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WKB ANALYSIS FOR THE NONLINEAR SCHRÖDINGER EQUATION AND INSTABILITY RESULTS

RÉMI CARLES

ABSTRACT. For the semi-classical limit of the cubic, defocusing nonlinear Schrödinger equation with an external potential, we explain the notion of criticality before a caustic is formed. In the sub-critical and critical cases, we justify the WKB approximation. In the super-critical case, the WKB analysis provides a new phenomenon for the (classical) cubic, defocusing nonlinear Schrödinger equation, which can be compared to the loss of regularity established for the nonlinear wave equation by G. Lebeau. We also show some instabilities at the semi-classical level.

1. INTRODUCTION

We consider the solution to the nonlinear Schrödinger equation (NLS) with a cubic, defocusing nonlinearity, and an external potential:

$$(1.1) \quad i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + \varepsilon^\kappa |u^\varepsilon|^2 u^\varepsilon, \quad (t, x) \in [0, T] \times \mathbb{R}^n,$$

$$(1.2) \quad u^\varepsilon(0, x) = a_0^\varepsilon(x)e^{i\phi_0(x)/\varepsilon}.$$

We are interested in the behavior of the solution u^ε as the positive parameter ε goes to zero. According to the cultural background, this field goes under the name of semi-classical limit, WKB analysis (which is a little bit more specific, see below), or geometrical optics. The general idea is to describe the asymptotic behavior of u^ε with a simplified model, which involves geometric quantities. In the case of (1.1), these quantities are called either classical trajectories (in view of classical mechanics), or rays (in view of geometric optics).

There are at least two motivations for such a study. We outline them here, and refer to the survey [22] for a broader discussion on this subject. The first one comes from the applied mathematics, and may find its origins in physics. In the case of (1.1), suppose that ε represents the (rescaled) Planck constant. It may be small compared to the other parameters at stake. In this case, it is sensible to consider that the asymptotic behavior of u^ε as $\varepsilon \rightarrow 0$ provides a reliable approximation of the exact solution. Hopefully, the asymptotic model is easier to describe than the initial one (1.1)–(1.2). Another motivation stems from the propagation of singularities for equation where the small parameter ε is not necessarily present initially. Most of the studies in this direction concern hyperbolic equations. However, the belief according to which Schrödinger equations share properties with hyperbolic equations in the semi-classical limit is a first hint that this field is applicable to Schrödinger equations as well (see e.g. [16, 24]). To illustrate this statement, we give a result, whose proof will be straightforward after the analysis of (1.1)–(1.2).

Theorem 1.1 ([6], Cor. 1.7). *Let $n \geq 3$. Consider the cubic, defocusing NLS:*

$$(1.3) \quad i\partial_t u + \frac{1}{2}\Delta u = |u|^2 u \quad ; \quad u|_{t=0} = u_0.$$

Denote $s_c = \frac{n}{2} - 1$. Let $0 < s < s_c$. We can find a family $(u_0^\varepsilon)_{0 < \varepsilon \leq 1}$ in $\mathcal{S}(\mathbb{R}^n)$ with

$$\|u_0^\varepsilon\|_{H^s(\mathbb{R}^n)} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

and $0 < t^\varepsilon \rightarrow 0$ such that the solution u^ε to (1.3) associated to u_0^ε satisfies:

$$\|u^\varepsilon(t^\varepsilon)\|_{H^k(\mathbb{R}^n)} \rightarrow +\infty \text{ as } \varepsilon \rightarrow 0, \quad \forall k \in \left] \frac{s}{\frac{n}{2} - s}, s \right].$$

This result is in the same spirit as the initial breakthrough by G. Lebeau ([17, 18], see also [21]). These former results also rely on geometrical optics (in a super-critical régime; see below for this notion in the case of (1.1)). For (1.3), the above result was first established by M. Christ, J. Colliander and T. Tao [10] in the case $k = s$ (see also [3, Appendix] and [6, Appendix B]). The fact that we can go strictly below the value $k = s$ stems from an analysis of (1.1) in a case where the nonlinearity should be considered as quasilinear, and not semilinear. This result is then a consequence of the original idea of E. Grenier [14].

The rest of this text is organized as follows. In §2, we introduce the notion of criticality for (1.1) at a formal level. In §3, we explain how to justify this notion, and describe the asymptotic behavior of u^ε in different cases. The proof of Theorem 1.1 is given in §4. More instability results are given in §5.

2. WKB ANALYSIS AND THE NOTION OF CRITICALITY

This section remains at a formal level only. It is a preparation to the forthcoming rigorous justifications. In a WKB analysis, one assumes for instance that the initial profile a_0^ε can be expanded as a power series in ε :

$$a_0^\varepsilon \sim a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \dots$$

Note that the modulation factor ε^κ in front of the nonlinearity in (1.1) could be taken equal to one, up to replacing a_0^ε with $\varepsilon^{\kappa/2} a_0^\varepsilon$. In particular, it is clear that for large $\kappa > 0$, the nonlinearity is expected to be negligible in the limit $\varepsilon \rightarrow 0$ (at least locally in time). WKB analysis consists in seeking an approximation of the form

$$u^\varepsilon(t, x) \underset{\varepsilon \rightarrow 0}{\sim} a(t, x) e^{\phi(t, x)/\varepsilon},$$

where the amplitude a and the phase ϕ are independent of ε . Plugging this approximate solution into (1.1) and canceling the first powers of ε yields:

$$(2.1) \quad \mathcal{O}(\varepsilon^0) : \quad \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + V = 0 \quad ; \quad \phi|_{t=0} = \phi_0 \quad \text{if } \kappa > 0,$$

$$(2.2) \quad \mathcal{O}(\varepsilon^1) : \quad \partial_t a + \nabla \phi \cdot \nabla a + \frac{1}{2} a \Delta \phi = \begin{cases} 0 & \text{if } \kappa > 1, \\ -i|a|^2 a & \text{if } \kappa = 1. \end{cases} \quad ; \quad a|_{t=0} = a_0$$

This shows that the minimal value of κ for which nonlinear effects affect the solution at leading order is $\kappa = 1$. We shall consider here the following three cases: $\kappa > 1$ (sub-critical case), $\kappa = 1$ (critical case), and $\kappa = 0$ (a super-critical case). We refer to [7] for the case $0 < \kappa < 1$. The first step consists in solving (2.1). The potential V may be time dependent: $V = V(t, x)$.

Lemma 2.1. *Assume that V and ϕ_0 are smooth and sub-quadratic:*

- $V \in C^\infty(\mathbb{R}_t \times \mathbb{R}_x^n)$, and $\partial_x^\alpha V \in L_{\text{loc}}^\infty(\mathbb{R}_t; L^\infty(\mathbb{R}_x^n))$ as soon as $|\alpha| \geq 2$.
- $\phi_0 \in C^\infty(\mathbb{R}^n)$, and $\partial_x^\alpha \phi_0 \in L^\infty(\mathbb{R}^n)$ as soon as $|\alpha| \geq 2$.

