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# Convergence of weighted power variations of fractional Brownian motion 

Mihai Gradinaru ${ }^{1}$ and Ivan Nourdin ${ }^{2}$


#### Abstract

The first part of the paper contains the study of the convergence for some weighted power variations of a fractional Brownian motion $B$ with Hurst index $H \in(0,1)$. The behaviour is different when $H<1 / 2$ and powers are odd, compared with the case when $H=1 / 2$ or when $H>1 / 2$ and powers are even. In the second part, one applies the results of the first part to compute the exact rate of convergence of some approximating schemes associated to scalar stochastic differential equations driven by $B$. The limit of the error between the exact solution and the considered scheme (whose size depends on the Hurst index $H$ ) is computed explicitly.


Key words: Fractional Brownian motion - weighted power variations - stochastic differential equation - approximating scheme - exact rate of convergence.

2000 Mathematics Subject Classification: 60F15, 60G15, 60H05, 60H10, 60H35, 60G18, 60H20.

## 1 Introduction

Let $B=\left(B_{t}\right)_{t \in[0,1]}$ be the linear fractional Brownian motion with Hurst index $H \in(0,1)$. In other words, $B$ is a centred Gaussian process with covariance function given by

$$
K(s, t)=\operatorname{Cov}\left(B_{s}, B_{t}\right)=\frac{1}{2}\left(s^{2 H}+t^{2 H}-|t-s|^{2 H}\right), \quad s, t \in[0,1]
$$

For $H=1 / 2, B$ is the standard Brownian motion, while for $H \neq 1 / 2$, it is neither a semimartingale, nor a Markov process. Moreover, it holds, for any $p>1$ :

$$
\mathrm{E}\left|B_{t}-B_{s}\right|^{p}=c_{p}|t-s|^{p H}, \quad s, t \in[0,1], \quad \text { with } \quad c_{p}=\mathrm{E}\left(|G|^{p}\right), \quad G \sim \mathcal{N}(0,1)
$$

and, consequently, almost all sample paths of $B$ are Hölder continuous of any order $\alpha \in(0, H)$.
The study of path behaviour of stochastic processes is often based on the study of theirs power variations and there exists a very extensive literature on the subject. Recall that, given a real $p>1$, the $p$-power variation of a process $X$ with respect to a subdivision $\pi_{n}=\left\{0=t_{n, 0}<t_{n, 1}<\ldots<t_{n, n}=1\right\}$ of $[0,1]$, such that $\lim _{n \rightarrow \infty} \max _{1 \leq \ell \leq n}\left(t_{n, \ell}-t_{n, \ell-1}\right)=0$, is defined to be the sum

$$
\sum_{\ell=0}^{n-1}\left|X_{t_{n, \ell+1}}-X_{t_{n, \ell}}\right|^{p}
$$

Very recently, power variations for processes related to the fractional Brownian motion $B$ have been studied in [3].

For simplicity, consider from now on the case where $t_{n, \ell}=\ell / n$, for $n \in \mathbb{N}^{*}$ and $\ell \in$ $\{0, \ldots, n\}$. In this paper we shall point out some interesting phenomena when $X=B$ is

[^0]the fractional Brownian motion and $p=\kappa \geq 2$ is an integer. In fact, we will also drop the absolute value (when $\kappa$ is odd) and we will introduce some weights. More precisely, we will consider:
$$
\sum_{\ell=0}^{n-1} h\left(B_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa}, \quad h \text { being a smooth function. }
$$

These quantities appears in a natural manner when studying some approximation schemes as will be explained below. Before, let us recall some results concerning $\kappa$-power variations which are more or less classical.

First, assume that the Hurst index is $H=1 / 2$, that is $B$ is the standard Brownian motion. Let $\mu_{\kappa}$ be the $\kappa$-moment of a standard Gaussian random variable $G \sim \mathscr{N}(0,1)$. It is classical, by using central limit theorem that, as $n \rightarrow \infty$,

$$
\begin{equation*}
\sqrt{n}\left(n^{\kappa / 2-1} \sum_{\ell=0}^{n-1}\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa}-\mu_{\kappa}\right) \xrightarrow{\text { Law }} \mathscr{N}\left(0, \mu_{2 \kappa}-\mu_{\kappa}^{2}\right) . \tag{1.1}
\end{equation*}
$$

When weights are introduced, an interesting phenomenon appears: instead of Gaussian random variables, we rather obtain mixing random variables as limit in (1.1). More precisely (see [1], 12] for a complete study of these phenomena, and also [8, 9] for related results with different proofs), as $n \rightarrow \infty$,

$$
\begin{equation*}
\sqrt{n}\left(n^{\kappa / 2-1} \sum_{\ell=0}^{n-1} h\left(B_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa}-\mu_{\kappa} \int_{0}^{1} h\left(B_{s}\right) d s\right) \xrightarrow{\text { Law }}\left(\mu_{2 \kappa}-\mu_{\kappa}\right) \int_{0}^{1} h\left(B_{s}\right) d W_{s} . \tag{1.2}
\end{equation*}
$$

Here, $W$ denotes another standard Brownian motion, independent of $B$. In fact we shall see in $\S 2$ that the convergence in (1.2) can be improved: the convergence in law can be replaced by the stable convergence. This remark will play a crucial role in the proof of Proposition 2.1 below.

