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Galois coverings and simple connectedness of piecewise hereditary algebras

Patrick Le Meur ^{*†}

2nd May 2007

Abstract

Let A a basic connected and finite dimensional piecewise hereditary k -algebra of type Q . We prove that A admits a universal Galois covering with group $\pi_1(Q)$. As a corollary, we deduce that A is simply connected if and only if Q is a tree, if and only if $HH^1(A) = 0$. As an application, we prove that if $\mathcal{C} \rightarrow A$ is a Galois covering with group G then \mathcal{C} is piecewise hereditary of type a Galois covering with group G of Q .

Introduction

Let A be a basic connected finite dimensional algebra over an algebraically closed field k . In the study of the category $\text{mod}(A)$ of (right) A -modules, a Galois covering $F: \mathcal{C} \rightarrow A$ allows one to reduce part of this study to the one of $\text{mod}(\mathcal{C})$, where \mathcal{C} is a locally bounded k -category easier to handle than A is. This idea is due to [7] and to [25] (see also [13]) and a special case appears when \mathcal{C} is simply connected. Following [4], we shall say that A is simply connected if and only if for any presentation $kQ/I \simeq A$ with quiver and admissible relations (see [5]), the fundamental group $\pi_1(Q, I)$ (see [24]) is trivial (without assuming that A is triangular or not). Equivalently (see [26] and [23]), A is simply connected if and only if there is no proper connected Galois covering $\mathcal{C} \rightarrow A$ (where connected means that \mathcal{C} is a connected locally bounded k -category). The simple connectedness of an algebra has been studied a lot (see for example [6], [2], [4]) and more particularly, there are many studies on the invariance of simple connectedness under tilting. Indeed, for a (basic) tilting A -module T (of projective dimension 1), [1] proved that if A is simply connected and of finite representation type, then $\text{End}_A(T)$ is simply connected; [4] proved that if $A = kQ$ with Q a finite euclidean quiver, then A is simply connected if and only if $\text{End}_A(T)$ is; the same equivalence was proved in [3] under the assumption that $\text{End}_A(T)$ is tame; finally, the author proved in [22] that if $A = kQ$ with Q a finite quiver without oriented cycle and if $\text{End}_A(T)$ is simply connected, then A is simply connected (i.e. Q is a tree).

In this text, we shall consider a more general problem than the invariance of simple connectedness under tilting namely, the invariance of the Galois coverings of A under derived equivalence in the case A is piecewise hereditary of type Q . Recall that A is called piecewise hereditary if there exists an abelian hereditary category \mathcal{H} such that $\mathcal{D}^b(A)$ and $\mathcal{D}^b(\mathcal{H})$ are equivalent. If moreover $\mathcal{H} = \text{mod}(kQ)$ with Q a locally finite quiver without oriented cycle, then A is called of type Q . This problem has been studied in [22] (for A not necessarily piecewise hereditary) where the following result was proved: if T is a basic tilting A -module (of arbitrary finite projective dimension) such that T and A lie in the same connected component of the Hasse diagram $\vec{\mathcal{K}}_A$ of basic tilting A -modules (see [17]), then A admits a connected Galois covering with group G if and only if the same holds for $\text{End}_A(T)$, for any group G . When A is of finite representation type, $\vec{\mathcal{K}}_A$ is connected. However, this is not the case in general. On the other hand, one can define ([9]) the cluster category C_A of A . This Krull-Schmidt category is triangulated ([20]) when A is piecewise hereditary of type Q . In such a case, the basic tilting objects of C_A (which comprise the basic tilting A -modules of projective dimension 1) are the vertices of the *tilting graph* of C_A which is not oriented yet defined similarly to $\vec{\mathcal{K}}_A$ and which is always connected. Hence, one can expect that the cited above correspondence of [22] holds for any tilting module of projective dimension 1 in the case A is piecewise hereditary of type Q . In this text we prove the following main result.

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Theorem 1. *Assume that A is piecewise hereditary of type Q . Then A admits a universal Galois covering with group $\pi_1(Q)$ (in the sense of [23]).*

The above theorem shows that for piecewise hereditary algebras, the data of the connected Galois coverings is an invariant of the derived category. The universal factorisation property that we use here to define the universal cover is weaker than the usual one (see Section 1). However, recall that a universal cover exists for A in the following cases: A is of finite representation type ([12], the universal property is the usual one in this case), A has no double bypass and k has characteristic zero ([23]), and A is monomial, triangular and without multiple arrows ([21]). Hence, Theorem 1 gives a new class of algebras for which a universal cover exists. From the cited above characterisation of simple connectedness in terms of Galois coverings, we deduce the following corollary on the invariance of simple connectedness under derived equivalence. This corollary generalises the cited above results of [4], [3] and [22]. In particular it solves the conjecture of [3] which asserts that a tilted algebra is simply connected if and only if its type is a tree:

Corollary 2. *Assume that A is piecewise hereditary of type Q . The following are equivalent:*

- (a) A is simply connected,
- (b) Q is a tree,
- (c) $HH^1(A) = 0$.

Notice also that Theorem 1 answers positively the problem of A. Skowronski ([27]) for piecewise hereditary algebras of Q and which asks whether a tame algebra A is simply connected if and only if its Hochschild cohomology group $HH^1(A)$ vanishes.

The techniques used to prove Theorem 1 allow us to prove the third main result of this text.

Theorem 3. *Assume that A is piecewise hereditary of type Q . Let $F: \mathcal{C} \rightarrow A$ be a connected Galois covering with group G . Then \mathcal{C} is piecewise hereditary of type Q' such that there exists a Galois covering of quivers $Q' \rightarrow Q$ with group G .*

It is natural to ask if the converse in the preceding theorem holds. This problem was successfully treated in [11, Thm. 3] where the following result was proven: if A is a piecewise hereditary algebra (of type some quiver Q or derived equivalent to a hereditary abelian category \mathcal{H}) and if G is a finite group whose order is invertible in k and acting on A by algebra automorphisms, then the skew group algebra $A[G]$ is piecewise hereditary. This result is indeed a converse for Theorem 3 because ([10, Thm. 2.8]) $A[G]$ and A/G are Morita equivalent if G acts freely on the objects of A .

Let us give some explanations to motivate the techniques used to prove Theorem 1. This proof is based on ideas from [22]. The cited above comparison of *loc. cit* was obtained using hypotheses on tilting A -modules. These hypotheses are satisfied for $T = A$, are preserved along a connected component of $\overline{\mathcal{K}}_A$, and are expressed using the push-down functor $F_\lambda: \text{mod}(\mathcal{C}) \rightarrow \text{mod}(A)$ associated to a Galois covering $F: \mathcal{C} \rightarrow A$. Here, we shall replace $\text{mod}(\mathcal{C})$ and $\text{mod}(A)$ by the cluster categories $C_{\mathcal{C}}$ and C_A and $\overline{\mathcal{K}}_A$ (which not always connected) by the (connected) tilting graph of C_A . Hence, we need a functor $F_\lambda: C_{\mathcal{C}} \rightarrow C_A$ to play the role of $F_\lambda: \text{mod}(\mathcal{C}) \rightarrow \text{mod}(A)$. Unfortunately, the push-down functor is exact whereas $C_{\mathcal{C}}$ is not triangulated *a priori* (even though C_A is). For this reason, we shall first work on the triangulated hull $\mathcal{M}_{\mathcal{C}}$ and \mathcal{M}_A of $C_{\mathcal{C}}$ and of C_A respectively (see [20]) rather than on the cluster categories themselves. In particular, we ought to construct a triangle functor $F_\lambda: \mathcal{M}_{\mathcal{C}} \rightarrow \mathcal{M}_A$. Let us briefly recall the construction of $\mathcal{M}_{\mathcal{C}}$ (see [20] for more details). Denote by $\mathcal{A}_{\mathcal{C}}$ the dg category of bounded complexes of finite dimensional projective \mathcal{C} -modules and by $\Sigma: \mathcal{A}_{\mathcal{C}} \rightarrow \mathcal{A}_{\mathcal{C}}$ the shift functor on complexes. If $\text{gldim } A < \infty$, $H^0(\mathcal{A}_{\mathcal{C}}) \simeq \mathcal{D}^b(A)$ and if $\theta_{\mathcal{C}}: \mathcal{A}_{\mathcal{C}} \rightarrow \mathcal{A}_{\mathcal{C}}$ is a dg functor commuting with Σ and such that $H^0(\theta_{\mathcal{C}}): H^0(\mathcal{A}_{\mathcal{C}}) \rightarrow H^0(\mathcal{A}_{\mathcal{C}})$ defines a Serre functor of $\mathcal{D}^b(A)$, then one defines the dg category $\mathcal{B}_{\mathcal{C}} := \mathcal{A}_{\mathcal{C}}/(\theta_{\mathcal{C}} \circ \Sigma^{-2})$. Then $C_{\mathcal{C}} \simeq H^0(\mathcal{B}_{\mathcal{C}})$ and $H^0(\mathcal{B}_{\mathcal{C}})$ is naturally a full subcategory of the derived category $\mathcal{D}(\mathcal{B}_{\mathcal{C}})$ thanks to the Yoneda embedding $H^0(\mathcal{B}_{\mathcal{C}}) \hookrightarrow \mathcal{D}(\mathcal{B}_{\mathcal{C}})$. The triangulated hull $\mathcal{M}_{\mathcal{C}}$ of $C_{\mathcal{C}}$ is by definition the triangulated closure of $H^0(\mathcal{B}_{\mathcal{C}})$ in $\mathcal{D}(\mathcal{B}_{\mathcal{C}})$. Hence, in order to construct a triangle functor $F_\lambda: \mathcal{M}_{\mathcal{C}} \rightarrow \mathcal{M}_A$ induced by $F: \mathcal{C} \rightarrow A$, we ought to construct functors induced by F on $\mathcal{A}_{\mathcal{C}}$ and on $\mathcal{B}_{\mathcal{C}}$.

The text is organised as follows. In Section 1 we recall some useful definitions. From Section 2 to Section 5, we describe constructions associated to a Galois covering $F: \mathcal{C} \rightarrow A$ and whose objective is to get the triangle functor $F_\lambda: \mathcal{M}_{\mathcal{C}} \rightarrow \mathcal{M}_A$. In Section 2, we construct a triangle functor $F_\lambda: \mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(A)$ and a dg functor $F_\lambda: \mathcal{A}_{\mathcal{C}} \rightarrow \mathcal{A}_A$. In Section 3, we construct dg functors $\theta_{\mathcal{C}}: \mathcal{A}_{\mathcal{C}} \rightarrow \mathcal{A}_{\mathcal{C}}$ and $\theta_A: \mathcal{A}_A \rightarrow \mathcal{A}_A$ defining Serre dualities on $\mathcal{D}^b(\mathcal{C})$ and $\mathcal{D}^b(A)$ and which are compatible with $F_\lambda: \mathcal{A}_{\mathcal{C}} \rightarrow \mathcal{A}_A$ (more precisely: $F_\lambda \circ \theta_{\mathcal{C}}$ and $\theta_A \circ F_\lambda$ are isomorphic dg functors). In Section 4, we use the construction of Section 2 to construct an additive functor $F_\lambda: C_{\mathcal{C}} \rightarrow C_A$ and we use the construction of Section 3 to construct a dg functor $F_\lambda: \mathcal{B}_{\mathcal{C}} \rightarrow \mathcal{B}_A$. This allows us to construct the triangle functor $F_\lambda: \mathcal{M}_{\mathcal{C}} \rightarrow \mathcal{M}_A$. In Section 5

we prove a useful criterion for an object in $\mathcal{M}_{\mathcal{C}}$ to lie in $C_{\mathcal{C}}$. In Section 6 we prove some useful properties on tilting A -modules with respect to Galois coverings of A when A is piecewise hereditary. These results are expressed in terms of the module category $mod(A)$ but use the results of all the preceding sections by considering a tilting A -module as a tilting object of C_A . Finally, in Section 6, we prove Theorem 1, Corollary 2 and Theorem 3. As an application of Theorem 3 we establish a bijection between the set of tilting objects of C_A and the set of G -invariant tilting objects of $C_{A'}$ when $A' \rightarrow A$ is a Galois covering with finite group G such that A is piecewise hereditary of type some quiver Q .

I would like to express my gratitude to Bernhard Keller for his helpful explanations on his construction of the triangulated hull in [20].

1 Basic definitions

Notations on k -categories. For a reminder on k -categories and on their properties (connected, locally bounded), we refer the reader to [7]. All functors between k -categories are assumed to be k -linear. If \mathcal{C} is a k -category, $ob(\mathcal{C})$ will denote the set of objects and ${}_y\mathcal{C}_x$ will denote the space of morphisms from x to y . If A is a basic finite dimensional k -algebra, we consider A as a locally bounded k -category whose set of objects is a complete set $\{e_1, \dots, e_n\}$ of primitive pairwise orthogonal idempotents and such that $e_j A e_i = e_j A e_i$ for any i, j .

Modules over k -categories (see [7]). If \mathcal{C} is a k -category, a left (resp. a right) \mathcal{C} -module is a k -linear functor $M: \mathcal{C} \rightarrow MOD(k)$ (resp. $M: \mathcal{C}^{op} \rightarrow MOD(k)$) where $MOD(k)$ is the category of k -vector spaces. Unless otherwise specified, we shall always consider \mathcal{C} -modules as right \mathcal{C} -modules. A morphism of \mathcal{C} -modules $M \rightarrow N$ is a k -linear natural transformation of functors. We denote by $MOD(\mathcal{C})$ the category of \mathcal{C} -modules and by $mod(\mathcal{C})$ the full subcategory of finite dimensional \mathcal{C} -modules ($M \in MOD(\mathcal{C})$ is called locally finite dimensional if $dim_k M(x) < \infty$ for any $x \in ob(\mathcal{C})$ and it is called finite dimensional if $\sum_{x \in ob(\mathcal{C})} dim_k M(x) < \infty$). A module is called basic if it is the direct sum of pairwise non isomorphic

indecomposable modules. If \mathcal{C}' is another k -category, a $\mathcal{C} - \mathcal{C}'$ -bimodule is a left $\mathcal{C} \times \mathcal{C}'^{op}$ -module. For example, \mathcal{C} is naturally a $\mathcal{C} - \mathcal{C}$ -bimodule for the mapping $(x, y) \mapsto {}_y\mathcal{C}_x$ for any $(x, y) \in ob(\mathcal{C}) \times ob(\mathcal{C}^{op})$. Notice that \mathcal{C} is a locally finite dimensional $\mathcal{C} - \mathcal{C}$ -bimodule if \mathcal{C} is locally bounded. The usual duality $X \mapsto Hom(X, k)$ is denoted by $D: MOD(\mathcal{C}) \rightarrow MOD(\mathcal{C}^{op})$.

If \mathcal{C} is locally bounded, $proj(\mathcal{C})$ will denote the full subcategory of $mod(\mathcal{C})$ of projective \mathcal{C} -modules. Recall that $\{{}_x\mathcal{C}_? \mid x \in ob(\mathcal{C})\}$ is a skeleton of the category of indecomposable projective \mathcal{C} -modules. Finally, $\mathcal{D}(MOD(\mathcal{C}))$ (resp. $\mathcal{D}^b(\mathcal{C})$) will denote the derived category of complexes of \mathcal{C} -modules (resp. of bounded complexes of finite dimensional \mathcal{C} -modules). The shift functor will be denoted by Σ .

Galois coverings of k -categories. Let G be a group. A **free G -category** is a k -category \mathcal{E} endowed with a group morphism $G \rightarrow Aut(\mathcal{E})$ such that G acts freely on $Ob(\mathcal{E})$. In this case, there exists a (unique) quotient $\mathcal{E} \rightarrow \mathcal{E}/G$ of \mathcal{E} by G in the category of k -categories. A Galois covering of \mathcal{B} with group G is by definition a functor $F: \mathcal{E} \rightarrow \mathcal{B}$ endowed with a group morphism $G \rightarrow Aut(F) = \{g \in Aut(\mathcal{E}) \mid F \circ g = F\}$ such that \mathcal{E} is a free G -category, and such that the functor $\mathcal{E}/G \xrightarrow{\bar{F}} \mathcal{B}$ induced by F is an isomorphism. Notice that if \mathcal{B} and \mathcal{E} are locally bounded and if $gldim \mathcal{B} < \infty$, then $gldim \mathcal{E} < \infty$. If \mathcal{E} is connected, the morphism $G \rightarrow Aut(F)$ is an isomorphism, so that F is a connected Galois covering with group $Aut(F)$.

Throughout this text, we shall write $Gal(\mathcal{B})$ for the category whose objects are the connected Galois coverings of \mathcal{B} , such that a morphism $u: F \rightarrow F'$ from $F: \mathcal{E} \rightarrow \mathcal{B}$ to $F': \mathcal{E}' \rightarrow \mathcal{B}$ is a morphism $u: \mathcal{E} \rightarrow \mathcal{E}'$ of k -categories such that $F' \circ u = F$ and with the composition defined in the obvious way.

If $F: \mathcal{E} \rightarrow \mathcal{B}$ and $F': \mathcal{E}' \rightarrow \mathcal{B}$ are Galois coverings, F and F' are called **equivalent** ($F \sim F'$) if and only if there exists a commutative diagram of k -categories:

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\sim} & \mathcal{E}' \\ F \downarrow & & \downarrow F' \\ \mathcal{B} & \xrightarrow{\sim} & \mathcal{B} \end{array}$$

where the horizontal arrows are isomorphisms of k -categories and where the bottom horizontal arrow extends the identity map on objects.