Then there exist $T > 0$ and a unique solution $\phi_{\text{eik}} \in C^\infty([0, T] \times \mathbb{R}^n)$ to (2.1). This solution is sub-quadratic: $\partial_x^\alpha \phi_{\text{eik}} \in L^\infty([0, T] \times \mathbb{R}^n)$ as soon as $|\alpha| \geq 2$.

The proof of this lemma relies on Hamilton–Jacobi theory. Note that the time of existence T is uniform with respect to $x \in \mathbb{R}^n$: a global inversion theorem is needed, which can be found in [23] or [11]. We refer to [7] for the proof of this lemma, and for a discussion on the optimality of the assumptions. In particular, one should not expect the above solution to remain smooth for all time. The appearance of singularities corresponds to the formation of caustics. The aim of WKB analysis is to describe the solution u^ε before a caustic is formed. See e.g. [5] for the description of a solution to (1.1)–(1.2) beyond a focal point.

To analyze (2.2), we introduce the Hamiltonian flow, on which the proof of Lemma 2.1 relies:

$$(2.3) \quad \begin{cases} \partial_t x(t, y) = \xi(t, y) & ; & x(0, y) = y, \\ \partial_t \xi(t, y) = -\nabla_x V(t, x(t, y)) & ; & \xi(0, y) = \nabla \phi_0(y). \end{cases}$$

The time T is such that the map $y \mapsto x(t, y)$ is a diffeomorphism of \mathbb{R}^n for $t \in [0, T]$. The key observation is that (2.2) is a transport equation, which turns out to be an ordinary differential equation along the classical trajectories. Introduce the Jacobi determinant

$$J_t(y) = \det \nabla_y x(t, y).$$

Denote

$$A(t, y) := a(t, x(t, y)) \sqrt{J_t(y)}.$$

For $t \in [0, T]$, (2.2) is equivalent to:

$$\partial_t A = \begin{cases} 0 & \text{if } \kappa > 1, \\ -i J_t(y)^{-1} |A|^2 A & \text{if } \kappa = 1. \end{cases} ; \quad A(0, y) = a_0(y).$$

This ordinary differential equation along the rays of geometrical optics can be solved explicitly: we see that $\partial_t |A|^2 = 0$, hence, for $\kappa = 1$,

$$A(t, y) = a_0(y) \exp \left(-i \int_0^t J_s(y)^{-1} |a_0(y)|^2 ds \right).$$

Inverting the map $y \mapsto x(t, y)$ yields $a(t, x)$. We see that the critical nonlinear effect is a self-modulation of the amplitude. In the context of laser physics, this phenomenon is known as *phase self-modulation* (see e.g. [25, 2, 12]).

We now turn to the super-critical case $\kappa = 0$. To illustrate the difficulty of this case, seek a more precise asymptotic expansion of u^ε :

$$u^\varepsilon(t, x) \sim (\mathbf{a}_0(t, x) + \varepsilon \mathbf{a}_1(t, x) + \varepsilon^2 \mathbf{a}_2(t, x) + \dots) e^{i\phi(t, x)/\varepsilon}.$$

Plugging such an asymptotic expansion into (1.1) yields a shifted cascade of equations:

$$\begin{aligned} \mathcal{O}(\varepsilon^0) : \quad & \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + V + |\mathbf{a}_0|^2 = 0 ; \quad \phi|_{t=0} = \phi_0. \\ \mathcal{O}(\varepsilon^1) : \quad & \partial_t \mathbf{a}_0 + \nabla \phi \cdot \nabla \mathbf{a}_0 + \frac{1}{2} \mathbf{a}_0 \Delta \phi = -2i \text{Re}(\mathbf{a}_0 \overline{\mathbf{a}_1}) \mathbf{a}_0 ; \quad \mathbf{a}_0|_{t=0} = a_0. \end{aligned}$$

Two comments are in order. First, we see that there is a strong coupling between the phase and the main amplitude: \mathbf{a}_0 is present in the equation for ϕ . Second, the above system is not closed: ϕ is determined in function of \mathbf{a}_0 , and \mathbf{a}_0 is determined in function of \mathbf{a}_1 . Even if we pursued the cascade of equations, this phenomenon would remain: no matter how many terms are computed, the system is never closed (see [13]). This is a typical feature of super-critical cases in nonlinear geometrical optics (see [8, 9]).

In the case when $V \equiv 0$ and $\phi_0 \in H^s$, this problem was resolved by E. Grenier [14], by modifying the usual WKB methods (see §3.2). Note that even though \mathbf{a}_1 is not determined by the above system, the pair $(\rho, v) := (|\mathbf{a}_0|^2, \nabla\phi)$ solves a compressible Euler equation:

$$(2.4) \quad \begin{aligned} \partial_t v + v \cdot \nabla v + \nabla V + \nabla \rho &= 0; & v|_{t=0} &= \nabla \phi_0 \\ \partial_t \rho + \nabla \cdot (\rho v) &= 0; & \rho|_{t=0} &= |\mathbf{a}_0|^2. \end{aligned}$$

3. RIGOROUS WKB ANALYSIS

We outline here some results presented in details in [7].

3.1. Sub-critical and critical cases. Let $\kappa \geq 1$. Under the assumptions of Lemma 2.1, we change the Cauchy problem (1.1)–(1.2): define a^ε by

$$u^\varepsilon(t, x) = a^\varepsilon(t, x) e^{i\phi_{\text{eik}}(t, x)/\varepsilon}.$$

Then for $t \in [0, T]$, (1.1)–(1.2) is equivalent to:

$$(3.1) \quad \begin{aligned} \partial_t a^\varepsilon + \nabla \phi_{\text{eik}} \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi_{\text{eik}} &= i \frac{\varepsilon}{2} \Delta a^\varepsilon - i \varepsilon^{\kappa-1} |a^\varepsilon|^2 a^\varepsilon, \\ a^\varepsilon|_{t=0} &= a_0^\varepsilon. \end{aligned}$$

Two things must be noticed: first, the potential V and the initial phase ϕ_0 do not appear in this new problem. Second, the factors involving ϕ_{eik} have the following features: in view of Lemma 2.1, the term $\Delta \phi_{\text{eik}}$ is in $L^\infty([0, T] \times \mathbb{R}^n)$, and the operator $\partial_t + \nabla \phi_{\text{eik}} \cdot \nabla$ is a transport operator. We can then obtain energy estimates in Sobolev spaces, and establish:

Proposition 3.1. *Under the assumptions of Lemma 2.1, assume moreover that a_0^ε is bounded in $H^s(\mathbb{R}^n)$ uniformly for $\varepsilon \in]0, 1]$, for all $s \geq 0$. Let $\kappa \geq 1$. Then for all $\varepsilon \in]0, 1]$, (3.1) has a unique solution $a^\varepsilon \in C^\infty([0, T] \times \mathbb{R}^n) \cap C([0, T]; H^s)$ for all $s > n/2$. Moreover, a^ε is bounded in $L^\infty([0, T]; H^s)$ uniformly in $\varepsilon \in]0, 1]$, for all $s \geq 0$.*

These uniform estimates allow us to neglect the term $\varepsilon \Delta a^\varepsilon$ on the right hand side of (3.1). Assume moreover the following convergence:

$$(3.2) \quad a_0^\varepsilon \rightarrow a_0 \text{ in } H^s(\mathbb{R}^n), \quad \forall s \geq 0.$$

Corollary 3.2. *Let $\kappa \geq 1$. Under the above assumptions,*

$$\|a^\varepsilon - \tilde{a}^\varepsilon\|_{L^\infty([0, T]; H^s)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad \forall s \geq 0,$$

where \tilde{a}^ε solves:

$$(3.3) \quad \partial_t \tilde{a}^\varepsilon + \nabla \phi_{\text{eik}} \cdot \nabla \tilde{a}^\varepsilon + \frac{1}{2} \tilde{a}^\varepsilon \Delta \phi_{\text{eik}} = -i \varepsilon^{\kappa-1} |\tilde{a}^\varepsilon|^2 \tilde{a}^\varepsilon \quad ; \quad \tilde{a}^\varepsilon|_{t=0} = a_0.$$

Proceeding as in §2, denote

$$A^\varepsilon(t, y) := \tilde{a}^\varepsilon(t, x(t, y)) \sqrt{J_t(y)}.$$

We see that so long as $y \mapsto x(t, y)$ defines a global diffeomorphism (which is guaranteed for $t \in [0, T]$ by construction), (3.3) is equivalent to:

$$\partial_t A^\varepsilon = -i\varepsilon^{\kappa-1} J_t(y)^{-1} |A^\varepsilon|^2 A^\varepsilon \quad ; \quad A^\varepsilon(0, y) = a_0(y).$$

This ordinary differential equation along the rays of geometrical optics can be solved explicitly:

$$A^\varepsilon(t, y) = a_0(y) \exp\left(-i\varepsilon^{\kappa-1} \int_0^t J_s(y)^{-1} |a_0(y)|^2 ds\right).$$

Back to the initial solution u^ε , we conclude:

Proposition 3.3. *Let $\kappa \geq 1$. Under the above assumptions, for all $\varepsilon \in]0, 1]$, (1.1)-(1.2) has a unique solution $u^\varepsilon \in C^\infty([0, T] \times \mathbb{R}^n) \cap C([0, T]; H^s)$ for all $s > n/2$. Moreover, there exist $a, G \in C^\infty([0, T] \times \mathbb{R}^n)$, independent of $\varepsilon \in]0, 1]$, where $a \in C([0, T]; L^2 \cap L^\infty)$, and G is real-valued with $G \in C([0, T]; L^\infty)$, such that:*

$$\left\| u^\varepsilon - a e^{i\varepsilon^{\kappa-1} G} e^{i\phi_{\text{eik}}/\varepsilon} \right\|_{L^\infty([0, T]; L^2 \cap L^\infty)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

The profile a solves the initial value problem:

$$(3.4) \quad \partial_t a + \nabla \phi_{\text{eik}} \cdot \nabla a + \frac{1}{2} a \Delta \phi_{\text{eik}} = 0 \quad ; \quad a|_{t=0} = a_0,$$

and G depends nonlinearly on a :

$$a(t, x) = \frac{1}{\sqrt{J_t(y(t, x))}} a_0(y(t, x)),$$

$$G(t, x) = - \int_0^t J_s(y(t, x))^{-1} |a_0(y(t, x))|^2 ds.$$

In particular, if $\kappa > 1$, then

$$\left\| u^\varepsilon - a e^{i\phi_{\text{eik}}/\varepsilon} \right\|_{L^\infty([0, T]; L^2 \cap L^\infty)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

and no nonlinear effect is present in the leading order behavior of u^ε . If $\kappa = 1$, nonlinear effects are present at leading order, measured by G .

3.2. Super-critical case: $\kappa = 0$. In this case, we recall the beautiful idea of E. Grenier, which makes it possible to consider the case $V \equiv 0$ and $\phi_0 \in H^s(\mathbb{R}^n)$ for sufficiently large s . The approach consists in reversing the steps of the WKB analysis: usually, one seeks an approximate solution, and then tries to show that stability arguments imply that the exact solution is well approximated by this process. To overcome the issues mentioned in §2, the idea in [14] consists in first writing the unknown as:

$$u^\varepsilon(t, x) = a^\varepsilon(t, x) e^{i\phi^\varepsilon(t, x)/\varepsilon},$$

where the amplitude a^ε is *complex-valued* (even if a_0^ε is real-valued), and ϕ^ε is real-valued. Doing so, one introduces one degree of freedom to rewrite (1.1)–(1.2). The

usual approach in physics (see e.g. [15]) consists in writing

$$\begin{cases} \partial_t \phi^\varepsilon + \frac{1}{2} |\nabla \phi^\varepsilon|^2 + |a^\varepsilon|^2 = \varepsilon^2 \frac{\Delta a^\varepsilon}{2a^\varepsilon} & ; \quad \phi^\varepsilon|_{t=0} = \phi_0, \\ \partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon = 0 & ; \quad a^\varepsilon|_{t=0} = a_0^\varepsilon. \end{cases}$$

Obviously, this approach is delicate when a^ε has zeroes; see the discussion in [13] on this subject. The choice in [14] is to write:

$$(3.5) \quad \begin{cases} \partial_t \phi^\varepsilon + \frac{1}{2} |\nabla \phi^\varepsilon|^2 + |a^\varepsilon|^2 = 0 & ; \quad \phi^\varepsilon|_{t=0} = \phi_0, \\ \partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon = i \frac{\varepsilon}{2} \Delta a^\varepsilon & ; \quad a^\varepsilon|_{t=0} = a_0^\varepsilon. \end{cases}$$

Inspired by the fact that the expected limit is related to the compressible Euler equation (see §2), introduce the “velocity” $v^\varepsilon = \nabla \phi^\varepsilon$. Then (3.5) yields:

$$(3.6) \quad \begin{cases} \partial_t v^\varepsilon + v^\varepsilon \cdot \nabla v^\varepsilon + 2 \operatorname{Re}(\overline{a^\varepsilon} \nabla a^\varepsilon) = 0 & ; \quad v^\varepsilon|_{t=0} = \nabla \phi_0, \\ \partial_t a^\varepsilon + v^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \nabla \cdot v^\varepsilon = i \frac{\varepsilon}{2} \Delta a^\varepsilon & ; \quad a^\varepsilon|_{t=0} = a_0^\varepsilon. \end{cases}$$

