{D}^b(\text{mod } A)$ is an equivalence, we also have:

(iii) Let $T, T' \in \mathcal{T}$ be such that T' is obtained from T by a transformation of the first kind. Then the proposition holds true for T if and only if it does for T' .

Now, assume that $T, T' \in \mathcal{T}'$ are such that $\Theta^{-1}(T')$ is obtained from $\Theta^{-1}(T)$ by a transformation of the second kind. We prove that the proposition holds true for T if and only if it does for T' . There exist $X, Y \in \mathcal{D}^b(\text{mod } A)$ indecomposable and $\overline{T} \in \mathcal{D}^b(\text{mod } A)$ such that $T = X \oplus \overline{T}$ and $T' = Y \oplus \overline{T}$. Also, there exists a triangle in $\mathcal{D}^b(\text{mod } A)$ of one the two following forms:

1. $X \rightarrow M \rightarrow Y \rightarrow X[1]$ where $M \in \text{add}(\overline{T})$, where $X \rightarrow M$ is a left minimal $\text{add}(\overline{T})$ -approximation and $M \rightarrow Y$ is a right minimal $\text{add}(\overline{T})$ -approximation.
2. $Y \rightarrow M \rightarrow X \rightarrow Y[1]$ where $M \in \text{add}(\overline{T})$, where $Y \rightarrow M$ is a left minimal $\text{add}(\overline{T})$ -approximation and $M \rightarrow X$ is a right minimal $\text{add}(\overline{T})$ -approximation.

Assume that the proposition holds for T and that there is a triangle $X \rightarrow M \rightarrow Y \rightarrow X[1]$ (the other cases are dealt with using similar arguments). We only need to prove that $\psi_\lambda Y \simeq Y$. Apply ψ_λ to the triangle $X \rightarrow M \rightarrow Y \rightarrow X[1]$. Since ψ_λ is an equivalence and the proposition holds true for T , there exists a triangle $X \rightarrow M \rightarrow \psi_\lambda Y \rightarrow X[1]$ in $\mathcal{D}^b(\text{mod } A)$ where $X \rightarrow M$ is a left minimal $\text{add}(\overline{T})$ -approximation. Therefore, $\psi_\lambda Y \simeq Y$ in $\mathcal{D}^b(\text{mod } A)$. So we proved that:

(iv) If $T, T' \in \mathcal{T}'$ are such that $\Theta^{-1}(T')$ is obtained from $\Theta^{-1}(T)$ by a transformation of the first or of the second kind, then the proposition holds true for T if and only if it does for T' .

As in the proof of Proposition 4.1, the conclusion follows from (i), (ii), (iii), (iv) and Proposition 3.6. \square

Application

We end this section with an application of the preceding results of the section. We need the following lemma. If $T = T_1 \oplus \dots \oplus T_n \in \text{mod } A$ is an indecomposable decomposition of a multiplicity-free module T , then $\text{End}_A(T)$ is naturally a k -category, equal to the full subcategory of $\text{mod } A$ with objects T_1, \dots, T_n .

Lemma 4.8. *Let $F: \mathcal{C} \rightarrow A$ be a connected Galois covering with group G . Let $T \in \mathcal{D}^b(\text{mod } A)$ be a tilting complex, $B = \text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$ and $T = T_1 \oplus \dots \oplus T_n$ be an indecomposable decomposition. Let $\lambda_i: F_\lambda \tilde{T}_i \rightarrow T_i$ be an isomorphism where $\tilde{T}_i \in \mathcal{D}^b(\text{mod } \mathcal{C})$ is indecomposable for every i . This defines the bounded complex of (not necessarily finite dimensional) \mathcal{C} -modules $\tilde{T} := \bigoplus_{i,g} {}^g \tilde{T}_i$, where the sum runs over $g \in G$ and $i \in \{1, \dots, n\}$.*

Let \mathcal{C}' for the full subcategory of $\mathcal{D}^b(\text{mod } \mathcal{C})$ with objects the complexes ${}^g \tilde{T}_i$ (for $g \in G, i \in \{1, \dots, n\}$). Then the triangle functor $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$ induces a connected Galois covering with group G :

$$F_{\tilde{T}, \lambda}: \begin{array}{ccc} \mathcal{C}' & \rightarrow & B \\ {}^g \tilde{T}_i & \mapsto & T_i \\ {}^g \tilde{T}_i \xrightarrow{u} {}^h \tilde{T}_j & \mapsto & T_i \xrightarrow{\lambda_j^{-1} F_\lambda(u) \lambda_i} T_j \end{array} .$$

The complex \tilde{T} is naturally a bounded complex of $\mathcal{C}' - \mathcal{C}$ -bimodules: As a functor from $\text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C})}(\tilde{T}) \times \mathcal{C}^{op}$, it assigns the vector space ${}^g \tilde{T}_i(x)$ to the pair of objects $({}^g \tilde{T}_i, x)$. The total derived functor:

$$- \otimes_{\mathcal{C}'} \tilde{T}: \mathcal{D}^b(\text{mod } \mathcal{C}') \rightarrow \mathcal{D}^b(\text{mod } \mathcal{C})$$

is a G -equivariant triangle equivalence. Finally, if T is a tilting A -module and all the objects \tilde{T}_i are \mathcal{C} -modules (see Corollary 4.5), then:

- (a) $\text{Ext}_{\mathcal{C}}^1({}^g\tilde{T}_i, {}^h\tilde{T}_j) = 0$ for every $i, j \in \{1, \dots, n\}$ and $g, h \in G$.
- (b) $\text{pd}_{\mathcal{C}}({}^g\tilde{T}_i) \leq 1$ for every i, g .
- (c) If $P \in \text{mod } \mathcal{C}$ is an indecomposable projective \mathcal{C} -module, then there exists an exact sequence $0 \rightarrow P \rightarrow T^{(1)} \rightarrow T^{(2)} \rightarrow 0$ in $\text{mod } \mathcal{C}$ where $T^{(1)}, T^{(2)} \in \text{add}(\{{}^g\tilde{T}_i \mid i \in \{1, \dots, n\}, g \in G\})$.

