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# Weighted power variations of fractional Brownian motion and application to approximating schemes

Mihai Gradinaru<sup>1</sup> and Ivan Nourdin<sup>2</sup>

**Abstract:** The first part of the paper contains the study of the convergence for some weighted power variations of a fractional Brownian motion  $B$  with Hurst index  $H \in (0, 1)$ . The behaviour is different when  $H < 1/2$  and powers are odd, compared with the case when  $H = 1/2$ . In the second part, one applies the results of the first part to compute the exact rate of convergence of some approximating schemes associated to scalar stochastic differential equations driven by  $B$ . The limit of the error between the exact solution and the considered scheme (whose size depends on the Hurst index  $H$ ) is computed explicitly.

**Key words:** Fractional Brownian motion - weighted power variations - stochastic differential equation - approximating scheme - exact rate of convergence.

**2000 Mathematics Subject Classification:** 60F15, 60G15, 60H05, 60H10, 60H35, 60G18, 60H20.

## 1 Introduction

Let  $B = (B_t)_{t \in [0,1]}$  be a fractional Brownian motion with Hurst index  $H \in (0, 1)$ . That is,  $B$  is a centered Gaussian process with covariance function given by

$$K(s, t) = \text{Cov}(B_s, B_t) = \frac{1}{2}(s^{2H} + t^{2H} - |t - s|^{2H}), \quad s, t \in [0, 1].$$

For  $H = 1/2$ ,  $B$  is a standard Brownian motion, while for  $H \neq 1/2$ , it is neither a semimartingale, nor a Markov process. Moreover, it holds, for any  $p > 1$ :

$$\mathbb{E}|B_t - B_s|^p = c_p |t - s|^{pH}, \quad s, t \in [0, 1], \quad \text{with} \quad c_p = \mathbb{E}(|G|^p), \quad G \sim \mathcal{N}(0, 1),$$

and, consequently, almost all sample paths of  $B$  are Hölder continuous of any order  $\alpha \in (0, H)$ .

The study of path behaviour of stochastic processes is often based on the study of their power variations and there exists a very extensive literature on the subject. Recall that, given a real  $p > 1$ , the  $p$ -power variation of a process  $X$  with respect to a subdivision  $\pi_n = \{0 = t_{n,0} < t_{n,1} < \dots < t_{n,n} = 1\}$  of  $[0, 1]$  is defined to be the sum

$$\sum_{\ell=0}^{n-1} |X_{t_{n,\ell+1}} - X_{t_{n,\ell}}|^p.$$

Very recently, power variations for processes related to the fractional Brownian motion  $B$  have been studied in [2, 16].

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For simplicity, consider from now on the case where  $t_{n,\ell} = \ell/n$ , for  $n \geq 1$  and  $\ell \in \{0, \dots, n\}$ . In this paper we shall point out some interesting phenomena when  $X = B$  is the fractional Brownian motion and  $p = \kappa \geq 2$  is an integer. In fact, we will also drop the absolute value (when  $\kappa$  is odd) and we will introduce some weights. More precisely, we will consider:

$$\sum_{\ell=0}^{n-1} h(B_{\ell/n})(B_{(\ell+1)/n} - B_{\ell/n})^\kappa, \quad h \text{ being a smooth function.}$$

These quantities appears in a natural manner when studying some approximation schemes as will be explained below. Before, let us recall some results concerning  $\kappa$ -power variations which are more or less classical.

First, assume that the Hurst index is  $H = 1/2$ , that is  $B$  is the standard Brownian motion. Let  $\mu_\kappa$  be the  $\kappa$ -moment of a standard Gaussian random variable  $G \sim \mathcal{N}(0, 1)$ . It is classical, by using central limit theorem that, as  $n \rightarrow \infty$ ,

$$\sqrt{n} \left( n^{\kappa/2-1} \sum_{\ell=0}^{n-1} (B_{(\ell+1)/n} - B_{\ell/n})^\kappa - \mu_\kappa \right) \xrightarrow{\text{Law}} \mathcal{N}(0, \mu_{2\kappa} - \mu_\kappa^2). \quad (1.1)$$

When weights are introduced, an interesting phenomenon appears: instead of Gaussian random variables, we rather obtain mixing random variables as limit in (1.1). For instance, when  $\kappa$  is even, it is a very particular case of a more general result by Jacod [13] (see also [8, 9] for related results with different proofs) that we have, as  $n \rightarrow \infty$ :

$$\frac{1}{\sqrt{n}} \sum_{\ell=0}^{n-1} h(B_{\ell/n}) \left( n^{\kappa/2} (B_{(\ell+1)/n} - B_{\ell/n})^\kappa - \mu_\kappa \right) \xrightarrow{\text{Law}} \sqrt{\mu_{2\kappa} - \mu_\kappa^2} \int_0^1 h(B_s) dW_s. \quad (1.2)$$

Here,  $W$  denotes another standard Brownian motion, independent of  $B$ . In fact we shall see in §2 that the convergence in (1.2) can be improved: the convergence in law can be replaced by the stable convergence with respect to the  $\sigma$ -field generated by  $(B_t)_{t \in [0,1]}$ . This remark will play a crucial role in the proof of Proposition 2.1 below.

Second, assume that  $H \neq 1/2$ , that is the case where the fractional Brownian motion  $B$  has not independent increments anymore. Then (1.1) has been extended by [1, 12] (see also [29]) and two cases are considered according to the evenness of  $\kappa$ :

- if  $\kappa$  is even and  $H < 3/4$ , as  $n \rightarrow \infty$ ,

$$\sqrt{n} \left( n^{\kappa H-1} \sum_{\ell=0}^{n-1} (B_{(\ell+1)/n} - B_{\ell/n})^\kappa - \mu_\kappa \right) \xrightarrow{\text{Law}} \mathcal{N}(0, \sigma_{H,\kappa}^2); \quad (1.3)$$

- if  $\kappa$  is odd and  $H < 1/2$ , as  $n \rightarrow \infty$ ,

$$n^{\kappa H-1/2} \sum_{\ell=0}^{n-1} (B_{(\ell+1)/n} - B_{\ell/n})^\kappa \xrightarrow{\text{Law}} \mathcal{N}(0, \sigma_{H,\kappa}^2). \quad (1.4)$$

Here  $\sigma_{H,\kappa} > 0$  is an explicit constant depending only on  $H$  and  $\kappa$  (see, *e.g.*, the second point of Remark 4 in [21] for a method of computation of this constant). In fact, one can relax the restrictive conditions made on  $H$  in (1.3)-(1.4), but on one hand the normalizations are

not the same anymore and, on the other hand, for (1.3) one obtains limits which are not Gaussian but the value at 1 of the so-called Hermite process (see for instance [4, 34]). We will not consider these cases in the present paper.

Now, let us proceed with the results concerning the weighted power variations in the case where  $H \neq 1/2$ . When  $\kappa$  is even and  $H \in (1/2, 3/4)$ , then by Theorem 2, p. 280 in León and Ludeña [16] we have that:

$$\sqrt{n} \left( n^{\kappa H - 1} \sum_{\ell=0}^{n-1} h(B_{\ell/n}) (B_{(\ell+1)/n} - B_{\ell/n})^\kappa - \mu_\kappa \int_0^1 h(B_s) ds \right) \xrightarrow{\text{Law}} c_{H,\kappa} \int_0^1 h(B_s) dW_s, \quad (1.5)$$

as  $n \rightarrow \infty$ , where again  $W$  is a standard Brownian motion independent of  $B$ . In other words, (1.5) shows a similar behaviour to that observed in the standard Brownian case, (1.2) (see also [2] for related results on the asymptotic behavior of the  $p$ -variation of stochastic integrals with respect to  $B$ ). When  $\kappa$  is odd and  $H < 1/2$ , in the light of (1.2), (1.4) and (1.5), one could expect to obtain, as  $n \rightarrow \infty$ :

$$n^{\kappa H - 1/2} \sum_{\ell=0}^{n-1} h(B_{\ell/n}) (B_{(\ell+1)/n} - B_{\ell/n})^\kappa \xrightarrow{\text{Law}} c_{H,\kappa} \int_0^1 h(B_s) dW_s,$$

with  $c_{H,\kappa}$  a certain constant depending only on  $H$  and  $\kappa$ . Surprisingly, and it is the first main result of the present paper (see Theorem 2.3 below for a precise statement), we rather obtain a convergence in probability, instead of the latter convergence in law:

$$n^{(\kappa+1)H-1} \sum_{\ell=0}^{n-1} h(B_{\ell/n}) (B_{\ell+1/n} - B_{\ell/n})^\kappa \xrightarrow{\text{Prob}} -\frac{\mu_{\kappa+1}}{2} \int_0^1 h'(B_s) ds. \quad (1.6)$$

In particular, the limit does not involve an independent standard Brownian motion anymore, as in (1.5). Moreover, observe that (1.6) is not in contradiction with (1.4), since if  $H < 1/2$ , we have  $(\kappa + 1)H - 1 < \kappa H - 1/2$  and (1.6) with  $h \equiv 1$  is actually a corollary of (1.4).

Let us now turn to the study of some approximation schemes for stochastic differential equations driven by the fractional Brownian motion. The study of such equations was recently considered by using several methods. For instance, in [30] (see also references therein) one uses fractional calculus of same type as in [35]; in [3, 6, 22] (and references therein) one uses the rough paths theory introduced in [18] and in [5, 24, 26] (and references therein) one uses the regularization method used firstly in [32].