Separate real and imaginary parts of a^ε , $a^\varepsilon = a_1^\varepsilon + i a_2^\varepsilon$. Then we have

$$(3.7) \quad \partial_t \mathbf{u}^\varepsilon + \sum_{j=1}^n A_j(\mathbf{u}^\varepsilon) \partial_j \mathbf{u}^\varepsilon = \frac{\varepsilon}{2} L \mathbf{u}^\varepsilon,$$

$$\text{with } \mathbf{u}^\varepsilon = \begin{pmatrix} a_1^\varepsilon \\ a_2^\varepsilon \\ v_1^\varepsilon \\ \vdots \\ v_n^\varepsilon \end{pmatrix}, \quad L = \begin{pmatrix} 0 & -\Delta & 0 & \dots & 0 \\ \Delta & 0 & 0 & \dots & 0 \\ 0 & 0 & & & 0_{n \times n} \end{pmatrix},$$

$$\text{and } A(\mathbf{u}, \xi) = \sum_{j=1}^n A_j(\mathbf{u}) \xi_j = \begin{pmatrix} v \cdot \xi & 0 & \frac{a_1}{2} t \xi \\ 0 & v \cdot \xi & \frac{a_2}{2} t \xi \\ 2a_1 \xi & 2a_2 \xi & v \cdot \xi I_n \end{pmatrix}.$$

The matrix $A(\mathbf{u}, \xi)$ can be symmetrized by

$$S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{1}{4} I_n \end{pmatrix}.$$

The important point to notice is that the operator L is skew-symmetric: it is invisible in the energy estimates, so the loss of derivative (it is the only operator of order two in (3.7)) is avoided. Denote $H^\infty = \bigcap_{s \geq 0} H^s(\mathbb{R}^n)$. Classical theory on symmetric hyperbolic systems yields a solution $(v^\varepsilon, a^\varepsilon)$ to (3.6). Once v^ε is known, we note that it is irrotational, so there exists ϕ^ε such that $v^\varepsilon = \nabla \phi^\varepsilon$. Up to adding a function of time only, $(\phi^\varepsilon, a^\varepsilon)$ solves (3.5).

Proposition 3.4 ([14], Th. 1.1). *Let $\kappa = 0$. Suppose that $\phi_0 \in H^\infty$, and that a_0^ε is bounded in $H^s(\mathbb{R}^n)$ uniformly for $\varepsilon \in]0, 1]$, for all $s \geq 0$. Let $s > 2 + n/2$. There exist $T_s > 0$ independent of $\varepsilon \in]0, 1]$ and $u^\varepsilon = a^\varepsilon e^{i\phi^\varepsilon/\varepsilon}$ solution to (1.1)–(1.2) on $[0, T_s]$. Moreover, a^ε and ϕ^ε are bounded in $L^\infty([0, T_s]; H^s)$, uniformly in $\varepsilon \in]0, 1]$.*

Assume moreover that (3.2) holds. The solution to (3.5) formally converges to the solution of:

$$(3.8) \quad \begin{cases} \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + |a|^2 = 0 & ; \quad \phi|_{t=0} = \phi_0, \\ \partial_t a + \nabla \phi \cdot \nabla a + \frac{1}{2} a \Delta \phi = 0 & ; \quad a|_{t=0} = a_0. \end{cases}$$

Under the above assumptions, (3.8) has a unique solution $(a, \phi) \in L^\infty([0, T_*]; H^m)^2$ for all $m > 0$ for some $T_* > 0$ independent of m (see e.g. [1, 19]).

Remark 3.5. More general nonlinearity. Suppose that we consider a more general nonlinearity:

$$i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon = f(|u^\varepsilon|^2) u^\varepsilon.$$

Then following the same lines as above, the symmetrizer naturally becomes

$$S = \begin{pmatrix} I_2 & 0 \\ 0 & \frac{1}{4f'(|a^\varepsilon|^2)} I_n \end{pmatrix}.$$

For this matrix to be positive, and to be able to estimate its time derivative, it is natural to assume $f' > 0$. This corresponds to the assumption made in [14], and in [7]. For the above analysis to be valid, the nonlinearity has to be defocusing, and cubic at the origin. In particular, the WKB analysis for the quintic defocusing NLS is still an open problem. Note however that it is possible to construct solutions to the limit problem in that case (the analogue of (3.8)), thanks to the result of [20] and the geometrical analysis of §3.1. Yet, the nonlinear change of variable of [20] is apparently incompatible with the above remark that L is skew-symmetric.

If we suppose in addition that there exists $a_0, a_1 \in H^\infty$ such that

$$(3.9) \quad a_0^\varepsilon = a_0 + \varepsilon a_1 + o(\varepsilon) \quad \text{in } H^s, \quad \forall s \geq 0,$$

then we infer:

Proposition 3.6. *Let $s \in \mathbb{N}$. Then $T_s \geq T_*$, and there exists C_s independent of ε such that for every $0 \leq t \leq T_*$,*

$$\|a^\varepsilon(t) - a(t)\|_{H^s} \leq C_s \varepsilon \quad ; \quad \|\phi^\varepsilon(t) - \phi(t)\|_{H^s} \leq C_s \varepsilon t.$$

Note that this suffices to describe u^ε for very small time only:

$$\begin{aligned} u^\varepsilon - a e^{i\phi/\varepsilon} &= a^\varepsilon e^{i\phi^\varepsilon/\varepsilon} - a e^{i\phi/\varepsilon} \\ &= (a^\varepsilon - a) e^{i\phi^\varepsilon/\varepsilon} + 2ia e^{i(\phi^\varepsilon + \phi)/2\varepsilon} \sin\left(\frac{\phi^\varepsilon - \phi}{2\varepsilon}\right). \end{aligned}$$

The first term of the right hand side is of order $\mathcal{O}(\varepsilon)$ in $L^2 \cap L^\infty$, but the second one is of order $\mathcal{O}(t)$ only: therefore, we only have

$$u^\varepsilon(t, x) \sim a(t, x) e^{i\phi(t, x)/\varepsilon} \quad \text{for } 0 \leq t \ll 1.$$

To have a better error estimate, it is necessary to compute the next term in the asymptotic expansion of $(\phi^\varepsilon, a^\varepsilon)$ in powers of ε . For times of order $\mathcal{O}(1)$, the initial corrector a_1 must be taken into account:

Proposition 3.7. Define $(a^{(1)}, \phi^{(1)})$ by

$$\begin{aligned} \partial_t \phi^{(1)} + \nabla \phi \cdot \nabla \phi^{(1)} + 2 \operatorname{Re}(\bar{a} a^{(1)}) &= 0, \\ \partial_t a^{(1)} + \nabla \phi \cdot \nabla a^{(1)} + \nabla \phi^{(1)} \cdot \nabla a + \frac{1}{2} a^{(1)} \Delta \phi + \frac{1}{2} a \Delta \phi^{(1)} &= \frac{i}{2} \Delta a, \\ \phi^{(1)}|_{t=0} &= 0 \quad ; \quad a^{(1)}|_{t=0} = a_1. \end{aligned}$$

