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CATALYTIC MAJORIZATION AND ℓ_p NORMS

GUILLAUME AUBRUN AND ION NECHITA

ABSTRACT. The possible transformations of jointly held quantum states under local operations and classical communication (LOCC) are described using majorization, which is a fairly well-known relation. This LOCC protocol can be generalized in two ways: using an auxiliary entangled state as a catalyst (Entanglement-assisted LOCC) or performing simultaneously several identical transformations (Multiple-copy LOCC). These protocols are encoded by relations on probability vectors that generalize the majorization relation. We study the geometry of the set of vectors dominated by a fixed vector for these two relations. We show that both closures in the ℓ_1 norm coincide and characterize it using inequalities involving the ℓ_p norms, $p \geq 1$. Our proof uses a variant of Cramér’s theorem on large deviations.

1. INTRODUCTION

The increasing interest that quantum entanglement has received in the past decade is due, in part, to its use as a *resource* in quantum information processing. We investigate the problem of entanglement transformation: under which conditions can an entangled state $|\phi\rangle$ be transformed into another entangled state $|\psi\rangle$? We restrict ourselves to LOCC protocols: Alice and Bob share $|\phi\rangle$ and have at their disposal only *local operations* (such as unitaries $U_A \otimes I_B$ for Alice) and *classical communication*. Nielsen showed in [11] that such a transformation is possible if and only if $\lambda_\phi \prec \lambda_\psi$, where “ \prec ” is the *majorization* relation and $\lambda_\phi, \lambda_\psi$ are the Schmidt coefficients vectors of $|\phi\rangle$ and $|\psi\rangle$ respectively. Practically in the same time, Jonathan and Plenio [7] discovered a very interesting phenomenon: entanglement can help LOCC communication, without even being consumed. Precisely, they have found states $|\phi\rangle$ and $|\psi\rangle$ such that $|\phi\rangle$ cannot be transformed into $|\psi\rangle$, but, with the help of a *catalyst* state $|\chi\rangle$, the transformation $|\phi\rangle \otimes |\chi\rangle \rightarrow |\psi\rangle \otimes |\chi\rangle$ is possible. When such a catalyst exists, we say that the state $|\phi\rangle$ is *trumped* by $|\psi\rangle$ and we write $\lambda_\phi \prec_T \lambda_\psi$. It turns out that the trumping relation is much more complicated than the majorization relation; one can easily check on two given states $|\phi\rangle$ and $|\psi\rangle$ whether $\lambda_\phi \prec \lambda_\psi$ is satisfied or not, but there is no direct way to determine if $\lambda_\phi \prec_T \lambda_\psi$. Later, Bandyopadhyay et al. [1] discovered a similar situation occurs when trying to transform by LOCC *multiple copies* of $|\phi\rangle$ into $|\psi\rangle$. It may happen that the transformation $|\phi\rangle \rightarrow |\psi\rangle$ is not possible, but when considering n copies, one can transform $|\phi\rangle^{\otimes n}$ into $|\psi\rangle^{\otimes n}$. The phenomenon of multiple simultaneous LOCC transformations, or MLOCC, has been intensively studied in the last years and many similarities with ELOCC have been found [6].

In this note, we make some progress towards a complete characterization of both ELOCC and MLOCC. We show that a set of inequalities involving ℓ_p norms (see the remark on Nielsen’s Conjecture 1 at the end of the paper) is equivalent to the fact that $|\phi\rangle$ can be approached by a sequence of states $|\phi_n\rangle$ which are MLOCC/ELOCC-dominated by $|\psi\rangle$. An important point is that we allow the dimension of $|\phi_n\rangle$ to exceed the dimension of $|\phi\rangle$. Our proof uses probabilistic tools; we introduce probability measures associated to $|\phi\rangle$ and $|\psi\rangle$ and we use large deviation techniques to show the desired result.

The paper is organized as follows: in Section 2 we introduce the notation and the general framework of entanglement transformation of bipartite states. We also state our main result, Theorem 1. The theorem is proved in the next section. Conclusions and some directions for further study are sketched

in Section 4. The appendix at the end of the paper contains basic results from large deviation theory needed in the proof of the main theorem.