In the present paper, we consider the easiest stochastic differential equation involving fractional Brownian motion, that is

$$dX_t = \sigma(X_t) dB_t, \quad t \in [0, 1], \quad X_0 = x \in \mathbb{R}. \quad (1.7)$$

Notice that we assume here and all along of this paper that  $\sigma \in C^\infty(\mathbb{R})$  is a real function which is bounded together with its derivatives. Let us denote by  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$  the flow associated to  $\sigma$ , that is the unique solution to

$$\phi(x, y) = x + \int_0^y \sigma(\phi(x, z)) dz, \quad x, y \in \mathbb{R}. \quad (1.8)$$

Then, it is an immediate consequence (see also (3.6) below) that the following process:

$$X_t^x = \phi(x, B_t), \quad t \in [0, 1], \quad (1.9)$$

is a natural solution of equation (1.7), as soon as the integral with respect to  $B$  we consider verifies the following Itô-Stratonovich type formula:

$$f(B_t) = f(0) + \int_0^t f'(B_s)dB_s, \quad t \in [0, 1], \quad f : \mathbb{R} \rightarrow \mathbb{R} \text{ smooth enough.}$$

Approximating schemes for stochastic differential equations of the type

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt, \quad t \in [0, 1], \quad X_0 = x \in \mathbb{R}, \quad (1.10)$$

have been considered only in few articles. The first work in that direction has been proposed in [17]: when  $H > 1/2$  one shows that the Euler approximation of equation (1.10) – but in the particular case where  $\sigma(X_t)$  is replaced by  $\sigma(t)$ , that is the so-called additive case – converges uniformly in probability. In [23] one introduces (see also [33]) some approximating schemes for the analogue of (1.10) where  $B$  is replaced by any Hölder continuous function: one determines upper error bounds and, in particular, these results apply almost surely when the driving Hölder continuous function is a path of the fractional Brownian motion  $B$ , and this for any Hurst index  $H \in (0, 1)$ .

Results on lower error bounds are available only since very recently: see [19] for the additive case, and [21] for equation (1.10) (see also [20] where approximation methods with respect to a mean square error are analysed). It is proved in the latter reference that the Euler scheme  $\tilde{X} = \{\tilde{X}^{(n)}\}_{n \in \mathbb{N}^*}$  associated to (1.10) verifies, under some classical assumptions on  $\sigma$  and  $b$  and when  $H \in (1/2, 1)$ , that, as  $n \rightarrow \infty$ ,

$$n^{2H-1} \left[ \tilde{X}_1^{(n)} - X_1 \right] \xrightarrow{\text{a.s.}} -\frac{1}{2} \int_0^1 \sigma'(X_s) D_s X_1 ds. \quad (1.11)$$

Here  $D_s X_1$  denotes the Malliavin derivatives of  $X_1$  with respect to  $B$ . Notice that the upper and lower error bounds are obtained from an almost sure convergence, which is somewhat surprising when compared with the case  $H = 1/2$ . In [21], it is proved that, for the so-called Crank-Nicholson scheme  $\bar{X} = \{\bar{X}^{(n)}\}_{n \in \mathbb{N}^*}$  associated to (1.7) and defined by

$$\begin{cases} \bar{X}_0^n = x, \\ \bar{X}_{(\ell+1)/n}^{(n)} = \bar{X}_{\ell/n}^{(n)} + \frac{1}{2} \left( \sigma(\bar{X}_{\ell/n}^{(n)}) + \sigma(\bar{X}_{(\ell+1)/n}^{(n)}) \right) (B_{(\ell+1)/n} - B_{\ell/n}), \quad \ell \in \{0, \dots, n-1\}, \end{cases} \quad (1.12)$$

the following convergence holds, as  $n \rightarrow \infty$ , for  $\sigma$  regular enough and when  $H \in (1/3, 1/2)$ :

$$n^\alpha \left[ \bar{X}_1^{(n)} - X_1 \right] \xrightarrow{\text{Prob}} 0, \quad \forall \alpha < 3H - 1/2. \quad (1.13)$$

In the particular case where the diffusion coefficient  $\sigma$  verifies  $\sigma(x)^2 = \alpha x^2 + \beta x + \gamma$ , for some  $\alpha, \beta, \gamma \in \mathbb{R}$ , one can derive the exact rate of convergence and one proves that, as  $n \rightarrow \infty$ :

$$n^{3H-1/2} \left[ \bar{X}_1^{(n)} - X_1 \right] \xrightarrow{\text{Law}} \frac{\alpha}{12} \sigma(X_1) G. \quad (1.14)$$

Here  $G$  is a centered Gaussian random variable independent of  $X_1$ , whose variance depends only on  $H$ . In particular, the upper and lower error bounds are obtained here from a convergence in law.

As was already said, the convergence in (1.11) is somewhat surprising, since there is no analogue for the case of the standard Brownian motion. More precisely, when  $H = 1/2$ , it is

proved in [15] that the Euler scheme (1.11) verifies (by denoting  $X^{\text{Ito}}$  the solution of (1.10) in the Ito sense), as  $n \rightarrow \infty$ :

$$n^{1/2} \left[ \tilde{X}_1^{(n)} - X_1^{\text{Ito}} \right] \xrightarrow{\text{Law}} -\frac{1}{\sqrt{2}} Y_1 \int_0^1 \sigma(X_s^{\text{Ito}}) \sigma'(X_s^{\text{Ito}}) Y_s^{-1} dW_s. \quad (1.15)$$

Here  $W$  is a Brownian motion independent of the Brownian motion  $B$  and

$$Y_t = \exp \left( \int_0^t b'(X_s^{\text{Ito}}) ds - \int_0^t \frac{1}{2} \sigma(X_s^{\text{Ito}}) \sigma'(X_s^{\text{Ito}}) ds + \int_0^t \sigma'(X_s^{\text{Ito}}) dB_s \right), \quad s \in [0, 1].$$

On the other hand, it can be proved (see Remark 3.8.2) that for the Crank-Nicholson scheme (1.12) we have, as  $n \rightarrow \infty$ :

$$n \left[ \bar{X}_1^{(n)} - X_1^{\text{Str}} \right] \xrightarrow{\text{Law}} \frac{1}{24} \int_0^1 (\sigma^2)''(X_s^{\text{Str}}) [\sqrt{15} dW_s + 3dB_s] + \frac{1}{16} \int_0^1 \sigma(\sigma^2)'''(X_s^{\text{Str}}) ds, \quad (1.16)$$

where  $X^{\text{Str}}$  denotes the solution of (1.7) in the Stratonovich sense.

In the present paper, we are interested by a better understanding of this phenomenon: which type of convergence allows to derive the upper and lower error bounds for some natural scheme of Milstein's type (see also [23])? To be more precise, let us define, by induction, the family of differential operators  $(\mathcal{D}^j)_{j \in \mathbb{N}}$  by:

$$\mathcal{D}^0 f = f, \quad \mathcal{D}^1 f = f' \sigma \quad \text{and, for } j \geq 2, \quad \mathcal{D}^j f = \mathcal{D}^1(\mathcal{D}^{j-1} f). \quad (1.17)$$

For instance, the first  $\mathcal{D}^j \sigma$ 's are given by:

$$\mathcal{D}^0 \sigma = \sigma, \quad \mathcal{D}^1 \sigma = \sigma \sigma', \quad \mathcal{D}^2 \sigma = \sigma \sigma'^2 + \sigma^2 \sigma'', \quad \mathcal{D}^3 \sigma = \sigma \sigma'^3 + 4\sigma^2 \sigma' \sigma'' + \sigma^3 \sigma''', \quad \text{etc.}$$

Then, let us consider the following scheme:

$$\begin{cases} \hat{X}_0^{(n)} = x \\ \hat{X}_{(\ell+1)/n}^{(n)} = \hat{X}_{\ell/n}^{(n)} + \sum_{j=0}^m \frac{1}{(j+1)!} \mathcal{D}^j \sigma(\hat{X}_{\ell/n}^{(n)}) \Delta^{j+1} B_{\ell/n}, \quad \ell \in \{0, \dots, n-1\}, \end{cases} \quad (1.18)$$

the integer  $m \in \mathbb{N}$  being the size of  $\hat{X} = \{\hat{X}^{(n)}\}_{n \in \mathbb{N}^*}$ . Here, for  $j, n \in \mathbb{N}^*$ ,  $\ell \in \{0, \dots, n-1\}$  and a process  $Z = (Z_t)_{t \in [0,1]}$ , we set  $\Delta^j Z_{\ell/n} = (Z_{(\ell+1)/n} - Z_{\ell/n})^j$ . When  $j = 1$ , we denote  $\Delta Z_{\ell/n}$  instead of  $\Delta^1 Z_{\ell/n}$  for simplicity. The idea of introducing these schemes will be better explained in §3.1 below. Let us note that when the size of  $\hat{X}$  is  $m = 0$  (resp.  $m = 1$ ),  $\hat{X}$  is the classical Euler (resp. Milstein) scheme.

The second aim of the present paper is to answer to the following questions: does the sequence  $\{\hat{X}_1^{(n)}\}_{n \in \mathbb{N}^*}$  converge? Is the limit  $X_1^x$  given by (1.9) as expected? What is the rate of convergence? Upper and lower error bounds for the approximating scheme (1.18) are obtained from a convergence in law or from a pathwise type convergence?

The paper is organized as follows: the next section contains the results concerning the rates of convergence of weighted power variations for the fractional Brownian motion. In Section 3, after some recalls concerning the definition and main properties of the so-called Newton-Côtes integral and the study of the equation (1.7), we state the result on the rate of convergence of the approximating scheme (1.18). In the last section one gives the proofs of the most technical results.