If \mathcal{B} is a connected k -category, a **universal cover of \mathcal{B}** (see [23]) is a connected Galois covering $\tilde{F}: \tilde{\mathcal{B}} \rightarrow \mathcal{B}$ such that for any connected Galois covering $F: \mathcal{E} \rightarrow \mathcal{B}$, there exists a commutative diagram

of k -categories:

$$\begin{array}{ccc} \tilde{\mathcal{B}} & \longrightarrow & \mathcal{E} \\ \tilde{F} \downarrow & & \downarrow F \\ \mathcal{B} & \xrightarrow{\sim} & \mathcal{B} \end{array}$$

where the bottom horizontal arrow is an isomorphism extending the identity map on objects. Notice that this is equivalent to say that any $F \in \text{Gal}(\mathcal{B})$ is equivalent to some F' such that there exists a morphism $\tilde{F} \rightarrow F'$ in $\text{Gal}(\mathcal{B})$.

Recall ([23]) that if $F, F' \in \text{Gal}(\mathcal{B})$ and if u, u' are morphisms $F \rightarrow F'$, then there exists a unique $g \in \text{Aut}(F')$ such that $g \circ u = u'$. Recall also (loc. cit.) that to any morphism $u: F \rightarrow F'$ is associated a unique (surjective) group morphism $\lambda^u: \text{Aut}(F) \rightarrow \text{Aut}(F')$ such that:

$$(\forall g \in \text{Aut}(F)) \quad \lambda^u(g) \circ u = u \circ g$$

Covering properties on module categories. (see [7] and [25]). Let $F: \mathcal{E} \rightarrow \mathcal{B}$ be a Galois covering with group G . The G -action on \mathcal{E} gives rise to a G -action on $\text{MOD}(\mathcal{E})$: if $M \in \text{MOD}(\mathcal{E})$ and $g \in G$, then ${}^g M := F \circ g^{-1} \in \text{MOD}(\mathcal{E})$. Moreover, F defines two exact functors $F_\lambda: \text{MOD}(\mathcal{E}) \rightarrow \text{MOD}(\mathcal{B})$ (the push-down functor) and $F: \text{MOD}(\mathcal{B}) \rightarrow \text{MOD}(\mathcal{E})$ (the pull-up functor) as follows. For any $M \in \text{MOD}(\mathcal{B})$, $F.M = M \circ F$. If $M \in \text{MOD}(\mathcal{E})$, then $(F_\lambda M)(x) = \bigoplus_{x' \in F^{-1}(x)} M(x')$, for any $x \in \text{Ob}(\mathcal{B})$. If $u \in {}_y \mathcal{E}_x$, then the restriction of $(F_\lambda M)(F(u))$ to $M(g.x)$ ($g.x \in F^{-1}(F(x)) = G.x$) is equal to $M(g.u): M(g.x) \rightarrow M(g.y)$. The functors F and F_λ are exact, they preserve projective modules, $F_\lambda(\text{mod}(\mathcal{C})) \subseteq \text{mod}(A)$, $D \circ F_\lambda|_{\text{mod}(\mathcal{E})} \simeq F_\lambda \circ D|_{\text{mod}(\mathcal{E})}$ and F_λ is G -invariant ($F_\lambda \circ g = F_\lambda$ for any $g \in G$). Moreover, $F.F_\lambda M \simeq \bigoplus_{g \in G} {}^g M$ for any $M \in \text{MOD}(\mathcal{E})$. Finally, for any $M, N \in \text{MOD}(\mathcal{E})$, the following mappings induced by F_λ are bijective:

$$\bigoplus_{g \in G} \text{Hom}_{\mathcal{E}}({}^g M, N) \rightarrow \text{Hom}_{\mathcal{B}}(F_\lambda M, F_\lambda N) \quad \text{and} \quad \bigoplus_{g \in G} \text{Hom}_{\mathcal{E}}(M, {}^g N) \rightarrow \text{Hom}_{\mathcal{B}}(F_\lambda M, F_\lambda N) \quad (\star)$$

For more details and properties on F and F_λ , we refer the reader to [7]. If $X \in \text{mod}(\mathcal{E})$, the **stabiliser of X** is the subgroup $G_X := \{g \in G \mid {}^g X \simeq X\}$ of G . An indecomposable module $X \in \text{mod}(\mathcal{B})$ is called **of the first kind w.r.t. F** if and only if $F_\lambda \tilde{X} \simeq X$ for some $\tilde{X} \in \text{mod}(\mathcal{E})$ (necessarily indecomposable). In such a case, one may choose \tilde{X} such that $F_\lambda \tilde{X} = X$. More generally, $X \in \text{mod}(\mathcal{B})$ is called of the first kind w.r.t. F if and only if it is the direct sum of indecomposable \mathcal{B} -modules of the first kind w.r.t. F .

Covering properties on additive categories. Let \mathcal{E} and \mathcal{B} be additive categories. Let the group G act on \mathcal{E} and let $\theta: \mathcal{E} \rightarrow \mathcal{B}$ be a G -invariant additive functor. We shall say that θ **has the covering property** if for any $M, N \in \mathcal{E}$, the mappings of (\star) are bijective (after replacing F_λ by θ).

The cluster category (see [9] and [20]). If \mathcal{C} is a locally bounded k -category, a Serre functor is a triangle equivalence $\nu_{\mathcal{C}}: \mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(\mathcal{C})$ such that for any $X, Y \in \mathcal{D}^b(\mathcal{C})$, there is an isomorphism $D\text{Hom}_{\mathcal{D}^b(\mathcal{C})}(X, Y) \simeq \text{Hom}_{\mathcal{D}^b(\mathcal{C})}(Y, \nu_{\mathcal{C}}(X))$ functorial in X and Y . If $\nu_{\mathcal{C}}$ is a Serre functor, the cluster category $C_{\mathcal{C}}$ is the additive category $\mathcal{D}^b(\mathcal{C})/(\nu_{\mathcal{C}}\Sigma^{-2})$ whose objects are those of $\mathcal{D}^b(\mathcal{C})$, such that:

$$\text{Hom}_{C_{\mathcal{C}}} = \bigoplus_{n \in \mathbb{Z}} \text{Hom}((\nu_{\mathcal{C}}\Sigma^{-2})^n X, Y)$$

for any $X, Y \in \mathcal{D}^b(\mathcal{C})$ and whose composition of morphisms is induced by the one in $\mathcal{D}^b(\mathcal{C})$. Then $C_{\mathcal{C}}$ is Krull-Schmidt and an object is indecomposable in $\mathcal{D}^b(\mathcal{C})$ if and only if it is in $C_{\mathcal{C}}$ ([9, Prop. 1.2]). Moreover, if \mathcal{C} is piecewise hereditary, then ([20]) it is triangulated.

Triangulated hull for the cluster category. Let \mathcal{C} be a locally bounded k -category. For a reminder on dg categories, we refer the reader to [18]. Following *loc. cit.*, we shall use the following notations for a dg category \mathcal{A}

1. $H^0(\mathcal{A})$ is the additive category whose objects are those of \mathcal{A} and such that $H^0(\mathcal{A})(X, Y) = H^0(\mathcal{A}(X, Y))$ for any objects X, Y . The construction H^0 is functorial,
2. $\text{Dif}\mathcal{A}$ is the dg category of dg \mathcal{A} -modules. In particular, if $X \in \mathcal{A}$, then X^\wedge will denote the dg \mathcal{A} -module such that $X^\wedge(Y) = \mathcal{A}(Y, X)$ and $X^\wedge(f)(g) = (-1)^{n_d} g \circ f$ if $f \in \mathcal{A}^n(Y, Z)$ and $g \in \mathcal{A}^d(Z, X)$. This way, we get the Yoneda (fully faithful) embedding $\mathcal{A} \hookrightarrow \text{Dif}\mathcal{A}$, $X \mapsto X^\wedge$.

3. $\mathcal{H}\mathcal{A}$ is the additive category $H^0(\text{Dif}\mathcal{A})$. In particular, $X \mapsto X^\wedge$ defines the Yoneda embedding $H^0\mathcal{A} \hookrightarrow \mathcal{H}\mathcal{A}$.
4. $\mathcal{D}\mathcal{A}$ is the derived category of \mathcal{A} , i.e. the triangulated category obtained from $\mathcal{H}\mathcal{A}$ by formally inverting the quasi-isomorphisms.

Let \mathcal{A}_C be the dg category whose objects are the bounded complexes of finite dimensional projective C -modules, whose space of morphisms of degree d from X to Y is $\prod_{n \in \mathbb{Z}} \text{Hom}_C(X^n, Y^{n+d})$ and such that the differential of $f = (f^n)_{n \in \mathbb{Z}} \in \mathcal{A}_C^d(X, Y)$ is $(d_Y^{n+d} \circ f^n - (-1)^d f^{n+1} \circ d_X^n)_{n \in \mathbb{Z}}$. The shift functor Σ on complexes naturally defines a dg functor $\Sigma: \mathcal{A}_C \rightarrow \mathcal{A}_C$ and the natural functor $H^0(\mathcal{A}_C) \rightarrow \mathcal{D}^b(C)$ is an equivalence of additive categories if $\text{gldim} C < \infty$.

Let $\theta_C: \mathcal{A}_C \rightarrow \mathcal{A}_C$ be a dg functor commuting with the shift and assume that $\text{gldim} C < \infty$. We shall say that θ_C induces a Serre functor if $H^0(\theta): H^0(\mathcal{A}_C) \rightarrow H^0(\mathcal{A}_C)$ induces a Serre functor $\mathcal{D}^b(C) \rightarrow \mathcal{D}^b(C)$ using the natural equivalence $H^0(\mathcal{A}_C) \simeq \mathcal{D}^b(C)$. In general, θ_C defines the dg category \mathcal{B}_C whose objects are those of \mathcal{A}_C , such that

$$\mathcal{B}^d(X, Y) := \text{colim}_p \bigoplus_{n \geq 0} \mathcal{A}^d((\theta_C \Sigma^{-2})^n X, (\theta_C \Sigma^{-2})^p Y)$$

for any $X, Y \in \mathcal{A}_C$ and any $d \in \mathbb{Z}$ (the transition maps are induced by $\nu_C \Sigma^{-2}$). The differential is induced by the one in \mathcal{A}_C and so is the composition. Assume that θ_C induces a Serre functor, then, the equivalence $H^0(\mathcal{A}) \xrightarrow{\sim} \mathcal{D}^b(C)$ defines an equivalence of additive categories $H^0(\mathcal{B}_C) \xrightarrow{\sim} C_C$. The triangulated closure of $H^0(\mathcal{B}_C)$ in $\mathcal{D}(\mathcal{B}_C)$ is called **the triangulated hull of C_C and denoted by \mathcal{M}_C** . It was shown in [20] that if C is piecewise hereditary, then $H^0(\mathcal{B}_C) = \mathcal{M}_C$ (whence the triangulated structure on C_C). To summarise these constructions, we draw a commutative diagram:

$$\begin{array}{ccccc} H^0(\mathcal{A}_C) & \longrightarrow & H^0(\mathcal{B}_C) & \longrightarrow & \mathcal{M}_C \\ & & \downarrow \sim & & \downarrow \sim \\ \text{mod}(C) & \longrightarrow & \mathcal{D}^b(C) & \longrightarrow & C_C \end{array}$$

Unless otherwise specified, any unlabelled functor ($\mathcal{D}^b(C) \rightarrow C_C$, $H^0(\mathcal{A}_C) \simeq \mathcal{D}^b(C)$, $\mathcal{A}_C \rightarrow \mathcal{B}_C$, $H^0(\mathcal{B}_C) \simeq C_C$, $H^0(\mathcal{B}_C) \hookrightarrow \mathcal{D}(\mathcal{B}_C)$) will always denote the natural additive functor introduced in this section.

G -actions on derived and cluster categories. Assume that \mathcal{C} is a free G -category. Then the G -action on $\text{mod}(C)$ naturally defines a G -action on $\mathcal{D}^b(C)$, on \mathcal{A}_C and on $H^0(\mathcal{A}_C)$. Assume that $\nu_C: \mathcal{D}^b(C) \rightarrow \mathcal{D}^b(C)$ is a G -equivariant Serre functor (i.e. a Serre functor such that $\nu_C \circ g = g \circ \nu_C$ for any $g \in G$). Then, the G -action on $\mathcal{D}^b(C)$ naturally defines a G -action on C_C . If $\theta_C: \mathcal{A}_C \rightarrow \mathcal{A}_C$ is a G -equivariant dg functor inducing a Serre functor on $\mathcal{D}^b(C)$, then the G -action on \mathcal{A}_C naturally defines a G -action on \mathcal{B}_C , on $H^0(\mathcal{B}_C)$, on $\mathcal{D}(\mathcal{B}_C)$ and on \mathcal{M}_C .

Tilting modules. Let A be a basic finite dimensional k -algebra. A tilting A -module (of projective dimension at most one) is a module $T \in \text{mod}(A)$ verifying the following conditions (see [8], and [16]):

- (T1) T has projective dimension at most one,
- (T2) $\text{Ext}_A^1(T, T) = 0$ (i.e. T is selforthogonal),
- (T3) there is an exact sequence in $\text{mod}(A)$: $0 \rightarrow A \rightarrow T_1 \rightarrow T_2 \rightarrow 0$ with $T_1, T_2 \in \text{add}(T)$.

Recall that if S is a set of objects in an additive category, then $\text{add}(S)$ is the smallest full subcategory containing S , stable under (finite) direct sums and under direct summands. For simplicity, all tilting modules will be assumed to be basic. Also, recall ([8]) that if T is tilting A -module, then, T is naturally a tilting $\text{End}_A(T)^{\text{op}}$ -module and that A and $\text{End}_{\text{End}_A(T)^{\text{op}}}(T)^{\text{op}}$ are isomorphic k -categories.

Tilting objects in the cluster category. Let \mathcal{C} be a locally bounded k -category. The shift functor Σ on $\mathcal{D}^b(C)$ naturally defines an automorphism of C_C still denoted by Σ . Following [9], we shall write $\text{Ext}_{C_C}^1(X, Y)$ instead of $\text{Hom}_{C_C}(X, \Sigma Y)$. A set S of objects of C_C is called **exceptional** is $\text{Ext}_{C_C}^1(X, Y) = 0$ for any $X, Y \in S$. It is called **maximal exceptional**, if it is exceptional and if $X \in C_C$ is isomorphic to an object of S as soon as $S \cup \{X\}$ is exceptional. Finally, it is called **tilting** if it is maximal exceptional and if distinct objects in S are not isomorphic. Assume that $\mathcal{C} = A$ is a basic finite dimensional piecewise

hereditary k -algebra, then ([9, Thm. 3.3]) any tilting set of C_A has exactly n elements where n is the rank of the Grothendieck group of A . With this setting, a **tilting object** of C_A is an object of the form $\bigoplus_{i=1}^n T_i$ where $\{T_1, \dots, T_n\}$ is a tilting set (hence, a tilting object is always basic). Moreover, if T is a tilting A -module, then T is also a tilting object of C_A . The isomorphism classes of tilting objects of C_A are the vertices of a connected unoriented graph (see [9]) such that there is an edge $T - T'$ if and only if $T = X \oplus \overline{T}$ and $T' = Y \oplus \overline{T}$ with X, Y indecomposable. In such a case, there exists triangles in C_A :

$$\begin{aligned} X &\rightarrow B \rightarrow Y \rightarrow \Sigma X \\ Y &\rightarrow B' \rightarrow X \rightarrow \Sigma Y \end{aligned}$$

where $X \rightarrow B$ and $Y \rightarrow B'$ (resp. $B \rightarrow Y$ and $B' \rightarrow X$) are minimal left (resp. right) $\text{add}(\overline{T})$ -approximation.

Comparison of Galois coverings (see [22]). Let A and B be basic connected finite dimensional k -algebras and let G be a group. Let $T = T_1 \oplus \dots \oplus T_n$ be a basic tilting A -module (with T_i indecomposable), let $B = \text{End}_A(T)$. Hence, B is a k -category with set of objects $\{T_1, \dots, T_n\}$ and with space of morphisms from T_i to T_j equal to $\text{Hom}_A(T_i, T_j)$. We recall here the comparison between $\text{Gal}(A)$ and $\text{Gal}(B)$ obtained in [22]. Assume that the following conditions hold and that so do the corresponding conditions hold for T viewed as a B^{op} -module:

- (H₁) T is of the first kind w.r.t. any connected Galois covering of A ,
- (H₂) if X is an indecomposable direct summand of T , if $F: \mathcal{C} \rightarrow A$ is a connected Galois covering and if $\tilde{X} \in \text{mod}(\mathcal{C})$ is such that $F_\lambda \tilde{X} \simeq X$, then $G_{\tilde{X}} := \{g \in G \mid {}^g \tilde{X} \simeq X\} = 1$,
- (H₃) for any automorphism $\psi: A \rightarrow A$ extending the identity map on objects, we have $\psi.T \simeq T$.

For every connected Galois covering $F: \mathcal{C} \rightarrow A$ with group G and every i , fix $T_i^F \in \text{mod}(\mathcal{C})$ such that $F_\lambda T_i^F = T_i$. This defines a connected Galois covering $F_T: \mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g \in G, i} T_i^F) \rightarrow B$ with group G as follows:

1. $\mathcal{E}nd_{\mathcal{C}}(\bigoplus_{g \in G, i} T_i^F)$ is the k -category whose set of objects is $\{{}^g T_i^F \mid g \in G, i \in \{1, \dots, n\}\}$, whose space of morphisms from ${}^g T_i^F$ to ${}^h T_j^F$ is equal to $\text{Hom}_{\mathcal{C}}({}^g T_i^F, {}^h T_j^F)$ and whose composition of morphisms is induced the one in $\text{mod}(\mathcal{C})$,
2. $F_T({}^g T_i^F) = T_i$ for every $g \in G, i \in \{1, \dots, n\}$,
3. $F_T(f) = F_\lambda f$ for every $f \in \text{Hom}_{\mathcal{C}}({}^g T_i^F, {}^h T_j^F)$.

