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BAHADUR REPRESENTATION OF SAMPLE QUANTILES FOR  
FUNCTIONAL OF GAUSSIAN DEPENDENT SEQUENCES UNDER A  
MINIMAL ASSUMPTION

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We obtain a Bahadur representation for sample quantiles of nonlinear functional of Gaussian sequences with correlation function decreasing as  $k^{-\alpha}$  for some  $\alpha > 0$ . This representation is derived under a minimal assumption.

## 1 Introduction

We consider the problem of obtaining a Bahadur representation of sample quantiles in a certain dependence context. Before stating in what a Bahadur representation consists, let us specify some general notation. Given some random variable  $Y$ ,  $F(\cdot) = F_Y(\cdot)$  is referred as the cumulative distribution function of  $Y$ ,  $\xi(p) = \xi_Y(p)$  for some  $0 < p < 1$  as the quantile of order  $p$ . If  $F(\cdot)$  is absolutely continuous with respect to Lebesgue measure, the probability density function is denoted by  $f(\cdot) = f_Y(\cdot)$ . Based on the observation of a vector  $\mathbf{Y} = (Y(1), \dots, Y(n))$  of  $n$  random variables distributed as  $Y$ , the sample cumulative distribution function and the sample quantile of order  $p$  are respectively denoted by  $\widehat{F}_Y(\cdot; \mathbf{Y})$  and  $\widehat{\xi}_Y(p; \mathbf{Y})$  or simply by  $\widehat{F}(\cdot; \mathbf{Y})$  and  $\widehat{\xi}(p; \mathbf{Y})$ .

Let  $\mathbf{Y} = (Y(1), \dots, Y(n))$  a vector of  $n$  i.i.d. random variables such that  $F''(\xi(p))$  exists and is bounded in a neighborhood of  $\xi(p)$  and such that  $F'(\xi(p)) > 0$ , Bahadur proved that as  $n \rightarrow +\infty$ ,

$$\widehat{\xi}(p) - \xi(p) = \frac{p - \widehat{F}(p)}{f(\xi(p))} + r_n,$$

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with  $r_n = \mathcal{O}_{a.s.}(n^{-3/4} \log(n)^{3/4})$  where a sequence of random variables  $U_n$  is said to be  $\mathcal{O}_{a.s.}(v_n)$  if  $U_n/v_n$  is almost surely bounded. Kiefer obtained the exact rate  $n^{-3/4} \log \log(n)^{3/4}$ . Under an Assumption on  $F(\cdot)$  which is quite similar to the one done by Bahadur, extensions of above results to dependent random variables have been pursued in Sen and Ghosh (1972) for  $\phi$ -mixing variables, in Yoshihara (1995) for strongly mixing variables, and recently in Wu (2005) for short-range and long-range dependent linear processes, following works of Hesse (1990) and Ho and Hsing (1996). Finally, such a representation has been obtained by Coeurjolly (2007) for nonlinear functional of Gaussian sequences with correlation function decreasing as  $k^{-\alpha}$  for some  $\alpha > 0$ .

Ghosh (1971) proposed in the i.i.d. case a much simpler proof of Bahadur's result which suffices for many statistical applications. He established under a weaker assumption on  $F(\cdot)$  ( $F'(\cdot)$  exists and is bounded in a neighborhood of  $\xi(p)$  and  $f(\xi(p)) > 0$ ) that the remainder term satisfies  $r_n = o_p(n^{-1/2})$ , which means that  $n^{1/2}r_n$  tends to 0 in probability. This result is sufficient for example to establish a central limit theorem for the sample quantile. Our goal is to extend Ghosh's result to nonlinear functional of Gaussian sequences with correlation function decreasing as  $k^{-\alpha}$ . The Bahadur representation is presented in Section 2 and is applied to a central limit theorem for the sample quantile. Proofs are deferred in Section 3.

## 2 Main result

Let  $\{Y(i)\}_{i=1}^{+\infty}$  be a stationary (centered) gaussian process with variance 1, and correlation function  $\rho(\cdot)$  such that, as  $i \rightarrow +\infty$

$$|\rho(i)| \sim i^{-\alpha} \tag{1}$$

for some  $\alpha > 0$ .