## 2 Weighted power variations for fractional Brownian motion

In this section we study the rate of convergence of weighted power variations for the process  $B$  in order to complete the picture described above. As usually, we start with the standard Brownian motion case and then we proceed with the fractional Brownian motion case. As we shall see, the results and the methods of proofs are somehow different for odd order power variations.

Recall that we denote  $\mu_{2n}$  ( $n \geq 1$ ) the  $2n$ -moment of a random variable  $G \sim \mathcal{N}(0, 1)$ .

**Proposition 2.1** *Let  $\kappa \in \mathbb{N} \setminus \{0, 1\}$  and  $g, h \in C_b(\mathbb{R})$ . Assume that  $H = 1/2$ , that is  $B$  is the standard Brownian motion.*

1. *If  $\kappa$  is even then, as  $n \rightarrow \infty$ ,*

$$n^{\kappa/2-1} \sum_{\ell=0}^{n-1} h(B_{\ell/n}) (B_{(\ell+1)/n} - B_{\ell/n})^\kappa \xrightarrow{\text{Prob}} \mu_\kappa \int_0^1 h(B_s) ds. \quad (2.1)$$

2. *If  $\kappa$  is odd then, as  $n \rightarrow \infty$ ,*

$$\left( B_1, n^{\frac{\kappa-1}{2}} \sum_{\ell=0}^{n-1} g(B_{\ell/n}) (B_{(\ell+1)/n} - B_{\ell/n})^\kappa, n^{\frac{\kappa-1}{2}} \sum_{\ell=0}^{n-1} h(B_{\ell/n}) (B_{(\ell+1)/n} - B_{\ell/n})^{\kappa+1} \right) \\ \xrightarrow{\text{Law}} \left( B_1, \int_0^1 g(B_s) (\sqrt{\mu_{2\kappa}} dW_s + \mu_{\kappa+1} dB_s), \mu_{\kappa+1} \int_0^1 h(B_s) ds \right), \quad (2.2)$$

with  $W$  another standard Brownian motion independent of  $B$ .

### Remark 2.2

We shall see that, for the study of Milstein's type schemes one needs the behaviour of the triplet (2.2) and not only the behaviour of  $\sum_{\ell=0}^{n-1} g(B_{\ell/n}) (B_{(\ell+1)/n} - B_{\ell/n})^\kappa$ , with  $\kappa$  odd.  $\square$

**Proof.** The first part of the proposition is a particular case of the first part of Theorem 2.3 below and which proof will be given in §4. So, let us admit (2.1), and let us prove only (2.2).

We shall use the notion of stable convergence for random variables, so let us recall its definition (see for instance, the original paper [31] or the survey work [7]). Let  $\mathcal{F} \subset \overline{\mathcal{F}}$  be two  $\sigma$ -fields. A sequence of  $\mathcal{F}$ -measurable random variables  $\{X_n\}$  converges  $\mathcal{F}$ -stably to a  $\overline{\mathcal{F}}$ -measurable random variable  $X$  if, for any continuous bounded function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and any bounded  $\overline{\mathcal{F}}$ -measurable random variable  $Z$ , we have, as  $n \rightarrow \infty$ ,

$$\mathbb{E}(f(X_n)Z) \rightarrow \mathbb{E}(f(X)Z). \quad (2.3)$$

Let  $\mathcal{F}^B$  be the  $\sigma$ -field generated by  $\{B_t : 0 \leq t \leq 1\}$  and  $W$  be a standard Brownian motion independent of  $\mathcal{F}^B$ . We denote  $\overline{\mathcal{F}} = \mathcal{F}^B \vee \mathcal{F}^W$ , where  $\mathcal{F}^W$  is the  $\sigma$ -field generated by  $\{W_t : 0 \leq t \leq 1\}$ . Set

$$X_n := n^{-1/2} \sum_{\ell=0}^{n-1} g(B_{\ell/n}) (n^{\kappa/2} (B_{(\ell+1)/n} - B_{\ell/n})^\kappa - \mu_{\kappa+1} n^{1/2} (B_{(\ell+1)/n} - B_{\ell/n}))$$

and

$$X := \int_0^1 g(B_s)(\sqrt{\mu_{2\kappa}} dW_s + \mu_{\kappa+1} dB_s).$$

The *Hermite rank* of a polynomial  $P$  is the first non-zero index in the expansion of  $P$  in Hermite polynomials. Since  $x^\kappa - \mu_{\kappa+1}x$  has Hermite rank  $\geq 3$  (see also (4.5) below), it is a consequence of [25] (see also Theorem 1.1, p. 3, in Jacod [13]) that, as  $n \rightarrow \infty$ ,

$$X_n \xrightarrow{\mathcal{F}^B\text{-stably}} X. \quad (2.4)$$

Furthermore, we obviously have

$$Y_n := \mu_{\kappa+1} \sum_{\ell=0}^{n-1} g(B_{\ell/n})(B_{(\ell+1)/n} - B_{\ell/n}) \xrightarrow{\text{Prob}} \mu_{\kappa+1} \int_0^1 g(B_s) dB_s =: Y, \quad \text{as } n \rightarrow \infty, \quad (2.5)$$

while, by (2.1):

$$Z_n := n^{\frac{\kappa-1}{2}} \sum_{\ell=0}^{n-1} h(B_{\ell/n})(B_{(\ell+1)/n} - B_{\ell/n})^{\kappa+1} \xrightarrow{\text{Prob}} \mu_{\kappa+1} \int_0^1 h(B_s) ds =: Z, \quad \text{as } n \rightarrow \infty. \quad (2.6)$$

We stress that  $Y$  and  $Z$  are  $\mathcal{F}^B$ -measurable random variables. On the other hand, for any  $(t_1, t_2, t_3) \in \mathbb{R}^3$ , we can write

$$\begin{aligned} & \mathbb{E} \left[ e^{it_1 B_1 + it_2 (X_n + Y_n) + it_3 Z_n} \right] - \mathbb{E} \left[ e^{it_1 B_1 + it_2 (X + Y) + it_3 Z} \right] \\ &= \mathbb{E} \left[ (e^{it_2 X_n} - e^{it_2 X}) e^{it_1 B_1 + it_2 Y + it_3 Z} \right] + \mathbb{E} \left[ (e^{it_2 Y_n} - e^{it_2 Y}) e^{it_1 B_1 + it_2 X_n + it_3 Z} \right] \\ & \quad + \mathbb{E} \left[ (e^{it_3 Z_n} - e^{it_3 Z}) e^{it_1 B_1 + it_2 (X_n + Y_n)} \right] \end{aligned}$$

By using (2.3) and (2.4), the first term on the right hand side of the latter equality tends to zero, while the modulus of the second and the third terms tend to zero by (2.5) and (2.6). The proof of (2.2) is done.  $\square$

The following result concerns the fractional Brownian motion and it is the first main result of this paper. Its proof is postponed in Section 4.

**Theorem 2.3** *Let  $\kappa \in \mathbb{N} \setminus \{0, 1\}$  and assume that  $B$  is a fractional Brownian motion with Hurst index  $H$ .*

1. *If  $\kappa$  is even and  $h \in C_b(\mathbb{R})$  then, for any  $H \in (0, 1)$ , as  $n \rightarrow \infty$ :*

$$n^{\kappa H - 1} \sum_{\ell=0}^{n-1} h(B_{\ell/n})(B_{(\ell+1)/n} - B_{\ell/n})^\kappa \xrightarrow{\text{Prob}} \mu_\kappa \int_0^1 h(B_s) ds. \quad (2.7)$$

2. *If  $\kappa$  is odd and  $h \in C_b^{2\kappa}(\mathbb{R})$  then, for any  $H \in (0, 1/2)$ , as  $n \rightarrow \infty$ :*

$$n^{(\kappa+1)H - 1} \sum_{\ell=0}^{n-1} h(B_{\ell/n})(B_{(\ell+1)/n} - B_{\ell/n})^\kappa \xrightarrow{\text{Prob}} -\frac{\mu_{\kappa+1}}{2} \int_0^1 h'(B_s) ds. \quad (2.8)$$

**Remark 2.4**

In fact we will prove that the convergence (2.7), and consequently the convergence (2.1), are almost sure.  $\square$

### 3 Approximating schemes for fractional stochastic differential equations

Here we present an application of the previous results to the approximation schemes for fractional stochastic differential equations.

#### 3.1 Newton-Côtes integral and fractional stochastic differential equations

In this paper, we will use, as integral with respect to  $B$ , the so-called Newton-Côtes integral introduced in [10] and studied further in [24]:

**Definition 3.1** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function,  $X, Y$  be two continuous processes on  $[0, 1]$  and  $N \in \mathbb{N}$ . The  $N$ -order Newton-Côtes integral of  $f(Y)$  with respect to  $X$  is defined by:*

$$\int_0^t f(Y_s) d^{\text{NC}, N} X_s := \lim_{\varepsilon \downarrow 0} \text{prob} \frac{1}{\varepsilon} \int_0^t ds (X_{s+\varepsilon} - X_s) \int_0^1 f(Y_s + \alpha(Y_{s+\varepsilon} - Y_s)) \nu_N(d\alpha), \quad t \in [0, 1], \quad (3.1)$$

provided the limit exists. Here  $\nu_0 = \delta_0$ ,  $\nu_1 = (\delta_0 + \delta_1)/2$  and, for  $N \geq 2$ ,

$$\nu_N = \sum_{j=0}^{2N-2} \left( \int_0^1 \prod_{k \neq j} \frac{2(N-1)u - k}{j - k} du \right) \delta_{j/(2N-2)},$$

$\delta_a$  being the Dirac measure at point  $a$ .