With this construction, the mapping $F \mapsto F_T$ defines a bijective mapping from the set of equivalence classes of connected Galois coverings with group G of A to the set of equivalence classes of connected Galois coverings with group G of B , for any group G .

Throughout this text, all the considered locally bounded k -categories are supposed to be of finite global dimension.

2 Covering properties on the derived category

Let $F: \mathcal{C} \rightarrow A$ be a Galois covering with group G and with \mathcal{C} and A locally bounded. The aim of this section is to extend $F_\lambda: \text{mod}(\mathcal{C}) \rightarrow \text{mod}(A)$ to a triangle functor $F_\lambda: \mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(A)$ and to obtain similar features in terms of dg functors $\mathcal{A}_{\mathcal{C}} \rightarrow \mathcal{A}_A$ and additive functors $H^0(\mathcal{A}_{\mathcal{C}}) \rightarrow H^0(\mathcal{A}_A)$. We begin with the following useful lemma.

Lemma 2.1. *There exists a triangle functor $F_\lambda: \mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(A)$ such that the following diagram commutes:*

$$\begin{array}{ccc} \text{mod}(\mathcal{C}) & \longrightarrow & \mathcal{D}^b(\mathcal{C}) \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ \text{mod}(A) & \longrightarrow & \mathcal{D}^b(A) \end{array} \tag{D_1}$$

Moreover, $F_\lambda: \mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(A)$ has the covering property.

Proof: The first assertion is due to the fact that $F_\lambda: \text{mod}(\mathcal{C}) \rightarrow \text{mod}(A)$ is an exact functor. On the other hand, if we denote by $\mathcal{K}^b(?)$ the homotopy category of bounded complexes, then, it is easily checked that $F_\lambda: \mathcal{K}^b(\text{proj}(\mathcal{C})) \rightarrow \mathcal{K}^b(\text{proj}(A))$ has the covering property (recall that $F_\lambda: \text{mod}(\mathcal{C}) \rightarrow \text{mod}(A)$ sends projective modules to projective modules). Since A and \mathcal{C} have finite global dimension, we get the last assertion. \blacksquare

Since $F_\lambda: \text{mod}(\mathcal{C}) \rightarrow \text{mod}(A)$ sends projective \mathcal{C} -modules to projective A -modules, hence, it naturally defines a dg functor $F_\lambda: \mathcal{A}_\mathcal{C} \rightarrow \mathcal{A}_A$ commuting with the shift and such that the following diagram commutes:

$$\begin{array}{ccc} \text{proj}(\mathcal{C}) & \longrightarrow & \mathcal{A}_\mathcal{C} \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ \text{proj}(A) & \longrightarrow & \mathcal{A}_A \end{array}$$

The following lemma is a direct consequence of the fact that $F_\lambda: \text{mod}(\mathcal{C}) \rightarrow \text{mod}(A)$ has the covering property and of the fact that we are dealing with bounded complexes.

Lemma 2.2. $F_\lambda: \mathcal{A}_\mathcal{C} \rightarrow \mathcal{A}_A$ and $H^0(F_\lambda): H^0(\mathcal{A}_\mathcal{C}) \rightarrow H^0(\mathcal{A}_A)$ have the covering property. Moreover, the following diagram is commutative:

$$\begin{array}{ccc} H^0(\mathcal{A}_\mathcal{C}) & \xrightarrow{\sim} & \mathcal{D}^b(\mathcal{C}) \\ H^0(F_\lambda) \downarrow & & \downarrow F_\lambda \\ H^0(\mathcal{A}_A) & \xrightarrow{\sim} & \mathcal{D}^b(A) \end{array} \tag{D_2}$$

For short, we shall write $F_\lambda: H^0(\mathcal{A}_\mathcal{C}) \rightarrow H^0(\mathcal{A}_A)$ instead of $H^0(F_\lambda): H^0(\mathcal{A}_\mathcal{C}) \rightarrow H^0(\mathcal{A}_A)$.

3 Dg functors inducing Serre functors and compatible with push-down functors

Let $F: \mathcal{C} \rightarrow A$ be a Galois covering with group G and with \mathcal{C} and A locally bounded. The aim of this section is to construct dg functors $\theta_\mathcal{C}: \mathcal{A}_\mathcal{C} \rightarrow \mathcal{A}_\mathcal{C}$ and $\theta_A: \mathcal{A}_A \rightarrow \mathcal{A}_A$ and triangle functors $\nu_A: \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(A)$ and $\nu_\mathcal{C}: \mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(\mathcal{C})$ with the following requirements:

1. $\theta_\mathcal{C}$ and $\nu_\mathcal{C}$ are G -equivariant,
2. ν_A and $\nu_\mathcal{C}$ are Serre functors,
3. θ_A induces ν_A and $\theta_\mathcal{C}$ induces $\nu_\mathcal{C}$,
4. $\theta_A \circ F_\lambda \simeq F_\lambda \circ \theta_\mathcal{C}$ as dg functors $\mathcal{A}_\mathcal{C} \rightarrow \mathcal{A}_A$.

3.1 The G -equivariant dg functor $\theta_\mathcal{C}$ and the G -equivariant Serre functor $\nu_\mathcal{C}$

Since G acts on \mathcal{C} , it also acts on $\mathcal{C} \times \mathcal{C}^{op}$ via $(x, y) \mapsto (gx, gy)$, for any $g \in G$. This defines a G -action on $\mathcal{C} - \mathcal{C}$ -bimodules as follows: ${}^g X(x, y) = X(g^{-1}x, g^{-1}y)$ for X a $\mathcal{C} - \mathcal{C}$ -bimodule, $g \in G$ and $x, y \in \text{ob}(\mathcal{C})$.

Recall that any Serre functor $\nu_\mathcal{C}: \mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(\mathcal{C})$ is isomorphic to the total derived functor $? \otimes_{\mathcal{C}}^{\mathbb{L}} X$ where $p: X \rightarrow D(\mathcal{C})$ is any projective resolution of the locally finite dimensional $\mathcal{C} - \mathcal{C}$ -bimodule $D(\mathcal{C})$. The following lemma proves that X may be chosen to be G -invariant.

Lemma 3.1. *There exists a projective resolution $p: X \rightarrow D(\mathcal{C})$ of $\mathcal{C} - \mathcal{C}$ -bimodule where X is a (right bounded) complex of locally finite dimensional projective $\mathcal{C} - \mathcal{C}$ -bimodules such that ${}^g X = X$ for any $g \in G$.*

Proof: The category of locally finite dimensional $\mathcal{C} - \mathcal{C}$ -bimodules X such that ${}^g X = X$ for any $g \in G$ is an abelian category and contains \mathcal{C} and $\mathcal{C} \times \mathcal{C}^{op}$. Whence the lemma. \blacksquare

Let θ_C be the dg functor $? \otimes_C X: \mathcal{A}_C \rightarrow \mathcal{A}_C$ where X is as in Lemma 3.1. Also, let $\nu_C: \mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(\mathcal{C})$ be the triangle functor $? \otimes_C X$. By construction, the following diagram is commutative:

$$\begin{array}{ccc} H^0(\mathcal{A}_C) & \xrightarrow{\sim} & \mathcal{D}^b(\mathcal{C}) \\ H^0(\theta_C) \downarrow & & \downarrow \nu_C \\ H^0(\mathcal{A}_C) & \xrightarrow{\sim} & \mathcal{D}^b(\mathcal{C}) \end{array} \quad (D_3)$$

Lemma 3.2. ν_C is a G -equivariant Serre functor and θ_C is G -equivariant and commutes with the shift.

Proof: By construction, ν_C is a Serre functor. For short, let us denote by \mathcal{A} the dg category whose objects are the bounded complexes of \mathcal{C} -modules, whose space of morphisms $X \rightarrow Y$ of degree d are the families $(f_n: X^n \rightarrow Y^{n+d})_{n \in \mathbb{Z}}$ of morphisms of \mathcal{C} -modules and whose differential of morphisms is induced by the differential of the complexes. Hence, we have a dg functor $? \otimes_C X: \mathcal{A} \rightarrow \mathcal{A}$. The G -action on $\text{mod}(\mathcal{C})$ naturally defines a G -action on \mathcal{A} and in order to prove the lemma, it suffices to prove that $? \otimes_C X: \mathcal{A} \rightarrow \mathcal{A}$ is G -equivariant. Let M be a bounded complex of \mathcal{C} -modules and let $g \in G$. For sake of simplicity, we shall make the verifications of equivariance assuming that M and X are stalk complexes (the general case is dealt with the same way). Then, for any $x \in \text{ob}(\mathcal{C})$, the G -equivariance of X implies (with the notation $\otimes_k := \otimes$):

$$\begin{aligned} {}^g \left(M \otimes_C X \right) (x) &= \left(M \otimes_C X \right) (g^{-1}x) \\ &= \left(\bigoplus_{y_1, y_2 \in \text{ob}(\mathcal{C})} M(y_1) \otimes X(y_2, g^{-1}x) \right) \left/ \left\langle ma \otimes n - m \otimes an \left| \begin{array}{l} y_1, y_2 \in \text{ob}(\mathcal{C}), \\ m \in M(y_1), \\ n \in X(y_2, g^{-1}x), \\ a \in {}_{y_1} \mathcal{C}_{y_2} \end{array} \right. \right. \right\rangle \\ &= \left(\bigoplus_{y_1, y_2 \in \text{ob}(\mathcal{C})} M(y_1) \otimes X(gy_2, x) \right) \left/ \left\langle ma \otimes n - m \otimes g(a)n \left| \begin{array}{l} y_1, y_2 \in \text{ob}(\mathcal{C}), \\ m \in M(y_1), \\ n \in X(gy_2, x), \\ a \in {}_{y_1} \mathcal{C}_{y_2} \end{array} \right. \right. \right\rangle \\ &= \left(\bigoplus_{y_1, y_2 \in \text{ob}(\mathcal{C})} M(g^{-1}y_1) \otimes X(y_2, x) \right) \left/ \left\langle ma \otimes n - m \otimes g(a)n \left| \begin{array}{l} y_1, y_2 \in \text{ob}(\mathcal{C}), \\ m \in M(g^{-1}y_1), \\ n \in X(y_2, x), \\ a \in {}_{g^{-1}y_1} \mathcal{C}_{g^{-1}y_2} \end{array} \right. \right. \right\rangle \\ &= \left(\bigoplus_{y_1, y_2 \in \text{ob}(\mathcal{C})} M(g^{-1}y_1) \otimes X(y_2, x) \right) \left/ \left\langle mg^{-1}(a) \otimes n - m \otimes an \left| \begin{array}{l} y_1, y_2 \in \text{ob}(\mathcal{C}), \\ m \in M(g^{-1}y_1), \\ n \in X(y_2, x), \\ a \in {}_{y_1} \mathcal{C}_{y_2} \end{array} \right. \right. \right\rangle \\ &= \left({}^g M \otimes_C X \right) (x) \end{aligned}$$

The same arguments show that if $u \in {}_y \mathcal{C}_x$, then ${}^g \left(M \otimes_C X \right) (u) = \left({}^g M \otimes_C X \right) (u)$ and that for any $f \in \mathcal{A}^d(M, N)$, then ${}^g \left(f \otimes_C X \right) = {}^g f \otimes_C X$. Hence $? \otimes_C X: \mathcal{A} \rightarrow \mathcal{A}$ is G -equivariant. So the lemma is proved. \blacksquare

3.2 The dg functor θ_A compatible with F_λ and θ_C

Now we are going to construct the dg functor $\theta_A: \mathcal{A}_A \rightarrow \mathcal{A}_A$ such that $F_\lambda \circ \theta_C \simeq \theta_A \circ F_\lambda$ as dg functors from \mathcal{A}_C to \mathcal{A}_A . In this purpose, we introduce some useful data.

Fix $L: \text{ob}(A) \rightarrow \text{ob}(\mathcal{C})$ a section of $F: \text{ob}(\mathcal{C}) \rightarrow \text{ob}(A)$ and let \mathcal{P}_C be the full subcategory of $\text{proj}(\mathcal{C})$ such that $\text{ob}(\mathcal{P}_C) = \{ {}^g {}_{L(x)} \mathcal{C}_? \mid g \in G \text{ and } x \in \text{ob}(A) \}$. Hence:

- $\text{add}(\mathcal{P}_C)$ is a skeleton of $\text{proj}(\mathcal{C})$,
- $\text{ob}(\mathcal{P}_C)$ is G -stable (i.e. ${}^g X \in \text{ob}(\mathcal{P}_C)$ if $g \in G$ and $X \in \text{ob}(\mathcal{P}_C)$).

Let \mathcal{P}_A be the full subcategory of $\text{proj}(A)$ such that $\text{ob}(\mathcal{P}_A) = \{F_\lambda X \mid X \in \text{ob}(\mathcal{P}_C)\}$. One checks the following facts easily:

- \mathcal{C} and \mathcal{P}_C (resp. A and \mathcal{P}_A) are isomorphic k -categories, more precisely, there exists a commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\sim} & \mathcal{P}_C \\ F \downarrow & & \downarrow F_\lambda \\ A & \xrightarrow{\sim} & \mathcal{P}_A \end{array}$$

where $F_\lambda: \mathcal{P}_C \rightarrow \mathcal{P}_A$ is the restriction of $F_\lambda: \text{mod}(\mathcal{C}) \rightarrow \text{mod}(A)$, where $\mathcal{C} \rightarrow \mathcal{P}_C$ is $gL(x) \mapsto \overset{g}{L(x)}\mathcal{C}_?$ and where $A \rightarrow \mathcal{P}_A$ is $x \mapsto F_\lambda L(x)\mathcal{C}_{?,s}$

- $F_\lambda: \mathcal{P}_C \rightarrow \mathcal{P}_A$ is a Galois covering with group G .
- $\text{add}(\mathcal{P}_A)$ is a skeleton of $\text{proj}(A)$.

Throughout this subsection, we shall write \mathcal{A}'_A (resp. \mathcal{A}'_C) for the full dg subcategory of \mathcal{A}_A (resp. of \mathcal{A}_C) of bounded complexes in $\text{add}(\mathcal{P}_A)$ (resp. $\text{add}(\mathcal{P}_C)$).

In order to construct θ_A we will use a formal construction presented in the following lemma. This construction will be used later in order to prove that θ_A induces a Serre functor. In this lemma, we consider \mathcal{P}_A as a full dg subcategory (resp. as a full additive subcategory) of \mathcal{A}_A (resp. $H^0(\mathcal{A}_A)$).

Lemma 3.3. *Let $\varphi: \mathcal{P}_A \rightarrow \mathcal{A}_A$ (resp. $\varphi: \mathcal{P}_A \rightarrow H^0(\mathcal{A}_A)$) be a dg functor (resp. an additive functor). Then, there exists a dg functor $\overline{\varphi}: \mathcal{A}'_A \rightarrow \mathcal{A}_A$ (resp. an additive functor $\overline{\varphi}: H^0(\mathcal{A}'_A) \rightarrow H^0(\mathcal{A}_A)$) commuting with Σ and extending φ . The construction of $\overline{\varphi}$ is such that if φ is the restriction to \mathcal{P}_A of $? \otimes_A Y$ (resp. of $H^0(? \otimes_A Y)$) for some complex Y of locally finite dimensional projective $A - A$ -bimodules, then $\overline{\varphi}$ is the restriction of $? \otimes_A Y$ to \mathcal{A}'_A (resp. of $H^0(? \otimes_A Y)$ to $H^0(\mathcal{A}'_A)$).*

Proof: We shall prove the lemma for $\varphi: \mathcal{P}_A \rightarrow \mathcal{A}_A$, the proof for the second situation is similar. First, φ naturally extends to a dg functor $\varphi: \text{add}(\mathcal{P}_A) \rightarrow \mathcal{A}_A$. Let M be a complex in $\text{add}(\mathcal{P}_A)$. We define $\overline{\varphi}(M)$ as follows:

1. $\overline{\varphi}(M)^n = \bigoplus_{i \in \mathbb{Z}} \varphi(M^i)^{n-i}$, for any $n \in \mathbb{Z}$,
2. $d_{\overline{\varphi}(M)}^n: \overline{\varphi}(M)^n \rightarrow \overline{\varphi}(M)^{n+1}$ is the morphism whose restriction to $\varphi(M^i)^{n-i}$ ($i \in \mathbb{Z}$) is equal to:

$$\varphi(M^i)^{n-i} \xrightarrow{\begin{bmatrix} (-1)^i d_{\varphi(M^i)}^{n-i} \\ \varphi(d_M^i)^{n-i} \end{bmatrix}} \varphi(M^i)^{n+1-i} \oplus \varphi(M^{i+1})^{n-i}$$

It is a basic exercise on complexes to check that $\overline{\varphi}(M) \in \mathcal{A}_A$.

Let $u \in \mathcal{A}^0(M, N)$ where M, N are bounded complexes in $\text{add}(\mathcal{P}_A)$. We define $\overline{\varphi}(u): \overline{\varphi}(M) \rightarrow \overline{\varphi}(N)$ to be the morphism of graded \mathcal{C} -modules defined as follows. For any $n \in \mathbb{Z}$, the restriction of $\overline{\varphi}(u)^n: \bigoplus_{i \in \mathbb{Z}} \varphi(M^i)^{n-i} \rightarrow \bigoplus_{i \in \mathbb{Z}} \varphi(N^i)^{n-i}$ to $\varphi(M^i)^{n-i}$ is equal to $\varphi(u^i)^{n-i}: \varphi(M^i)^{n-i} \rightarrow \varphi(N^i)^{n-i}$. Hence $\overline{\varphi}(u) \in \mathcal{A}^0(\overline{\varphi}(M), \overline{\varphi}(N))$.