Then $a^{(1)}, \phi^{(1)} \in L^\infty([0, T_*]; H^s)$ for every $s \geq 0$, and

$$\|a^\varepsilon - a - \varepsilon a^{(1)}\|_{L^\infty([0, T_*]; H^s)} + \|\phi^\varepsilon - \phi - \varepsilon \phi^{(1)}\|_{L^\infty([0, T_*]; H^s)} \leq C_s (\varepsilon^2 + o(\varepsilon)), \quad \forall s \geq 0.$$

Despite the notations, it seems unadapted to consider $\phi^{(1)}$ as being part of the phase. Indeed, we infer from Proposition 3.7 that

$$\left\| u^\varepsilon - a e^{i\phi^{(1)}} e^{i\phi/\varepsilon} \right\|_{L^\infty([0, T_*]; L^2 \cap L^\infty)} = o(1).$$

Remark 3.8. If the term $o(\varepsilon)$ in (3.9) is controlled more precisely as a $\mathcal{O}(\varepsilon^2)$, then the above $o(1)$ becomes a $\mathcal{O}(\varepsilon)$.

Relating this information to the WKB methods presented at the end of §2, we would have:

$$\mathbf{a}_0 = a e^{i\phi^{(1)}}.$$

Since $\phi^{(1)}$ depends on a_1 while a does not, we retrieve the fact that in super-critical régimes, the leading order amplitude in WKB methods depends on the initial first corrector a_1 .

Remark 3.9. The term $e^{i\phi^{(1)}}$ does not appear in the Wigner measure of $a e^{i\phi^{(1)}} e^{i\phi/\varepsilon}$. Thus, from the point of view of Wigner measures, the asymptotic behavior of the exact solution is described by the Euler-type system (2.4).

Remark 3.10. If we assume that a_0 is real-valued, then so is a . If moreover a_1 is purely imaginary (for instance, if $a_1 = 0$), then we see that $a^{(1)}$ is purely imaginary, hence, $\phi^{(1)} \equiv 0$.

So far we have assumed $V \equiv 0$ and $\phi_0 \in H^\infty$. If we try to mimic the approach of [14] for a non-trivial external potential for instance, we have to consider:

$$\begin{aligned} \partial_t \phi^\varepsilon + \frac{1}{2} |\nabla \phi^\varepsilon|^2 + V + |a^\varepsilon|^2 &= 0 \quad ; \quad \phi^\varepsilon|_{t=0} = \phi_0, \\ \partial_t a^\varepsilon + \nabla \phi^\varepsilon \cdot \nabla a^\varepsilon + \frac{1}{2} a^\varepsilon \Delta \phi^\varepsilon &= i \frac{\varepsilon}{2} \Delta a^\varepsilon \quad ; \quad a^\varepsilon|_{t=0} = a_0^\varepsilon. \end{aligned}$$

The analysis of [14] works in the same way only when $\nabla V \in L_{\text{loc}}^\infty(\mathbb{R}_t; H^s(\mathbb{R}^n))$ for a sufficiently large s . To be able to consider general sub-quadratic potentials (including the harmonic oscillator), resume the assumption of Lemma 2.1, and write

$$\phi^\varepsilon = \phi_{\text{eik}} + \varphi^\varepsilon.$$

Working with the unknown $(\varphi^\varepsilon, a^\varepsilon)$, we see that we are now rid of the external potential V , and of the possibly unbounded initial phase ϕ_0 . The price to pay is that extra terms have appeared. The good news however is that these extra terms are *semilinear* (as in §3.1), and can be treated by perturbative methods in energy estimates. We conclude:

Theorem 3.11. *Let $\kappa = 0$. Under the above assumptions, there exists $T_* > 0$ independent of $\varepsilon \in]0, 1]$ and a unique solution $u^\varepsilon \in C^\infty([0, T_*] \times \mathbb{R}^n) \cap C([0, T_*]; H^s)$ for all $s > n/2$ to (1.1)–(1.2). Moreover, there exist $a, \varphi \in C([0, T_*]; H^s)$ for every $s \geq 0$, such that:*

$$\limsup_{\varepsilon \rightarrow 0} \left\| u^\varepsilon - a e^{i(\varphi + \phi_{\text{eik}})/\varepsilon} \right\|_{L^2 \cap L^\infty} = \mathcal{O}(t) \quad \text{as } t \rightarrow 0.$$

Here, a and φ are nonlinear functions of ϕ_{eik} and a_0 . Finally, there exists $\varphi^{(1)} \in C([0, T_*]; H^s)$ for every $s \geq 0$, real-valued, such that:

$$\limsup_{\varepsilon \rightarrow 0} \sup_{0 \leq t \leq T_*} \left\| u^\varepsilon - a e^{i\varphi^{(1)}} e^{i(\varphi + \phi_{\text{eik}})/\varepsilon} \right\|_{L^2 \cap L^\infty} = 0.$$

The phase shift $\varphi^{(1)}$ is a nonlinear function of ϕ_{eik}, a_0 and a_1 .

Remark 3.12. In [7], some assumptions on the momentum of a_0^ε are made, and not here. This is due to the fact that here, the nonlinearity that we consider is exactly cubic. When it is cubic at the origin only (see Remark 3.5), extra estimates are needed, which apparently impose some extra decay at infinity for a_0^ε, a_0 and a_1 .

4. PROOF OF THEOREM 1.1

Theorem 1.1 is a straightforward consequence of Proposition 3.7. For $a_0 \in \mathcal{S}(\mathbb{R}^n)$, let

$$u_0(x) = \lambda^{-\frac{n}{2}+s} a_0 \left(\frac{x}{\lambda} \right).$$

Let $\varepsilon = \lambda^{\frac{n}{2}-1-s}$: ε and λ go simultaneously to zero, since $s < s_c$. Define

$$\psi^\varepsilon(t, x) = u^\lambda(\varepsilon t, x) = \lambda^{\frac{n}{2}-s} u \left(\lambda^{\frac{n}{2}+1-s} t, \lambda x \right).$$

It solves:

$$(4.1) \quad i\varepsilon \partial_t \psi^\varepsilon + \frac{\varepsilon^2}{2} \Delta \psi^\varepsilon = |\psi^\varepsilon|^2 \psi^\varepsilon \quad ; \quad \psi^\varepsilon|_{t=0} = a_0(x).$$

The idea of the proof is that for times of order $\mathcal{O}(1)$, ψ^ε has become ε -oscillatory.