2. NOTATION AND STATEMENT OF THE RESULTS

For $d \in \mathbf{N}^*$, let P_d be the set of d -dimensional probability vectors : $P_d = \{x \in \mathbf{R}^d \text{ s.t. } x_i \geq 0, \sum x_i = 1\}$. If $x \in P_d$, we write x^\downarrow for the decreasing rearrangement of x , i.e. the vector $x^\downarrow \in P_d$ such that x and x^\downarrow have the same coordinates up to permutation, and $x_i^\downarrow \geq x_{i+1}^\downarrow$. We shall also write x_{\max} for x_1^\downarrow and x_{\min} for the smallest nonzero coordinate of x .

There are two operations on probability vectors that are important in what follows: the direct sum \oplus and the tensor product \otimes . If $x = (x_1, \dots, x_d) \in P_d$ and $x' = (x'_1, \dots, x'_{d'}) \in P_{d'}$, then $x \oplus x'$ denotes the concatenated vector $(x_1, \dots, x_d, x'_1, \dots, x'_{d'}) \in \mathbf{R}^{d+d'}$. The tensor product $x \otimes x'$ is the vector $(x_i x'_j)_{ij} \in P_{dd'}$; the way we order the coordinates of $x \otimes x'$ is immaterial for our purposes.

If $x \in P_d$ satisfies $x_d = 0$, it will be useful to identify x with the truncated vector $(x_1, \dots, x_{d-1}) \in P_{d-1}$. This identification induces a canonical inclusion $P_{d-1} \subset P_d$. Thus, every vector $x \in P_d$ can be thought of as a vector of $P_{d'}$, for all $d' \geq d$ by appending $d' - d$ null elements to x . We consider thus the set of all probability vectors $P_c = \bigcup_{d>0} P_d$. In other words, P_c is the set of finitely supported probability vectors.

Let us now introduce the classical majorization relation [9, 2]. If $x, y \in P_d$, we write $x \prec y$ (we say that y majorizes x) if for any $k \in \{1, \dots, d\}$, we have

$$\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow.$$

Note that for $k = d$ we have equality in the preceding inequality. We write $S(y)$ for the set of vectors x in P_d which are majorized by y . It is well-known that $S(y)$ is a compact convex set whose extreme points are the vectors obtained by permuting the coordinates of y ; many other characterizations of $S(y)$ are known [12, 3]. This relation behaves well with respect to direct sums and tensor products: $x \prec y$ implies $x \oplus z \prec y \oplus z$ and $x \otimes z \prec y \otimes z$ for any $z \in P_c$. The majorization relation has been shown to have a very important role in quantum information. Nielsen has proved [11] that a state $|\phi\rangle$ belonging to Alice and Bob can be transformed into the state $|\psi\rangle$ by using local operations and classical communication (LOCC) if and only if

$$\lambda_\phi \prec \lambda_\psi,$$

where λ_ϕ (respectively λ_ψ) is the vector of eigenvalues of the density matrix for Alice's system when the joint system is in the state $|\phi\rangle$ (respectively $|\psi\rangle$). Not long after Nielsen's theorem, Jonathan and Plenio have discovered a very intriguing phenomenon: there exist states $|\phi\rangle$ and $|\psi\rangle$ such that the transformation $|\phi\rangle \rightarrow |\psi\rangle$ is impossible by LOCC, but, with the aid of a *catalyst state* $|\chi\rangle$, the transformation $|\phi\rangle \otimes |\chi\rangle \rightarrow |\psi\rangle \otimes |\chi\rangle$ becomes possible; we say that $|\phi\rangle$ can be transformed into $|\psi\rangle$ by *Entanglement-assisted LOCC* or ELOCC. This result has motivated a more complex relation between probability vectors: if $x, y \in P_d$, we say that y trumps x and write $x \prec_T y$ if there exists $z \in P_c$ such that $x \otimes z \prec y \otimes z$. It is important to require that the auxiliary vector z (called the catalyst) is finitely supported (see Remark 2). Given $y \in P_d$, we write $T(y)$ for the set of d -dimensional vectors trumped by y , that is

$$T(y) = \{x \in P_d \text{ s.t. } x \prec_T y\}.$$

The set $T(y)$ is in general larger than $S(y)$ [4] and much more complicated to describe. Up to now, there is no known simple procedure to decide whether $x \in T(y)$ or not. Hence, finding a tractable characterization of the relation \prec_T (or, equivalently, of the set $T(y)$) has become an important open