Let us recall some background on Hermite polynomials: the Hermite polynomials form an orthogonal system for the Gaussian measure and are in particular such that  $\mathbf{E}(H_j(Y)H_k(Y)) = j! \delta_{j,k}$ , where  $Y$  is referred to a standard Gaussian variable. For some measurable function  $g(\cdot)$  defined on  $\mathbb{R}$  such that  $\mathbf{E}(g(Y)^2) < +\infty$ , the following expansion holds

$$g(t) = \sum_{j \geq \tau} \frac{c_j}{j!} H_j(t) \quad \text{with} \quad c_j = \mathbf{E}(g(Y)H_j(Y)),$$

where the integer  $\tau$  defined by  $\tau = \inf \{j \geq 0, c_j \neq 0\}$ , is called the Hermite rank of the function  $g$ . Note that this integer plays an important role. For example, it is related to the correlation of  $g(Y_1)$  and  $g(Y_2)$ , for  $Y_1$  and  $Y_2$  two standard gaussian variables with correlation  $\rho$ , since  $\mathbf{E}(g(Y_1)g(Y_2)) = \sum_{k \geq \tau} \frac{(c_k)^2}{k!} \rho^k = \mathcal{O}(\rho^\tau)$ .

Our result is based on the assumption that  $F'_{g(Y)}(\cdot)$  exists and is bounded in a neighborhood of  $\xi(p)$ . This is achieved if the function  $g(\cdot)$  satisfies the following assumption (see *e.g.* Dacunha-Castelle and Dufflo (1982), p.33).

**Assumption A( $\xi(p)$ )** : there exist  $U_i$ ,  $i = 1, \dots, L$ , disjoint open sets such that  $U_i$  contains a unique solution to the equation  $g(t) = \xi_{g(Y)}(p)$ , such that  $F'_{g(Y)}(\xi(p)) > 0$  and such that  $g$  is a  $\mathcal{C}^1$ -diffeomorphism on  $\cup_{i=1}^L U_i$ .

Note that this assumption allows us to obtain

$$F'_{g(Y)}(\xi_{g(Y)}(p)) = f_{g(Y)}(\xi_{g(Y)}(p)) = \sum_{i=1}^L \frac{\phi(g_i^{-1}(t))}{g'(g_i^{-1}(t))},$$

where  $g_i(\cdot)$  is the restriction of  $g(\cdot)$  on  $U_i$  and where  $\phi(\cdot)$  is referred to the probability density function of a standard Gaussian variable.

Now, define, for some real  $u$ , the function  $h_u(\cdot)$  by:

$$h_u(t) = \mathbf{1}_{\{g(t) \leq u\}}(t) - F_{g(Y)}(u). \quad (2)$$

We denote by  $\tau(u)$  the Hermite rank of  $h_u(\cdot)$ . For the sake of simplicity, we set  $\tau_p = \tau(\xi_{g(Y)}(p))$ . For some function  $g(\cdot)$  satisfying Assumption **A( $\xi(p)$ )**, we denote by

$$\bar{\tau}_p = \inf_{\gamma \in \cup_{i=1}^L g(U_i)} \tau(\gamma), \quad (3)$$

that is the minimal Hermite rank of  $h_u(\cdot)$  for  $u$  in a neighborhood of  $\xi_{g(Y)}(p)$ .

**Theorem 1** *Under Assumption **A( $\xi(p)$ )**, the following result holds as  $n \rightarrow +\infty$*

$$\hat{\xi}(p; \mathbf{g}(\mathbf{Y})) - \xi_{g(Y)}(p) = \frac{p - \hat{F}(\xi_{g(Y)}(p); \mathbf{g}(\mathbf{Y}))}{f_{g(Y)}(\xi_{g(Y)}(p))} + o_p(r_n(\alpha, \bar{\tau}_p)), \quad (4)$$

where  $\mathbf{g}(\mathbf{Y}) = (g(Y(1)), \dots, g(Y(n)))$ , for  $i = 1, \dots, n$  and where the sequence  $(r_n(\alpha, \bar{\tau}_p))_{n \geq 1}$  is defined by

$$r_n(\alpha, \bar{\tau}_p) = \begin{cases} n^{-1/2} & \text{if } \alpha \bar{\tau}_p > 1, \\ n^{-1/2} \log(n)^{1/2} & \text{if } \alpha \bar{\tau}_p = 1, \\ n^{(1-\alpha \bar{\tau}_p)/2} & \text{if } \alpha \bar{\tau}_p < 1. \end{cases} \quad (5)$$