Proof: Thanks to Proposition 2.1, the functor $F_{\tilde{T}, \lambda}$ is a well-defined Galois covering. Proposition 4.6 implies that \mathcal{C}' is a locally bounded k -category (see [21, 2.1], for more details on the construction of $F_{\tilde{T}, \lambda}$). We prove that \mathcal{C}' is connected. By definition of \tilde{T} , we have ${}^g\tilde{T} = \tilde{T}$ for every $g \in G$. Hence, the functor $-\overset{\mathbb{L}}{\otimes}_{\mathcal{C}'} \tilde{T}$ is G -equivariant. On the other hand, $-\overset{\mathbb{L}}{\otimes}_{\mathcal{C}'} \tilde{T}$ is a triangle equivalence. Indeed, by Proposition 4.1, (d), and classical arguments on derived equivalences (see [14, III.2], for instance), this functor is full and faithful. Moreover its image contains the complexes ${}^g\tilde{T}_i$ (for $g \in G$ and $i \in \{1, \dots, n\}$). So Proposition 4.1, (b), implies that this functor is dense and, therefore, a triangle equivalence $\mathcal{D}^b(\text{mod } \mathcal{C}') \rightarrow \mathcal{D}^b(\text{mod } \mathcal{C})$. In particular, \mathcal{C}' is connected.

Now we assume that T is a tilting A -module. Assertion (a) follows from Proposition 4.1, (d). Assertion (b) follows from the fact that $\text{pd}_A(T) \leq 1$ and $F_\lambda: \text{mod } \mathcal{C} \rightarrow \text{mod } A$ is exact. We prove assertion (c). Let $P \in \text{mod } \mathcal{C}$ be indecomposable projective. Since $F_\lambda P$ is projective, there exists an exact sequence $0 \rightarrow F_\lambda P \rightarrow X \rightarrow Y \rightarrow 0$ in $\text{mod } A$ with $X, Y \in \text{add}(T)$. Lemma 4.2 implies that the triangle $F_\lambda P \rightarrow X \rightarrow Y \rightarrow F_\lambda P[1]$ is isomorphic to the image under F_λ of a triangle $P \rightarrow X' \rightarrow Y' \rightarrow P[1]$ where $X', Y' \in \text{add}(\{{}^g\tilde{T}_i \mid g \in G, i \in \{1, \dots, n\}\})$. Since F_λ is exact, the sequence $0 \rightarrow P \rightarrow X' \rightarrow Y' \rightarrow 0$ is an exact sequence in $\text{mod } \mathcal{C}$. \square

Remark 4.9. Keep the hypotheses and notations of the preceding lemma. If G is finite and if T is a tilting A -module, then the lemma expresses that $\bigoplus_{g,i} {}^g\tilde{T}_i$ is a tilting \mathcal{C} -module.

Now, we can prove Theorem 3 which was stated in the introduction.

Proof of Theorem 3: Thanks to [14, Cor. 5.5], there exists a sequence of algebras:

$$A_0 = kQ, A_1 = \text{End}_{A_0}(T^{(0)}), \dots, A_l = \text{End}_{A_{l-1}}(T^{(l-1)}) = A,$$

such that $T^{(i)} \in \text{mod } A_{l-1}$ is tilting for every i . We prove the theorem by induction on l . If $l = 0$, then $A = kQ$. For any connected Galois covering $\mathcal{C} \rightarrow A$ with group G there exists a Galois covering of quivers $Q' \rightarrow Q$ with group G such that $\mathcal{C} \simeq kQ'$ (see [20, Prop. 4.4]). Assume that $l > 0$ and the conclusion of the theorem holds true for A_{l-1} . Let $\mathcal{C} \rightarrow A$ be a connected Galois covering with group G . Note that $T^{(l-1)}$ is a tilting A^{op} -module. So, the preceding lemma yields a connected Galois covering $\mathcal{C}' \rightarrow \text{End}_{A^{op}}(T^{(l-1)})$ with group G such that $\mathcal{D}^b(\text{mod } \mathcal{C}^{op})$ and $\mathcal{D}^b(\text{mod } \mathcal{C}')$ are triangle equivalent. On the other hand, $A_{l-1} \simeq \text{End}_{A^{op}}(T^{(l-1)})^{op}$. Therefore, the induction hypothesis implies that $\mathcal{D}^b(\text{mod } \mathcal{C}'^{op})$ is triangle equivalent to $\mathcal{D}^b(\text{mod } kQ')$ where Q' is a quiver such that there exists a Galois covering of quivers $Q' \rightarrow Q$ with group G . \square

Remark 4.10. Let A be a finite dimensional algebra endowed with a (non necessarily free) G -action. Then:

1. If the G -action on A is free, then the quotient algebra A/G is well-defined. The proof of Theorem 3 shows that if A/G is tilted (or, more generally, piecewise hereditary), then so is A .
2. It is proved in [10, Thm. 3] that if the order of G is invertible in k and if A is piecewise hereditary, then so is the skew-group algebra $A[G]$. Recall that if G acts freely on A , then the algebras $A[G]$ and A/G are Morita equivalent (see [9, Thm.2.8]).