Let $u \in \mathcal{A}^d(M, N)$ where M, N are complexes in $\text{add}(\mathcal{P}_A)$. Then $u \in \mathcal{A}^0(M, \Sigma^d N)$. Thus $\overline{\varphi}(u)$, as defined above, lies in $\mathcal{A}^0(M, \Sigma^d N) = \mathcal{A}^d(M, N)$. Hence, we have defined a graded functor $\overline{\varphi}: \mathcal{A}'_A \rightarrow \mathcal{A}_A$ which extends φ . Moreover, the following facts are easily verified:

1. $\overline{\varphi}$ is a dg functor and commutes with the shift,
2. if φ is the restriction of $? \otimes_A Y$ to \mathcal{P}_A , then $\overline{\varphi}$ is the restriction of $? \otimes_A Y$ to \mathcal{A}'_A .

■

Remark 3.4. *The proof of the preceding lemma shows that if \mathcal{C} and \mathcal{C}' are locally bounded categories with $\text{gldim } \mathcal{C} < \infty$, if \mathcal{P} is an additive full subcategory of $\text{mod}(\mathcal{C})$ containing one representative of each isomorphism class of indecomposable projective \mathcal{C} -module, then any additive functor $\mathcal{P} \rightarrow \mathcal{C}^b(\mathcal{C}')$ (where $\mathcal{C}^b(?)$ is the bounded category of complexes) defines a triangle functor $\mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(\mathcal{C}')$ extending the composition $\mathcal{P} \rightarrow \mathcal{C}^b(\mathcal{C}') \rightarrow \mathcal{D}^b(\mathcal{C}')$.*

Using the fact that $F_\lambda: \mathcal{P}_C \rightarrow \mathcal{P}_A$ is a Galois covering, we are able to exhibit a dg functor $\theta_A: \mathcal{A}_A \rightarrow \mathcal{A}_A$ compatible with θ_C :

Lemma 3.5. *There exists a dg functor $\theta_A: \mathcal{A}_A \rightarrow \mathcal{A}_A$ which commutes with the shift and such that the following diagram commutes up to isomorphism of dg functors:*

$$\begin{array}{ccc} \mathcal{A}_C & \xrightarrow{\theta_C} & \mathcal{A}_C \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ \mathcal{A}_A & \xrightarrow{\theta_A} & \mathcal{A}_A \end{array} \quad (D_4)$$

Proof: First, let us define θ_A on \mathcal{P}_A . Since $\theta_C: \mathcal{A}_C \rightarrow \mathcal{A}_C$ (resp. $F_\lambda: \mathcal{A}_C \rightarrow \mathcal{A}_A$) is G -equivariant (resp. G -invariant) and since $F_\lambda: \mathcal{P}_C \rightarrow \mathcal{P}_A$ is a Galois covering with group G , we infer that there exists a graded functor $\theta_A: \text{add}(\mathcal{P}_A) \rightarrow \mathcal{A}_A$ such that the following diagram is commutative:

$$\begin{array}{ccc} \text{add}(\mathcal{P}_C) & \xrightarrow{\theta_C} & \mathcal{A}_C \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ \text{add}(\mathcal{P}_A) & \xrightarrow{\theta_A} & \mathcal{A}_A \end{array}$$

Notice that if $f = F_\lambda(h)$ with h a morphism in \mathcal{P}_C , then $d(\theta_A(f)) = d(F_\lambda(\theta_C(h))) = 0$ because $F_\lambda: \mathcal{A}_C \rightarrow \mathcal{A}_A$ and θ_C are dg functors. Moreover, since $F_\lambda: \mathcal{P}_C \rightarrow \mathcal{P}_A$ is a Galois covering, we also know that any morphism in \mathcal{P}_A is the sum of (finitely many) morphisms of the form $F_\lambda(h)$. This proves that $\theta_A: \mathcal{P}_A \rightarrow \mathcal{A}_A$ is a dg functor. From Lemma 3.3 applied to $\varphi = \theta_A$, we deduce that θ_A is the restriction of a dg functor $\theta_A: \mathcal{A}'_A \rightarrow \mathcal{A}_A$ commuting with the shift. Also, Lemma 3.3 applied to φ equal to the restriction of θ_C to \mathcal{P}_C , proves that the preceding commutative diagram extends to the following one:

$$\begin{array}{ccc} \mathcal{A}'_C & \xrightarrow{\theta_C} & \mathcal{A}_C \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ \mathcal{A}'_A & \xrightarrow{\theta_A} & \mathcal{A}_A \end{array}$$

Recall that the embedding $\text{add}(\mathcal{P}_C) \hookrightarrow \text{proj}(\mathcal{C})$ (resp. $\text{add}(\mathcal{P}_A) \hookrightarrow \text{proj}(A)$) is an equivalence of additive categories. Thus, the embedding $\mathcal{A}'_C \hookrightarrow \mathcal{A}_C$ (resp. $\mathcal{A}'_A \hookrightarrow \mathcal{A}_A$) is an equivalence of dg categories. Therefore, if $\mathcal{A}_A \xrightarrow{\sim} \mathcal{A}'_A$ is an inverse equivalence of $\mathcal{A}'_A \hookrightarrow \mathcal{A}_A$, then the composition $\mathcal{A}_A \xrightarrow{\sim} \mathcal{A}'_A \xrightarrow{\theta_A} \mathcal{A}_A$ fits the lemma. \blacksquare

3.3 The Serre functor ν_A induced by θ_A

Let us fix once and for all the section a dg functor θ_A as in Lemma 3.5 and let ν_A be a triangle functor $\mathcal{D}^b(A) \rightarrow \mathcal{D}^b(A)$ such that the following diagram commutes:

$$\begin{array}{ccc} H^0(\mathcal{A}_A) & \xrightarrow{\sim} & \mathcal{D}^b(A) \\ H^0(\theta_A) \downarrow & & \downarrow \nu_A \\ H^0(\mathcal{A}_A) & \xrightarrow{\sim} & \mathcal{D}^b(A) \end{array} \quad (D_5)$$

We shall keep the notations $(\mathcal{P}_A, \mathcal{P}_C, \mathcal{A}'_A$ and $\mathcal{A}'_C)$ introduced in the preceding subsection. In order to prove that ν_A is a Serre functor, we first prove the following lemma.

Lemma 3.6. *There exists an isomorphism of functors $\mathcal{P}_C \rightarrow \text{mod}(A)$:*

$$F_\lambda(?) \otimes_A DA \simeq F_\lambda(?) \otimes_C DC$$

Proof: We shall use some well known isomorphisms of functors. Hence, the necessary verifications are left to the reader. The two following functors from \mathcal{C} to $\text{mod}(\mathcal{C})$ are isomorphic:

$$x \mapsto {}_x\mathcal{C} \otimes_C DC \quad \text{and} \quad x \mapsto D({}_? \mathcal{C}_x)$$

Moreover, the following functors from \mathcal{C} to $mod(A)$ are isomorphic:

$$x \mapsto F_\lambda D({}_? \mathcal{C}_x) \quad \text{and} \quad x \mapsto D({}_? A_{F(x)})$$

From these two isomorphisms, we deduce that the following functors from \mathcal{C} to $mod(A)$ are isomorphic:

$$x \mapsto F_\lambda ({}_x \mathcal{C}_? \otimes_{\mathcal{C}} DC) \quad \text{and} \quad x \mapsto D({}_? A_{F(x)}) \quad (i)$$

On the other hand, the following functors from \mathcal{C} to $mod(A)$ are isomorphic:

$$x \mapsto F_\lambda {}_x \mathcal{C}_? \quad \text{and} \quad x \mapsto {}_{F(x)} A_?$$

Moreover, we have an isomorphism between the following functors from the full subcategory $\{ {}_x A_? \mid x \in ob(A) \}$ of $mod(A)$ to $mod(A)$:

$${}_x A_? \mapsto {}_x A_? \otimes_A DA \quad \text{and} \quad x \mapsto D({}_? A_x)$$

From these two isomorphisms, we deduce that the following functors from \mathcal{C} to $mod(A)$ are isomorphic:

$$x \mapsto F_\lambda {}_x \mathcal{C}_? \otimes_A DA \quad \text{and} \quad x \mapsto D({}_? A_{F(x)}) \quad (ii)$$

From (i), (ii) and from the isomorphism $x \mapsto {}_x \mathcal{C}_?$ from \mathcal{C} to $\mathcal{P}_{\mathcal{C}}$ we deduce the announced isomorphism. \blacksquare

Lemma 3.7. $\nu_A: \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(A)$ is a Serre functor.

Proof: Recall from Lemma 3.1 that $p: X \twoheadrightarrow DC$ is a projective resolution of DC viewed as a $\mathcal{C} - \mathcal{C}$ -bimodule and that $\theta_{\mathcal{C}} = ? \otimes_{\mathcal{C}} X: \mathcal{A}_{\mathcal{C}} \rightarrow \mathcal{A}_{\mathcal{C}}$. Let us fix $q: Y \twoheadrightarrow DA$ a projective resolution of D viewed

as an $A - A$ -bimodule so that $? \otimes_A^{\mathbb{L}} Y: \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(A)$ is a Serre functor. In order to prove that ν_A and $? \otimes_A^{\mathbb{L}} Y$ are isomorphic, we shall first construct a morphism $\psi: H^0(\theta_A) \rightarrow H^0({}_? \otimes_A Y)$ between functors from $H^0(\mathcal{A}_A)$ to $H^0(\mathcal{A}_A)$. In this purpose, we shall proceed in two steps: first construct a natural

transformation from $\mathcal{P}_A \hookrightarrow H^0(\mathcal{A}_A) \xrightarrow{H^0(\theta_A)} H^0(\mathcal{A}_A)$ to $\mathcal{P}_A \hookrightarrow H^0(\mathcal{A}_A) \xrightarrow{H^0({}_? \otimes_A Y)} H^0(\mathcal{A}_A)$, then, extend this natural transformation to $H^0(\mathcal{A}_A)$ using the construction of θ_A . Thanks to Lemma 3.6, and because $q: Y \twoheadrightarrow DA$ is a projective resolution, we have a natural transformation ψ where the unlabelled arrows are the natural inclusions:

$$\begin{array}{ccccccc} \mathcal{P}_{\mathcal{C}} & \longrightarrow & H^0(\mathcal{A}_{\mathcal{C}}) & \xrightarrow{H^0(\theta_{\mathcal{C}})} & H^0(\mathcal{A}_{\mathcal{C}}) & \xrightarrow{F_\lambda} & H^0(\mathcal{A}_A) \\ \psi \downarrow & & & & & & \\ \mathcal{P}_{\mathcal{C}} & \longrightarrow & H^0(\mathcal{A}_A) & \xrightarrow{F_\lambda} & H^0(\mathcal{A}_A) & \xrightarrow{H^0({}_? \otimes_A Y)} & H^0(\mathcal{A}_A) \end{array}$$

Notice that since $\theta_{\mathcal{C}} = ? \otimes_{\mathcal{C}} X$, since $p: X \twoheadrightarrow DC$ and $q: Y \twoheadrightarrow DA$ are projective resolutions, and since $F_\lambda({}_? \otimes_A DA) \simeq F_\lambda({}_? \otimes_{\mathcal{C}} DC)$ as functors $\mathcal{P}_{\mathcal{C}} \rightarrow mod(A)$, the above natural transformation is an isomorphism of functors.

Using the fact that the dg functors $\theta_A \circ F_\lambda$ and $F_\lambda \circ \theta_{\mathcal{C}}$ are isomorphic, we deduce the following isomorphism of functors, still denoted by ψ , notice that two involved functors are G -invariant:

$$\begin{array}{ccccccc} \mathcal{P}_{\mathcal{C}} & \xrightarrow{F_\lambda} & \mathcal{P}_A & \longrightarrow & H^0(\mathcal{A}_A) & \xrightarrow{H^0(\theta_A)} & H^0(\mathcal{A}_A) \\ \psi \downarrow & & & & & & \\ \mathcal{P}_{\mathcal{C}} & \xrightarrow{F_\lambda} & \mathcal{P}_A & \longrightarrow & H^0(\mathcal{A}_A) & \xrightarrow{H^0({}_? \otimes_A Y)} & H^0(\mathcal{A}_A) \end{array}$$

Since $F_\lambda: \mathcal{P}_C \rightarrow \mathcal{P}_A$ is a Galois covering, this defines the following isomorphism of functors:

$$\begin{array}{ccccc} & \text{add}(\mathcal{P}_A) & \longrightarrow & H^0(\mathcal{A}_A) & \xrightarrow{H^0(\theta_A)} & H^0(\mathcal{A}_A) \\ \psi \downarrow & & & & & \\ & \text{add}(\mathcal{P}_A) & \longrightarrow & H^0(\mathcal{A}_A) & \xrightarrow{H^0(? \otimes_A Y)} & H^0(\mathcal{A}_A) \end{array}$$

Recall from the proof of Lemma 3.5, that $\mathcal{A}'_A \xrightarrow{\theta_A} \mathcal{A}_A$ is constructed starting from its restriction to $\text{add}(\mathcal{P}_A)$ and that if one performs this construction to the restriction of $\mathcal{A}_A \xrightarrow{? \otimes_A Y} \mathcal{A}_A$ to $\text{add}(\mathcal{P}_A)$, then, one gets the restriction of this dg functor to \mathcal{A}'_A . Therefore, the above isomorphism of functors gives rise to the following natural transformation, where the unlabelled arrows are the natural inclusions:

$$\begin{array}{ccccc} & H^0(\mathcal{A}'_A) & \longrightarrow & H^0(\mathcal{A}_A) & \xrightarrow{H^0(\theta_A)} & H^0(\mathcal{A}_A) \\ \psi \downarrow & & & & & \\ & H^0(\mathcal{A}'_A) & \longrightarrow & H^0(\mathcal{A}_A) & \xrightarrow{H^0(? \otimes_A Y)} & H^0(\mathcal{A}_A) \end{array}$$

Since the embedding $\mathcal{A}'_A \hookrightarrow \mathcal{A}_A$ and the functor $H^0(\mathcal{A}) \rightarrow \mathcal{D}^b(A)$ are equivalences, we deduce a natural transformation $\psi: \nu_A \rightarrow ? \otimes_A Y$ between triangle functors $\mathcal{D}^b(A) \rightarrow \mathcal{D}^b(A)$. Moreover, ψ_M is an isomorphism for any $M \in \mathcal{P}_A$ as observed above. Since the smallest triangulated subcategory of $\mathcal{D}^b(A)$ containing \mathcal{P}_A is $\mathcal{D}^b(A)$ (recall that $\text{gldim } A < \infty$ and that $\text{add}(\mathcal{P}_A) \hookrightarrow \text{proj}(A)$ is an equivalence), we deduce that $\psi: \nu_A \rightarrow ? \otimes_A Y$ is an isomorphism. So ν_A is a Serre functor. \blacksquare

4 Covering properties on the cluster category and on the triangulated hull

Let $F: \mathcal{C} \rightarrow A$ be a Galois covering with group G and with \mathcal{C} and A locally bounded. Let us fix dg functors θ_A, θ_C and Serre functors ν_A, ν_C as obtained in the preceding section.

4.1 The additive functor $F_\lambda: C_C \rightarrow C_A$

We have a G -equivariant Serre functor $\nu_C: \mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(\mathcal{C})$, hence G acts naturally on C_C (see Section 1). The following lemma shows the compatibility between $F_\lambda: \mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(A)$ and $\nu_C: \mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(\mathcal{C})$.

Lemma 4.1. $F_\lambda \circ \nu_C \simeq \nu_A \circ F_\lambda$ and therefore $F_\lambda \circ \tau_C \simeq \tau_A \circ F_\lambda$ where τ_τ denotes the Auslander-Reiten translation on $\mathcal{D}^b(?)$.

Proof: From (D_4) , we deduce a commutative diagram of additive categories

$$\begin{array}{ccc} H^0(\mathcal{A}_C) & \xrightarrow{H^0(\theta_C)} & H^0(\mathcal{A}_C) \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ H^0(\mathcal{A}_A) & \xrightarrow{H^0(\theta_A)} & H^0(\mathcal{A}_A) \end{array}$$

This diagram together with (D_3) and (D_5) imply the first assertion. Since $\tau_\tau \simeq \nu_\tau[-1]$ they imply the second assertion also. \blacksquare

Tanks to Lemma 4.1, we deduce that there exists an additive functor $F_\lambda: C_C \rightarrow C_A$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{D}^b(\mathcal{C}) & \longrightarrow & C_C \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ \mathcal{D}^b(A) & \longrightarrow & C_A \end{array} \tag{D_6}$$

Lemma 4.2. $F_\lambda: C_C \rightarrow C_A$ has the covering property.