**Remark 1** *The sequence  $r_n(\alpha, \bar{\tau}_p)$  is related to the behaviour short-range or long-range dependent behaviour of the sequence  $h_u(Y(1)), \dots, h_u(Y(n))$  for  $u$  in a neighborhood of  $\xi(p)$ . More precisely, it corresponds to the asymptotic behaviour of the sequence*

$$\left( \frac{1}{n} \sum_{|i| < n} \rho(i)^{\bar{\tau}_p} \right)^{1/2}.$$

**Corollary 2** *Under Assumption  $\mathbf{A}(\xi(\mathbf{p}))$ , then if  $\alpha \bar{\tau}_p > 1$ , the following convergence in distribution holds as  $n \rightarrow +\infty$*

$$\sqrt{n} \left( \widehat{\xi}(p; \mathbf{g}(\mathbf{Y})) - \xi_{g(Y)}(p) \right) \xrightarrow{d} \mathcal{N}(0, \sigma_p^2), \quad (6)$$

where  $\sigma_p^2 = \sum_{i \in \mathbb{Z}} \sum_{j \geq \tau_p} \frac{(c_j)^2}{j!} \rho(i)^j$

The proof of this result is omitted since it is a direct application of Theorem 1 and a central limit theorem adapted to nonlinear functional of Gaussian sequences, *e.g.* Breuer and Major (1983).

## 3 Proofs

### 3.1 Auxiliary Lemma

**Lemma 3** *For every  $j \geq 1$  and for all positive sequence  $(u_n)_{n \geq 1}$  such that  $u_n \rightarrow 0$ , as  $n \rightarrow +\infty$ , we have, under Assumption  $\mathbf{A}(\xi(\mathbf{p}))$*

$$I = \int_{\mathbb{R}} H_j(t) \phi(t) \mathbf{1}_{\{|g(t) - \xi_{g(Y)}(p)| \leq u_n\}} dt \sim u_n \kappa_j, \quad (7)$$

where  $\kappa_j$  is defined, for every  $j \geq 1$ , by

$$\kappa_j = \begin{cases} -2 \sum_{i=1}^L \frac{\phi'(g_i^{-1}(\xi(p)))}{g'(g_i^{-1}(\xi(p)))} & \text{if } j = 1, \\ 2(-1)^j \sum_{i=1}^L \frac{\phi^{(j-2)}(g_i^{-1}(\xi(p)))}{g'(g_i^{-1}(\xi(p)))} & \text{if } j > 1. \end{cases} \quad (8)$$

**Proof.** Under Assumption  $\mathbf{A}(\xi(\mathbf{p}))$ , there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ ,

$$I = \sum_{i=1}^L I_i \quad \text{with} \quad I_i = \int_{U_i} H_j(t) \phi(t) \mathbf{1}_{\{\xi(p) - u_n \leq g(t) \leq \xi(p) + u_n\}} dt. \quad (9)$$

Assume without loss of generality that the restriction of  $g(\cdot)$  on  $U_i$  (denoted by  $g_i(\cdot)$ ) is an increasing function, we have

$$\begin{aligned} I_i &= \int_{U_i} H_j(t) \phi(t) \mathbf{1}_{\{\xi(p)-u_n \leq g(t) \leq \xi(p)+u_n\}} dt \\ &= \int_{g_i^{-1}(\xi(p)-u_n)}^{g_i^{-1}(\xi(p)+u_n)} H_j(t) \phi(t) dt \\ &= \begin{cases} \phi(m_{i,n}) - \phi(M_{i,n}) = (m_{i,n} - M_{i,n}) & \text{if } j = 1 \\ (-1)^j (\phi^{(j-1)}(M_{i,n}) - \phi^{(j-1)}(m_{i,n})) & \text{if } j > 1, \end{cases} \end{aligned}$$

where  $M_{i,n} = g_i^{-1}(\xi(p) + u_n)$  and  $m_{i,n} = g_i^{-1}(\xi(p) - u_n)$ . Then, there exists  $\omega_{n,i,j} \in [m_{i,n}, M_{i,n}]$  for every  $j \geq 1$  such that

$$I_i = \begin{cases} (m_{i,n} - M_{i,n}) \phi^{(1)}(\omega_{n,i,1}) & \text{if } j = 1 \\ (-1)^j (M_{i,n} - m_{i,n}) \phi^{(j-2)}(\omega_{n,i,j}) & \text{if } j > 1. \end{cases},$$