We infer from Proposition 3.7 that there exist $T > 0$ independent of $\varepsilon \in]0, 1]$, and $a, \phi, \phi_1 \in C([0, T]; H^m)$ for any $m \geq 0$, such that:

$$\left\| \psi^\varepsilon - a e^{i\phi_1} e^{i\phi/\varepsilon} \right\|_{L^\infty([0, T]; H^m)} \leq C_m \varepsilon^{1-m}.$$

Since the \dot{H}^m -norm of $a e^{i\phi_1} e^{i\phi/\varepsilon}$ is of order ε^{-m} (when ϕ is not stationary), we deduce that there exists $t \in]0, T]$ such that for any $m \geq 0$:

$$\|\psi^\varepsilon(t)\|_{\dot{H}^m} \approx \varepsilon^{-m}.$$

This implies:

$$\|u \left(\lambda^{\frac{n}{2}+1-s} t \right)\|_{\dot{H}^k} \approx \lambda^{s-k} \|\psi^\varepsilon(t)\|_{\dot{H}^k} \approx \lambda^{s-k} \varepsilon^{-k} = \lambda^{s-k-k(\frac{n}{2}-1-s)}.$$

The result then follows when considering the limit $\lambda \rightarrow 0$. We get exactly the statement of the theorem by replacing a_0 by $|\log \lambda|^{-1} a_0$ for instance.

Remark 4.1. The proof of ill-posed presented in [10] (see also [3, Appendix], [6, Appendix B]) consists in neglecting the Laplacian in (4.1) for very small times, and

integrating explicitly an ordinary differential equation. Proving that the Laplacian is negligible stems from Gronwall lemma. Essentially, the error satisfies an inequality of the form

$$\|w^\varepsilon(t)\|_X \lesssim \varepsilon + \frac{1}{\varepsilon} \int_0^t \|w^\varepsilon(s)\|_X ds,$$

for some space X that we do not describe. The singular factor ε^{-1} is due to the ε in front of the time derivative, and to the fact that no power of ε is present in front of the nonlinearity. Therefore, Gronwall lemma yields no better than:

$$\|w^\varepsilon(t)\|_X \lesssim \varepsilon e^{Ct/\varepsilon},$$

for some $C > 0$. The error is small on an interval of the form $[0, \varepsilon |\log \varepsilon|^\theta]$ for some $\theta > 0$. This is enough to prove Theorem 1.1 for $k = s$. This analysis considers (4.1) as a semilinear equation, since the nonlinearity is viewed as a perturbation of the linear equation. To prove Theorem 1.1 for $k < s$, it seems necessary to consider (4.1) as a quasilinear equation, as was done by E. Grenier. Note also that the quasilinear approach shows that the Laplacian in (4.1) is negligible for $0 < t^\varepsilon \ll \varepsilon^{1/3}$, that is a ‘‘much larger’’ interval than $[0, \varepsilon |\log \varepsilon|^\theta]$ (but still very small!). See the next section.

Remark 4.2. On the other hand, Theorem 1.1 is valid only for cubic, defocusing nonlinear Schrödinger equations, while the results in [10] are valid for more general equations. This is due to the fact that the justification of super-critical nonlinear geometric optics for times $\mathcal{O}(1)$ is available only for nonlinearities which are defocusing, and cubic at the origin (see Remark 3.5). Since the proofs of ill-posedness rely on an homogeneous change of unknown function, we are left with the only possibility of an exactly cubic, defocusing nonlinearity. However, it is very likely that (an analogue of) Theorem 1.1 should be true under more general assumptions.

5. INSTABILITY FOR THE SEMI-CLASSICAL EQUATION

The results we present in the paragraph are taken from [6]. We assume $V = \phi_0 = 0$ for the sake of concision. We first fix some notations.

Notation. Let $(\alpha^\varepsilon)_{0 < \varepsilon \leq 1}$ and $(\beta^\varepsilon)_{0 < \varepsilon \leq 1}$ be two families of positive real numbers.

- We write $\alpha^\varepsilon \ll \beta^\varepsilon$ if $\limsup_{\varepsilon \rightarrow 0} \alpha^\varepsilon / \beta^\varepsilon = 0$.
- We write $\alpha^\varepsilon \lesssim \beta^\varepsilon$ if $\limsup_{\varepsilon \rightarrow 0} \alpha^\varepsilon / \beta^\varepsilon < \infty$.
- We write $\alpha^\varepsilon \approx \beta^\varepsilon$ if $\alpha^\varepsilon \lesssim \beta^\varepsilon$ and $\beta^\varepsilon \lesssim \alpha^\varepsilon$.

A typical result of [6] is the following:

Theorem 5.1. *Let $n \geq 1$, $a_0, \tilde{a}_0^\varepsilon \in \mathcal{S}(\mathbb{R}^n)$, where a_0 is independent of ε . Let u^ε and v^ε solve the initial value problems:*

$$\begin{aligned} i\varepsilon \partial_t u^\varepsilon + \frac{\varepsilon^2}{2} \Delta u^\varepsilon &= |u^\varepsilon|^2 u^\varepsilon; \quad u^\varepsilon|_{t=0} = a_0. \\ i\varepsilon \partial_t v^\varepsilon + \frac{\varepsilon^2}{2} \Delta v^\varepsilon &= |v^\varepsilon|^2 v^\varepsilon; \quad v^\varepsilon|_{t=0} = \tilde{a}_0^\varepsilon. \end{aligned}$$

Assume that there exists $N \in \mathbb{N}$ and $\varepsilon^{1-\frac{1}{N}} \ll \delta^\varepsilon \ll 1$ such that:

$$(5.1) \quad \|a_0 - \tilde{a}_0^\varepsilon\|_{H^s} \approx \delta^\varepsilon, \quad \forall s \geq 0; \quad \limsup_{\varepsilon \rightarrow 0} \left\| \frac{\operatorname{Re}(a_0 - \tilde{a}_0^\varepsilon) \overline{a_0}}{\delta^\varepsilon} \right\|_{L^\infty(\mathbb{R}^n)} \neq 0.$$

Then we can find $0 < t^\varepsilon \ll 1$ such that: $\|u^\varepsilon(t^\varepsilon) - v^\varepsilon(t^\varepsilon)\|_{L^2} \gtrsim 1$. More precisely, this mechanism occurs as soon as $t^\varepsilon \delta^\varepsilon \gtrsim \varepsilon$. In particular, for all $s \geq 0$,

$$\frac{\|u^\varepsilon - v^\varepsilon\|_{L^\infty([0, t^\varepsilon]; L^2)}}{\|u^\varepsilon|_{t=0} - v^\varepsilon|_{t=0}\|_{H^s}} \rightarrow +\infty \quad \text{as } \varepsilon \rightarrow 0.$$

Example. Consider $a_0, b_0 \in \mathcal{S}(\mathbb{R}^n)$ independent of ε , such that $\operatorname{Re}(\overline{a_0}b_0) \neq 0$, and take $\tilde{a}_0^\varepsilon = a_0 + \delta^\varepsilon b_0$.