Proof: Let $X, Y \in C_C$. Then, the mapping $\iota: \bigoplus_{g \in G} \text{Hom}_{C_C}({}^g X, Y) \rightarrow \text{Hom}_{C_A}(F_\lambda X, F_\lambda Y)$ induced by F_λ is the direct sum of the mappings:

$$\iota_n: \bigoplus_{g \in G} \text{Hom}_{\mathcal{D}^b(C)}((\nu_C \Sigma^{-2})^n {}^g X, Y) \rightarrow \text{Hom}_{\mathcal{D}^b(A)}((\nu_A \Sigma^{-2})^n F_\lambda X, F_\lambda Y)$$

for $n \in \mathbb{Z}$. Using the isomorphism $\nu_A \circ F_\lambda \simeq F_\lambda \circ \nu_C$ of Lemma 4.1, using the fact that $F_\lambda: \mathcal{D}^b(C) \rightarrow \mathcal{D}^b(A)$ has the covering property (see Lemma 2.1) and using the G -equivariance of ν_C (and the one of Σ), we deduce that ι_n is an isomorphism because the following mapping is an isomorphism:

$$\bigoplus_{g \in G} \text{Hom}_{\mathcal{D}^b(C)}({}^g ((\nu_C \Sigma^{-2})^n X), Y) \rightarrow \text{Hom}_{\mathcal{D}^b(A)}(F_\lambda ((\nu_C \Sigma^{-2})^n X), Y)$$

Hence, ι is a bijective mapping. With similar arguments, we also prove that the mapping $\bigoplus_{g \in G} \text{Hom}_{C_C}(X, {}^g Y) \rightarrow \text{Hom}_{C_A}(F_\lambda X, F_\lambda Y)$ induced by F_λ is bijective. \blacksquare

The preceding lemma gives an additive functor $C_C \rightarrow C_A$ with the covering property. On the other hand, thanks to (D_4) we know that there exists a dg functor $F_\lambda: \mathcal{B}_C \rightarrow \mathcal{B}_A$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_C & \longrightarrow & \mathcal{B}_C \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ \mathcal{A}_A & \longrightarrow & \mathcal{B}_A \end{array}$$

Hence, the following diagram commutes also:

$$\begin{array}{ccc} H^0(\mathcal{A}_C) & \longrightarrow & H^0(\mathcal{B}_C) \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ H^0(\mathcal{A}_A) & \longrightarrow & H^0(\mathcal{B}_A) \end{array} \quad (D_7)$$

For short, we shall write $F_\lambda: H^0(\mathcal{B}_C) \rightarrow H^0(\mathcal{B}_A)$ instead of $H^0(F_\lambda): H^0(\mathcal{B}_C) \rightarrow H^0(\mathcal{B}_A)$.

Lemma 4.3. $F_\lambda: \mathcal{B}_C \rightarrow \mathcal{B}_A$ and $F_\lambda: H^0(\mathcal{B}_C) \rightarrow H^0(\mathcal{B}_A)$ have the covering property.

Proof: Using arguments similar to those of the proof of Lemma 4.2, we prove that $F_\lambda: \mathcal{B}_C \rightarrow \mathcal{B}_A$ has the covering property. Using basic linear algebra arguments one infers that $H^0(F_\lambda): H^0(\mathcal{B}_C) \rightarrow H^0(\mathcal{B}_A)$ also has the covering property. \blacksquare

The following lemma shows the compatibility between $F_\lambda: H^0(\mathcal{B}_C) \rightarrow H^0(\mathcal{B}_A)$ and $F_\lambda: C_C \rightarrow C_A$.

Lemma 4.4. The following diagram commutes:

$$\begin{array}{ccc} H^0(\mathcal{B}_C) & \xrightarrow{\sim} & C_C \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ H^0(\mathcal{B}_A) & \xrightarrow{\sim} & C_A \end{array} \quad (D_8)$$

Proof: Let us consider the following diagram:

$$\begin{array}{ccccc} H^0(\mathcal{A}_C) & \longrightarrow & H^0(\mathcal{B}_C) & & \\ \downarrow F_\lambda & \searrow & \downarrow & \searrow & \\ & & \mathcal{D}^b(C) & \xrightarrow{F_\lambda} & C_C \\ & & \downarrow & & \downarrow F_\lambda \\ H^0(\mathcal{A}_A) & \xrightarrow{F_\lambda} & H^0(\mathcal{B}_A) & & \\ & \searrow & \downarrow & \searrow & \\ & & \mathcal{D}^b(A) & \longrightarrow & C_A \end{array}$$

Thanks to (D_2) , (D_6) and (D_7) , we know that all the faces of the above diagram commute excepted the one of the lemma. By construction, $H^0(\mathcal{A}_C) \rightarrow H^0(\mathcal{B}_C)$ extends the identity map on objects and any morphism in $H^0(\mathcal{B}_C)$ is the sum of (finitely many) images of morphisms in $H^0(\mathcal{A}_C)$. This proves the announced commutativity. \blacksquare

4.2 The triangle functor $F_\lambda: \mathcal{M}_C \rightarrow \mathcal{M}_A$

The dg functor $F_\lambda: \mathcal{B}_C \rightarrow \mathcal{B}_A$ defines a triangle functor $\mathcal{D}(\mathcal{B}_C) \rightarrow \mathcal{D}(\mathcal{B}_A)$ which we will denote F_λ for simplicity. Since $F_\lambda: \mathcal{B}_C \rightarrow \mathcal{B}_A$ is a dg functor, we have the following commutative diagram:

$$\begin{array}{ccc} H^0(\mathcal{B}_C) & \longrightarrow & \mathcal{D}(\mathcal{B}_C) \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ H^0(\mathcal{B}_A) & \longrightarrow & \mathcal{D}(\mathcal{B}_A) \end{array} \quad (D_9)$$

Also, recall that \mathcal{M}_C (resp. \mathcal{M}_A) is the triangle closure of $H^0(\mathcal{B}_C)$ (resp. of $H^0(\mathcal{B}_A)$) in $\mathcal{D}(\mathcal{B}_C)$ (resp. in $\mathcal{D}(\mathcal{B}_A)$). Since $F_\lambda: \mathcal{D}(\mathcal{B}_C) \rightarrow \mathcal{D}(\mathcal{B}_A)$ is a triangle functor, we deduce that there exists a triangle functor $F_\lambda: \mathcal{M}_C \rightarrow \mathcal{M}_A$ such that the following diagram commutes:

$$\begin{array}{ccc} H^0(\mathcal{B}_C) & \longrightarrow & \mathcal{M}_C \\ F_\lambda \downarrow & & \downarrow F_\lambda \\ H^0(\mathcal{B}_A) & \longrightarrow & \mathcal{M}_A \end{array} \quad (D_{10})$$

Lemma 4.5. $F_\lambda: \mathcal{M}_C \rightarrow \mathcal{M}_A$ has the covering property.

Proof: For $X, Y \in \mathcal{D}(\mathcal{B}_C)$, let us denote by $\iota_{X,Y}$ the mapping:

$$\bigoplus_{g \in G} \text{Hom}_{\mathcal{D}(\mathcal{B}_C)}({}^g X, Y) \rightarrow \text{Hom}_{\mathcal{D}(\mathcal{B}_A)}(F_\lambda X, F_\lambda Y)$$

induced by F_λ . Obviously $\iota_{X,Y}$ is an isomorphism if and only if $\iota_{\Sigma X, Y}$ is an isomorphism, if and only if $\iota_{X, \Sigma Y}$ is an isomorphism. Moreover, using the fact that Hom -functors are cohomological, we infer that if $X \rightarrow X' \rightarrow X'' \rightarrow \Sigma X$ is a triangle in $\mathcal{D}(\mathcal{B}_C)$, then $\iota_{X,Y}$, $\iota_{X',Y}$ and $\iota_{X'',Y}$ are all isomorphisms as soon as two of them are isomorphisms. Finally, Lemma 4.3 implies that $\iota_{X,Y}$ is an isomorphism for any $X, Y \in H^0(\mathcal{B}_C)$. These three arguments prove that $\iota_{X,Y}$ is an isomorphism for any $X, Y \in \mathcal{M}_C$. Using similar arguments, one also proves that $\bigoplus_{g \in G} \text{Hom}_{\mathcal{D}(\mathcal{B}_C)}(X, {}^g Y) \rightarrow \text{Hom}_{\mathcal{D}(\mathcal{B}_A)}(F_\lambda X, F_\lambda Y)$ is an isomorphism for any $X, Y \in \mathcal{M}_C$. \blacksquare

To summarise the constructions made in this section and in the two preceding ones, we deduce from (D_1) , (D_2) , (D_6) , (D_7) , (D_8) and (D_{10}) that we have the following commutative diagram:

$$\begin{array}{ccccc} & & H^0(\mathcal{A}_C) & \longrightarrow & H^0(\mathcal{B}_C) & \longrightarrow & \mathcal{M}_C & & (D) \\ & & \downarrow F_\lambda & & \downarrow F_\lambda & & \downarrow F_\lambda & & \\ & & H^0(\mathcal{A}_A) & \longrightarrow & H^0(\mathcal{B}_A) & \longrightarrow & \mathcal{M}_A & & \\ \text{mod}(\mathcal{C}) & \longrightarrow & \mathcal{D}^b(\mathcal{C}) & \longrightarrow & \mathcal{C}_C & & & & \\ \downarrow F_\lambda & & \downarrow F_\lambda & & \downarrow F_\lambda & & & & \\ \text{mod}(A) & \longrightarrow & \mathcal{D}^b(A) & \longrightarrow & \mathcal{C}_A & & & & \end{array}$$

(Note: The diagram above is a simplified representation of the complex commutative diagram in the image, showing the relationships between the various categories and functors.)

where:

1. the horizontal and oblique arrows are the natural ones,
2. G acts on all the categories relative to \mathcal{C} ,
3. every vertical arrow has the covering property for the corresponding G -action,
4. $F_\lambda: \mathcal{D}^b(\mathcal{C}) \rightarrow \mathcal{D}^b(A)$ and $F_\lambda: \mathcal{M}_\mathcal{C} \rightarrow \mathcal{M}_A$ are triangle functors.

From now on, whenever A is a locally bounded k -category, we shall identify $\mathcal{D}^b(A)$ and $H^0(\mathcal{A}_A)$, C_A and $H^0(\mathcal{B}_A)$ and we shall denote by $X \mapsto X^\wedge$ the fully faithful additive functor $C_A \hookrightarrow \mathcal{M}_A$ induced by these identifications and by the Yoneda embedding $H^0(\mathcal{B}_A) \hookrightarrow \mathcal{M}_A$. In particular, \mathcal{M}_A is generated, as a triangulated category, by C_A . Also, whenever $F: \mathcal{C} \rightarrow A$ is a Galois covering, we will always assume that we are given the Serre functors ν_A and $\nu_\mathcal{C}$ and the dg functors θ_A and $\theta_\mathcal{C}$ as constructed in the previous section, so that (D) holds.

5 A criterion on representability of objects in the triangulated hull

Let A be a locally bounded k -category. In the present section, we aim at presenting a useful criterion for an object $X \in \mathcal{M}_A$ to be of the form M^\wedge with $M \in C_A$. We shall denote by τ_A the Auslander-Reiten translate on $\mathcal{D}^b(A)$. Recall ([14]) that $\tau_A: \mathcal{D}^b(A) \rightarrow \mathcal{D}^b(A)$ is a triangle equivalence isomorphic to $\nu_A \Sigma^{-1}$ for any Serre functor ν_A and that τ_A extends the Auslander-Reiten translation on $\text{mod}(A) \setminus \text{proj}(A)$. Recall also that if $X \in \text{mod}(A)$ and $i > 0$, then $(\tau_A \Sigma^{-1})^i X \simeq \Sigma^{-l} M$ (resp. $(\tau_A \Sigma^{-1})^{-i} X \simeq \Sigma^l M$) for some $M \in \text{mod}(A)$ and some $l \in \{i, i+1, \dots, 2i\}$. We begin with the following useful lemma:

Lemma 5.1. *Let $X \in \text{mod}(A)$ and let $P \in \text{mod}(A)$ be projective. Then, the natural inclusion mapping $\text{Hom}_{\mathcal{D}^b(A)}(P, X) \hookrightarrow \text{Hom}_{C_A}(P, X)$ is a bijection.*

Proof: It suffices to show that $\text{Hom}_{\mathcal{D}^b(A)}(P, (\tau_A \Sigma^{-1})^n X) = 0$ for any $n \in \mathbb{Z} \setminus \{0\}$. For any such n , we have $(\tau_A \Sigma^{-1})^n X \simeq \Sigma^{l_n} M_n$ where $M_n \in \text{mod}(A)$ and $l_n \neq 0$. Since P is projective, we deduce that $\text{Hom}_{\mathcal{D}^b(A)}(P, (\tau_A \Sigma^{-1})^n X) = 0$. ■

Later we shall compare A -modules which are isomorphic in C_A . In this purpose, the two following lemmas will be helpful.

Lemma 5.2. *Let $X, Y \in \text{mod}(A)$. If $X \simeq Y$ in C_A , then $X \simeq Y$ in $\text{mod}(A)$.*

Proof: The assumption implies that there exists $i \in \mathbb{Z}$ such that $\Sigma^i Y \simeq \tau_A^i X$ in $\mathcal{D}^b(A)$. Moreover, there exists $M \in \text{mod}(A)$ and $l \in \mathbb{Z}$ with the sign of i and such that $\tau_A^i X \simeq \Sigma^{-l} M$ in $\mathcal{D}^b(A)$. If $i \neq 0$, we infer that $\Sigma^{i+l} Y \simeq M$ in $\mathcal{D}^b(A)$, with $i+l \neq 0$. This is impossible, so $i = 0$ and $X \simeq Y$ in $\mathcal{D}^b(A)$ and therefore in $\text{mod}(A)$. ■

Lemma 5.3. *Assume that A is a piecewise hereditary finite dimensional algebra and let $F: \mathcal{C} \rightarrow A$ be a Galois covering with group G and with \mathcal{C} locally bounded. Let $X, Y \in \mathcal{M}_\mathcal{C}$ be such that $F_\lambda X \simeq F_\lambda Y$ in $\text{ind}(C_A)$. Then, there exists $g \in G$ such that $Y \simeq {}^g X$ in $\mathcal{M}_\mathcal{C}$.*

Proof: Recall that the assumption on A implies that C_A and \mathcal{M}_A coincide. Also, notice that since $F_\lambda X$ and $F_\lambda Y$ are indecomposable, X and Y are so. Since $F_\lambda: \mathcal{M}_\mathcal{C} \rightarrow C_A$ has the covering property and since $F_\lambda X \simeq F_\lambda Y$, there exist morphisms $\varphi_i \in \text{Hom}_{\mathcal{M}_\mathcal{C}}({}^{g_i} X, Y)$ (with $1 \leq i \leq t$, $g_i \in G$) such that $f := F_\lambda \varphi_1 + \dots + F_\lambda \varphi_t: F_\lambda X \rightarrow F_\lambda Y$ is an isomorphism in C_A . Thus, $\text{Id}_{F_\lambda X} = f^{-1} \circ F_\lambda \varphi_1 + \dots + f^{-1} \circ F_\lambda \varphi_t$. Since $F_\lambda X$ is indecomposable, $\text{End}_{C_A}(F_\lambda X)$ is a local algebra and therefore, at least one of the $f^{-1} \circ F_\lambda \varphi_i$ is invertible. Notice that if $f^{-1} F_\lambda \varphi_i$ is invertible, then $F_\lambda \varphi_i: F_\lambda X \rightarrow F_\lambda Y$ is a section and therefore an isomorphism (because $F_\lambda X$ and $F_\lambda Y$ are indecomposable). This proves that there exists $\varphi \in \text{Hom}_{\mathcal{M}_\mathcal{C}}({}^{g_0} X, Y)$ (with $g_0 \in G$) such that $F_\lambda \varphi: F_\lambda X \rightarrow F_\lambda Y$ is an isomorphism. Since $F_\lambda: \mathcal{M}_\mathcal{C} \rightarrow C_A$ has the covering property, we infer that there exists $(\psi_g)_{g \in G} \in \bigoplus_{g \in G} \text{Hom}_{\mathcal{M}_\mathcal{C}}(Y, {}^g X)$ such that $\text{Id}_{F_\lambda X} = \sum_{g \in G} F_\lambda \psi_g \circ F_\lambda \varphi$. Since $\psi_g \circ \varphi \in \text{Hom}_{\mathcal{M}_\mathcal{C}}({}^{g_0} X, {}^g X)$ for every g , the same argument proves that $\psi_g \varphi = 0$ if $g \neq g_0$ and that $\psi_{g_0} \varphi = \text{Id}_X$. Thus, φ is left invertible. Using similar arguments, one proves that φ is right invertible also. Hence, φ is invertible and $Y \simeq {}^{g_0} X$ in $\mathcal{M}_\mathcal{C}$. ■

If $X \in \mathcal{M}_A$, then $x \in \text{ob}(A) \mapsto \text{Hom}_{\mathcal{M}_A}(x A^\wedge, X)$ defines an A -module which we shall denote by \underline{X} . This way, we get an additive functor:

$$\begin{array}{ccc} \mathcal{M}_A & \longrightarrow & \text{mod}(A) \\ X & \longmapsto & \underline{X} \end{array}$$

Proposition 5.4. *Let $X \in \text{mod}(A)$. Then, there exists an isomorphism of A -modules: $X \xrightarrow{\sim} \underline{X}^\wedge$. This isomorphism is functorial in X .*

Proof: Let $x \in \text{ob}(A)$. Recall the natural isomorphism:

$$\begin{aligned} X(x) &\longrightarrow \text{Hom}_A({}_x A?, X) \\ m &\longmapsto (u \in {}_x A_y \mapsto mu \in X(y)) \end{aligned}$$

Moreover, $\text{Hom}_A({}_x A?, X) \rightarrow \text{Hom}_{\mathcal{D}^b(A)}({}_x A?, X)$ and $\text{Hom}_{C_A}({}_x A?, X) \rightarrow \text{Hom}_{\mathcal{M}_A}(({}_x A?)^\wedge, X^\wedge)$ are isomorphisms because $\text{mod}(A) \rightarrow \mathcal{D}^b(A)$ and $C_A \rightarrow \mathcal{M}_A$ are fully faithful. Finally, Lemma 5.1 proves that $\text{Hom}_{\mathcal{D}^b(A)}({}_x A?, X) \rightarrow \text{Hom}_{C_A}({}_x A?, X)$ is an isomorphism. Therefore, we have an isomorphism of vector spaces $X(x) \xrightarrow{\sim} \text{Hom}_{\mathcal{M}_A}(({}_x A?)^\wedge, X^\wedge)$. We let the reader verify that this defines a functorial isomorphism $X \xrightarrow{\sim} \underline{X}^\wedge$ of A -modules. \blacksquare

As a corollary, we deduce the following criterion.