Under Assumption **A**( $\xi(\mathbf{p})$ ), we have, as  $n \rightarrow +\infty$

$$\omega_{n,i,j} \sim g_i^{-1}(\xi(p)) \quad \text{and} \quad M_{i,n} - m_{i,n} \sim 2u_n \frac{1}{g'(g_i^{-1}(\xi(p)))},$$

which ends the proof.  $\blacksquare$

### 3.2 Proof of Theorem 1

For the sake of simplicity, we set  $\widehat{\xi}(p) = \widehat{\xi}(p; \mathbf{g}(\mathbf{Y}))$ ,  $\xi(p) = \xi_{g(Y)}(p)$ ,  $\widehat{F}(\cdot) = \widehat{F}(\cdot; \mathbf{g}(\mathbf{Y}))$ ,  $F(\cdot) = F_{g(Y)}(\cdot)$  et  $f(\cdot) = f_{g(Y)}(\cdot)$  and  $r_n = r_n(\alpha, \overline{\tau}_p)$ . Define,

$$V_n = r_n^{-1} \left( \widehat{\xi}(p) - \xi(p) \right) \quad \text{and} \quad W_n = r_n^{-1} \left( \frac{p - F(p)}{f(p)} \right).$$

The result is established if  $V_n - W_n \xrightarrow{\mathbb{P}} 0$  as  $n \rightarrow +\infty$ . It suffices to prove that  $V_n$  and  $W_n$  satisfy the conditions of Lemma 1 of Ghosh (1971):

- **condition (a)** : for all  $\delta > 0$ , there exists  $\varepsilon = \varepsilon(\delta)$  such that  $\mathbb{P}(|W_n| > \varepsilon) < \delta$ .
- **condition (b)** : for all  $y \in \mathbb{R}$  and for all  $\varepsilon > 0$

$$\lim_{n \rightarrow +\infty} \mathbb{P}(V_n \leq y, W_n \geq k + \varepsilon) \quad \text{and} \quad \lim_{n \rightarrow +\infty} \mathbb{P}(V_n \geq y + \varepsilon, W_n \geq k)$$

**condition (a)** : from Bienaymé-Tchebyshev's inequality it is sufficient to prove that  $\mathbf{E}W_n^2 = \mathcal{O}(1)$ . Rewrite  $W_n = \frac{r_n^{-1}}{n} \sum_{i=\ell+1}^n h_{\xi(p)}(Y(i))$ . Let  $c_j$  (for some  $j \geq 0$ ) denote the  $j$ -th Hermite coefficient of  $h_{\xi(p)}(\cdot)$ . Since  $h_{\xi(p)}(\cdot)$  has at least Hermite rank  $\bar{\tau}_p$ , then

$$\begin{aligned} \mathbf{E}W_n^2 &= \frac{r_n^{-2}}{n^2} \sum_{i_1, i_2=1}^n \mathbf{E} (h_{\xi(p)}(Y(i_1)) h_{\xi(p)}(Y(i_2))) \\ &= \frac{r_n^{-2}}{n^2} \sum_{i_1, i_2=1}^n \sum_{j_1, j_2 \geq \bar{\tau}_p} c_{j_1} c_{j_2} \mathbf{E} (H_{j_1}(Y(i_1)) H_{j_2}(Y(i_2))) \\ &= \frac{r_n^{-2}}{n^2} \sum_{i_1, i_2=1}^n \sum_{j \geq \bar{\tau}_p} \frac{(c_j)^2}{(j)!} \rho(i_2 - i_1)^j \\ &= \mathcal{O} \left( r_n^{-2} \times \frac{1}{n} \sum_{|i| < n} \rho(i)^{\bar{\tau}_p} \right) = \mathcal{O}(1), \end{aligned}$$

from Remark 1.

**condition (b)** : let  $y \in \mathbb{R}$ , we have

$$\begin{aligned} \{V_n \leq y\} &= \left\{ \widehat{\xi}(p) \leq y \times r_n + \xi(p) \right\} \\ &= \left\{ p \leq \widehat{F}(y \times r_n + \xi(p)) \right\} = \{Z_n \leq y_n\}, \end{aligned} \quad (10)$$

with

$$Z_n = \frac{r_n^{-1}}{f(\xi(p))} \left( F(y \times r_n + \xi(p)) - \widehat{F} \left( \frac{y}{\sqrt{r_n}} + \xi(p) \right) \right)$$

and

$$y_n = \frac{r_n^{-1}}{f(\xi(p))} \left( F(y \times r_n + \xi(p)) - p \right)$$

Under Assumption  $\mathbf{A}(\xi(p))$ , we have  $y_n \rightarrow y$ , as  $n \rightarrow +\infty$ . Now, prove that  $Z_n - W_n \xrightarrow{\mathbb{P}} 0$ .