**Remark 3.2**

1. Note that the 0- and 1-order Newton-Côtes integrals are nothing but the forward integral and the symmetric integral in the sense of Russo-Vallois [32], respectively:

$$\int_0^t f(Y_s) d^{\text{NC}, 0} X_s = \int_0^t f(Y_s) d^- X_s = \lim_{\varepsilon \downarrow 0} \text{prob} \frac{1}{\varepsilon} \int_0^t f(Y_s) (X_{s+\varepsilon} - X_s) ds,$$

and

$$\int_0^t f(Y_s) d^{\text{NC}, 1} X_s = \int_0^t f(Y_s) d^\circ X_s = \lim_{\varepsilon \downarrow 0} \text{prob} \frac{1}{\varepsilon} \int_0^t \frac{f(Y_{s+\varepsilon}) + f(Y_s)}{2} (X_{s+\varepsilon} - X_s) ds.$$

2. Another way to define  $\nu_N$  is to view it as the unique discrete signed probability carried by  $j/(2N-2)$  ( $j = 0, \dots, 2N-2$ ), which coincides with Lebesgue measure on polynomials of degree smaller than  $2N-1$ . □

The Newton-Côtes integral defined by (3.1) is actually a special case of so-called  $N$ -order  $\nu$ -integrals introduced in [10], p. 789. Moreover, in the same cited paper, p. 795, one proves that the  $N$ -order Newton-Côtes integral of  $f(B)$  with respect to  $B$  exists for any  $f \in C^{4N+1}$

if and only if  $H \in (1/(4N+2), 1)$ . In this case, an Ito's type change of variables formula holds: for any primitive  $F$  of  $f$ , we can write

$$F(B_t) - F(0) = \int_0^t f(B_s) d^{\text{NC},N} B_s, \quad t \in [0, 1]. \quad (3.2)$$

Moreover, as a consequence of (3.2), let us note that

$$\int_0^t f(B_s) d^{\text{NC},N} B_s = \int_0^t f(B_s) d^{\text{NC},n} B_s = F(B_t) - F(0),$$

as soon as  $f \in C^{4N+1}$ ,  $n < N$  and  $H \in (1/(4n+2), 1)$ . Therefore, for  $f$  regular enough, it is possible to define the Newton-Côtes integral without ambiguity by:

$$\int_0^t f(B_s) d^{\text{NC}} B_s := \int_0^t f(B_s) d^{\text{NC},n} B_s \quad \text{if } H \in (1/(4n+2), 1). \quad (3.3)$$

Set  $n_H := \inf\{n \geq 1 : H > 1/(4n+2)\}$ . Hence, an immediate consequence of (3.2) and (3.3) is that, for any  $H \in (0, 1)$  and any  $f : \mathbb{R} \rightarrow \mathbb{R}$  of class  $C^{4n_H+1}$ , the following Ito's type change of variables formula holds:

$$F(B_t) = F(0) + \int_0^t f(B_s) d^{\text{NC}} B_s, \quad \text{for any primitive } F \text{ of } f. \quad (3.4)$$

**Remark 3.3**

In the sequel we will only use the fact that the Newton-Côtes integral verifies the classical change of variable formula (3.4). Consequently, any other stochastic integral with respect to the fractional Brownian motion, verifying (3.4), could be used in the following.  $\square$

All along this paper we will keep an ellipticity assumption and also regularity for the function  $\sigma$ . More precisely, we suppose that

$$(\mathcal{E}) \quad \inf_{\mathbb{R}} |\sigma| > 0 \quad \text{and } \sigma \in C^\infty(\mathbb{R}) \text{ is bounded together with its derivatives.}$$

Under hypothesis  $(\mathcal{E})$ , the flow  $\phi$  associated to  $\sigma$ , given by (1.8), is well-defined and verifies the semigroup property:

$$\forall x, y, z \in \mathbb{R}, \quad \phi(\phi(x, y), z) = \phi(x, y + z). \quad (3.5)$$

Note that the process  $X^x$  given by (1.9) verifies:

$$X_t^x = x + \int_0^t \sigma(X_s^x) d^{\text{NC}} B_s, \quad t \in [0, 1], \quad (3.6)$$

as we can see immediately, by applying (3.4).

**Remark 3.4**

In [24] (see also [26]), one studies a notion of solution for (3.6) and also the existence and the uniqueness of solution. Note however that, in the present work, we will only use the fact that there exists a natural solution to (3.6) given by (1.9).  $\square$

The following result explains the definition (1.18). By using (1.17), the process  $X^x$  defined by (1.9) can be expanded as follows:

**Lemma 3.5** *For any integers  $m \geq 0$ ,  $n \geq 1$  and  $\ell \in \{0, \dots, n-1\}$ , we have*

$$\begin{aligned} X_{(\ell+1)/n}^x &= X_{\ell/n}^x + \sum_{j=0}^m \frac{1}{(j+1)!} \mathcal{D}^j \sigma(X_{\ell/n}^x) \Delta^{j+1} B_{\ell/n} \\ &\quad + \int_{\ell/n}^{(\ell+1)/n} d^{\text{NC}} B_{t_1} \int_{\ell/n}^{t_1} d^{\text{NC}} B_{t_2} \dots \int_{\ell/n}^{t_m} d^{\text{NC}} B_{t_{m+1}} \int_{\ell/n}^{t_{m+1}} \mathcal{D}^{m+1} \sigma(X_{t_{m+2}}^x) d^{\text{NC}} B_{t_{m+2}}. \end{aligned} \quad (3.7)$$

**Proof.** We proceed by induction on  $m$ . By applying (3.6), and then using (1.9) and (3.4), we can write:

$$\begin{aligned} X_{(\ell+1)/n}^x &= X_{\ell/n}^x + \sigma(X_{\ell/n}^x) \Delta B_{\ell/n} + \int_{\ell/n}^{(\ell+1)/n} (\sigma(X_{t_1}^x) - \sigma(X_{\ell/n}^x)) d^{\text{NC}} B_{t_1} \\ &= X_{\ell/n}^x + \sigma(X_{\ell/n}^x) \Delta B_{\ell/n} + \int_{\ell/n}^{(\ell+1)/n} d^{\text{NC}} B_{t_1} \int_{\ell/n}^{t_1} \sigma \sigma'(X_{t_2}^x) d^{\text{NC}} B_{t_2}, \end{aligned}$$

which is exactly (3.7) for  $m = 0$ .

Let us assume that (3.7) is true for  $m \in \mathbb{N}$ . Then we can write

$$\begin{aligned} X_{(\ell+1)/n}^x &= X_{\ell/n}^x + \sum_{j=0}^m \frac{1}{(j+1)!} \mathcal{D}^j \sigma(X_{\ell/n}^x) \Delta^{j+1} B_{\ell/n} \\ &\quad + \mathcal{D}^{m+1} \sigma(X_{\ell/n}^x) \int_{\ell/n}^{(\ell+1)/n} d^{\text{NC}} B_{t_1} \dots \int_{\ell/n}^{t_m} d^{\text{NC}} B_{t_{m+1}} \int_{\ell/n}^{t_{m+1}} d^{\text{NC}} B_{t_{m+2}} \\ &\quad + \int_{\ell/n}^{(\ell+1)/n} d^{\text{NC}} B_{t_1} \dots \int_{\ell/n}^{t_m} d^{\text{NC}} B_{t_{m+1}} \int_{\ell/n}^{t_{m+1}} (\mathcal{D}^{m+1} \sigma(X_{t_{m+2}}^x) - \mathcal{D}^{m+1} \sigma(X_{\ell/n}^x)) d^{\text{NC}} B_{t_{m+2}}. \end{aligned} \quad (3.8)$$

On one hand, using (3.4) repeatedly, it is easy to compute that

$$\int_{\ell/n}^{(\ell+1)/n} d^{\text{NC}} B_{t_1} \dots \int_{\ell/n}^{t_m} d^{\text{NC}} B_{t_{m+1}} \int_{\ell/n}^{t_{m+1}} d^{\text{NC}} B_{t_{m+2}} = \frac{1}{(m+2)!} \Delta^{m+2} B_{\ell/n}.$$

On the other hand, using (1.9) and again (3.4), we can write

$$\mathcal{D}^{m+1} \sigma(X_{t_{m+2}}^x) - \mathcal{D}^{m+1} \sigma(X_{\ell/n}^x) = \int_{\ell/n}^{t_{m+2}} \sigma(\mathcal{D}^{m+1} \sigma)'(X_{t_{m+3}}^x) d^{\text{NC}} B_{t_{m+3}}.$$

Finally, putting these latter two equalities in (3.8) and noting that  $\sigma(\mathcal{D}^{m+1} \sigma)' = \mathcal{D}^{m+2} \sigma$  by definition, we obtain that (3.7) is true for  $m+1$ .  $\square$

**Remark 3.6**

Clearly (1.18) is the natural scheme constructed from (3.7), by considering the  $(m+2)$ -order iterated integral in the right hand side of (3.7) as a remainder.  $\square$