Example. Consider $a_0 \in \mathcal{S}(\mathbb{R}^n)$ independent of ε and $x^\varepsilon \in \mathbb{R}^n$. We can take $\tilde{a}_0^\varepsilon(x) = a_0(x - x^\varepsilon)$, provided that $|x^\varepsilon| = \delta^\varepsilon$ and

$$\limsup_{\varepsilon \rightarrow 0} \left\| \frac{x^\varepsilon}{|x^\varepsilon|} \cdot \nabla (|a_0|^2) \right\|_{L^\infty} \neq 0.$$

This example and the analysis of [6] make it possible to refine some results of [4].

The general idea consists in using the WKB analysis in this super-critical case. Roughly speaking, we have seen that (3.8) provides a good approximation of u^ε for very small time only. Since we are interested in instabilities occurring for very small time, this is not a problem for us now. The coupling in (3.8) shows that a small perturbation of a_0 yields a small perturbation of ϕ . But when we write

$$u^\varepsilon(t, x) \sim a(t, x)e^{i\phi(t, x)/\varepsilon},$$

we see that this small perturbation is divided by ε , which goes to zero. The result may not be small...

Technically, our approach consists in resuming the result provided by Proposition 3.4. Instead of letting $\varepsilon \rightarrow 0$ in the initial data of (3.5), just neglect the skew-symmetric term (recall that we assume $\phi_0 = 0$):

$$(5.2) \quad \begin{aligned} \partial_t \Phi^\varepsilon + \frac{1}{2} |\nabla \Phi^\varepsilon|^2 + |\mathbf{a}^\varepsilon|^2 &= 0 \quad ; \quad \Phi^\varepsilon|_{t=0} = 0, \\ \partial_t \mathbf{a}^\varepsilon + \nabla \Phi^\varepsilon \cdot \nabla \mathbf{a}^\varepsilon + \frac{1}{2} \mathbf{a}^\varepsilon \Delta \Phi^\varepsilon &= 0 \quad ; \quad \mathbf{a}^\varepsilon|_{t=0} = a_0^\varepsilon. \end{aligned}$$

Assuming that a_0^ε is bounded in H^s for all $s \geq 0$, (5.2) has a unique solution $(\Phi^\varepsilon, \mathbf{a}^\varepsilon) \in L^\infty([0, T_*]; H^m)^2$ for all $m > 0$ for some $T_* > 0$ independent of ε and m .

Proposition 5.2. *Let $s \in \mathbb{N}$. Then $T_s \geq T_*$, and there exists C_s independent of ε such that for every $0 \leq t \leq T_*$,*

$$\|a^\varepsilon(t) - \mathbf{a}^\varepsilon(t)\|_{H^s} \leq C_s \varepsilon t \quad ; \quad \|\phi^\varepsilon(t) - \Phi^\varepsilon(t)\|_{H^s} \leq C_s \varepsilon t^2.$$

The second idea consists in considering the Taylor expansion in time of $(\Phi^\varepsilon, \mathbf{a}^\varepsilon)$:

$$\Phi^\varepsilon(t, x) \sim \sum_{j \geq 1} t^{2j-1} \Phi_j^\varepsilon(x) \quad ; \quad \mathbf{a}^\varepsilon(t, x) \sim \sum_{j \geq 1} t^{2j} \mathbf{a}_j^\varepsilon(x).$$

Note that only odd powers of t are present in the expansion of Φ^ε , and even powers in that of \mathbf{a}^ε . This is because we have assumed $\phi_0 = 0$. Plugging these expansions into (5.2), we get formally:

$$\mathbf{a}_0^\varepsilon = a_0^\varepsilon \quad ; \quad \Phi_1^\varepsilon = -|a_0^\varepsilon|^2.$$

We can then check that a perturbation of order δ^ε of a_0^ε yields a perturbation of order δ^ε of Φ_1^ε , provided that the polarization condition (5.1) is satisfied. By

induction, we see that this perturbs the other Φ_j^ε 's and \mathbf{a}_j^ε 's by a $\mathcal{O}(\delta^\varepsilon)$. Consider the approximate solution defined by

$$u_K^\varepsilon(t, x) = a_0^\varepsilon(x) \exp\left(i \sum_{j=1}^K t^{2j-1} \Phi_j^\varepsilon(x)/\varepsilon\right).$$

Formally, we have:

$$\begin{aligned} \mathbf{a}^\varepsilon(t, x) e^{i\Phi^\varepsilon(t, x)/\varepsilon} - u_K^\varepsilon(t, x) &= (\mathbf{a}^\varepsilon(t, x) - a_0^\varepsilon(x)) e^{i\Phi^\varepsilon(t, x)/\varepsilon} \\ &\quad + a_0^\varepsilon(x) \left(\exp\left(i\Phi^\varepsilon(t, x)/\varepsilon\right) - \exp\left(i \sum_{j=1}^K t^{2j-1} \Phi_j^\varepsilon(x)/\varepsilon\right) \right) \\ &= \mathcal{O}(t^2) + \mathcal{O}\left(\left(\Phi^\varepsilon(t, x) - \sum_{j=1}^K t^{2j-1} \Phi_j^\varepsilon(x)\right)/\varepsilon\right) \\ &= \mathcal{O}(t^2) + \mathcal{O}(t^{2K+1}/\varepsilon). \end{aligned}$$

We infer that the above quantity is small for times such that $t^\varepsilon \ll 1$ and $t^\varepsilon \ll \varepsilon^{\frac{1}{2K+1}}$. On the other hand, Proposition 5.2 shows that $\mathbf{a}^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$ is a good approximation of u^ε for times such that $t^\varepsilon \ll 1$. Therefore, we expect

$$\|u^\varepsilon(t^\varepsilon, \cdot) - u_K^\varepsilon(t^\varepsilon, \cdot)\|_{L^2 \cap L^\infty} \ll 1 \quad \text{for } t^\varepsilon \ll \varepsilon^{\frac{1}{2K+1}}.$$

This can be proved by the analysis presented in §3.2.

The case $K = 1$ is of special interest. Indeed, the Laplacian plays no role in the definition of u_1^ε , and we check that it solves:

$$i\varepsilon \partial_t u_1^\varepsilon = |u_1^\varepsilon|^2 u_1^\varepsilon \quad ; \quad u_1^\varepsilon|_{t=0} = a_0^\varepsilon.$$

This is the solution of the ordinary differential equation considered in [10] and [4]. The above analysis shows that it is a reasonable approximation of u^ε for $0 < t^\varepsilon \ll \varepsilon^{1/3}$: see Remark 4.1.