Second, assume that $H \neq 1 / 2$, that is the case where the fractional Brownian motion $B$ has not independent increments any more. Then (1.1) has been extended by [2] (see also [25]) and two cases are considered according to the evenness of $\kappa$ :

- if $\kappa$ is even and $H<3 / 4$, as $n \rightarrow \infty$,

$$
\begin{equation*}
\sqrt{n}\left(n^{\kappa H-1} \sum_{\ell=0}^{n-1}\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa}-\mu_{\kappa}\right) \xrightarrow{\text { Law }} \mathscr{N}\left(0, \sigma_{H, \kappa}^{2}\right) ; \tag{1.3}
\end{equation*}
$$

- if $\kappa$ is odd and $H<1 / 2$, as $n \rightarrow \infty$,

$$
\begin{equation*}
n^{\kappa H-1 / 2} \sum_{\ell=0}^{n-1}\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa} \xrightarrow{\text { Law }} \mathscr{N}\left(0, \sigma_{H, \kappa}^{2}\right) . \tag{1.4}
\end{equation*}
$$

Here $\sigma_{H, \kappa}>0$ is an explicit constant depending only on $H$ and $\kappa$ (see, e.g., the second point of Remark 4 in [20] for a method of computation of this constant). In fact, one can relax the restrictive conditions made on $H$ in (1.3)-(1.4), but on one hand the normalization is not the same any more and, on the other hand, one obtains limits which are not Gaussian but the
value at 1 of the so-called Hermite process (see for instance [2]). For this reason we will not consider this case in the present paper.

Now, let us proceed with the results concerning the weighted power variations in the case where $H \neq 1 / 2$. Following the ideas in [34 $]^{1}$ (see also [15]), one could prove that, when $\kappa$ is even and $1 / 2<H<3 / 4$, as $n \rightarrow \infty$ :

$$
\begin{equation*}
\sqrt{n}\left(n^{\kappa H-1} \sum_{\ell=0}^{n-1} h\left(B_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa}-\mu_{\kappa} \int_{0}^{1} h\left(B_{s}\right) d s\right) \stackrel{\text { Law }}{\longrightarrow} c_{H, \kappa} \int_{0}^{1} h\left(B_{s}\right) d W_{s}, \tag{1.5}
\end{equation*}
$$

where again $W$ is a standard Brownian motion independent of $B$. In other words, (1.5) shows a similar behaviour to that observed in the standard Brownian case, see (1.2). When $\kappa$ is odd and $H<1 / 2$, in the light of (1.2), (1.4) and (1.5), one could expect to obtain, as $n \rightarrow \infty$ :

$$
n^{\kappa H-1 / 2} \sum_{\ell=0}^{n-1} h\left(B_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa} \xrightarrow{\text { Law }} c_{H, \kappa} \int_{0}^{1} h\left(B_{s}\right) d W_{s},
$$

with $c_{H, \kappa}$ a certain constant depending only on $H$ and $\kappa$. Very surprisingly, and it is first main results of the present paper (see Theorem 2.3 below for a precise statement), we rather obtain a convergence in probability, instead of the latter convergence in law:

$$
\begin{equation*}
n^{(\kappa+1) H-1} \sum_{\ell=0}^{n-1} h\left(B_{\ell / n}\right)\left(B_{\ell+1 / n}-B_{\ell / n}\right)^{\kappa} \xrightarrow{\text { Prob }}-\frac{\mu_{\kappa+1}}{2} \int_{0}^{1} h^{\prime}\left(B_{s}\right) d s . \tag{1.6}
\end{equation*}
$$

In particular, the limit does not involve an independent standard Brownian motion any more, as in (1.5). Moreover, notice that (1.6) is not in contradiction with (1.4), since if $H<1 / 2$, we have $(\kappa+1) H-1<\kappa H-1 / 2$ and (1.6) with $h \equiv 1$ is in fact a corollary of (1.4).

Let us now turn to the study of some approximation schemes for stochastic differential equations driven by the fractional Brownian motion. The study of such equations was recently considered by using several methods. For instance, in [26] (see also references therein) one uses fractional calculus of same type as in [30]; in [4, 6, 21] (and references therein) one uses the rough paths theory introduced in [17] and in [5, 23, 24] (and references therein) one uses the regularization method used firstly in [28].