### 3.2 Rate of convergence

Recall that we denote by  $\mu_{2n}$  ( $n \geq 1$ ) the  $2n$ -moment of a random variable  $G \sim \mathcal{N}(0, 1)$ . For  $m \in \mathbb{N}$ , let us introduce the functions  $f_m, g_m, h_m : \mathbb{R} \rightarrow \mathbb{R}$  given by:

$$f_m = \begin{cases} \mu_{m+2} h_m & \text{if } m \text{ is even,} \\ \mu_{m+3} (g_m - \sigma h'_m/2) & \text{if } m \text{ is odd,} \end{cases} \quad (3.9)$$

$$g_m = -\sigma' h_m + h_{m+1} \quad \text{and} \quad h_m = -\frac{\sigma^{m+1} \sigma / \sigma}{(m+2)!}. \quad (3.10)$$

For instance, the first  $f_m$ 's are

$$f_0 = -\frac{\sigma'}{2}, \quad f_1 = \frac{1}{8}(3\sigma'^3 + 6\sigma\sigma'\sigma'' + \sigma^2\sigma'''), \quad \text{etc.}$$

Our second main result of this paper contains the answer to the questions in the introduction and can be stated as follows:

**Theorem 3.7** *Assume that hypothesis  $(\mathcal{E})$  holds and let  $m \in \mathbb{N}$ . Then, for any  $H \in (1/(m+2), 1)$ , the sequence  $\{\widehat{X}_1^{(n)}\}$  defined by (1.18) converges, as  $n \rightarrow \infty$ , almost surely toward  $X_1^x = \phi(x, B_1)$ . Moreover, as  $n \rightarrow \infty$ :*

- when  $m$  is even and  $H \in (1/(m+2), 1)$ ,

$$n^{(m+2)H-1} \left[ \widehat{X}_1^{(n)} - X_1^x \right] \xrightarrow{\text{Prob}} \sigma(X_1^x) \int_0^1 f_m(X_s^x) ds; \quad (3.11)$$

- when  $m$  is odd and  $H \in (1/(m+2), 1/2)$ ,

$$n^{(m+3)H-1} \left[ \widehat{X}_1^{(n)} - X_1^x \right] \xrightarrow{\text{Prob}} \sigma(X_1^x) \int_0^1 f_m(X_s^x) ds; \quad (3.12)$$

- when  $m$  is odd and  $H = 1/2$ ,

$$n^{(m+1)/2} \left[ \widehat{X}_1^{(n)} - X_1^x \right] \xrightarrow{\text{Law}} \sigma(X_1^x) \left( \int_0^1 h_m(X_s^x) [\sqrt{\mu_{2m+4}} dW_s + \mu_{m+3} dB_s] + \mu_{m+3} \int_0^1 g_m(X_s^x) ds \right), \quad (3.13)$$

with  $W$  a Brownian motion independent of  $B$ .

#### Remark 3.8

1. When  $m = 0$  and  $H > 1/2$ , one recovers the convergence (1.11). One should also compare it with (1.15) when  $m = 0$  and  $H = 1/2$ .
2. With the same method used to obtain (3.13), one could prove (1.16) with the help of Lemma 3.4 in [24]. Details are left to the reader.
3. In fact we shall prove that, in (3.11), the convergence is almost sure. Also the convergence in (3.12) is certainly almost sure, but the method of proof we have used here does not allow to obtain it. Thus it remains an open question.

4. According to Theorem 3.7, the scheme  $\widehat{X}$  of size  $m = 2\kappa - 1$  has the same rate of convergence than the scheme  $\widehat{X}$  of size  $m = 2\kappa$ , namely  $n^{(2\kappa+2)H-1}$ . Thus, it is a priori better to use odd-size schemes.

5. With the same method (see also Theorems 2 and 4 in [21]), one could also derive the exact rate of convergence for the global error on the whole interval  $[0, 1]$ . More precisely, as analogue of (3.11), as  $n \rightarrow \infty$ , one would have

$$n^{(m+2)H-1} \max_{\ell=0, \dots, n} \left| \widehat{X}_{\ell/n}^{(n)} - X_{\ell/n}^x \right| \xrightarrow{\text{Prob}} \sup_{t \in [0, 1]} \left| \sigma(X_t^x) \int_0^t f_m(X_s^x) ds \right|,$$

while as analogue of (3.12), as  $n \rightarrow \infty$ , one would have

$$n^{(m+3)H-1} \max_{\ell=0, \dots, n} \left| \widehat{X}_{\ell/n}^{(n)} - X_{\ell/n}^x \right| \xrightarrow{\text{Prob}} \sup_{t \in [0, 1]} \left| \sigma(X_t^x) \int_0^t f_m(X_s^x) ds \right|.$$

□

Notice that, under  $(\mathcal{E})$ , the convergences (3.11) and (3.12) give the right lower error bound when

$$P \left( \int_0^1 f_m(X_s^x) ds \neq 0 \right) > 0.$$

Due to (1.9) and the fact that  $B_t$  has a Gaussian density for any  $t \in [0, 1]$ , it is easy to see that it is equivalent to have  $f_m \neq 0$ . Indeed, if  $\int_0^1 f(B_s) ds = 0$  almost surely for a certain  $f \in C_b^1(\mathbb{R}, \mathbb{R})$ , then  $0 = D_u \int_0^1 f(B_s) ds = \int_u^1 f'(B_s) ds$ , for any  $u \in [0, 1]$  (here  $D$  denotes the Malliavin derivative with respect to  $B$ ). We deduce that  $f'(B_u) = 0$ , for any  $u \in [0, 1]$ , and, since the support of the law of  $B_1$  (for instance) is  $\mathbb{R}$ , we obtain  $f' \equiv 0$ . The conclusion follows easily.

Solving  $f_m \equiv 0$  seems, in general, complicated. Nevertheless, when  $m \in \{0, 1\}$ , we can state:

### Proposition 3.9

Assume  $(\mathcal{E})$  and let  $f_m$  ( $m \in \mathbb{N}$ ) be given by (3.9).

1. The function  $f_0 = -\sigma'/2$  is identically zero if and only if  $\sigma$  is a constant function.
2. Assume that  $\sigma$  verifies the following additional hypothesis :

$$\exists j_0 = -\infty < j_1 < \dots < j_{r-1} < j_r = \infty \quad \text{s.t.} \quad \sigma'_{(j_i, j_{i+1})} \equiv 0 \quad \text{or} \quad \sigma'_{(j_i, j_{i+1})} \neq 0.$$

Then the function  $f_1 = (3\sigma'^3 + 6\sigma\sigma'\sigma'' + \sigma^2\sigma''')/8$  is identically zero if and only if  $\sigma$  is a constant function.

### Remark 3.10

1. When  $\sigma(x) = \sigma$  is constant, we have  $\widehat{X}_1^{(n)} = X_1^x = x + \sigma B_1$ . Consequently, the study of the rate of convergence in the case where  $\sigma$  is a constant function is not interesting.
2. A corollary of Theorem 3.7 and Proposition 3.9 is that upper and lower error bounds come from a convergence in probability, of course except when  $\sigma$  is constant. In particular, we never observe a phenomenon of the type (1.14). □

**Proof.** The first point is obvious so we focus on the second one. It is clear that  $\sigma(x) = \sigma$  implies  $f_1 = 0$ . Conversely, assume that  $f_1 = 0$ . Assume for a moment that  $\sigma'$  does not vanish on  $(j_{r-1}, \infty)$ . Then, on  $(j_{r-1}, \infty)$  we have either  $\sigma' > 0$  or  $\sigma' < 0$ . Assume, for instance, that  $\sigma' > 0$ , the proof for the other situation being similar. Since  $f_1 = 0$ , we have  $3\sigma'(\sigma^2)'' = -\sigma(\sigma^2)'''$ . We deduce that the derivative of  $\sigma^3(\sigma^2)''$  is zero on  $I := [1 + j_{r-1}, \infty)$  and then  $(\sigma^2)'' = \alpha \sigma^{-3}$  on  $I$ , for some  $\alpha \neq 0$ . Set  $h = \sigma^2$ ; we have  $h'' h' = \alpha h' h^{-3/2}$  on  $I$  or, equivalently,  $h'^2 = -4\alpha h^{-1/2} + \beta$  on  $I$ , for some  $\beta \in \mathbb{R}$ . In particular, we have  $\beta - 4\alpha y^{-1/2} > 0$ , for any  $y \in h(I)$ . Let  $F$  be defined on  $h(I)$  by

$$F(y) = \int_{h(1+j_{r-1})}^y \frac{dz}{\sqrt{\beta - 4\alpha z^{-1/2}}}.$$

For all  $x \in I$ , we have

$$F(\sigma(x)^2) = F(h(x)) = x + \gamma, \text{ for some } \gamma \in \mathbb{R}. \quad (3.14)$$

The function  $\sigma^2$  being bounded, we necessarily have  $h(x) \rightarrow (4\alpha/\beta)^2$ , as  $x \rightarrow \infty$ . Then  $h''(x) \rightarrow \beta^3/(4^3\alpha^2)$ , as  $x \rightarrow \infty$ , which is a contradiction with the fact that  $h = \sigma^2$  is bounded. Consequently,  $\sigma' = 0$  on  $(j_{r-1}, \infty)$ . With the same reasoning, we have either  $\sigma' = 0$  on  $(j_{r-2}, j_{r-1})$ , or  $\sigma$  verifies an analogue of (3.14) on  $(j_{r-2}, j_{r-1})$ . But, due to the fact that  $\sigma$  is assumed  $C^\infty$  and that we already obtained that  $\sigma' = 0$  on  $(j_{r-1}, \infty)$ , we necessarily have  $\sigma' = 0$  on  $(j_{r-2}, j_{r-1})$ . Finally, by an induction argument, we have that  $\sigma' = 0$  on  $\mathbb{R}$ , that is  $\sigma$  is constant.  $\square$

## 4 Proofs of main results

### Proof of Theorem 2.3.

**Proof of (2.7).** When  $h \equiv 1$ , it is a classical result (see [14] when  $\kappa = 2$ ). More precisely, if one fixes  $t \in [0, 1]$ , then, as  $n \rightarrow \infty$ ,

$$n^{\kappa H-1} \sum_{\ell=0}^{[nt]-1} \Delta^\kappa B_{\ell/n} = n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbf{1}_{[0,t]}(\ell/n) \Delta^\kappa B_{\ell/n} \longrightarrow \mu_\kappa t, \quad \text{a.s.} \quad (4.1)$$

By (4.1) we deduce that, almost surely, (4.1) holds for any  $t \in [0, 1] \cap \mathbb{Q}$ . In other words, there exists  $\Omega^*$  with  $P(\Omega^*) = 1$  verifying that, for every  $\omega \in \Omega^*$  and  $t \in [0, 1] \cap \mathbb{Q}$ , (4.1) holds.