problem in quantum information theory [14]. The geometry of $T(y)$ has been studied in [3, 4]: it is a bounded convex set that it is neither closed, nor open (provided y is not too simple). We shall introduce now another important extension of LOCC transformations. Bandyopadhyay et al [1] found an example of entangled states $|\phi\rangle$ and $|\psi\rangle$ with the property that the LOCC transformation $|\phi\rangle \rightarrow |\psi\rangle$ is impossible but, when one tries to transform multiple copies of the states, the transformation $|\phi\rangle^{\otimes n} \rightarrow |\psi\rangle^{\otimes n}$ becomes possible. We say that $|\psi\rangle$ MLOCC-dominates $|\phi\rangle$. We introduce the analogue of the trumping relation for probability vectors:

$$x \prec_M y \quad \text{iff} \quad \exists n \geq 1 \text{ s.t. } x^{\otimes n} \prec y^{\otimes n},$$

and the set of probability vectors MLOCC-dominated by a given vector y :

$$M(y) = \{x \in P_d \text{ s.t. } x \prec_M y\}.$$

Not much is known about the set $M(y)$. It has been studied in [6] and shown to have many similarities with the set $T(y)$: for example it is neither closed nor open in general. One important point is that, for all y , we have $M(y) \subseteq T(y)$ (see [6]).

We report progress towards a description of the sets of $M(y)$ and $T(y)$. The main ingredient of our approach is the following observation. Consider two vectors $x, y \in P_d$. Whether $x \prec y$, $x \prec_M y$, $x \prec_T y$ or not depends only on the non-zero coordinates of x and y . Thus, it is possible to $\prec/\prec_M/\prec_T$ -compare vectors of different sizes by appending the necessary amount of zero coordinates to the end of one of them. Hence, it seems more natural (at least from a mathematical point of view) to consider the sets

$$T_c(y) = \{x \in P_c \text{ s.t. } x \prec_T y\} = \{x \in P_c \text{ s.t. } \exists z \in P_c \text{ s.t. } x \otimes z \prec y \otimes z\}$$

and

$$M_c(y) = \{x \in P_c \text{ s.t. } x \prec_M y\} = \{x \in P_c \text{ s.t. } \exists n \geq 1 \text{ s.t. } x^{\otimes n} \prec y^{\otimes n}\}.$$

The important point here is that both $T_c(y)$ and $M_c(y)$ do not depend anymore on the size of y , but only on the non-null coordinates of y . This was not the case for $T(y)$ or $M(y)$: $T((1, 0)) = M((1, 0)) = P_2$, whereas $T((1, 0, 0)) = M((1, 0, 0)) = P_3$.

Of course, if $y \in P_d$, $T(y) = T_c(y) \cap P_d$ and $M(y) = M_c(y) \cap P_d$; this shows that the sets $T_c(y)$ and $M_c(y)$ are not closed either in general. We then write $\overline{T_c(y)}$ and $\overline{M_c(y)}$ to denote the closure taken with respect to the ℓ_1 -norm, the natural topology in this setting. Recall that for $p \geq 1$, the ℓ_p norm of a vector $x \in P_d$ is defined as

$$(1) \quad \|x\|_p = \left(\sum_{i=1}^d x_i^p \right)^{1/p}$$

and $\|x\|_\infty = \max x_i$. We now come to our main result:

Theorem 1. *Consider two vectors $x, y \in P_c$. The following assertions are equivalent:*

- (a) $x \in \overline{M_c(y)}$,
- (b) $x \in \overline{T_c(y)}$,
- (c) $\|x\|_p \leq \|y\|_p \quad \forall p \geq 1$.

Remark 1. *Note that instead of demanding that $\|x\|_p \leq \|y\|_p$ for all $p \geq 1$, it suffices to ask for $x, y \in P_d$ that the inequality holds for all $p \in [1, p_{\max}(x, y)]$, where $p_{\max}(x, y) = \log d / (\log y_{\max} - \log x_{\max})$. The inequalities for $p > p_{\max}$ follow by simple computation. For such results in a more general setting, see [10].*