5 Correspondence between Galois coverings

Let $T \in \mathcal{D}^b(\text{mod } A)$ be a tilting complex and $B = \text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$. In this section, we construct a correspondence between the Galois coverings of A and those of B . This work has been done in [21] in the particular case where T is a tilting A -module. In order to compare the Galois coverings of A and those of B , it is convenient to use the notion of equivalent Galois covering. Given two Galois coverings $F: \mathcal{C} \rightarrow A$ and $F': \mathcal{C}' \rightarrow A$, we say that F and F' are *equivalent* if there exists a commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\sim} & \mathcal{C}' \\ F \downarrow & & \downarrow F' \\ A & \xrightarrow[\varphi]{\sim} & A \end{array}$$

where the horizontal arrows are isomorphisms and $\varphi: A \rightarrow A$ is an automorphism such that $\varphi(x) = x$ for every $x \in \text{ob}(A)$.

Equivalence classes of Galois coverings of A associated to equivalence classes of Galois coverings of B

In Lemma 4.8, we have associated a Galois covering $F_{\tilde{T},\lambda}$ of B to any Galois covering of A and to any data consisting of isomorphisms $(\lambda_i: F_\lambda \tilde{T}_i \rightarrow T_i)_{i=1,\dots,n}$ in $\mathcal{D}^b(\text{mod } \mathcal{C})$. The following lemma shows that different choices for these data give rise to equivalent Galois coverings.

Lemma 5.1. [21, § 2] *Let $F: \mathcal{C} \rightarrow A$ be a connected Galois covering with group G . Let $T \in \mathcal{D}^b(\text{mod } A)$ be a tilting complex, and $T = T_1 \oplus \dots \oplus T_n$ be an indecomposable decomposition.*

- (a) *Let $(\lambda_i: F_\lambda \tilde{T}_i \rightarrow T_i)_{i=1,\dots,n}$ and $(\mu_i: F_\lambda \hat{T}_i \rightarrow T_i)_{i=1,\dots,n}$ be isomorphisms in $\mathcal{D}^b(\text{mod } A)$ defining the Galois coverings $F_{\tilde{T},\lambda}: \text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C})}(\tilde{T}) \rightarrow \text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$ and $F_{\hat{T},\mu}: \text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C})}(\hat{T}) \rightarrow \text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$, respectively. Then $F_{\tilde{T},\lambda}$ and $F_{\hat{T},\mu}$ are equivalent. We write $[F]_T$ for the corresponding equivalence class of Galois coverings of $\text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$.*
- (b) *Let $F': \mathcal{C}' \rightarrow A$ be a connected Galois covering with group G and equivalent to F . Then the equivalence classes $[F]_T$ and $[F']_T$ coincide.*

Proof: In the case of tilting modules, (a) and (b) were proved in [21, Lem. 2.4] and [21, Lem. 2.5], respectively. Using Lemma 2.3 and Proposition 4.7, it is easily checked that the same arguments apply to prove (a) and (b) in the present situation. \square

In the sequel, we keep the notation $[F]_T$ introduced in the preceding lemma.

Galois coverings of A induced by Galois coverings of B

We now express any Galois covering of A as induced by a Galois covering of B as in Lemma 4.8. The tilting complex T is naturally a complex of $B - A$ -bimodules. Also, it defines a triangle equivalence:

$$-\overset{\mathbb{L}}{\otimes}_B X: \mathcal{D}^b(\text{mod } B) \rightarrow \mathcal{D}^b(\text{mod } A).$$

Fix a connected Galois covering $F: \mathcal{C} \rightarrow A$ with group G , an indecomposable decomposition $T = T_1 \oplus \dots \oplus T_n$ and isomorphisms $(\mu_i: F_\lambda \tilde{T}_i \xrightarrow{\sim} T_i)_{i=1,\dots,n}$. According to Lemma 4.8, these data define the Galois covering $F_{\tilde{T},\mu}: \text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C})}(\tilde{T}) \rightarrow B$, which we denote by $F': \mathcal{C}' \rightarrow B$ for simplicity.

Lemma 5.2. *The following diagram commutes up to an isomorphism of functors.*

$$\begin{array}{ccc} \mathcal{D}^b(\text{mod } \mathcal{C}') & \xrightarrow{-\overset{\mathbb{L}}{\otimes}_{\mathcal{C}'} \tilde{T}} & \mathcal{D}^b(\text{mod } \mathcal{C}) \\ F'_\lambda \downarrow & & \downarrow F_\lambda \\ \mathcal{D}^b(\text{mod } B) & \xrightarrow{-\overset{\mathbb{L}}{\otimes}_B T} & \mathcal{D}^b(\text{mod } A) \end{array},$$

Proof: Recall that $F_\lambda: \text{mod } \mathcal{C} \rightarrow \text{mod } A$ (or $F'_\lambda: \text{mod } \mathcal{C}' \rightarrow \text{mod } B$) is equal to $-\overset{\mathbb{L}}{\otimes}_{\mathcal{C}} A$ (or to $-\overset{\mathbb{L}}{\otimes}_{\mathcal{C}'} B$, respectively). Since these two functors are exact and map projective modules to projective modules and the horizontal arrows of the diagram are triangle equivalences (see Lemma 5.1), we deduce that:

1. The composition $\mathcal{D}^b(\text{mod } \mathcal{C}') \xrightarrow{F'_\lambda} \mathcal{D}^b(B) \xrightarrow{-\overset{\mathbb{L}}{\otimes}_B T} \mathcal{D}^b(\text{mod } A)$ is isomorphic to $-\overset{\mathbb{L}}{\otimes}_{\mathcal{C}'} \left(B \overset{\mathbb{L}}{\otimes}_B T \right)$.
2. The composition $\mathcal{D}^b(\text{mod } \mathcal{C}') \xrightarrow{-\overset{\mathbb{L}}{\otimes}_{\mathcal{C}'} \tilde{T}} \mathcal{D}^b(\text{mod } \mathcal{C}) \xrightarrow{F_\lambda} \mathcal{D}^b(\text{mod } A)$ is isomorphic to $-\overset{\mathbb{L}}{\otimes}_{\mathcal{C}'} \left(\tilde{T} \overset{\mathbb{L}}{\otimes}_{\mathcal{C}} A \right)$.

On the other hand, the isomorphisms $\mu_i: F_\lambda \tilde{T}_i \xrightarrow{\sim} T_i$ (for $i \in \{1, \dots, n\}$) define an isomorphism $B \overset{\mathbb{L}}{\otimes}_B T \xrightarrow{\sim} \tilde{T} \overset{\mathbb{L}}{\otimes}_{\mathcal{C}} A$ of $\mathcal{C}' - A$ -bimodules. This proves that the diagram commutes up to an isomorphism of functors. \square

Since $-\overset{\mathbb{L}}{\otimes}_B T$ is an equivalence, there exists an isomorphism $\varphi_x: X_x \overset{\mathbb{L}}{\otimes}_B T \rightarrow A(-, x)$ in $\mathcal{D}^b(\text{mod } A)$ with $X_x \in \mathcal{D}^b(\text{mod } B)$ for every $x \in \text{ob}(A)$. In particular, $\bigoplus_{x \in \text{ob}(A)} X_x$ is an indecomposable decomposition of tilting complex in $\mathcal{D}^b(\text{mod } B)$. Then by the preceding section, there exists an isomorphism $\nu_x: F'_\lambda \tilde{X}_x \xrightarrow{\sim} X_x$ in $\mathcal{D}^b(\text{mod } B)$

with $\tilde{X}_x \in \mathcal{D}^b(\text{mod } \mathcal{C}')$ for every $x \in \text{ob}(A)$. By Lemma 4.8, the datum $(\nu_x)_{x \in \text{ob}(A)}$ defines a connected Galois covering with group G :

$$\begin{aligned} F'_{\tilde{X}, \nu}: \quad \text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C}')}(\tilde{X}) &\rightarrow \text{End}_{\mathcal{D}^b(\text{mod } B)}(X) \\ g \tilde{X}_x &\mapsto X_x \\ g \tilde{X}_x \xrightarrow{u} h \tilde{X}_y &\mapsto X_x \xrightarrow{\nu_y F'_\lambda(u) \nu_x^{-1}} X_y \quad . \end{aligned}$$

On the other hand, the isomorphisms φ_x (for $x \in \text{ob}(A)$) define the following isomorphism of k -categories:

$$\begin{aligned} \rho_{X, \varphi}: \quad \text{End}_{\mathcal{D}^b(\text{mod } B)}(X) &\rightarrow A \\ X_x &\mapsto x \\ X_x \xrightarrow{u} X_y &\mapsto (\varphi_y \circ (u \otimes T) \circ \varphi_x^{-1}) (\text{Id}_x) \in A(x, y) \quad . \end{aligned}$$

Thus, we have a connected Galois covering $\rho_{X, \varphi} \circ F'_{\tilde{X}, \nu}: \text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C}')}(\tilde{X}) \rightarrow A$ with group G which we denote by F'' . The following lemma relates F and F'' .

Lemma 5.3. *The Galois coverings F and F'' are equivalent.*

Proof: We need to construct a commutative diagram:

$$\begin{array}{ccc} \text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C}')}(\tilde{X}) & \xrightarrow{\sim} & \mathcal{C} \\ F'' \downarrow & & \downarrow F \\ A & \xrightarrow{\sim} & A \end{array} \quad (D)$$

where the horizontal arrows are isomorphisms and the bottom horizontal isomorphism extends the identity map on objects. For this purpose, we proceed in two steps.

Step 1: We express F as a functor between subcategories of $\mathcal{D}^b(\text{mod } \mathcal{C})$ and $\mathcal{D}^b(\text{mod } A)$. Given $x \in \text{ob}(\mathcal{C})$, the A -module $F_\lambda(\mathcal{C}(-, x))$ does depend only on $F(x)$ (and not on x) because F_λ is G -invariant. Besides, there is a canonical isomorphism $\iota_x: F_\lambda(\mathcal{C}(-, x)) \xrightarrow{\sim} A(-, F(x))$ of A -modules induced by F : If $y \in \text{ob}(A)$, then $(F_\lambda(\mathcal{C}(-, x)))(y) = \bigoplus_{y' \in F^{-1}(F(y))} \mathcal{C}(y', x)$ and an element $(u_{y'})_{y'}$ of this vector space is mapped by ι_x to $\sum_{y'} F(u_{y'}) \in A(F(y), F(x))$.