Now, fix  $\omega \in \Omega^*$  and  $t \in [0, 1]$ . Let  $\{t_m\}, \{t'_m\} \subset \mathbb{Q} \cap [0, 1]$  such that  $t_m \uparrow t$  and  $t'_m \downarrow t$ , as  $m \rightarrow \infty$ . Then

$$n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbf{1}_{[0,t_m]}(\ell/n) \Delta^\kappa B_{\ell/n} \leq n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbf{1}_{[0,t]}(\ell/n) \Delta^\kappa B_{\ell/n} \leq n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbf{1}_{[0,t'_m]}(\ell/n) \Delta^\kappa B_{\ell/n}.$$

Letting  $n \rightarrow \infty$ , we obtain

$$\mu_\kappa t_m \leq \liminf_{n \rightarrow \infty} n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbf{1}_{[0,t]}(\ell/n) \Delta^\kappa B_{\ell/n} \leq \limsup_{n \rightarrow \infty} n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbf{1}_{[0,t]}(\ell/n) \Delta^\kappa B_{\ell/n} \leq \mu_\kappa t'_m.$$

Letting  $m \rightarrow \infty$ , we obtain that, for every  $\omega \in \Omega^*$  and  $t \in [0, 1]$ , (4.1) holds. Thanks to second Dini's theorem<sup>1</sup>, we obtain that, for every  $\omega \in \Omega^*$ , as  $n \rightarrow \infty$ :

$$\sup_{t \in [0, 1]} \left| n^{\kappa H - 1} \sum_{\ell=0}^{n-1} \mathbb{1}_{[0, t]}(\ell/n) \Delta^\kappa B_{\ell/n} - \mu_\kappa t \right| \longrightarrow 0.$$

The following (deterministic) lemma allows us to finish the proof of (2.7).

**Lemma 4.1** *Let  $b, g : [0, 1] \rightarrow \mathbb{R}$  be two continuous functions and  $\kappa$  be an even integer. If*

$$\sup_{t \in [0, 1]} \left| n^{\kappa H - 1} \sum_{\ell=0}^{n-1} \mathbb{1}_{[0, t]}(\ell/n) \Delta^\kappa b_{\ell/n} - \mu_\kappa t \right| \longrightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (4.2)$$

then

$$\sup_{t \in [0, 1]} \left| n^{\kappa H - 1} \sum_{\ell=0}^{n-1} g_{\ell/n} \Delta^\kappa b_{\ell/n} - \mu_\kappa \int_0^t g_s ds \right| \longrightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.3)$$

**Proof.** If  $g = \mathbb{1}_{[0, t]}$  then (4.2) and (4.3) are the same. By linearity, (4.3) holds for piecewise functions. An argument of density allows us to finish easily the proof.  $\square$

**Proof of (2.8).** We begin with the statement and the proof of the following simple result:

**Lemma 4.2**

1. *Let  $s < t$  and  $v$  belong to  $[0, 1]$ . Then*

$$|E(B_v(B_t - B_s))| \leq (t - s)^{2H}.$$

2. *For every fixed  $r \geq 1$  and  $H < 1/2$ , the quantity*

$$n^{-1} \sum_{k, \ell=0}^{n-1} \left| |k - \ell + 1|^{2H} + |k - \ell - 1|^{2H} - 2|k - \ell|^{2H} \right|^r$$

*is bounded.*

**Proof of Lemma 4.2.** 1. We have

$$E(B_v(B_t - B_s)) = \frac{1}{2}(t^{2H} - s^{2H}) + \frac{1}{2}(|s - v|^{2H} - |t - v|^{2H}).$$

For  $u \geq 0$ , we can write

$$(1 + u)^{2H} - u^{2H} = 2H \int_0^1 \frac{dx}{(x + u)^{1-2H}} \leq 2H \int_0^1 \frac{dx}{x^{1-2H}} = 1.$$

By choosing  $u = a^{\wedge b}/|b-a|$ , we get  $|b^{2H} - a^{2H}| \leq |b - a|^{2H}$  for any  $0 \leq a < b \leq 1$ , because  $H < \frac{1}{2}$ . The desired result follows.

2. We can write

$$n^{-1} \sum_{k, \ell=0}^{n-1} \left| |k - \ell + 1|^{2H} + |k - \ell - 1|^{2H} - 2|k - \ell|^{2H} \right|^r \leq \sum_{i=-\infty}^{\infty} \left| |i + 1|^{2H} + |i - 1|^{2H} - 2|i|^{2H} \right|^r,$$

and this series is convergent since  $H < 1/2$ .  $\square$

---

<sup>1</sup>Second Dini's theorem: Let  $(f_n)$  be a sequence of real functions defined on  $[0, 1]$ . We assume, for every  $x \in [0, 1]$ , that  $f_n(x) \rightarrow f(x)$ , as  $n \rightarrow \infty$ , with  $f : [0, 1] \rightarrow \mathbb{R}$  a continuous function. Moreover, we assume, for any  $n \in \mathbb{N}$ , that  $f_n$  is an increasing function. Then  $\sup_{x \in [0, 1]} |f_n(x) - f(x)| \rightarrow 0$  as  $n \rightarrow \infty$ .

We can start now the proof of (2.8). Fix  $H \in (0, 1/2)$ , an odd integer  $\kappa \geq 3$  and  $h \in C_b^{2\kappa}$ , and set

$$V_n^{(\kappa)}(h) := n^{(\kappa+1)H-1} \sum_{\ell=0}^{n-1} h(B_{\ell/n}) \Delta^\kappa B_{\ell/n}.$$

We shall prove the following control:

$$E \left| V_n^{(\kappa)}(h) - \mu_{\kappa+1} n^{2H-1} \sum_{\ell=0}^{n-1} h(B_{\ell/n}) \Delta B_{\ell/n} \right|^2 = O(n^{(2H-1)\vee(-4H)}). \quad (4.4)$$

Suppose for a moment that (4.4) is proved, and let us finish the proof of (2.8). Using the mean value theorem, we can write:

$$n^{2H-1} \sum_{\ell=0}^{n-1} h(B_{\ell/n}) \Delta B_{\ell/n} = n^{2H-1} \left( \int_0^{B_1} h(x) dx - \frac{1}{2} \sum_{\ell=0}^{n-1} h'(B_{\theta_{\ell/n}}) \Delta^2 B_{\ell/n} \right),$$

for some  $\theta_{\ell/n}$  lying between  $\ell/n$  and  $(\ell+1)/n$ . Note that  $n^{2H-1} \int_0^{B_1} h(x) dx \xrightarrow{\text{Prob}} 0$  as  $n \rightarrow \infty$ , since  $H < 1/2$ . On the other hand, using a slight modification of (2.7), we have

$$n^{2H-1} \sum_{\ell=0}^n h'(B_{\theta_{\ell/n}}) \Delta^2 B_{\ell/n} \xrightarrow{\text{Prob}} \int_0^1 h'(B_s) ds, \quad \text{as } n \rightarrow \infty.$$

Thus, (2.8) is a consequence of (4.4).

Let us proceed with the proof of (4.4). Since  $\kappa$  is odd, the monomial  $x^\kappa$  may be expanded in terms of the Hermite polynomials as follows:

$$x^\kappa = \mu_{\kappa+1} x + \sum_{p=1}^{\frac{\kappa-1}{2}} a_{2p+1, \kappa} H_{2p+1}(x), \quad \text{for some universal constants } a_{2p+1, \kappa}. \quad (4.5)$$

For instance, we have  $x^3 = 3x + H_3(x)$ . Thus,

$$\begin{aligned} V_n^{(\kappa)}(h) - \mu_{\kappa+1} n^{2H-1} \sum_{\ell=0}^{n-1} h(B_{\ell/n}) \Delta B_{\ell/n} \\ = \sum_{p=1}^{\frac{\kappa-1}{2}} \frac{a_{2p+1, \kappa}}{(2p+1)!} n^{2pH+2H-1} \sum_{\ell=0}^{n-1} h(B_{\ell/n}) I_{2p+1}(\delta_{\ell/n}^{\otimes 2p+1}), \end{aligned}$$

where, for simplicity, we write  $\delta_{\ell/n}$  instead of  $\mathbf{1}_{[\ell/n, (\ell+1)/n]}$ . Recall that  $I_m$  denotes the  $m$ th multiple stochastic integral, and that we have  $I_m(h^{\otimes m}) = m! H_m(B(h))$ . Here  $h \mapsto B(h)$  denotes the natural isometry between the Hilbert space  $\mathfrak{H}$ , the closure of the set of step  $\mathbb{R}$ -valued functions on  $[0, 1]$  with respect to the scalar product  $\langle \mathbf{1}_{[0, t]}, \mathbf{1}_{[0, s]} \rangle_{\mathfrak{H}} = E(B_t B_s)$ , and the first Wiener chaos of  $B$ . Moreover the mapping  $I_m(\cdot)$  provides a linear isometry between the symmetric tensor product  $\mathfrak{H}^{\odot m}$  and  $m$ th Wiener chaos of  $B$ .