Remark 2. *It is important to see at this point how the set $\overline{T_c(y)}$ is related to the set $T(y)$. First of all, note that if we drop the closure, we have equality: $T_c(y) \cap P_d = T(y)$ for $y \in P_d$. However, when taking the ℓ_1 closure of the left hand side, we obtain a strict inclusion: $\overline{T(y)} \subsetneq \overline{T_c(y)} \cap P_d$. An example for such a vector is provided by the phenomenon of infinite-dimensional catalysis, discovered by Daftuar [3]. Take $y = (0.5, 0.25, 0.25)$ and $x = (0.4, 0.4, 0.2)$. It is obvious that $x \notin \overline{T(y)}$ because $x_3 < y_3$ and the condition $x_d \geq y_d$ is necessary for $x \in \overline{T(y)}$. However, there exist an infinite-dimensional catalyst $z = (1 - \alpha)(1, \alpha, \alpha^2, \dots, \alpha^k, \dots)$, where $\alpha = 2^{-\frac{1}{8}}$, such that $x \otimes z \prec y \otimes z$. Note that z is ℓ_p -bounded and thus $\|x\|_p \leq \|y\|_p$ for all $p \geq 1$. By the preceding theorem, we have that $x \in \overline{T_c(y)} \cap P_3$. For further remarks on this topic, see Section 4.*

Remark 3. *Let us explain here how the ℓ_1 norm used in the statement of the theorem relates to other physically motivated distances between the approaching states $|\phi_n\rangle$ and the original state $|\phi\rangle$. Recall that x is the eigenvalue vector of the reduced density matrix corresponding to Alice's (or, equivalently to Bob's) part of the system. From the details of the proof (see also Section 4), one sees that the size of the approaching vectors x_n increases with n . So, in order to compare ρ and ρ_n , we have to realize them as density matrices on the same Hilbert space \mathcal{H} . Moreover, we can suppose that the two states are diagonalizable in the same basis (Alice can achieve this by applying a local unitary basis change). As usually, we append the necessary number of zero eigenvalues to x in order to have the same size as x_n . We obtain the following equality:*

$$\|x - x_n\|_1 = \|\rho - \rho_n\|_{tr}.$$

So, for Alice's part of the system, we obtain a convergence in the trace norm sense. It is well known that the trace norm distance is related to the probability that the two states can be distinguished by some measurement. Moreover, by using some classical inequalities (see [13], Chapter 9), the fidelity $F(\rho, \rho_n)$ can be shown to converge to 1.

3. THE PROOF OF THE THEOREM

We shall prove the sequence of implications (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a). The first two are well known; we sketch their proof for completeness. The third is the most difficult one and represents our contribution to the theorem.

(a) \Rightarrow (b) Because the closure is taken with respect to the same topology (ℓ_1) for both $\overline{M_c(y)}$ and $\overline{T_c(y)}$, it is enough to show $M_c(y) \subset T_c(y)$. Let $x \in M_c(y)$ and consider n such that $x^{\otimes n} \prec y^{\otimes n}$. The trick here (see [6]) is to use the following z as a catalyst

$$z = x^{\otimes(n-1)} \oplus x^{\otimes(n-2)} \otimes y \oplus \dots \oplus x \otimes y^{\otimes(n-2)} \oplus y^{\otimes(n-1)}.$$

For simplicity we do not normalize z , but this is irrelevant. The vector z has been constructed such that

$$x \otimes z = x^{\otimes n} \oplus w \text{ and } y \otimes z = y^{\otimes n} \oplus w,$$

where w is the same in both expressions. This implies that $x \otimes z \prec y \otimes z$, i.e. $x \in T_c(y)$.

(b) \Rightarrow (c) Let $z \in P_c$ be the catalyst for $x \prec_T y$: $x \otimes z \prec y \otimes z$. A function $\varphi : \mathbf{R}^d \rightarrow \mathbf{R}$ is said to be Schur-convex if $a \prec b$ implies $\varphi(a) \leq \varphi(b)$. It is well-known (see [9, 12]) that if $h : \mathbf{R} \rightarrow \mathbf{R}$ is a convex function, then $\varphi : x \mapsto \sum_{i=1}^d h(x_i)$ is Schur-convex. Consequently, the functions $x \mapsto \|x\|_p^p$ are Schur-convex for $p \geq 1$. Moreover, they satisfy the identity $\|x \otimes z\|_p = \|x\|_p \|z\|_p$, and similarly for y . Since $\|z\|_p$ is finite, we get that $\|x\|_p \leq \|y\|_p$. To show that the same is true for $x \in \overline{T_c(y)}$, it suffices to check that the set of $x \in \ell_1$ such that $\|x\|_p \leq \|y\|_p$ is norm-closed; this follows from the inequality $\|\cdot\|_p \leq \|\cdot\|_1$.