Clearly, this isomorphism does depend only on $F(x)$ (and not on x) whence the notation ι_x . Now, let \mathcal{P}_A and $\mathcal{P}_\mathcal{C}$ be the full subcategories of $\mathcal{D}^b(\text{mod } A)$ and $\mathcal{D}^b(\text{mod } \mathcal{C})$ with objects sets $\{A(-, x) \mid x \in \text{ob}(A)\}$ and $\{\mathcal{C}(-, x) \mid x \in \text{ob}(\mathcal{C})\}$, respectively. Hence, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\sim} & \mathcal{P}_\mathcal{C} \\ F \downarrow & & \downarrow \\ A & \xrightarrow{\sim} & \mathcal{P}_A \end{array} \quad (D_1)$$

where the unlabelled functors are as follows:

1. The functor $\mathcal{C} \rightarrow \mathcal{P}_\mathcal{C}$ is the following isomorphism:

$$\begin{aligned} \mathcal{C} &\rightarrow \mathcal{P}_\mathcal{C} \\ x \in \text{ob}(\mathcal{C}) &\mapsto \mathcal{C}(-, x) \\ u \in \mathcal{C}(x, y) &\mapsto \mathcal{C}(-, u): \mathcal{C}(-, x) \rightarrow \mathcal{C}(-, y) \quad . \end{aligned}$$

2. The functor $A \rightarrow \mathcal{P}_A$ is the following isomorphism:

$$\begin{aligned} A &\rightarrow \mathcal{P}_A \\ x \in \text{ob}(A) &\mapsto A(-, x) \\ u \in A(x, y) &\mapsto A(-, u): A(-, x) \rightarrow A(-, y) \quad . \end{aligned}$$

3. The functor $\mathcal{P}_\mathcal{C} \rightarrow \mathcal{P}_A$ is as follows:

$$\begin{aligned} \mathcal{P}_\mathcal{C} &\rightarrow \mathcal{P}_A \\ \mathcal{C}(-, x) &\mapsto A(-, F(x)) \\ \mathcal{C}(-, x) \xrightarrow{u} \mathcal{C}(-, y) &\mapsto A(-, F(x)) \xrightarrow{\iota_y \circ F_\lambda(u) \circ \iota_x^{-1}} A(-, F(y)) \quad . \end{aligned} \quad (i)$$

In particular, $\mathcal{P}_C \rightarrow \mathcal{P}_A$ is a Galois covering with group G .

Step 2: We now relate F'' to the Galois covering $\mathcal{P}_C \rightarrow \mathcal{P}_A$. We first construct an isomorphism $\text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C}')}(\tilde{X}) \xrightarrow{\sim} \mathcal{P}_C$. Let $\Theta: F_\lambda(-) \overset{\mathbb{L}}{\otimes} \tilde{T} \xrightarrow{\sim} F_\lambda(- \overset{\mathbb{L}}{\otimes} \tilde{T})$ be an isomorphism of functors (see Lemma 5.2). Let $x \in \text{ob}(A)$. So, we have a composition of isomorphisms in $\mathcal{D}^b(\text{mod } A)$:

$$F_\lambda \left(\tilde{X}_x \overset{\mathbb{L}}{\otimes} \tilde{T} \right) \xrightarrow{\Theta_{\tilde{X}_x}^{-1}} F'_\lambda \tilde{X}_x \overset{\mathbb{L}}{\otimes} T \xrightarrow{\nu_x \otimes T} X_x \overset{\mathbb{L}}{\otimes} T \xrightarrow{\varphi_x} A(-, x).$$

By Lemma 2.3, there exists an isomorphism $\psi_x: \tilde{X}_x \overset{\mathbb{L}}{\otimes} \tilde{T} \xrightarrow{\sim} \mathcal{C}(-, L(x))$ in $\mathcal{D}^b(\text{mod } \mathcal{C})$ with $L(x) \in F^{-1}(x)$. We deduce that the following is an isomorphism of k -categories because $- \overset{\mathbb{L}}{\otimes} \tilde{T}$ is a G -equivariant functor (see Lemma 4.8):

$$\begin{aligned} \text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C}')}(\tilde{X}) &\rightarrow \mathcal{P}_C \\ g\tilde{X}_x &\mapsto \mathcal{C}(-, gL(x)) \\ g\tilde{X}_x \xrightarrow{u} h\tilde{X}_y &\mapsto \mathcal{C}(-, gL(x)) \xrightarrow{h\psi_y \circ (u \otimes \tilde{T}) \circ (g\psi_x)^{-1}} \mathcal{C}(-, hL(y)). \end{aligned} \quad (ii)$$

We now construct another isomorphism between A and \mathcal{P}_A . We have the following composition of isomorphisms in $\mathcal{D}^b(\text{mod } A)$, which we denote by α_x :

$$\alpha_x: A(-, x) \xrightarrow{\varphi_x^{-1}} X_x \overset{\mathbb{L}}{\otimes} T \xrightarrow{(\nu_x \otimes T)^{-1}} F'_\lambda \tilde{X}_x \overset{\mathbb{L}}{\otimes} T \xrightarrow{\Theta_{\tilde{X}_x}} F_\lambda \left(\tilde{X}_x \overset{\mathbb{L}}{\otimes} \tilde{T} \right) \xrightarrow{F_\lambda(\psi_x)} F_\lambda(\mathcal{C}(-, L(x))) \xrightarrow{\iota_x} A(-, x).$$

Note that $\alpha_x: A(-, x) \xrightarrow{\sim} A(-, x)$ is necessarily equal to the multiplication by a scalar in k^* because $A(-, x)$ is an indecomposable projective A -module and A is piecewise hereditary. Therefore, we have an isomorphism of categories:

$$\begin{aligned} A &\rightarrow \mathcal{P}_A \\ x &\mapsto A(-, x) \\ u \in A(x, y) &\mapsto \alpha_y \circ A(-, u) \circ \alpha_x^{-1}. \end{aligned} \quad (iii)$$