To obtain (4.4) it is then sufficient to prove that, for every fixed  $p \in \{1, \dots, \frac{\kappa-1}{2}\}$ , the quantities

$$R_n^{(p)}(h) := n^{2pH+2H-1} \sum_{\ell=0}^{n-1} h(B_{\ell/n}) I_{2p+1}(\delta_{\ell/n}^{\otimes 2p+1})$$

verify  $E \left| R_n^{(p)}(h) \right|^2 = O(n^{(2H-1)\vee(-4H)})$ . Thus, let us compute

$$E \left| R_n^{(p)}(h) \right|^2 = n^{4pH+4H-2} \sum_{k,\ell=0}^{n-1} E \left\{ h(B_{k/n})h(B_{\ell/n})I_{2p+1}(\delta_{k/n}^{\otimes 2p+1})I_{2p+1}(\delta_{\ell/n}^{\otimes 2p+1}) \right\}.$$

Since the product formula for multiple stochastic integrals (see, *e.g.*, Proposition 1.1.3, p. 12 in [28]) yields

$$\begin{aligned} I_{2p+1}(\delta_{k/n}^{\otimes 2p+1})I_{2p+1}(\delta_{\ell/n}^{\otimes 2p+1}) \\ = \sum_{r=0}^{2p+1} r! \binom{2p+1}{r}^2 I_{4p+2-2r}(\delta_{k/n}^{\otimes 2p+1-r} \tilde{\otimes} \delta_{\ell/n}^{\otimes 2p+1-r}) \langle \delta_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}^r, \end{aligned}$$

it suffice to prove that, for every fixed  $r \in \{0, \dots, 2p+1\}$ , the quantities

$$S_n^{(r,p)}(h) := n^{4pH+4H-2} \sum_{k,\ell=0}^{n-1} E \left\{ h(B_{k/n})h(B_{\ell/n})I_{4p+2-2r}(\delta_{k/n}^{\otimes 2p+1-r} \tilde{\otimes} \delta_{\ell/n}^{\otimes 2p+1-r}) \right\} \langle \delta_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}^r$$

verify  $E \left| S_n^{(r,p)}(h) \right|^2 = O(n^{(2H-1)\vee(-4H)})$ .

In the sequell we shall denote by  $D$  the Malliavin derivative with respect to  $B$  and by  $I = I_1$  the divergence operator, the adjoint of the operator  $D$ , defined by the duality relationship  $E(FI(u)) = E(\langle DF, u \rangle_{\mathfrak{H}})$ , for every smooth random variable  $F$  (we refer to [27, 28] for basic facts and details about stochastic calculus with respect to the fractional Brownian motion).

By writing, for simplicity,  $\varepsilon_{k/n}$  instead of  $\mathbf{1}_{[0,k/n]}$ , we have

$$\begin{aligned} & \left| E \left\{ h(B_{k/n})h(B_{\ell/n})I_{4p+2-2r}(\delta_{k/n}^{\otimes 2p+1-r} \tilde{\otimes} \delta_{\ell/n}^{\otimes 2p+1-r}) \right\} \right| \\ &= \left| E \left\langle D^{4p+2-2r}(h(B_{k/n})h(B_{\ell/n})), \delta_{k/n}^{\otimes 2p+1-r} \tilde{\otimes} \delta_{\ell/n}^{\otimes 2p+1-r} \right\rangle_{\mathfrak{H}^{\otimes 4p+2-2r}} \right| \\ &\leq \sum_{a+b=4p+2-2r} (4p+2-2r)! \left| E \left\{ h^{(a)}(B_{k/n})h^{(b)}(B_{\ell/n}) \right\} \right| \left| \left\langle \varepsilon_{k/n}^{\otimes a} \tilde{\otimes} \varepsilon_{\ell/n}^{\otimes b}, \delta_{k/n}^{\otimes 2p+1-r} \tilde{\otimes} \delta_{\ell/n}^{\otimes 2p+1-r} \right\rangle_{\mathfrak{H}^{\otimes 4p+2-2r}} \right| \\ &= O(n^{-4(2p+1-r)H}), \end{aligned}$$

by using the first point of the Lemma 4.2. When  $r = 0$ , we deduce that  $E \left| S_n^{(r,p)}(h) \right|^2 = O(n^{-4H})$  because  $p \geq 1$ . Moreover, by the second point of the same lemma, we have, when  $r \geq 1$ :

$$\sum_{k,\ell=0}^{n-1} |\langle \delta_{k/n}, \delta_{\ell/n} \rangle_{\mathfrak{H}}^r| = n^{-2rH} \sum_{k,\ell=0}^{n-1} \left| |k-\ell+1|^{2H} + |k-\ell-1|^{2H} - 2|k-\ell|^{2H} \right| = O(n^{1-2rH}).$$

Thus, when  $1 \leq r \leq 2p+1$ , we have that  $E \left| S_n^{(r,p)}(h) \right|^2 = O(n^{-4pH-1+2rH}) = O(n^{2H-1})$ . The equality (4.4) follows.

The proof of (2.8) is done. □

**Proof of Theorem 3.7.** We assume that  $H \in (1/(m+2), 1)$  and we denote in the following

$$\Delta^p(B) = \max_{k=0, \dots, n-1} |\Delta^p B_{k/n}|,$$

for any integer  $p \geq 1$ . We split the proof in several steps.

1. *General computations.* The following lemma can be showed by using the same method as in the proof of Lemma 3.5, but with Lebesgue integral instead of Newton-Côtes integral:

**Lemma 4.3** *For any  $x, y \in \mathbb{R}$ , we have, as  $y \rightarrow 0$ ,*

$$\phi(x, y) = x + \sum_{j=0}^{m+2} \frac{1}{(j+1)!} \mathcal{D}^j \sigma(x) y^{j+1} + O(y^{m+4}).$$

By applying this lemma with  $x = \widehat{X}_{k/n}^{(n)}$  and  $y = \Delta B_{k/n}$ , we obtain

$$\begin{aligned} \widehat{X}_{(k+1)/n}^{(n)} &= \phi(\widehat{X}_{k/n}^{(n)}, \Delta B_{k/n}) - \mathcal{D}^{m+1} \sigma(\widehat{X}_{k/n}^{(n)}) \frac{\Delta^{m+2} B_{k/n}}{(m+2)!} \\ &\quad - \mathcal{D}^{m+2} \sigma(\widehat{X}_{k/n}^{(n)}) \frac{\Delta^{m+3} B_{k/n}}{(m+3)!} + O(\Delta^{m+4}(B)). \end{aligned}$$

By straightforward computations, we get<sup>2</sup>

$$\widehat{X}_{(k+1)/n}^{(n)} = \phi \left( \widehat{X}_{k/n}^{(n)}, \Delta B_{k/n} + h_m(\widehat{X}_{k/n}^{(n)}) \Delta^{m+2} B_{k/n} + g_m(\widehat{X}_{k/n}^{(n)}) \Delta^{m+3} B_{k/n} + O(\Delta^{m+4}(B)) \right), \quad (4.6)$$

with  $g_m$  and  $h_m$  given by (3.10). By the semi-group property (3.5), we finally obtain that, for any  $\ell \in \{1, \dots, n\}$ :

$$\widehat{X}_{\ell/n}^{(n)} = \phi \left( x, B_{\ell/n} + \sum_{k=0}^{\ell-1} h_m(\widehat{X}_{k/n}^{(n)}) \Delta^{m+2} B_{k/n} + \sum_{k=0}^{\ell-1} g_m(\widehat{X}_{k/n}^{(n)}) \Delta^{m+3} B_{k/n} + O(n \Delta^{m+4}(B)) \right). \quad (4.7)$$

In particular, we deduce, as  $n \rightarrow \infty$ ,

$$\sup_{\ell \in \{1, \dots, n\}} \left| \widehat{X}_{\ell/n}^{(n)} - X_{\ell/n}^x \right| = \sup_{\ell \in \{1, \dots, n\}} \left| \widehat{X}_{\ell/n}^{(n)} - \phi(x, B_{\ell/n}) \right| = O(n \Delta^{m+2}(B)), \quad (4.8)$$

and  $\widehat{X}_1^{(n)}$  converges almost surely to  $X_1^x$ , as  $n \rightarrow \infty$ , since  $H > 1/(2m+2)$ .