(c) \Rightarrow (a) We will adapt some techniques used by G. Kuperberg in a slightly different context [8]. In our proof, we allow deficient vectors, i.e. vectors with total mass smaller than 1. Note that the definitions of $\|x\|_p$ and $x \prec y$ extend to any vectors with nonnegative coordinates. As in [8], we associate to a positive vector $x \in \mathbf{R}^d$ the measure $\mu_x = \sum_{i=1}^d x_i \delta_{\log x_i}$, where δ_z is the Dirac measure at point z . The basic property is that the tensor product operation of vectors corresponds to the convolution of associated measures:

$$\mu_{x \otimes y} = \mu_x * \mu_y.$$

The following lemma gives a way to prove majorization using comparison of the tails of the associated measures

Lemma 1. *Let x and y be two vectors of \mathbf{R}^d with non-negative components. Consider the measures μ_x and μ_y associated with x and y . Assume that, for all $t \in \mathbf{R}$, $\mu_x[t, \infty) \leq \mu_y[t, \infty)$. Then $x \prec y$.*

Proof. Note that

$$\mu_x[t, \infty) = \sum_{i: \log x_i \geq t} x_i = \sum_{i: x_i \geq \exp(t)} x_i.$$

Thus, for all $u > 0$, $\sum_{i: x_i \geq u} x_i \leq \sum_{i: y_i \geq u} y_i$. For simplicity, we assume first that all coordinates of y are distinct. We will show by induction on $k \in \{1, \dots, d\}$ that $\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow$. For the first step; use $u = y_1^\downarrow$ to conclude that $x_1^\downarrow \leq y_1^\downarrow$. Now, fix $k \in \{1, \dots, d-1\}$ and suppose that $\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow$. If $x_{k+1}^\downarrow \leq y_{k+1}^\downarrow$, the induction step is obvious. If $x_{k+1}^\downarrow > y_{k+1}^\downarrow$, we use $u = x_{k+1}^\downarrow$ to get

$$\sum_{i=1}^{k+1} x_i^\downarrow \leq \sum_{i: x_i \geq x_{k+1}^\downarrow} x_i \leq \sum_{i: y_i \geq x_{k+1}^\downarrow} y_i \leq \sum_{i: y_i \geq y_{k+1}^\downarrow} y_i = \sum_{i=1}^{k+1} y_i^\downarrow.$$

This completes the induction when y has distinct coordinates. The general case follows by approximating y by $y + \varepsilon_n$, where (ε_n) is a suitable sequence of positive vectors tending to 0. The approximation is possible since the set of vectors y majorizing a fixed x is closed. \square

We now get to the key lemma in our argument. We shall use a slightly modified version of Cramér large deviations theorem — see Appendix.

Lemma 2. *Let x, y in \mathbf{R}^d , with nonnegative coordinates. Assume that for any $1 \leq p \leq \infty$, we have the **strict inequality** $\|x\|_p < \|y\|_p$. Then there exists an integer n such that $x^{\otimes n} \prec y^{\otimes n}$.*

Proof. Consider x and y satisfying the hypotheses of the lemma. We can assume by multiplying both vectors by a positive constant K that $\|y\|_1 = 1$. Let $p = 1 - \|x\|_1 > 0$. We introduce the measures μ_x and μ_y associated to x and y ; μ_y is a probability measure but μ_x is not, so we add a mass at $-\infty$ by setting $\bar{\mu}_x = \mu_x + p\delta_{-\infty}$. Let X and Y be random variables distributed according to $\bar{\mu}_x$ and μ_y respectively. We denote by (X_n) (resp. (Y_n)) a sequence of i.i.d. copies of X (resp. Y). We are going to show that for n large enough

$$(2) \quad \forall t \in \mathbf{R}, \mathbf{P}(X_1 + \dots + X_n \geq nt) \leq \mathbf{P}(Y_1 + \dots + Y_n \geq nt).$$

This is equivalent to showing that

$$\int_{nt}^{\infty} d\mu_x^{*n} = \int_{nt}^{\infty} d\bar{\mu}_x^{*n} \leq \int_{nt}^{\infty} d\mu_y^{*n},$$

which, by the previous lemma implies $x^{\otimes n} \prec y^{\otimes n}$. Note that the asymptotic behavior of the quantities appearing in (2) is governed by Cramér's theorem. Let $f_n(t) = \mathbf{P}(X_1 + \dots + X_n \geq nt)^{1/n}$ and