Corollary 5.5. *Let $F: \mathcal{C} \rightarrow A$ be a Galois covering. Let $X \in \text{mod}(A)$ and assume that there exists $Y \in \mathcal{M}_{\mathcal{C}}$ such that $F_\lambda Y \simeq X^\wedge$ in C_A . Then, $X \simeq F_\lambda \underline{Y}$ in $\text{mod}(A)$. If moreover A is a piecewise hereditary finite dimensional k -algebra and if X is indecomposable, then there exists $g \in G$ such that $({}^g \underline{Y})^\wedge \simeq Y$ (so that Y is necessarily a \mathcal{C} -module).*

Proof: Let us fix a lifting $L: \text{ob}(A) \rightarrow \text{ob}(\mathcal{C})$ of $F: \text{ob}(\mathcal{C}) \rightarrow \text{ob}(A)$. Let $x \in \text{ob}(A)$. Since $F_\lambda: \mathcal{M}_{\mathcal{C}} \rightarrow \mathcal{M}_A$ has the covering property, we deduce that the following mapping defined by F_λ is bijective:

$$\lambda_x: \bigoplus_{g \in G} \text{Hom}_{\mathcal{M}_{\mathcal{C}}}({}^g ({}_{L(x)} \mathcal{C}?)^\wedge, Y) \rightarrow \text{Hom}_{\mathcal{M}_A}((F_\lambda {}_{L(x)} \mathcal{C}?)^\wedge, F_\lambda Y)$$

For every $x \in \text{ob}(A)$, let $\iota_x: F_\lambda {}_{L(x)} \mathcal{C}? \xrightarrow{\sim} {}_x A?$ be the isomorphism in $\text{mod}(A)$ such that for any $y \in \text{ob}(A)$, $\iota_x(y): (F_\lambda {}_{L(x)} \mathcal{C}?) (y) \rightarrow ({}_x A?) (y)$ is the bijective mapping $\bigoplus_{y' \in F^{-1}(y)} {}_{L(x)} \mathcal{C}_{y'} \rightarrow {}_x A_y$ induced by $F: \mathcal{C} \rightarrow A$.

The isomorphisms ι_x and λ_x define the following isomorphism:

$$\mu_x: \bigoplus_{g \in G} \text{Hom}_{\mathcal{M}_{\mathcal{C}}}({}^g ({}_{L(x)} \mathcal{C}?)^\wedge, Y) \rightarrow \text{Hom}_{\mathcal{M}_A}(({}_x A?)^\wedge, F_\lambda Y)$$

Hence, for every x , we have an isomorphism of vector spaces $\mu_x: F_\lambda \underline{Y}(x) \xrightarrow{\sim} \underline{F_\lambda Y}(x)$ and the reader may easily check that these isomorphisms define an isomorphism $F_\lambda \underline{Y} \simeq \underline{F_\lambda Y}$ in $\text{mod}(A)$. From the isomorphism $X^\wedge \simeq F_\lambda Y$ in \mathcal{M}_A and from Proposition 5.4, we deduce the isomorphisms in $\text{mod}(A)$: $X \simeq \underline{X}^\wedge \simeq \underline{F_\lambda Y} \simeq F_\lambda \underline{Y}$. The last assertion is a consequence of these isomorphisms and of Lemma 5.3. \blacksquare

6 Tilting modules of the first kind with respect to Galois coverings

Throughout this section, A is assumed to be a basic connected piecewise hereditary of type Q finite dimensional k -algebra and $F: \mathcal{C} \rightarrow A$ is a Galois covering with group G and with \mathcal{C} locally bounded. Recall ([20]) that the assumption on A implies that $C_A \simeq H^0(\mathcal{B}_A) = \mathcal{M}_A$. For simplicity, if $X \in C_{\mathcal{C}}$, we shall write X instead of X^\wedge for the corresponding element of $\mathcal{M}_{\mathcal{C}}$. The aim of this section is to prove the following facts for any tilting A -module T :

- (H₁) T is of the first kind w.r.t. $F_\lambda: \text{mod}(\mathcal{C}) \rightarrow \text{mod}(A)$,
- (H₂) for any indecomposable direct summand $X \in \text{mod}(A)$ of T and any $\tilde{X} \in \text{mod}(\mathcal{C})$ such that $F_\lambda \tilde{X} \simeq X$, we have $G_{\tilde{X}} = 1$,
- (H₃) if $\psi: A \xrightarrow{\sim} A$ is an automorphism extending the identity map on objects, then $\psi.T \simeq T$ in $\text{mod}(A)$.

We will devote a subsection to each point. All the results and proofs presented in this paragraph are nothing but an adaptation to the triangulated setting of $\mathcal{M}_{\mathcal{C}}$ and $\mathcal{M}_A = C_A$ of the results and proofs of [22, Sect. 3] in the setting of abelian categories ($\text{mod}(A)$ and $\text{mod}(\mathcal{C})$).

For short, we shall say that $X \in \text{ind}(C_A)$ is of the first kind w.r.t. $F_\lambda: \mathcal{M}_{\mathcal{C}} \rightarrow C_A$ if and only if there exists $Y \in \mathcal{M}_{\mathcal{C}}$ such that $F_\lambda Y \simeq X$ (notice that we do not require that $Y \in C_{\mathcal{C}}$). More generally, $X \in C_A$ will be called of the first kind w.r.t. $F_\lambda: \mathcal{M}_{\mathcal{C}} \rightarrow C_A$ if and only if it is the direct sum of (finitely many) indecomposable objects in C_A of the first kind w.r.t. $F_\lambda: \mathcal{M}_{\mathcal{C}} \rightarrow C_A$.

6.1 Proof of assertion (H₁)

First, we prove the following.

Proposition 6.1. *Let $T \in C_A$ be a tilting object. Then T is of the first kind w.r.t. $F_\lambda: \mathcal{M}_C \rightarrow C_A$.*

In order to prove Proposition 6.1, we will prove the two following dual lemmas.

Lemma 6.2. *Let $\Delta: X \xrightarrow{u} M \rightarrow Y \rightarrow \Sigma X$ be triangle in C_A verifying the following hypotheses:*

- . $X \in \text{ind}(C_A)$ and $X = F_\lambda \widehat{X}$ (with $\widehat{X} \in \text{ind}(\mathcal{M}_C)$),
- . $M = M_1 \oplus \dots \oplus M_t$ where $M_i = F_\lambda \widehat{M}_i \in \text{ind}(C_A)$ (with $\widehat{M}_i \in \text{ind}(\mathcal{M}_C)$), for every i ,
- . $\text{Ext}_{C_A}^1(Y, M) = 0$.

Then, (Δ) is isomorphic to a triangle:

$$X \xrightarrow{\begin{bmatrix} F_\lambda u'_1 \\ \vdots \\ F_\lambda u'_t \end{bmatrix}} M_1 \oplus \dots \oplus M_t \rightarrow Y \rightarrow \Sigma X$$

where $u'_i \in \text{Hom}_{\mathcal{M}_C}(\widehat{X}, {}^{g_i} \widehat{M}_i)$ for some $g_i \in G$, for every i .

Lemma 6.3. *Let $\Delta: X \rightarrow M \xrightarrow{v} Y \rightarrow \Sigma X$ be triangle in C_A verifying the following hypotheses:*

- . $Y \in \text{ind}(C_A)$ and $Y = F_\lambda \widehat{Y}$ (with $\widehat{Y} \in \text{ind}(\mathcal{M}_C)$),
- . $M = M_1 \oplus \dots \oplus M_t$ where $M_i = F_\lambda \widehat{M}_i \in \text{ind}(C_A)$ (with $\widehat{M}_i \in \text{ind}(\mathcal{M}_C)$), for every i ,
- . $\text{Ext}_{C_A}^1(M, X) = 0$.

Then, (Δ) is isomorphic to a triangle:

$$X \rightarrow M_1 \oplus \dots \oplus M_t \xrightarrow{\begin{bmatrix} F_\lambda v'_1 \\ \vdots \\ F_\lambda v'_t \end{bmatrix}} Y \rightarrow \Sigma X$$

where $v'_i \in \text{Hom}_{\mathcal{M}_C}({}^{g_i} \widehat{M}_i, \widehat{Y})$ for some $g_i \in G$, for every i .

Proof of Lemma 6.2: Let us say that $u \in \text{Hom}_{C_A}(X, M_i)$ is homogeneous of degree $g \in G$ if and only if $u = F_\lambda u'$ with $u' \in \text{Hom}_{\mathcal{M}_C}(\widehat{X}, {}^g \widehat{M}_i)$. Since $F_\lambda: \mathcal{M}_C \rightarrow C_A$ has the covering property, any $u \in \text{Hom}_{C_A}(X, M_i)$ is (uniquely) the sum of d homogeneous morphisms of pairwise different degrees (with

$d \geq 0$). Let us write $u = \begin{bmatrix} u_1 \\ \vdots \\ u_t \end{bmatrix}$ with $u_i: X \rightarrow M_i$ for each i . We may assume that $u_1: X \rightarrow M_1$ is not

homogeneous. Thus $u_1 = h_1 + \dots + h_d$ where $d \geq 2$ and $h_1, \dots, h_d: X \rightarrow M_1$ are non zero homogeneous morphisms of pairwise different degree. In order to prove the lemma, it suffices to prove that (Δ) is

isomorphic to a triangle $X \xrightarrow{u'} M \rightarrow Y \rightarrow \Sigma X$ where $u' = \begin{bmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_t \end{bmatrix}$ with u'_i equal to the sum of at most

$d-1$ nonzero homogeneous morphisms $X \rightarrow M_1$ of pairwise different degrees. For simplicity we adopt the following notations:

- . $\overline{M} = M_2 \oplus \dots \oplus M_t$ (so $M = M_1 \oplus \overline{M}$),
- . $\overline{u} = \begin{bmatrix} u_2 \\ \vdots \\ u_t \end{bmatrix}: X \rightarrow \overline{M}$ (so $u = \begin{bmatrix} u_1 \\ \overline{u} \end{bmatrix}: X \rightarrow M_1 \oplus \overline{M}$),
- . $\overline{h} = h_2 + \dots + h_d: X \rightarrow M_1$ (so $u_1 = h_1 + \overline{h}$).

From $\text{Hom}_{C_A}(\Delta, M_1)$ we get the exact sequence:

$$\text{Hom}_{C_A}(M_1 \oplus \overline{M}, M_1) \xrightarrow{u^*} \text{Hom}_{C_A}(X, M_1) \rightarrow \text{Ext}_{C_A}^1(Y, M_1) = 0$$

So there exists $[\lambda, \mu]: M_1 \oplus \overline{M} \rightarrow M_1$ such that $h_1 = [\lambda, \mu]u$. Hence:

$$h_1 = \lambda u_1 + \mu \overline{u} = \lambda h_1 + \lambda \overline{h} + \mu \overline{u} \quad (i)$$

Let us distinguish two cases whether $\lambda \in \text{End}_A(M_1)$ is invertible or nilpotent (recall that $M_1 \in \text{ind}(C_A)$):

• If λ is invertible then:

$$\theta := \begin{bmatrix} \lambda & \mu \\ 0 & Id_{\overline{M}} \end{bmatrix} : M_1 \oplus \overline{M} \rightarrow M_1 \oplus \overline{M}$$

is invertible. Using (i) we deduce an isomorphism of triangles:

$$\begin{array}{ccccc} X & \xrightarrow{\begin{bmatrix} u_1 \\ \overline{u} \end{bmatrix}} & M_1 \oplus \overline{M} & \longrightarrow & Y & \longrightarrow & \Sigma X & (\Delta) \\ \parallel & & \downarrow \theta & & \downarrow \sim & & & \\ X & \xrightarrow{\begin{bmatrix} h_1 \\ \overline{u} \end{bmatrix}} & M_1 \oplus \overline{M} & \longrightarrow & Y & \longrightarrow & \Sigma X & (\Delta') \end{array}$$

Since $h_1: X \rightarrow M_1$ is homogeneous, (Δ') fits our requirements.

• If $\lambda \in \text{End}_{C_A}(M_1)$ is nilpotent, let $p \geq 0$ be such that $\lambda^p = 0$. Using (i) we get the following equalities:

$$\begin{aligned} h_1 &= \lambda^2 h_1 + (\lambda^2 + \lambda) \overline{h} + (\lambda + Id_{M_1}) \mu \overline{u} \\ &\vdots \\ h_1 &= \lambda^t h_1 + (\lambda^t + \lambda^{t-1} + \dots + \lambda) \overline{h} + (\lambda^{t-1} + \dots + \lambda + Id_{M_1}) \mu \overline{u} \\ &\vdots \\ h_1 &= \lambda^p h_1 + (\lambda^p + \lambda^{p-1} + \dots + \lambda) \overline{h} + (\lambda^{p-1} + \dots + \lambda + Id_{M_1}) \mu \overline{u} \end{aligned}$$

Since $\lambda^p = 0$ and $u_1 = h_1 + \overline{h}$ we infer that:

$$u_1 = \lambda' \overline{h} + \lambda' \mu \overline{u}$$

where $\lambda' := Id_{M_1} + \lambda + \dots + \lambda^{p-1} \in \text{End}_{C_A}(M_1)$ is invertible. So we have an isomorphism:

$$\theta := \begin{bmatrix} \lambda' & \lambda' \mu \\ 0 & Id_{\overline{M}} \end{bmatrix} : M_1 \oplus \overline{M} \rightarrow M_1 \oplus \overline{M}$$

and consequently we have an isomorphism of triangles:

$$\begin{array}{ccccc} X & \xrightarrow{\begin{bmatrix} \overline{h} \\ \overline{u} \end{bmatrix}} & M_1 \oplus \overline{M} & \longrightarrow & Y & \longrightarrow & \Sigma X & (\Delta') \\ \parallel & & \downarrow \theta & & \downarrow \sim & & & \\ X & \xrightarrow{\begin{bmatrix} u_1 \\ \overline{u} \end{bmatrix}} & M_1 \oplus \overline{M} & \longrightarrow & Y & \longrightarrow & \Sigma X & (\Delta) \end{array}$$

where $\overline{h} = h_2 + \dots + h_p$ is the sum of $p-1$ non zero homogeneous morphisms of pairwise different degree. So (Δ') fits our requirements. This finishes the proof of the lemma. \blacksquare

The proof of Lemma 6.3 is the dual of the one of Lemma 6.2, so we omit it. Thanks to these two lemmas, we can prove Proposition 6.1.

Proof of Proposition 6.1: Since A is a tilting object in C_A and since it is of the first kind w.r.t. $F_\lambda: \text{mod}(\mathcal{C}) \rightarrow \text{mod}(A)$ (hence, w.r.t. $F_\lambda: \mathcal{M}_C \rightarrow C_A$) we only need to prove that for any edge $T - T'$ in the tilting graph of C_A , T' is of the first kind w.r.t. $F_\lambda: \mathcal{M}_C \rightarrow C_A$ if the same holds for T . Recall from Section 1, that there exists a triangle in C_A :

$$X \xrightarrow{u} M \rightarrow Y \rightarrow \Sigma X \quad (\Delta)$$

such that:

1. $T = X \oplus \overline{T}, T' = Y \oplus \overline{T}, X, Y \in \text{ind}(C_A)$,
2. $M \in \text{add}(\overline{T})$,
3. u is a minimal left $\text{add}(\overline{T})$ -approximation of X and $M \rightarrow Y$ is a minimal right $\text{add}(\overline{T})$ -approximation of Y .

By assumption on T , we know that X , M and \overline{T} are of the first kind w.r.t. $F_\lambda: \mathcal{M}_C \rightarrow C_A$. Thus, we only need to prove that the same holds for Y . Thanks to Lemma 6.2 and with the same notations, we know that (Δ) is isomorphic to a triangle:

$$X \xrightarrow{u'} \begin{bmatrix} F_\lambda u'_1 \\ \vdots \\ F_\lambda u'_t \end{bmatrix} M_1 \oplus \dots \oplus M_t \rightarrow Y \rightarrow \Sigma X \quad (\Delta')$$

where $u'_i \in \text{Hom}_{\mathcal{M}_C}(\widehat{X}, {}^{g_i}\widehat{M}_i)$ for some g_i , for every i . Let $v := \begin{bmatrix} u'_1 \\ \vdots \\ u'_t \end{bmatrix} \in \text{Hom}_{\mathcal{M}_C}(\widehat{X}, \bigoplus_{i=1}^t {}^{g_i}\widehat{M}_i)$, so that $u' = F_\lambda v$. Then, v may be completed into a triangle in \mathcal{M}_C :

$$\widehat{X} \xrightarrow{v} \bigoplus_{i=1}^t {}^{g_i}\widehat{M}_i \rightarrow \widehat{Y} \rightarrow \Sigma \widehat{X} \quad (\Delta'')$$

Since $u' = F_\lambda v$, we deduce that $F_\lambda(\Delta'')$ and (Δ) are isomorphic triangles of C_A . In particular, $F_\lambda \widehat{Y} \simeq Y$ in C_A . This proves the proposition. \blacksquare

Notice that the proof of Proposition 6.1 gives the following fact:

Remark 6.4. *If $T - T'$ is an edge of the tilting graph of C_A , then there exists a triangle in \mathcal{M}_C :*

$$X \xrightarrow{\iota} M \xrightarrow{\pi} Y \xrightarrow{\eta} \Sigma X \quad (\Delta)$$

such that:

- $F_\lambda X, F_\lambda Y \in \text{ind}(C_A)$ (hence $X, Y \in \text{ind}(\mathcal{M}_C)$),
- $T = F_\lambda X \oplus \overline{T}, T' = F_\lambda Y \oplus \overline{T}$ and $F_\lambda M \in \text{add}(\overline{T})$,
- $F_\lambda \iota: F_\lambda X \rightarrow F_\lambda M$ (resp. $F_\lambda \pi: F_\lambda M \rightarrow F_\lambda Y$) is a minimal left (resp. right) $\text{add}(\overline{T})$ -approximation.