Hence, the horizontal arrows of the following diagram are isomorphisms:

$$\begin{array}{ccc} \text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C}')}(\tilde{X}) & \xrightarrow{(ii)} & \mathcal{P}_C \\ F'' \downarrow & & \downarrow (i) \\ A & \xrightarrow{(iii)} & \mathcal{P}_A \end{array} \quad (D_2)$$

We claim that this diagram commutes. The commutativity is clearly satisfied on objects. Let $u: g\tilde{X}_x \rightarrow h\tilde{X}_y$ be a morphism in $\text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C}')}(\tilde{X})$. Denote by $u_1: A(-, x) \rightarrow A(-, y)$ the image of u under the composition of (i) and (ii). Then:

$$\begin{aligned} u_1 &= \iota_y \circ F_\lambda \left(h\psi_y \circ (u \otimes \tilde{T}) \circ (g\psi_x)^{-1} \right) \circ \iota_x^{-1} \\ &= \iota_y \circ F_\lambda(\psi_y) \circ F_\lambda \left(u \otimes \tilde{T} \right) \circ (F_\lambda(\psi_x))^{-1} \circ \iota_x^{-1} && \text{because } F_\lambda \text{ is } G\text{-invariant,} \\ &= \iota_y \circ F_\lambda(\psi_y) \circ \Theta_{\tilde{X}_y} \circ (F'_\lambda(u) \otimes T) \circ \Theta_{\tilde{X}_x}^{-1} \circ (F_\lambda(\psi_x))^{-1} \circ \iota_x^{-1} && \text{by definition of } \Theta, \\ &= \alpha_y \circ \varphi_y \circ (\nu_y \otimes T) \circ (F'_\lambda(u) \otimes T) \circ (\nu_x \otimes T)^{-1} \circ \varphi_x^{-1} \circ \alpha_x^{-1} && \text{by definition of } \alpha_x \text{ and } \alpha_y, \\ &= \alpha_y \circ \varphi_y \circ ((\nu_y \circ F'_\lambda(u) \circ \nu_x^{-1}) \otimes T) \circ \varphi_x^{-1} \circ \alpha_x^{-1}. \end{aligned}$$

Now, let $u_2 \in A(x, y)$ be the image of u under F'' , that is $u_2 = (\varphi_y \circ ((\nu_y \circ F'_\lambda(u) \circ \nu_x^{-1}) \otimes T) \circ \varphi_x^{-1}) (\text{Id}_x)$. Therefore, $A(-, u_2)$ is equal to the morphism $\varphi_y \circ ((\nu_y \circ F'_\lambda(u) \circ \nu_x^{-1}) \otimes T) \circ \varphi_x^{-1}: A(-, x) \rightarrow A(-, y)$ of \mathcal{P}_A . In particular, the image of u_2 under (iii) coincides with u_1 . Therefore, (D_2) is commutative. Since (D_1) also commutes, we deduce that so does (D) . Thus, F and F'' are equivalent. \square

Correspondence between the Galois coverings of A and those of B

Proposition 5.4. *Let A be a connected and piecewise hereditary algebra. Let $T \in \mathcal{D}^b(\text{mod } A)$ be a tilting complex and $B = \text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$. Let G be a group. With the notations of Lemma 5.1, the map $[F] \mapsto [F]_T$ is a well-defined bijection from the set of equivalence classes of connected Galois coverings with group G of A to the set of equivalence classes of Galois coverings with group G of B .*

Proof: Let $\text{Gal}_A(G)$ be the set of equivalence classes of connected Galois coverings with group G of A . By Lemma 5.1, there is a well-defined map:

$$\begin{array}{ccc} \gamma_A: & \text{Gal}_A(G) & \rightarrow \text{Gal}_B(G) \\ & [F] & \mapsto [F]_T \end{array} .$$

We keep the notations X_x, φ_x (for $x \in \text{ob}(A)$) introduced after the proof of Lemma 5.2. Then we also have a well-defined map:

$$\begin{array}{ccc} \gamma_B: & \text{Gal}_B(G) & \rightarrow \text{Gal}_{\text{End}_{\mathcal{D}^b(\text{mod } B)}(X)}(G) \\ & [F] & \mapsto [F]_X \end{array}$$

Lemma 5.3 shows that γ_A is injective and γ_B is surjective. Therefore, γ_A is bijective because A, T and B, X play symmetric rôles. \square

Proposition 5.4 gives information on the existence of a universal cover. Indeed, we have the following result.

Proposition 5.5. *Let A be a connected and piecewise hereditary algebra. Let $T \in \mathcal{D}^b(\text{mod } A)$ be a tilting complex. Assume that A admits a universal cover $\tilde{F}: \tilde{\mathcal{C}} \rightarrow A$. Then $\text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$ admits a universal cover with group isomorphic to the one of \tilde{F} .*

Proof: Fix an indecomposable decomposition $T = T_1 \oplus \dots \oplus T_n$. Let $B = \text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$. So B is the full subcategory of $\mathcal{D}^b(\text{mod } A)$ with objects T_1, \dots, T_n . Let $x_0 \in \text{ob}(A)$ be a base-point for the category $\text{Gal}(A, x_0)$ of pointed Galois coverings of A . We construct a (full and faithful) functor $\tilde{F}^\rightarrow \rightarrow \text{Gal}(B, T_1)$. Recall that \tilde{F}^\rightarrow was defined in Section 1 and there is at most one morphism between two pointed Galois coverings. We need the following data:

1. For every $i \in \{1, \dots, n\}$, let $\tilde{T}_i \in \text{mod } \tilde{\mathcal{C}}$ be such that $\tilde{F}_\lambda \tilde{T}_i \simeq T_i$. Therefore, the k -categories $B = \text{End}_{\mathcal{D}^b(\text{mod } A)}(T)$ and $\text{End}_{\mathcal{D}^b(\text{mod } A)}(\bigoplus_{i=1}^n F_\lambda \tilde{T}_i)$ are isomorphic. For simplicity, we assume that $\tilde{F}_\lambda \tilde{T}_i = T_i$ for every i .
2. If $F \in \tilde{F}^\rightarrow$, there exists a unique morphism $p: \tilde{F} \rightarrow F$ in $\text{Gal}(A, x_0)$. Since p is a Galois covering (see [20, Prop. 3.4]), we set $T_i^F = p_\lambda \tilde{T}_i$ for every i .

Then:

- (i) We have $T_i = F_\lambda(T_i^F)$ for every $i \in \{1, \dots, n\}$ and $F \in \tilde{F}^\rightarrow$. Indeed, there exists a unique morphism $p: \tilde{F} \rightarrow F$, so that $\tilde{F}_\lambda = F_\lambda \circ p_\lambda$.
- (ii) Let $u: F \rightarrow F'$ be a morphism in \tilde{F}^\rightarrow . Let G be the group of F and G' the group of F' . Then u is a Galois covering (see [20, Prop. 3.4]). Let $p: \tilde{F} \rightarrow F$ and $p': \tilde{F} \rightarrow F'$ be the unique morphisms in $\text{Gal}(A, x_0)$ from \tilde{F} to F and from \tilde{F} to F' , respectively. Then $p' = u \circ p$ because of [20, Lem. 3.1] and because we are dealing with pointed Galois coverings. Therefore, $u_\lambda(T_i^F) = \sigma_u(g) T_i^{F'}$ for every $i \in \{1, \dots, n\}$ and every $g \in G$. Here, $\sigma_u: G \rightarrow G'$ is the unique (surjective) morphism of groups such that $u \circ g = \sigma_u(g) \circ u$ for every $g \in G$ (see [20, Prop. 3.4]).

Now we can construct a functor $\tilde{F}^\rightarrow \rightarrow \text{Gal}(B, T_1)$. Let $F: (\mathcal{C}, x) \rightarrow (A, x_0)$ be in \tilde{F}^\rightarrow . Let G be the group of F . By (i) and Lemma 4.8, we have a pointed Galois covering with group G induced by $F_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } A)$:

$$\begin{array}{ccc} F_T: & \left(\text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C})} \left(\bigoplus_{g,i} {}^g T_i^F \right), T_1^F \right) & \rightarrow (B, T_1) \\ & {}^g T_i^F & \mapsto T_i \\ & {}^g T_i^F \xrightarrow{f} {}^h T_j^F & \mapsto T_i \xrightarrow{F_\lambda(f)} T_j \end{array} .$$

So $[F_T] = [F]_T$. Thus, we have associated a pointed Galois covering with group G of B to any pointed Galois covering with group G of A . We now associate a morphism of pointed Galois coverings of B to any morphism of pointed Galois coverings of A . Let $u: F \rightarrow F'$ be a morphism in \tilde{F}^\rightarrow where $F: (\mathcal{C}, x) \rightarrow (A, x_0)$ and $F': (\mathcal{C}', x') \rightarrow (A, x_0)$ have groups G and G' , respectively. By (ii), we have a well-defined k -linear functor induced by $u_\lambda: \mathcal{D}^b(\text{mod } \mathcal{C}) \rightarrow \mathcal{D}^b(\text{mod } \mathcal{C}')$:

$$\begin{array}{ccc} u_T: & \left(\text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C})} \left(\bigoplus_{g,i} {}^g T_i^F \right), T_1^F \right) & \rightarrow \left(\text{End}_{\mathcal{D}^b(\text{mod } \mathcal{C}')} \left(\bigoplus_{g',i} {}^{g'} T_i^{F'} \right), T_1^{F'} \right) \\ & {}^g T_i^F & \mapsto u_\lambda({}^g T_i^F) = \sigma_u(g) T_i^{F'} \\ & {}^g T_i^F \xrightarrow{f} {}^h T_j^F & \mapsto \sigma_u(g) T_i^{F'} \xrightarrow{u_\lambda f} \sigma_u(h) T_j^{F'} \end{array} .$$

The equality $u_\lambda({}^g T_i^F) = \sigma_u(g) T_i^{F'}$ follows from the equality $u \circ g = \sigma_u(g) \circ u$. Also, $u_\lambda(T_1^F) = T_1^{F'}$. Since $F' \circ u = F$ and F_T, F'_T and u_T are defined as restrictions of F_λ, F'_λ and u_λ respectively, $u_T: F_T \rightarrow F'_T$ is a morphism

in $\text{Gal}(B, T_1)$. Thus, to any morphism in \tilde{F}^\rightarrow , we have associated a morphism in $\text{Gal}(B, T_1)$. We let the reader check that the following is a functor:

$$\begin{aligned} \Psi: \quad \tilde{F}^\rightarrow &\rightarrow \text{Gal}(B, T_1) \\ F &\mapsto F_T \\ F \xrightarrow{u} F' &\mapsto F_T \xrightarrow{u_T} F'_T \end{aligned}$$

Also, it is not difficult to prove that Ψ is full and faithful, although we shall not use this fact in the sequel. Remark that the Galois covering F_T lies in $\Psi(\tilde{F}^\rightarrow)$ for every $F \in \tilde{F}^\rightarrow$.