$g_n(t) = \mathbf{P}(Y_1 + \dots + Y_n \geq nt)^{1/n}$. Applying Cramér's theorem (see Appendix), we obtain

$$f(t) := \lim_{n \rightarrow \infty} f_n(t) = \begin{cases} 1 - p & \text{if } t \leq \mathbf{E}(X|X \neq -\infty) \\ e^{-\Lambda_X^*(t)} & \text{otherwise.} \end{cases}$$

$$g(t) := \lim_{n \rightarrow \infty} g_n(t) = \begin{cases} 1 & \text{if } t \leq \mathbf{E}(Y) \\ e^{-\Lambda_Y^*(t)} & \text{otherwise.} \end{cases}$$

Note also that the log-Laplace of X , defined for $\lambda \in \mathbf{R}$ by $\Lambda_X(\lambda) = \log \mathbf{E}e^{\lambda X}$, is related to the ℓ_p norms of x :

$$\forall \lambda \geq 0, \quad \Lambda_X(\lambda) = \log \|x\|_{\lambda+1}^{\lambda+1}.$$

The same holds for Y : $\Lambda_Y(\lambda) = \log \|y\|_{\lambda+1}^{\lambda+1}$ and thus we have $\Lambda_X(\lambda) < \Lambda_Y(\lambda)$ for $\lambda \geq 0$.

Let $M_X = \text{esssup } X = \log \|x\|_\infty$ and $M_Y = \text{esssup } Y = \log \|y\|_\infty$; by hypothesis $M_X < M_Y$. First of all, note that $f_n(t) = 0$ for $t \geq M_X$, so it suffices to show that $f_n \leq g_n$ on $(-\infty, M_X]$, for n large enough. We claim that $f < g$ on $(-\infty, M_Y)$, and thus on $(-\infty, M_X]$. Indeed, for $\mathbf{E}(Y) \leq t < M_Y$, the supremum in the definition of $\Lambda_Y^*(t)$ is attained at a point $\lambda_0 \geq 0$ (cf Appendix), so we have that

$$f(t) \leq e^{-(\lambda_0 t - \Lambda_X(\lambda_0))} < e^{-(\lambda_0 t - \Lambda_Y(\lambda_0))} = g(t),$$

where the **strict** inequality follows from the fact that $\Lambda_X(\lambda) < \Lambda_Y(\lambda)$, for all $\lambda \geq 0$. For $t < \mathbf{E}(Y)$, $g(t) = 1$ and $f(t) \leq 1 - p < 1$. Moreover, the functions f and g admit finite limits in $-\infty$: $\lim_{t \rightarrow -\infty} f(t) = 1 - p$ and $\lim_{t \rightarrow -\infty} g(t) = 1$. Thus, on the compact set $[-\infty, M_X]$, the functions f and g are well-defined, non-increasing, continuous and satisfy $f < g$.

We now use the following elementary fact: if a sequence of non-increasing functions defined on a compact interval I converges pointwise towards a continuous limit, then the convergence is actually uniform on I (for a proof see [15] Part 2, Problem 127; this statement is attributed to Pólya or to Dini depending on authors). We apply this result to (f_n) and (g_n) on the interval $I = [-\infty, M_X]$ to conclude that the convergence is uniform for both sequences. As $f < g$, we can therefore find n large enough such that $f_n \leq g_n$ on I , and thus on \mathbf{R} . This is equivalent to (2) and completes the proof of the lemma. \square

Remark 4. *It is possible to avoid the use of Cramér's theorem by using low-technology estimates on large deviations probability instead, as done in [8]. This requires additional care to get the required uniform bounds and slightly obfuscates the argument. The only advantage is to give explicit bounds for the value of n in Lemma 2, which our compactness argument does not. These bounds are quite bad anyway, and for example do not allow to replace the ℓ_1 -closure in the main theorem by a ℓ_p -closure for some $p < 1$.*

Proof of (c) \Rightarrow (a) (continued) Recall that x and y are such that $\|x\|_p \leq \|y\|_p$ for any $p \geq 1$ and that we want to find, for any $\varepsilon > 0$, a vector $x_\varepsilon \in M_c(y)$ such that $\|x - x_\varepsilon\|_1 \leq \varepsilon$. Let d_x (resp. d_y) be the number of nonzero coordinates of x (resp. y). We proceed as follows: first consider the (deficient) vector x'_ε obtained from x by subtracting $\varepsilon/2d_x$ to each of its nonzero coordinates. This implies that $\|x - x'_\varepsilon\|_1 = \varepsilon/2$ and that x'_ε satisfies the hypotheses of Lemma 2. Applying the lemma, we obtain the existence of an integer n such that $(x'_\varepsilon)^{\otimes n} \prec y^{\otimes n}$.