Notice also that the proof of Proposition 6.1 gives the following proposition in a more general setting.

Proposition 6.5. *Let $\mathcal{T}, \mathcal{T}'$ be triangulated Krull-Schmidt k -linear categories such that endomorphism rings of indecomposables are local k -algebras. Assume that G acts by triangle isomorphisms on \mathcal{T}' and let $\Theta: \mathcal{T}' \rightarrow \mathcal{T}$ be a G -equivariant triangle functor having the covering property. Let X_2, Y, T_2, \dots, T_n be pairwise non isomorphic indecomposable objects such that $T_i \simeq \Theta T'_i$ in \mathcal{T} for some $T'_i \in \mathcal{T}'$. Set $\overline{T} := T_2 \oplus \dots \oplus T_n$, $T := X \oplus \overline{T}$, $T' := Y \oplus \overline{T}$. Assume that $\text{Ext}_{\mathcal{T}}^1(T, T) = \text{Ext}_{\mathcal{T}'}^1(T', T') = 0$ and assume that there exists a triangle in \mathcal{T} :*

$$X \rightarrow M \rightarrow Y \rightarrow \Sigma X$$

where $M \in \text{add}(\overline{T})$. Then $X \simeq \Theta \widetilde{X}$ for some $\widetilde{X} \in \mathcal{T}'$ if and only if $Y \simeq \Theta \widetilde{Y}$ for some $\widetilde{Y} \in \mathcal{T}'$. Moreover, if \widetilde{X} and \widetilde{Y} exist, then there exists a triangle in \mathcal{T}' :

$$\widetilde{X} \rightarrow \widetilde{M} \rightarrow {}^g \widetilde{Y} \rightarrow \Sigma \widetilde{X}$$

for some $g \in G$ and some $\widetilde{M} \in \text{add}(\{ {}^g \widetilde{T}_i \mid g \in G, i \in \{2, \dots, n\} \})$.

From Corollary 5.5 and Proposition 6.1, we deduce that (H_1) is satisfied:

Proposition 6.6. *Any tilting A -module is of the first kind w.r.t. $F_\lambda: \text{mod}(\mathcal{C}) \rightarrow \text{mod}(A)$.*

Proof: Let X be a direct summand of T . Since T is a tilting object of C_A , Proposition 6.1 implies that there exists $Y \in \mathcal{M}_C$ such that $X \simeq F_\lambda Y$ in C_A . Corollary 5.5 implies that there exists $Z \in \text{mod}(\mathcal{C})$ such that $X \simeq F_\lambda Z$ in $\text{mod}(A)$. \blacksquare

6.2 Proof of assertion (H₂)

Proposition 6.7. *Let T be a tilting object in C_A , let M be an indecomposable direct summand of T and let $\tilde{M} \in \mathcal{M}_C$ be such that $F_\lambda \tilde{M} \simeq M$ in \mathcal{M}_C (see Proposition 6.1). Then:*

$$g \in G \setminus \{1\} \implies {}^g \tilde{M} \not\cong \tilde{M} \text{ in } \mathcal{M}_C \quad (\star)$$

Proof: First, let us verify that (\star) is verified if $T = A$. Under this assumption, there is some $x \in \text{ob}(A)$ such that $X \simeq {}_x A?$ in \mathcal{M}_C . Thanks to Lemma 5.3, we infer that there exists $x' \in \text{ob}(C)$ such that $\tilde{X} \simeq {}_{x'} C?$ (with $F(x') = x$). If $g \in G$ is such that $\tilde{X} \simeq {}^g \tilde{X}$ in \mathcal{M}_C , then ${}^g {}_{x'} C? \simeq {}_{x'} C?$ in \mathcal{M}_C . From Lemma 5.2, we infer that ${}^g {}_{x'} C? \simeq {}_{x'} C?$ in $\text{mod}(A)$ and therefore, that $g = 1$.

Thus, in order to prove the proposition, it suffices to prove the following implication for any edge $T - T'$ in the tiling graph of C_A : (\star) holds for $T \implies (\star)$ holds for T' . Let us use Remark 6.4 from which we adopt the notations. Thanks to Lemma 5.3, and because (\star) holds for T , we only need to prove the implication ${}^g Y \simeq Y$ in $\mathcal{M}_C \implies g = 1$. Let $\varphi: Y \rightarrow {}^g Y$ be an isomorphism in \mathcal{M}_C (with $g \in G$), and let us prove that $g = 1$. To do this we will exhibit an isomorphism $\theta: X \rightarrow {}^g X$. Since T is tilting, we have $\text{Ext}_{C_A}^1(F_\lambda M, F_\lambda X) = \text{Ext}_{C_A}^1(F_\lambda X, F_\lambda M) = 0$. These equalities imply the following ones, since $F_\lambda: \mathcal{M}_C \rightarrow C_A$ has the covering property:

$$\text{Ext}_{C_C}^1({}^g M, X) = \text{Ext}_{C_C}^1(M, {}^g X) = 0 \quad (i)$$

With $\text{Hom}_{\mathcal{M}_C}(M, {}^g \Delta)$, this last equality gives the exact sequence:

$$\text{Hom}_{\mathcal{M}_C}(M, {}^g M) \xrightarrow{({}^g \pi)_*} \text{Hom}_{\mathcal{M}_C}(M, {}^g Y) \rightarrow \text{Ext}_{\mathcal{M}_C}^1(M, {}^g X) = 0$$

From this exact sequence, we deduce the existence of $\psi \in \text{Hom}_{\mathcal{M}_C}(M, {}^g M)$ such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\pi} & Y \\ \psi \downarrow & & \downarrow \varphi \\ {}^g M & \xrightarrow{{}^g \pi} & {}^g Y \end{array}$$

This implies the existence of $\theta \in \text{Hom}_{\mathcal{M}_C}(X, {}^g X)$ making commute the following morphism of triangles in \mathcal{M}_C :

$$\begin{array}{ccccccc} X & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y & \xrightarrow{\eta} & \Sigma X & (\Delta) \\ \theta \downarrow & & \psi \downarrow & & \downarrow \varphi & & & \\ {}^g X & \xrightarrow{{}^g \iota} & {}^g M & \xrightarrow{{}^g \pi} & {}^g Y & \xrightarrow{{}^g \eta} & \Sigma {}^g X & ({}^g \Delta) \end{array} \quad (ii)$$

We claim that $\theta: X \rightarrow {}^g X$ is an isomorphism. The arguments that have been used to get (ii) may be adapted (just replace the use of $\text{Hom}_{\mathcal{M}_C}(M, {}^g \Delta)$ and of $\varphi: Y \rightarrow {}^g Y$ by $\text{Hom}_{\mathcal{M}_C}({}^g M, \Delta)$ and $\varphi^{-1}: {}^g Y \rightarrow Y$) to get the following morphisms of triangles in \mathcal{M}_C :

$$\begin{array}{ccccccc} {}^g X & \xrightarrow{{}^g \iota} & {}^g M & \xrightarrow{{}^g \pi} & {}^g Y & \xrightarrow{{}^g \eta} & \Sigma {}^g X & ({}^g \Delta) \\ \theta' \downarrow & & \psi' \downarrow & & \downarrow \varphi^{-1} & & & \\ X & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y & \xrightarrow{\eta} & \Sigma X & (\Delta) \end{array} \quad (iii)$$

In order to show that $\theta: X \rightarrow {}^g X$ is an isomorphism, let us show that $\theta' \theta \in \text{End}_{C_C}(X)$ is an isomorphism. Notice (ii) and (iii) give the morphism of triangles in \mathcal{M}_C :

$$\begin{array}{ccccccc} X & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y & \xrightarrow{\eta} & \Sigma X & (\Delta) \\ \theta' \theta - \text{id}_X \downarrow & & \psi' \psi - \text{Id}_M \downarrow & & \downarrow 0 & & & \\ X & \xrightarrow{\iota} & M & \xrightarrow{\pi} & Y & \xrightarrow{\eta} & \Sigma X & (\Delta) \end{array} \quad (iv)$$

In particular we have $\pi(\psi' \psi - \text{Id}_M) = 0$, so there exists $\lambda \in \text{Hom}_{\mathcal{M}_C}(M, X)$ such that:

$$\psi' \psi - \text{Id}_M = \iota \lambda$$

Therefore:

$$\iota(\theta'\theta - Id_X) = \iota\lambda\iota$$

This implies that there exists $f \in Hom_{\mathcal{M}_C}(X, \Sigma^{-1}Y)$ such that $\theta'\theta - Id_X - \lambda\iota = \eta'f$, where $\eta' = \Sigma^{-1} = \eta$. Thus:

$$\theta'\theta = Id_X + \lambda\iota + \eta'f$$

Notice that:

1. $F_\lambda(\lambda\iota) \in End_{C_A}(F_\lambda X)$ factorises through $F_\lambda M$, and $F_\lambda X$ is indecomposable in C_A and is not a direct summand of $F_\lambda M$ (indeed, $T = F_\lambda X \oplus \overline{T}$ is basic and $F_\lambda M \in add(\overline{T})$). So $F_\lambda(\lambda\iota)$ is nilpotent. Since $F_\lambda: \mathcal{M}_C \rightarrow C_A$ is faithful (because it has the covering property), we infer that $\lambda\iota$ is nilpotent.
2. $F_\lambda(\eta'f) \in End_{C_A}(F_\lambda X)$ factorises through $F_\lambda Y$. Since $F_\lambda X$ is not a direct summand of $F_\lambda Y$ (because $T = F_\lambda X \oplus \overline{T}$ and $T' = F_\lambda Y \oplus \overline{T}$ are not isomorphic) we infer that $F_\lambda(\eta'f)$ is nilpotent. So $\eta'f$ is nilpotent.

These two arguments prove that $\theta'\theta = Id_X + \lambda\iota + \eta'f \in End_{\mathcal{M}_C}(X)$ is invertible. Let $\varphi: {}^g X \rightarrow X$ be such that $\varphi\theta = Id_X$. So $F_\lambda(\varphi)F_\lambda\theta = Id_{F_\lambda X}$. Since $F_\lambda X \in C_A$ is indecomposable, this implies that $F_\lambda\theta$ is invertible and that $F_\lambda\theta F_\lambda\varphi = Id_{F_\lambda X}$. Since $F_\lambda: \mathcal{M}_C \rightarrow C_A$ is faithful, we get $\theta\varphi = Id_{{}^g X}$. This proves that $\theta: X \rightarrow {}^g X$ is an isomorphism. Recall that we assumed (\star) is verified for T , so $g = 1$. This finishes the proof of the proposition. \blacksquare

Like for Proposition 6.1, the proof of Proposition 6.7 gives the following result in a more general setting:

Proposition 6.8. *Under the notations and hypotheses of Proposition 6.5, assume that $X \simeq \Theta\tilde{X}$ and $Y \simeq \Theta\tilde{Y}$ with $\tilde{X}, \tilde{Y} \in T'$. Then:*

$$G_{\tilde{X}} = 1 \Leftrightarrow G_{\tilde{Y}} = 1$$

where $G_M := \{g \in G \mid {}^g M \simeq M \text{ in } T'\}$.

Thanks to Proposition 6.7 and to the fact that a tilting A -module is a tilting object of C_A , we deduce that (H_2) is satisfied.

Proposition 6.9. *If T is a tilting A -module, then $G_{\tilde{X}} = 1$ for any $\tilde{X} \in mod(\mathcal{C})$ such that $F_\lambda\tilde{X}$ is an indecomposable direct summand of T .*

The proof is immediate.

6.3 Proof of assertion (H_3)

Notice that if $\psi: A \rightarrow A$ is an isomorphism extending the identity map on objects, then ψ and ψ^{-1} are Galois coverings such that $\psi = \psi_\lambda^{-1} = (\psi^{-1})_\lambda$ (see for example [22, Lem. 1.3]). Therefore, we have a well defined triangle functor $\psi: C_A \rightarrow C_A$ which is a triangle equivalence with inverse $\psi_\lambda: C_A \rightarrow C_A$.

Proposition 6.10. *Let $T \in C_A$ be a tilting object and let $\psi: A \xrightarrow{\sim} A$ be an automorphism extending the identity map on objects. Then $\psi.T \simeq T$ in C_A .*

Proof: The conclusion of the proposition holds if $T = A$ (see for example [22, Prop. 3.7]). Let $\psi: A \rightarrow A$ be an isomorphism extending the identity map on objects. In order to prove the proposition, it suffices to prove the following implication for any edge $T - T'$ in the tilting graph of C_A : $\psi.T \simeq T \Rightarrow \psi.T' \simeq T'$. For such an edge, we have the following data:

- $T = X \oplus \overline{T}$ with $X \in ind(A)$,
- $T' = Y \oplus \overline{T}$ with $Y \in ind(A)$,
- a non split triangle $X \xrightarrow{u} M \rightarrow Y \rightarrow \Sigma X$ where $M \in add(\overline{T})$ and where $u: X \rightarrow M$ is the minimal left $add(\overline{T})$ -approximation of X .

Notice that we only need to prove that $\psi.Y \simeq Y$. Since $\psi: C_A \rightarrow C_A$ is a triangle equivalence, we have a non split triangle $\psi.X \xrightarrow{\psi.u} \psi.M \rightarrow \psi.Y \rightarrow \Sigma\psi.X$ verifying: $\psi.M \in add(\psi.\overline{T})$ and $\psi.u: \psi.X \rightarrow \psi.M$ is the minimal left $add(\psi.\overline{T})$ -approximation of $\psi.X$. Moreover $\psi.X \simeq X$, $\psi.M \simeq M$ and $\psi.\overline{T} \simeq \overline{T}$ because $\psi.T \simeq T$. So, we have two triangles $X \xrightarrow{u} M \rightarrow Y \rightarrow \Sigma X$ and $X \xrightarrow{u'} M \rightarrow \psi.Y \rightarrow \Sigma X$ where u and u' are both minimal left $add(\overline{T})$ -approximations. This implies that $Y \simeq \psi.Y$ in C_A . So $\psi.T' \simeq T'$ in C_A . \blacksquare

Using Lemma 5.1 and Proposition 6.10, we deduce easily the following proposition.

Proposition 6.11. *Let T be a tilting A -module and let $\psi: A \rightarrow A$ be an isomorphism extending the identity map on objects. Then $\psi.T \simeq T$ in $mod(A)$.*

7 Proof of the main theorems

In this section, we prove Theorem 1, Corollary 2 and Theorem 3 that were announced in the introduction. We fix A a basic connected finite dimensional k -algebras (of finite global dimension). We shall freely use the construction of [22] recalled at the end of Section 1. Since we will use it many times, we recall the following theorem due to D. Happel:

Theorem 7.1. (*Happel, [14, Cor 5.5]*) *Let A be a finite dimensional k -algebra. Then, A is piecewise hereditary of type Q if and only if there exists a sequence $A_0 = kQ, A_1, \dots, A_l = A$ of finite dimensional k -algebras and a sequence $T^{(1)} \in \text{mod}(A_0), \dots, T^{(l)} \in \text{mod}(A_{l-1})$ of tilting modules such that $A_1 \simeq \text{End}_{A_0}(T_1), \dots, A_l \simeq \text{End}_{A_{l-1}}(T_l)$.*

In order to prove Theorem 1, we first prove the following proposition.

Proposition 7.2. *Let T be a tilting A -module. Assume that the assertions (H_1) , (H_2) and (H_3) are satisfied and that A admits a universal cover $\tilde{F}: \tilde{\mathcal{C}} \rightarrow A$ with group $\pi_1(A)$. Then $\text{End}_A(T)$ admits a universal cover with group $\pi_1(A)$.*

Proof: Let us write $T = T_1 \oplus \dots \oplus T_n$ with $T_i \in \text{mod}(A)$ indecomposable and let us set $B = \text{End}_A(T)$. Let us denote by C the full subcategory of $\text{Gal}(A)$ whose objects are the connected Galois coverings F such that there exists a morphism $\tilde{F} \rightarrow F$. Hence, any connected Galois covering of A is equivalent to an object in C . We shall construct a (fully faithful) functor $C \rightarrow \text{Gal}(\text{End}_A(T))$. In this purpose, we fix some data.

1. For every i , let us fix $\tilde{T}_i \in \text{mod}(\tilde{\mathcal{C}})$ such that $\tilde{F}_\lambda \tilde{T}_i = T_i$,
2. For every $F \in C$, let us fix a morphism $p^F: \tilde{F} \rightarrow F$ in C (with $p^{\tilde{F}} = \text{Id}_{\tilde{F}}$) and let us set $T_i^F := p_\lambda^F \tilde{T}_i$ for every $i \in \{1, \dots, n\}$. Since $F \circ p^F = \tilde{F}$, we deduce that $F_\lambda T_i^F = T_i$ for every i .
3. Let $u: F \rightarrow F'$ be a morphism in C . Thus, $p^{F'}, up^F \in \text{Hom}_C(\tilde{F}, F')$ so that there exists a unique $h^u \in \text{Aut}(F')$ such that: $h^u p^{F'} = up^F$.