Without loss of generality, assume  $y > 0$ . Then, we have

$$\begin{aligned} W_n - Z_n &= \frac{r_n^{-1}}{f(p)} \left( \widehat{F}(y \times r_n + \xi(p)) - F(y \times r_n + \xi(p)) - \widehat{F}(\xi(p)) + F(\xi(p)) \right) \\ &= \frac{r_n^{-1}}{n} \frac{1}{f(\xi(p))} \sum_{i=1}^n h_{\xi(p),n}(Y(i)) \end{aligned}$$

where  $h_{\xi(p),n}(\cdot)$  is the function defined for  $t \in \mathbb{R}$  by :

$$h_{\xi(p),n}(t) = \mathbf{1}_{\left\{ \xi(p) \leq g(t) \leq \xi(p) + y \times r_n \right\}}(t) - \mathbb{P} \left( \xi(p) \leq g(Y) \leq \xi(p) + y \times r_n \right).$$

For  $n$  sufficiently large, the function  $h_{\xi(p),n}(\cdot)$  has Hermite rank  $\bar{\tau}_p$ . Denote by  $c_{j,n}$  the  $j$ -th Hermite coefficient of  $h_{\xi(p),n}(\cdot)$ . From Lemma 3, there exists a sequence  $(\kappa_j)_{j \geq \bar{\tau}_p}$  such that, as  $n \rightarrow +\infty$

$$c_{j,n} \sim \kappa_j \times r_n.$$

Since, for all  $n \geq 1$   $\mathbf{E}(h_n(Y)^2) = \sum_{j \geq \bar{\tau}_p} (c_{j,n})^2 / j! < +\infty$ , it is clear that the sequence  $(\kappa_j)_{j \geq \bar{\tau}_p}$  is such that  $\sum_{j \geq \bar{\tau}_p} (\kappa_j)^2 / j! < +\infty$ . By denoting  $\lambda$  a positive constant, we get, as  $n \rightarrow +\infty$

$$\begin{aligned} \mathbf{E}(W_n - Z_n)^2 &= \frac{r_n^{-2}}{n^2} \frac{1}{f(\xi(p))^2} \sum_{i_1, i_2=1}^n \mathbf{E}(h_{\xi(p),n}(Y(i_1)) h_{\xi(p),n}(Y(i_2))) \\ &= \frac{r_n^{-2}}{n^2} \frac{1}{f(\xi(p))^2} \sum_{i_1, i_2=1}^n \sum_{j_1, j_2 \geq \bar{\tau}_p} c_{j_1, n} c_{j_2, n} \mathbf{E}(H_{j_1}(Y(i_1)) H_{j_2}(Y(i_2))) \\ &= \frac{r_n^{-2}}{n^2} \frac{1}{f(\xi(p))^2} \sum_{i_1, i_2=1}^n \sum_{j \geq \bar{\tau}_p} \frac{c_{j,n}^2}{j!} \rho(i_2 - i_1)^j \\ &\leq \lambda \frac{r_n^{-2}}{n} \sum_{j \geq \bar{\tau}_p} \frac{(\kappa_j)^2}{j!} r_n^2 \sum_{|i| < n} \rho(i)^j = \mathcal{O}\left(\frac{1}{n} \sum_{|i| < n} \rho(i)^{\bar{\tau}_p}\right) = \mathcal{O}(r_n^2), \end{aligned}$$

from Remark 1. Therefore,  $W_n - Z_n$  converges to 0 in probability, as  $n \rightarrow +\infty$ . Thus, for all  $\varepsilon > 0$ , we have, as  $n \rightarrow +\infty$ ,

$$\mathbb{P}(V_n \leq y, W_n \geq y + \varepsilon) = \mathbb{P}(Z_n \leq y_n, W_n \geq y + \varepsilon) \rightarrow 0.$$

Following the sketch of this proof, we also have  $\mathbb{P}(V_n \geq y + \varepsilon, W_n \leq y) \rightarrow 0$ , ensuring condition (b). Therefore,  $W_n - Z_n$  converges to 0 in probability, as  $n \rightarrow +\infty$ . Thus, for all  $\varepsilon > 0$ , we have, as  $n \rightarrow +\infty$ ,

$$\mathbb{P}(V_n \leq y, W_n \geq y + \varepsilon) = \mathbb{P}(Z_n \leq y_n, W_n \geq y + \varepsilon) \rightarrow 0.$$

Following the sketch of this proof, we also have  $\mathbb{P}(V_n \geq y + \varepsilon, W_n \leq y) \rightarrow 0$ , ensuring condition (b).

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