Now we can prove that $\Psi(\tilde{F}^\rightarrow)$ is a universal cover for B . Let F be a connected Galois covering of B . By Proposition 5.4, there exists a connected Galois covering F' of A such that $[F] = [F']_T$. Since \tilde{F} is a universal cover of A , the Galois covering F' of A is equivalent to some $F'' \in \tilde{F}^\rightarrow$, that is $[F'] = [F'']$. As noticed above, we have $[F'_T] = [F''_T]$. Therefore, $[F] = [F']_T = [F'']_T = [F''_T]$, that is, F is equivalent to a Galois covering of B lying in $\Psi(\tilde{F}^\rightarrow)$. So $\Psi(\tilde{F}^\rightarrow)$ is a universal Galois covering of B . \square

6 The main theorem and its corollary

In this section, we prove Theorem 1 and Corollary 2. We assume that A is a connected and piecewise hereditary algebra.

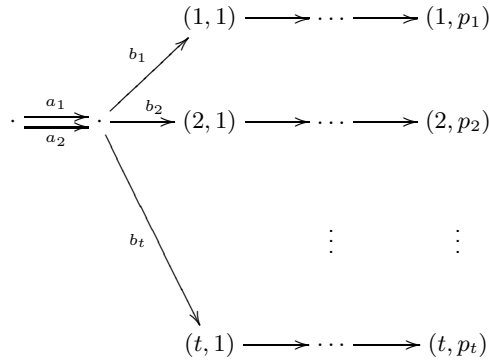
Two particular cases: paths algebras and squid algebras

We first check that our main results hold for paths algebras and for squid algebras.

Lemma 6.1. *Assume that $A = kQ$ where Q is a finite connected quiver with no oriented cycle. Then Theorem 1 and Corollary 2 hold true for A .*

Proof: Let $\tilde{Q} \rightarrow Q$ be the universal Galois covering of quivers (see [23]). It follows from [20, Prop. 4.4] that the induced Galois covering $k\tilde{Q} \rightarrow kQ$ with group $\pi_1(Q)$ is a universal cover of A . Whence Theorem 3. On the other hand, $\text{HH}^1(kQ) = 0$ if and only if Q is a tree (see [8]). Whence Corollary 2. \square

We now turn to the case of squid algebras. We refer the reader to [25] for more details on squid algebras. A squid algebra over an algebraically closed field k is defined by the following data: An integer $t \geq 2$, a sequence $p = (p_1, \dots, p_t)$ of non negative integers, and a sequence $\tau = (\tau_3, \dots, \tau_t)$ of pairwise distinct non zero elements of k . With this data, the squid algebra $S(t, p, \tau)$ is the k -algebra kQ/I where Q is the following quiver:



and I is the ideal generated by the following relations:

$$b_1 a_1 = b_2 a_2 = 0, \quad b_i a_2 = \tau_i b_i a_1 \quad \text{for } i = 3, \dots, t.$$

Using Happel's long exact sequence ([15]), one can compute $\text{HH}^1(S(t, p, \tau))$:

$$\dim_k \text{HH}^1(S(t, p, \tau)) = \begin{cases} 1 & \text{if } t = 2 \\ 0 & \text{if } t \geq 3. \end{cases}$$

Following [23], the bound quiver (Q, I) defines a Galois covering $k\tilde{Q}/\tilde{I} \rightarrow kQ/I$ with group isomorphic to \mathbb{Z} if $t = 2$, and with trivial group otherwise. One can easily check that this Galois covering is universal in the sense of Theorem 3. The above considerations give the following.

Lemma 6.2. *Let A be a squid algebra. Then Theorem 1 and Corollary 2 hold true for A .*

The general case

Thanks to Proposition 5.5, we can prove the two main results of this text.

Proof of Theorem 1 and Corollary 2: Assume first that A is piecewise hereditary of type Q , where Q is a finite connected quiver with no oriented cycle. So, there exists a tilting complex $T \in \mathcal{D}^b(\text{mod } kQ)$ such that $A \simeq \text{End}_{\mathcal{D}^b(\text{mod } kQ)}(T)$. By Proposition 5.5 and Lemma 6.1, the algebra A admits a universal Galois covering with group isomorphic to the fundamental group of Q . In particular, A is simply connected if and only if Q is a tree. On the other hand, Q is tree if and only if $\text{HH}^1(kQ) = 0$ (by [8]) and $\text{HH}^1(kQ) \simeq \text{HH}^1(A)$ (by [19]). Therefore, A is simply connected if and only if $\text{HH}^1(A) = 0$, or, if and only if Q is a tree.

Assume now that A is not derived equivalent to $\mathcal{D}^b(\text{mod } kQ)$ for any finite quiver Q . Then [16, Prop. 2.1, Thm. 2.6] implies that there exists a squid algebra $S = S(t, p, \tau)$ and a tilting complex $T \in \mathcal{D}^b(\text{mod } S)$ such that $A \simeq \text{End}_{\mathcal{D}^b(\text{mod } S)}(T)$. By Proposition 5.5 and Lemma 6.2, the algebra A has a universal cover with group isomorphic to the trivial group or to \mathbb{Z} according to whether $t \geq 3$ or $t = 2$. In particular, A is simply connected if and only if $t = 2$, that is, if and only if $\text{HH}^1(S) = 0$ (see Lemma 6.2). Since $\text{HH}^1(S) \simeq \text{HH}^1(A)$ (by [19]), we deduce that A is simply connected if and only if $\text{HH}^1(A) = 0$. \square

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