<sup>2</sup>In fact, we rather obtain

$$\widehat{X}_{(k+1)/n}^{(n)} = \phi \left( \widehat{X}_{k/n}^{(n)}, \Delta B_{k/n} + h_m(\widehat{X}_{k/n}^{(n)}) \Delta^{m+2} B_{k/n} + g_m(\widehat{X}_{k/n}^{(n)}) \Delta^{m+3} B_{k/n} \right) + O(\Delta^{m+4}(B)),$$

which is not exactly (4.6). But, in order to introduce  $O(\Delta^{m+4}(B))$  in the argument of  $\phi$ , we proceed as follows, by using hypothesis ( $\mathcal{E}$ ):

$$\phi(x, z) + O(\delta) = \phi(x, \phi^{-1}(x, \phi(x, z) + O(\delta))) = \phi(x, z + O(\delta)).$$

2. *Proof of (3.11).* Assume that  $m$  is an even integer. As a consequence of (4.7), (4.8) and the fact that  $H > 1/(m+2)$ , we can write

$$\widehat{X}_1^{(n)} = \phi \left( x, B_1 + \sum_{k=0}^{n-1} h_m(X_{k/n}^x) \Delta^{m+2} B_{k/n} + o(n\Delta^{m+2}(B)) \right).$$

(Observe that the main difference between (4.7) and the previous identity is that we have  $X_{k/n}^x$  as argument of  $h_m$ ). By using (2.7) with  $\kappa = m + 2$  and due to the fact that  $X_t^x = \phi(x, B_t)$  and  $(\partial\phi/\partial y)(x, y) = \sigma(\phi(x, y))$ , we finally obtain (3.11).

3. *Proof of (3.12) for  $H \in (2/(2m+3), 1/2)$ .* Assume that  $m$  is an odd integer and that  $H \in (2/(2m+3), 1/2)$ . Thanks to (4.8), (4.7) can be transformed in

$$\widehat{X}_{\ell/n}^{(n)} = \phi \left( x, B_{\ell/n} + \sum_{k=0}^{\ell-1} h_m(\widehat{X}_{k/n}^{(n)}) \Delta^{m+2} B_{k/n} + \sum_{k=0}^{\ell-1} g_m(X_{k/n}^x) \Delta^{m+3} B_{k/n} + o(n\Delta^{m+3}(B)) \right). \quad (4.9)$$

Due to the assumptions on  $\sigma$ , we have

$$\phi(x, y_2) = \phi(x, y_1) + \sum_{j=1}^M \frac{1}{j!} \frac{\partial^j \phi}{\partial y^j}(x, y_1) (y_2 - y_1)^j + O((y_2 - y_1)^{M+1}).$$

Thus we get

$$\widehat{X}_{k/n}^{(n)} = X_{k/n}^x + \sum_{j=1}^M \frac{1}{j!} \frac{\partial^j \phi}{\partial y^j}(x, B_{k/n}) \left( \sum_{k_1=0}^{k-1} h_m(\widehat{X}_{k_1/n}^{(n)}) \Delta^{m+2} B_{k_1/n} + O(n\Delta^{m+3}(B)) \right)^j + O(n^{M+1} \Delta^{(m+2)(M+1)}(B)). \quad (4.10)$$

By using (4.10) with  $M = 1$  and the equality  $(\partial\phi/\partial y)(x, y) = \sigma(\phi(x, y))$  once more, we get

$$\widehat{X}_{k/n}^{(n)} = X_{k/n}^x + \sigma(X_{k/n}^x) \sum_{k_1=0}^{k-1} h_m(\widehat{X}_{k_1/n}^{(n)}) \Delta^{m+2} B_{k_1/n} + O(n^2 \Delta^{2m+4}(B)) + O(n\Delta^{m+3}(B))$$

and then

$$\widehat{X}_{k/n}^{(n)} = X_{k/n}^x + \sigma(X_{k/n}^x) \sum_{k_1=0}^{k-1} h_m(X_{k_1/n}^x) \Delta^{m+2} B_{k_1/n} + O(n^2 \Delta^{2m+4}(B)) + O(n\Delta^{m+3}(B)).$$

By inserting the previous equality in (4.9) with  $\ell = n$ , we obtain

$$\begin{aligned} \widehat{X}_1^{(n)} &= \phi(x, B_1 + \sum_{k=0}^{n-1} h_m(X_{k/n}^x) \Delta^{m+2} B_{k/n} + \sum_{k=0}^{n-1} g_m(X_{k/n}^x) \Delta^{m+3} B_{k/n} \\ &\quad + \sum_{k=0}^{n-1} \sigma h'_m(X_{k/n}^x) \Delta^{m+2} B_{k/n} \sum_{j=0}^{k-1} h_m(X_{j/n}^x) \Delta^{m+2} B_{j/n} + o(n^3 \Delta^{3m+6}(B)) + o(n\Delta^{m+3}(B))). \end{aligned} \quad (4.11)$$

But we have due to (2.8) with  $\kappa = m + 2$ , as  $n \rightarrow \infty$ ,

$$n^{(m+3)H-1} \sum_{k=0}^{n-1} h_m(X_{k/n}^x) \Delta^{m+2} B_{k/n} \xrightarrow{\text{Prob}} -\frac{\mu_{m+3}}{2} \int_0^1 \sigma h'_m(X_s^x) ds$$

and also, due this time to (2.7) with  $\kappa = m + 3$ , as  $n \rightarrow \infty$ ,

$$n^{(m+3)H-1} \sum_{k=0}^{n-1} g_m(X_{k/n}^x) \Delta^{m+3} B_{k/n} \xrightarrow{\text{Prob}} \mu_{m+3} \int_0^1 g_m(X_s^x) ds.$$

Moreover, since  $H > 2/(2m+3)$  we have, as  $n \rightarrow \infty$ ,

$$n^{(m+3)H+2} \Delta^{3m+6}(B) \xrightarrow{\text{Prob}} 0.$$

On the other hand, (4.4) implies that

$$E \left| \sum_{j=0}^{k-1} h_m(X_{j/n}^x) \Delta^{m+2} B_{j/n} - \mu_{m+3} n^{-(m+1)H} \sum_{j=0}^{k-1} h_m(X_{j/n}^x) \Delta B_{j/n} \right|^2 = O(n^{(1-2mH-4H) \vee (2-2mH-10H)}).$$

Thus

$$E \left| n^{(m+3)H-1} \sum_{k=0}^{n-1} \sigma h'_m(X_{k/n}^x) \Delta^{m+2} B_{k/n} \sum_{j=0}^{k-1} h_m(X_{j/n}^x) \Delta^{m+2} B_{j/n} - \mu_{m+3} n^{2H-1} \sum_{k=0}^{n-1} \sigma h'_m(X_{k/n}^x) \Delta^{m+2} B_{k/n} \sum_{j=0}^{k-1} h_m(X_{j/n}^x) \Delta B_{j/n} \right|^2 = O(n^{(1-2H-2mH) \vee (2-8H-2mH)}).$$

Because  $H > 2/(2m+3)$ , we have  $H > 1/(2m+2)$  and  $H > 1/(m+4)$ . Consequently, this last term tends to zero as  $n \rightarrow \infty$ . But, by the mean value theorem:

$$\sum_{j=0}^{k-1} h_m(X_{j/n}^x) \Delta B_{j/n} = \int_0^{B_{k/n}} h_m(\phi(x, z)) dz - \frac{1}{2} \sum_{j=0}^{k-1} \sigma h'_m(X_{\theta_{j/n}}^x) \Delta^2 B_{j/n},$$

for some  $\theta_{j/n}$  between  $j/n$  and  $(j+1)/n$ . Still because  $H > 1/(2m+2)$ , we deduce, in particular by using (2.8) with  $\kappa = m + 2$ , that, as  $n \rightarrow \infty$ ,

$$n^{(m+3)H-1} \sum_{k=0}^{n-1} \sigma h'_m(X_{k/n}^x) \Delta^{m+2} B_{k/n} \sum_{j=0}^k h_m(X_{j/n}^x) \Delta^{m+2} B_{j/n} \xrightarrow{\text{Prob}} 0.$$

Finally, we obtain (3.12) when  $H > 2/(2m+3)$ .

4. *Proof of (3.12) for  $H \in (1/(m+2), 1/2)$ .* It suffices to use (4.10) with the appropriate  $M$  for the considered  $H$  and then to proceed as in the previous step. The remaining details are left to the reader.

5. *Proof of (3.13).* By (4.7)-(4.8), we have

$$\widehat{X}_1^{(n)} = \phi \left( x, B_1 + \sum_{k=0}^{n-1} h_m(X_{k/n}^x) \Delta^{m+2} B_{k/n} + \sum_{k=0}^{n-1} g_m(X_{k/n}^x) \Delta^{m+3} B_{k/n} + O(n^2 \Delta^{2m+4}(B)) \right).$$

When  $m \geq 3$  and since  $H = 1/2$ , we have, as  $n \rightarrow \infty$ :

$$n^{(m+1)/2+2} \Delta^{2m+4}(B) \xrightarrow{\text{Prob}} 0.$$

Hence, for  $m \geq 3$ , (3.13) is easily obtained by (2.2) and these latter two relations.

If  $m = 1$ , we rather use (4.11). Moreover, since  $H = 1/2$ , we have, as  $n \rightarrow \infty$ :

$$n^4 \Delta^9(B) \xrightarrow{\text{Prob}} 0 \quad \text{and} \quad o(n^2 \Delta^4(B)) \xrightarrow{\text{Prob}} 0.$$

Finally, using these convergences, (2.1), (2.2) and the fact that

$$\begin{aligned} & \mathbb{E} \left| n \sum_{k=0}^{n-1} \sigma h'_1(X_{k/n}^x) \Delta^3 B_{k/n} \sum_{j=0}^{k-1} h_1(X_{j/n}^x) \Delta^3 B_{j/n} \right|^2 \\ &= n^2 \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \mathbb{E} \left| \sigma h'_1(X_{k/n}^x) \Delta^3 B_{k/n} h_1(X_{j/n}^x) \Delta^3 B_{j/n} \right|^2 = O(n^{-2}) \longrightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned}$$

we obtain (3.13) also for  $m = 1$ . □

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