In view of Theorem 5.1, we define u_K^ε and v_K^ε in an obvious way, and we have:

$$\|u^\varepsilon(t^\varepsilon, \cdot) - u_K^\varepsilon(t^\varepsilon, \cdot)\|_{L^2 \cap L^\infty} + \|v^\varepsilon(t^\varepsilon, \cdot) - v_K^\varepsilon(t^\varepsilon, \cdot)\|_{L^2 \cap L^\infty} \ll 1 \quad \text{for } t^\varepsilon \ll \varepsilon^{\frac{1}{2K+1}}.$$

So to prove Theorem 5.1, we just have to compare u_K^ε and v_K^ε :

$$\begin{aligned} u_K^\varepsilon(t, x) - v_K^\varepsilon(t, x) &= a_0(x) \exp\left(i \sum_{j=1}^K t^{2j-1} \Phi_j(x)/\varepsilon\right) \\ &\quad - \tilde{a}_0^\varepsilon(x) \exp\left(i \sum_{j=1}^K t^{2j-1} \tilde{\Phi}_j^\varepsilon(x)/\varepsilon\right) \\ &= (a_0(x) - \tilde{a}_0^\varepsilon(x)) \exp\left(i \sum_{j=1}^K t^{2j-1} \Phi_j(x)/\varepsilon\right) \\ (5.3) \quad &\quad - \tilde{a}_0^\varepsilon(x) \left(\exp\left(i \sum_{j=1}^K t^{2j-1} \Phi_j(x)/\varepsilon\right) - \exp\left(i \sum_{j=1}^K t^{2j-1} \tilde{\Phi}_j^\varepsilon(x)/\varepsilon\right) \right). \end{aligned}$$

The first term is of order δ^ε by assumption. To estimate the second term, examine:

$$\sum_{j=1}^K t^{2j-1} \left(\Phi_j(x) - \tilde{\Phi}_j^\varepsilon(x) \right) / \varepsilon.$$

Since we consider times such that $t^\varepsilon \ll 1$ and that we have seen that $\Phi_j - \tilde{\Phi}_j^\varepsilon = \mathcal{O}(\delta^\varepsilon)$ for $j \geq 2$, the leading order term is simply:

$$\begin{aligned} t \left(\Phi_1(x) - \tilde{\Phi}_1^\varepsilon(x) \right) / \varepsilon &= \frac{t}{\varepsilon} \left(|\tilde{a}_0^\varepsilon(x)|^2 - |a_0^\varepsilon(x)|^2 \right) \\ &= \frac{t}{\varepsilon} \left(\operatorname{Re} (a_0 - \tilde{a}_0^\varepsilon) \overline{a_0} \right) + \mathcal{O} \left((\delta^\varepsilon)^2 \right). \end{aligned}$$

By assumption, the modulus of (5.3) behaves like

$$\left| a_0(x) \sin \left(\frac{t\delta^\varepsilon}{\varepsilon} f(x) \right) \right|,$$

for some non-trivial function f . The conclusion of Theorem 5.1 follows easily.

REFERENCES

- [1] S. Alinhac and P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, Savoirs Actuels, InterEditions, Paris, 1991.
- [2] R. W. Boyd, *Nonlinear optics*, Academic Press, New York, 1992.
- [3] N. Burq, P. Gérard, and N. Tzvetkov, *Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 2, 255–301.
- [4] N. Burq and M. Zworski, *Instability for the semiclassical non-linear Schrödinger equation*, Comm. Math. Phys. **260** (2005), no. 1, 45–58.
- [5] R. Carles, *Geometric optics with caustic crossing for some nonlinear Schrödinger equations*, Indiana Univ. Math. J. **49** (2000), no. 2, 475–551.
- [6] ———, *Geometric optics and instability for semi-classical Schrödinger equations*, Arch. Ration. Mech. Anal. **183** (2007), no. 3, 525–553.
- [7] ———, *WKB analysis for nonlinear Schrödinger equations with potential*, Comm. Math. Phys. **269** (2007), no. 1, 195–221.
- [8] C. Cheverry, *Cascade of phases in turbulent flows*, Bull. Soc. Math. France **134** (2006), no. 1, 33–82.
- [9] C. Cheverry and O. Guès, *Counter-examples to the concentration-cancellation property*, preprint, 2005.
- [10] M. Christ, J. Colliander, and T. Tao, *Ill-posedness for nonlinear Schrödinger and wave equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, to appear. See also [arXiv:math.AP/0311048](https://arxiv.org/abs/math/0311048)
- [11] J. Dereziński and C. Gérard, *Scattering theory of quantum and classical N-particle systems*, Texts and Monographs in Physics, Springer Verlag, Berlin Heidelberg, 1997.
- [12] P. Donnat, *Quelques contributions mathématiques en optique non linéaire*, Ph.D. thesis, École polytechnique, Palaiseau (France), 1994.
- [13] P. Gérard, *Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire*, Séminaire sur les Équations aux Dérivées Partielles, 1992–1993, École Polytech., Palaiseau, 1993, pp. Exp. No. XIII, 13.
- [14] E. Grenier, *Semiclassical limit of the nonlinear Schrödinger equation in small time*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 523–530.
- [15] L. Landau and E. Lifschitz, *Physique théorique (“Landau-Lifschitz”). Tome III: Mécanique quantique. Théorie non relativiste*, Éditions Mir, Moscow, 1967, Deuxième édition, Traduit du russe par Édouard Gloukhian.
- [16] G. Lebeau, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl. (9) **71** (1992), no. 3, 267–291.
- [17] ———, *Non linear optic and supercritical wave equation*, Bull. Soc. Roy. Sci. Liège **70** (2001), no. 4-6, 267–306 (2002), Hommage à Pascal Laubin.

- [18] ———, *Perte de régularité pour les équations d'ondes sur-critiques*, Bull. Soc. Math. France **133** (2005), 145–157.
- [19] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Applied Mathematical Sciences, vol. 53, Springer-Verlag, New York, 1984.
- [20] T. Makino, S. Ukai, and S. Kawashima, *Sur la solution à support compact de l'équation d'Euler compressible*, Japan J. Appl. Math. **3** (1986), no. 2, 249–257.
- [21] G. Métivier, *Exemples d'instabilités pour des équations d'ondes non linéaires (d'après G. Lebeau)*, Astérisque (2004), no. 294, vii, 63–75.
- [22] J. Rauch and M. Keel, *Lectures on geometric optics*, Hyperbolic equations and frequency interactions (Park City, UT, 1995), Amer. Math. Soc., Providence, RI, 1999, pp. 383–466.
- [23] J. T. Schwartz, *Nonlinear functional analysis*, Gordon and Breach Science Publishers, New York, 1969, Notes by H. Fattorini, R. Nirenberg and H. Porta, with an additional chapter by Hermann Karcher, Notes on Mathematics and its Applications.
- [24] J. Szeftel, *Propagation et réflexion des singularités pour l'équation de Schrödinger non linéaire*, Ann. Inst. Fourier (Grenoble) **55** (2005), no. 2, 573–671.
- [25] V. E. Zakharov and A. B. Shabat, *Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media*, Ž. Èksper. Teoret. Fiz. **61** (1971), no. 1, 118–134.

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