Remember that x'_ε is deficient; we now enlarge it into a vector $x_\varepsilon \in P_c$ by adding mass $\varepsilon/2$. But since we want to keep the property $x_\varepsilon^{\otimes n} \prec y^{\otimes n}$, a safe way to do this is to add a large number of coordinates, each of them being very small. More precisely, let $x_\varepsilon = x'_\varepsilon \oplus \delta^{\oplus D}$, where $\delta D = \varepsilon/2$ and δ

is a positive number such that $\delta(x'_\varepsilon)^{n-1} \leq \min((x'_\varepsilon)_{\min}^n, y_{\min}^n)$. We claim that $x_\varepsilon^{\otimes n} \prec y^{\otimes n}$, that is, for any $k \geq 1$,

$$(3) \quad \sum_{i=1}^k (x_\varepsilon^{\otimes n})_i^\downarrow \leq \sum_{i=1}^k (y^{\otimes n})_i^\downarrow.$$

Indeed, δ has been chosen so that the d_x^n largest coordinates of $x_\varepsilon^{\otimes n}$ are exactly the coordinates of $(x'_\varepsilon)^{\otimes n}$, so when $1 \leq k \leq d_x^n$, (3) follows from the relation $(x'_\varepsilon)^{\otimes n} \prec y^{\otimes n}$. If $d_x < k \leq d_y^n$, the inequality also holds since the choice of δ guarantees $(x_\varepsilon^{\otimes n})_k^\downarrow \leq (y^{\otimes n})_k^\downarrow$. Finally if $k \geq d_y^n$, (3) holds trivially since the right-hand side equals 1.

In conclusion, $x_\varepsilon^{\otimes n} \prec y^{\otimes n}$, and thus $x_\varepsilon \in M_c(y)$. But x_ε has been constructed such that $\|x - x_\varepsilon\|_1 \leq \varepsilon$ and thus $x \in \overline{M_c(y)}$ which completes the proof of the theorem.

4. CONCLUSION AND FURTHER REMARKS

In conclusion, we are able to give a nice description of the ℓ_1 -closure of the set $T_c(y)$. However, this closure may be substantially larger than the usual closure $\overline{T(y)}$ in P_d , and requires approximation by vectors with growing support. Our result can be seen as a contribution to a conjecture attributed to Nielsen [3]:

Conjecture 1 (Nielsen). *Fix a vector $y \in P_d$. Then a vector $x \in P_d$ belongs to $\overline{T(y)}$ if and only if the following conditions are verified.*

- (1) For $p \geq 1$, $\|x\|_p \leq \|y\|_p$.
- (2) For $0 < p \leq 1$, $\|x\|_p \geq \|y\|_p$.
- (3) For $p < 0$, $\|x\|_p \geq \|y\|_p$.

Note that the definition of $\|\cdot\|_p$ given in (1) is extended to any $p \in \mathbf{R}^*$. For $p < 1$, $\|\cdot\|_p$ is not a norm in the usual sense. We have shown that the condition (1) above is equivalent to $x \in \overline{T_c(y)}$. Notice however that $\overline{T_c(y)}$ is in general larger than $\overline{T(y)}$; note also that the set of $x \in P_d$ that satisfy conditions (1–3) is closed.

This question also appears in [6] where it is formulated using the Rényi entropies. For any real $p \neq 1$, the p -Rényi entropy is defined for $x \in P_d$ as

$$H_p(x) = \frac{\text{sgn}(p)}{p-1} \log_2 \left(\sum_{i=1}^d x_i^p \right).$$

The limit case $p = 1$ corresponds to the usual entropy. The conditions (1–3) of the conjecture can be concisely reformulated as “ $H_p(x) \leq H_p(y)$ for all p ”. The “only if” part of the conjecture follows from standard convexity/concavity properties of functionals $\|\cdot\|_p$, see [12, 3].

An intermediate notion is the following: for $y \in P_d$, let $\overline{T_c(y)}^b$ be the set of vectors $x \in P_d$ such that there is a sequence (x_n) in $T_c(y)$ tending to x , with a **uniform** bound on the size of the support of x_n . We think that a description of $\overline{T_c(y)}^b$ could be related to the set of vectors which satisfy conditions (1) and (2) — but not necessarily (3) — in Conjecture 1.