Now, let $u: F \rightarrow F'$ be a morphism in C where $F: \mathcal{C} \rightarrow A$ and $F': \mathcal{C}' \rightarrow A$. Notice that for any $g \in \text{Aut}(F)$ and any $i \in \{1, \dots, n\}$, we have:

$$u_\lambda({}^g T_i^F) = u_\lambda\left({}^g(p_\lambda^F \tilde{T}_i)\right) = \lambda^u(g) u_\lambda p_\lambda^F \tilde{T}_i = \lambda^u(g) h^u p_\lambda^{F'} \tilde{T}_i = \lambda^u(g) h^u T_i^{F'}$$

Recall from Section 1 that $\lambda^u: \text{Aut}(F) \rightarrow \text{Aut}(F')$ is the unique group morphism such that $\lambda^u(h)u = uh$ for any $h \in \text{Aut}(F)$. The above equality proves that we have a well defined functor:

$$\begin{array}{ccc} u_T: \text{End}_C \left(\bigoplus_{g \in \text{Aut}(F), i} {}^g T_i^F \right) & \longrightarrow & \text{End}_{C'} \left(\bigoplus_{g' \in \text{Aut}(F'), i} {}^{g'} T_i^{F'} \right) \\ & \longmapsto & u_\lambda({}^g T_i^F) = \lambda^u(g) h^u T_i^{F'} \\ & \xrightarrow{{}^g T_i^F \xrightarrow{f} {}^h T_j^F} & \longmapsto u_\lambda(f) \end{array}$$

Recall from Section 1 that F_T and F'_T are connected Galois coverings of B with group $\text{Aut}(F)$ and $\text{Aut}(F')$ respectively. Since F_T, F'_T and u_T are defined using F_λ, F'_λ and u_λ respectively and since $F'u = u$, we deduce that $F'_T u_T = F_T$. In other words, $u_T: F_T \rightarrow F'_T$ is a morphism of connected Galois coverings of B . Also, with the same arguments, one proves easily that $(\text{Id}_F)_T = \text{Id}_{F_T}$ and that $(u'u)_T = u'_T u_T$ for any object F of C and any morphisms u, u' in C . Hence, we have constructed a functor:

$$\begin{array}{ccc} \psi: C & \longrightarrow & \text{Gal}(B) \\ & \longmapsto & F_T \\ F \xrightarrow{u} F' & \longmapsto & F_T \xrightarrow{u_T} F'_T \end{array}$$

In particular, $\psi(C)$ is a subcategory of $\text{Gal}(B)$ such that any object of $\psi(C)$ is the target of a morphism in $\text{Gal}(B)$ with source $\psi(\tilde{F})$.

For every connected Galois covering $F: \mathcal{C} \rightarrow A$ not lying in C , let us fix $T_1^F, \dots, T_n^F \in \text{mod}(\mathcal{C})$ such that $F_\lambda T_i^F = T_i$. This defines the connected Galois covering F_T of B . From Section 1, we know that the mapping:

$$\begin{array}{ccc} \text{ob}(\text{Gal}(A)) & \longrightarrow & \text{ob}(\text{Gal}(B)) \\ F & \longmapsto & F_T \end{array}$$

induces a bijection between $\text{ob}(\text{Gal}(A))/\sim$ and $\text{ob}(\text{Gal}(B))/\sim$. Hence, any object in $\text{Gal}(B)$ is equivalent to an object in $\psi(C)$. This proves that $\psi(\tilde{F})$ is a universal cover of B , with group $\text{Aut}(\tilde{F}) = \pi_1(A)$. ■

Remark 7.3. *The reader may easily check that the functor $\psi: C \rightarrow \text{Gal}(B)$ is fully faithful.*

Thanks to Proposition 7.2, we are able to prove the first two main results of this text.

Proof of Theorem 1: Since kQ has a universal cover with group $\pi_1(Q)$ and since the assertions (H_1) , (H_2) and (H_3) are satisfied for any piecewise hereditary algebra of type Q and for any tilting module over this algebra, the theorem is a consequence of Theorem 7.1 and Proposition 7.2 using an immediate induction. ■

Proof of Corollary 2: Thanks to Theorem 1 and to the characterisation of the simple connectedness of A (see the introduction or [23, Cor. 4.5]) we infer that A is simply connected if and only if $\pi_1(Q)$ is trivial, i.e. Q is a tree. On the other hand, it was proved in [15] that $\dim_k HH^1(kQ) = rk(\pi_1(Q))$. Since $\dim_k HH^1(A) = \dim_k HH^1(kQ)$ (see [19]), we conclude that A is simply connected if and only if $HH^1(A) = 0$. ■

In order to prove Theorem 3, we will proceed in several steps. First, we prove the following lemma. This lemma is a direct consequence of tilting theory when the group of the considered Galois covering is infinite. However, recall that if $F: C \rightarrow A$ is a Galois covering with finite group, then $F.T$ is not finite dimensional if T is a nonzero A -module.

Lemma 7.4. *Let $F: C \rightarrow A$ be a connected Galois covering with group G . Let $T = T_1 \oplus \dots \oplus T_n$ be a tilting A -module (with T_i indecomposable) and assume that $T_i = F_\lambda \tilde{T}_i$ for some $\tilde{T}_i \in \text{mod}(C)$, for every i . Then:*

1. $\text{Ext}_C^1({}^g \tilde{T}_i, {}^h \tilde{T}_j) = 0$ for every $i, j \in \{1, \dots, n\}$ and every $g, h \in G$.
2. $\text{pd}_C({}^g \tilde{T}_i) \leq 1$ for every i and every g .
3. if $P \in \text{mod}(C)$ is an indecomposable projective C -module, then there exists an exact sequence $0 \rightarrow P \rightarrow T^{(1)} \rightarrow T^{(2)} \rightarrow 0$ where $T^{(1)}, T^{(2)} \in \text{add}(\{{}^g \tilde{T}_i \mid i \in \{1, \dots, n\}, g \in G\}) \subseteq \text{mod}(C)$.

Proof: The first two assertions are direct consequences of the fact that T is tilting and of Lemma 2.1. Let $P \in \text{mod}(C)$ be indecomposable projective. So $R := F_\lambda P \in \text{mod}(A)$ is indecomposable projective. Since A has finite global dimension and since T is tilting, we deduce that T is cotilting. Using the fact that $\text{Ext}_A^1(R, T) = 0$, we infer that there exists an exact sequence $0 \rightarrow R \rightarrow X \rightarrow Y \rightarrow 0$ with $X, Y \in \text{add}(T)$. Thanks to [22, Lem 7.4] (from which Lemma 6.2 is inspired), the fact that $Q = F_\lambda P$ is indecomposable, that X is of the first kind w.r.t. F and that $\text{Ext}_A^1(T, T) = 0$ imply that there exists an exact sequence $0 \rightarrow P \rightarrow X' \rightarrow Y' \rightarrow 0$ in $\text{mod}(C)$, with $X', Y' \in \text{add}(\{{}^g \tilde{T}_i \mid i \in \{1, \dots, n\}, g \in G\})$. ■

The preceding lemma allows us to prove the following proposition. Once again, this proposition is a direct consequence of tilting theory when the involved group is infinite. The proof of the following proposition is an adaptation of the proof of [14, III.2.10].

Proposition 7.5. *Let $T = T_1 \oplus \dots \oplus T_n$ be a basic tilting A -module (with T_i indecomposable) and set $B = \text{End}_A(T)$. Let $F: C \rightarrow A$ be a connected Galois covering with group G and assume that the two following conditions are satisfied:*

1. for every i , $\tilde{T}_i \in \text{mod}(C)$ is such that $F_\lambda \tilde{T}_i = T_i$,
2. $G_{\tilde{T}_i} = 1$ for every i .

Let $F_T: \text{End}_C(\bigoplus_{g,i} {}^g \tilde{T}_i) \rightarrow B$ be the associated connected Galois covering with group G . Then C and $\text{End}_C(\bigoplus_{g,i} {}^g \tilde{T}_i)$ are have equivalent bounded derived categories.

Proof: For short, we shall write $C' := \text{End}_C(\bigoplus_{g,i} {}^g \tilde{T}_i)$ and $\mathcal{X} := \text{add}(\{{}^g \tilde{T}_i \mid i \in \{1, \dots, n\}, g \in G\}) \subseteq \text{mod}(C)$. Since $B = \text{End}_A(T)$ with T tilting and since $C' \rightarrow B$ is a Galois covering, we infer that C' has finite global dimension. Thanks to Lemma 7.4, we may apply the proofs of [14, 2.1, 2.5] and deduce that the natural functor $\mathcal{K}^b(\mathcal{X}) \rightarrow \mathcal{D}^b(C)$ is fully faithful and dense. Here $\mathcal{K}^b(\mathcal{X})$ is the homotopy category of bounded complexes in \mathcal{X} . Moreover, by construction of C' , the functor $\text{Hom}_C(?, F.T): \mathcal{X} \rightarrow \text{proj}(C')$ is an equivalence and so is the induced functor $\mathcal{K}^b(\mathcal{X}) \rightarrow \mathcal{K}^b(\text{proj}(C'))$. Finally, $\mathcal{K}^b(\text{proj}(C')) \rightarrow \mathcal{D}^b(C')$ is an equivalence because C' has finite global dimension. Hence $\mathcal{D}^b(C) \simeq \mathcal{K}^b(\mathcal{X}) \simeq \mathcal{D}^b(C')$. ■

Proof of Theorem 3: Let us use Theorem 7.1 from which we keep the notations and let us prove the theorem by induction on l . If $l = 0$, then $A \simeq kQ$. Since $C \rightarrow A$ is a connected Galois covering with

group G , [23, Prop. 4.4] implies that there exists a quiver Q' and a Galois covering $Q' \rightarrow Q$ with group G such that $\mathcal{C} \simeq kQ'$. Whence the conclusion if $l = 0$. Now assume that $l > 0$ and that the conclusion of the theorem holds for A_{l-1} . Let us set $T := T^{(l)}$. Recall from the first section that T is a basic tilting A^{op} -module and that $A_{l-1} \simeq \text{End}_{A^{op}}(T)^{op}$. If $F: \mathcal{C} \rightarrow A$ is a connected Galois covering with group G , then so does $F^{op}: \mathcal{C}^{op} \rightarrow A^{op}$ and since A is piecewise hereditary of type Q , so does A^{op} . Hence, (H_1) , (H_2) and (H_3) are satisfied for A^{op} and $T \in \text{mod}(A^{op})$. From Proposition 7.5, we deduce that there exists a connected Galois covering $F': \mathcal{C}' \rightarrow \text{End}_{A^{op}}(T)$ with group G and that $\mathcal{D}^b(\mathcal{C}') \simeq \mathcal{D}^b(\mathcal{C}^{op})$ (and therefore $\mathcal{D}^b(\mathcal{C}) \simeq \mathcal{D}^b(\mathcal{C}'^{op})$) as triangulated categories. Since $F'^{op}: \mathcal{C}'^{op} \rightarrow \text{End}_{A^{op}}(T)^{op} \simeq A_{l-1}$ is also a connected Galois covering with group G , we infer, by induction hypothesis, that \mathcal{C}'^{op} is piecewise hereditary of type Q' such that there exists a Galois covering $Q' \rightarrow Q$ with group G . From the equivalence $\mathcal{D}^b(\mathcal{C}) \simeq \mathcal{D}^b(\mathcal{C}'^{op})$ we deduce that the same property holds for \mathcal{C} . \blacksquare

As an application of Theorem 3, we give the following corollary which compares the tilting objects of $C_{A'}$ and those of C_A when $A' \rightarrow A$ is a Galois covering with finite group (so that A' is finite dimensional).

Corollary 7.6. *Let $F: A' \rightarrow A$ be a connected Galois covering with finite group G , where A is piecewise hereditary of type Q . Let \mathcal{T} (resp. \mathcal{T}') be the set of isomorphism classes of tilting objects of C_A (resp. of $C_{A'}$). For every $T \in \mathcal{T}$, let $\tilde{T} \in C_{A'}$ such that $F_\lambda \tilde{T} \simeq T$ (see Proposition 6.1). Then, the following map is bijective:*

$$\begin{aligned} \mathcal{T} &\longrightarrow \mathcal{T}' \\ T &\longmapsto \bigoplus_{g \in G} {}^g \tilde{T} \end{aligned} \quad (\star)$$

This bijection restricts to a bijection:

$$T \longmapsto F.T \quad (\star\star)$$

from the set of isomorphism classes of tilting A -modules to the set of isomorphism classes of G -invariant tilting A' -modules.

Proof: Thanks to Theorem 3, we know that A' is piecewise hereditary. Hence, $C_{A'} = \mathcal{M}_{A'}$ is triangulated and $F_\lambda: C_{A'} \rightarrow C_A$ is a triangle functor with the covering property. For simplicity, we shall make no distinction between an object and its isomorphism class.

• Let $T = T_1 \oplus \dots \oplus T_n \in \mathcal{T}$ (with $T_i \in C_A$ indecomposable). For every i , let $\tilde{T}_i \in C_{A'}$ (necessarily indecomposable) such that $F_\lambda \tilde{T}_i \simeq T_i$. Then:

1. ${}^g \tilde{T}_i \simeq \tilde{T}_i \implies g = 1$ for every $g \in G$ (see Proposition 6.7),
2. $\bigoplus_{g \in G} {}^g \tilde{T} = \bigoplus_{g \in G, i} {}^g \tilde{T}_i$ (see Lemma 5.3),
3. $\bigoplus_{g \in G} \text{Ext}_{A'}^1({}^g \tilde{T}, \tilde{T}) \simeq \text{Ext}_A^1(T, T) = 0$ because T is tilting and F_λ has the covering property.

Hence, $\bigoplus_{g \in G} {}^g \tilde{T}$ is a basic exceptional object of $C_{A'}$ and is the direct sum of $n \cdot |G| = rk(K_0(A'))$ indecomposables. So $\bigoplus_{g \in G} {}^g \tilde{T} \in \mathcal{T}'$. This proves that (\star) is well-defined. This proves also that $(\star\star)$ is well defined because if $T \in \text{mod}(A)$, then all the introduced objects lie in $\text{mod}(A')$ (see Proposition 6.6).

• Let $T = T_1 \oplus \dots \oplus T_n \in \mathcal{T}$ and $T' = T'_1 \oplus \dots \oplus T'_n \in \mathcal{T}$ (with T_i, T'_i indecomposable for every i) and assume that $\bigoplus_{g \in G} {}^g \tilde{T} \simeq \bigoplus_{g \in G} {}^g \tilde{T}'$. For every i , fix $\tilde{T}_i, \tilde{T}'_i \in C_{A'}$ such that $F_\lambda \tilde{T}_i \simeq T_i$ and $\tilde{T}'_i \simeq T'_i$. So, up to a permutation of indices, there exists $g_i \in G$ such that $\tilde{T}'_i \simeq {}^{g_i} \tilde{T}_i$, for every i . So:

$$T' \simeq \bigoplus_{g, i} F_\lambda \tilde{T}'_i \simeq \bigoplus_{g, i} F_\lambda \tilde{T}_i \simeq F_\lambda \tilde{T} \simeq T'$$

Thus, (\star) and $(\star\star)$ are injective.

• Let $M \in \mathcal{T}'^G$. Let us write $M = \bigoplus_{i=1}^{n'} M_i$ (where $n' = rk(K_0(A')) = n \cdot |G|$) with $M_i \in C_{A'}$ indecomposable. Let us write $i \approx j$ if and only if $M_j \simeq {}^g M_i$ for some $g \in G$ and let $\{i_1, \dots, i_l\}$ be a complete set of representative of $\{1, \dots, n'\} / \approx$ in $\{1, \dots, n\}$. Since $M \in \mathcal{T}'^G$, we infer that M is a direct summand of $\bigoplus_{g \in G} {}^g M'$, where $M' := M_{i_1} \oplus \dots \oplus M_{i_l}$. In particular, $l \geq n$. Since M_{i_j} is a direct summand of M and since M is G -invariant, we also deduce that ${}^g M_{i_j}$ is a direct summand of M , for every j . This proves that $\text{add}(M) = \text{add}(\bigoplus_{g \in G} {}^g M')$, then:

1. $Ext_{A'}^1({}^g M', M') = 0$ because $Ext_{A'}^1({}^g M, M) = Ext_{A'}^1(M, M) = 0$,
2. $F_\lambda M'$ is basic by construction of M' and thanks to Lemma 5.3,
3. $Ext_A^1(F_\lambda M', F_\lambda M') \simeq \bigoplus_{g \in G} Ext_{A'}^1({}^g M', M') = 0$ because F_λ has the covering property.

These facts prove that $F_\lambda M'$ is basic and exceptional in C_A . Let $T' \in C_A$ be such that $T' \oplus F_\lambda M'$ is tilting and let $\widetilde{T}' \in C_{A'}$ be such that $F_\lambda \widetilde{T}' \simeq T'$. From the first point of the proof, we know that $\bigoplus_{g \in G} ({}^g \widetilde{T}' \oplus M')$ is tilting. Since $\bigoplus_{g \in G} {}^g M'$ is the direct sum of $l \cdot |G| \geq n \cdot |G| = rk(K_0(A'))$ indecomposables, we infer that $l = n$ and that $T' = 0$. Hence $F_\lambda M' \in \mathcal{T}$ and $M = \bigoplus_{g \in G} {}^g M'$. Thus (\star) is surjective and therefore bijective. Finally, notice that if $M \in mod(A')$, then $M' \in mod(A')$ and $F_\lambda M' \in mod(A)$. This proves that $(\star\star)$ is also bijective. ■

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