There is one more consequence of our main theorem we would like to discuss. Recall that when defining catalysis, we insisted on the fact that the catalyst should be finitely-supported. Let $P_\infty \subset \ell_1$ be the set of infinite-dimensional probability vectors, and for y in P_c , define the set $T_c^\infty(y)$ of (finitely supported) vectors trumped by y using infinite catalysts:

$$T_c^\infty(y) = \{x \in P_c \text{ s.t. } \exists z \in P_\infty \text{ s.t. } x \otimes z \prec y \otimes z\}.$$

As shown in [3] (Section 4.3), in general $T_c(y) \neq T_c^\infty(y)$. However, since $x \in T_c^\infty(y)$ implies $\|x\|_p \leq \|y\|_p$ for all $p \geq 1$, it follows from our main theorem that $\overline{T_c(y)} = \overline{T_c^\infty(y)}$.

5. APPENDIX: ON CRAMÉR'S THEOREM

We review here some facts from large deviations theory. A complete reference for all the material contained here is [5]. Let X be a random variable taking values in $[-\infty, \infty)$. We allow X to equal $-\infty$ with positive probability; this is a nonstandard hypothesis. We however exclude the trivial case $\mathbf{P}(X = -\infty) = 1$. We write \mathbf{E} for the expectation. We assume also that the conditional expectation $\mathbf{E}(X|X \neq -\infty)$ is finite. The cumulant generating function Λ_X of the random variable X is defined for any $\lambda \in \mathbf{R}$ by

$$\Lambda_X(\lambda) = \log \mathbf{E}e^{\lambda X}.$$

It is a convex function taking values in $(-\infty, +\infty]$. Its convex conjugate Λ_X^* , sometimes called the Cramér transform, is defined as

$$(4) \quad \Lambda_X^*(x) = \sup_{\lambda \in \mathbf{R}} \lambda x - \Lambda_X(\lambda).$$

Note that Λ_X is a smooth and strictly convex function on $[0, +\infty]$. Moreover, $\Lambda_X'(0) = \mathbf{E}(X|X \neq -\infty)$ and $\lim_{\lambda \rightarrow +\infty} \Lambda_X'(\lambda) = \text{esssup}(X)$. Consequently, for any x such that $\mathbf{E}(X|X \neq -\infty) < x < \text{esssup}(X)$, the supremum in (4) is attained at a unique point $\lambda \geq 0$. We now state Cramér's theorem in a suitable formulation

Proposition 1 (Cramér's theorem). *Let X be a $[-\infty, +\infty)$ -valued random variable such that $\Lambda_X(\lambda) < +\infty$ for any $\lambda \geq 0$. Let (X_i) be a sequence of i.i.d. copies of X . Then for any $t \in \mathbf{R}$*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}(X_1 + \dots + X_n \geq tn) = \begin{cases} \log \mathbf{P}(X \neq -\infty) & \text{if } t \leq \mathbf{E}(X|X \neq -\infty) \\ -\Lambda_X^*(t) & \text{otherwise.} \end{cases}$$

Proof. Let $p = \mathbf{P}(X = -\infty)$. The classical Cramér's theorem is the theorem above when $p = 0$, we refer to [5] for a proof. We show how it implies the general case: let \hat{X} denote the random variable X conditioned to be finite, that is for any Borel set $B \subset \mathbf{R}$

$$\mathbf{P}(\hat{X} \in B) = \frac{1}{1-p} \mathbf{P}(X \in B).$$

One checks that $\Lambda_{\hat{X}} = \Lambda_X - \log(1-p)$, and consequently

$$\Lambda_{\hat{X}}^* = \Lambda_X^* + \log(1-p).$$

Note also that

$$\mathbf{P}(X_1 + \dots + X_n \geq tn) = (1-p)^n \mathbf{P}(\hat{X}_1 + \dots + \hat{X}_n \geq tn).$$

Applying the classical Cramér's theorem to \hat{X} gives

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{P}(\hat{X}_1 + \dots + \hat{X}_n \geq tn) = \begin{cases} 0 & \text{if } t \leq \mathbf{E}(\hat{X}) \\ -\Lambda_{\hat{X}}^*(t) & \text{otherwise.} \end{cases}$$

The proposition follows from the last three equations. □

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