In the present paper, we consider the easiest stochastic differential equation involving fractional Brownian motion, that is

$$
\begin{equation*}
d X_{t}=\sigma\left(X_{t}\right) d B_{t}, t \in[0,1], \quad X_{0}=x \in \mathbb{R} \tag{1.7}
\end{equation*}
$$

Notice that we assume here and all along of this paper that $\sigma \in \mathrm{C}^{\infty}(\mathbb{R})$ is a real function which is bounded together with its derivatives. Let us denote by $\phi: \mathbb{R}^{2} \rightarrow \mathbb{R}$ the flow associated to $\sigma$, that is the unique solution to

$$
\begin{equation*}
\phi(x, y)=x+\int_{0}^{y} \sigma(\phi(x, z)) d z, \quad x, y \in \mathbb{R} . \tag{1.8}
\end{equation*}
$$

Then, it is an immediate consequence (see also (3.6) below) that the following process:

$$
\begin{equation*}
X_{t}^{x}=\phi\left(x, B_{t}\right), \quad t \in[0,1], \tag{1.9}
\end{equation*}
$$

[^1]is a natural solution of equation (1.7), as soon as the integral with respect to $B$ verifies an Itô-Stratonovich type formula
$$
f\left(B_{t}\right)=f(0)+\int_{0}^{t} f^{\prime}\left(B_{s}\right) d B_{s}, \quad t \in[0,1], \quad f: \mathbb{R} \rightarrow \mathbb{R} \text { smooth enough. }
$$

Approximating schemes for stochastic differential equations of the type

$$
\begin{equation*}
d X_{t}=\sigma\left(X_{t}\right) d B_{t}+b\left(X_{t}\right) d t, t \in[0,1], \quad X_{0}=x \in \mathbb{R} \tag{1.10}
\end{equation*}
$$

have been considered only in few articles. The first work in that direction has been proposed in [16]: when $H>1 / 2$ one shows that the Euler approximation of equation (1.10) - but in the particular case where $\sigma\left(X_{t}\right)$ is replaced by $\sigma(t)$, that is the so-called additive case converges uniformly in probability. In [22] one introduces (see also [29]) some approximating schemes for the analogue of (1.10) where $B$ is replaced by any Hölder continuous function: one determines upper error bounds and, in particular, these results apply almost surely when the driving Hölder continuous function is a path of the fractional Brownian motion $B$, and this for any Hurst index $H \in(0,1)$.

Results on lower error bounds are available only since very recently: see [18] for the additive case, and [20] for equation (1.10) (see also [19] where approximation methods with respect to a mean square error are analysed). It is proved in the latter reference that the Euler scheme $\widetilde{X}=\left\{\widetilde{X}^{(n)}\right\}_{n \in \mathbb{N}^{*}}$ associated to (1.10) verifies, under some classical assumptions on $\sigma$ and $b$ and when $H \in(1 / 2,1)$, that, as $n \rightarrow \infty$,

$$
\begin{equation*}
n^{2 H-1}\left[\widetilde{X}_{1}^{(n)}-X_{1}\right] \xrightarrow{\text { a.s. }}-\frac{1}{2} \int_{0}^{1} \sigma^{\prime}\left(X_{s}\right) D_{s} X_{1} d s . \tag{1.11}
\end{equation*}
$$

Here $D_{s} X_{1}$ denotes the Malliavin derivatives of $X_{1}$ with respect to $B$. Notice that the upper and lower error bounds are obtained from an almost sure convergence, which is somewhat surprising when compared with the case $H=1 / 2$. In [20], it is proved that, for the so-called Crank-Nicholson scheme $\bar{X}=\left\{\bar{X}^{(n)}\right\}_{n \in \mathbb{N}^{*}}$ associated to (1.7) and defined by

$$
\left\{\begin{array}{l}
\bar{X}_{0}^{n}=x,  \tag{1.12}\\
\bar{X}_{(\ell+1) / n}^{(n)}=\bar{X}_{\ell / n}^{(n)}+\frac{1}{2}\left(\sigma\left(\bar{X}_{\ell / n}^{(n)}\right)+\sigma\left(\bar{X}_{(\ell+1) / n}^{(n)}\right)\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right), \quad \ell \in\{0, \ldots, n-1\},
\end{array}\right.
$$

the following convergence holds, as $n \rightarrow \infty$, for $\sigma$ regular enough and when $H \in(1 / 3,1 / 2)$ :

$$
\begin{equation*}
n^{\alpha}\left[\bar{X}_{1}^{(n)}-X_{1}\right] \xrightarrow{\text { Prob }} 0, \quad \forall \alpha<3 H-1 / 2 . \tag{1.13}
\end{equation*}
$$

In the particular case where the diffusion coefficient $\sigma$ verifies $\sigma(x)^{2}=\alpha x^{2}+\beta x+\gamma$, for some $\alpha, \beta, \gamma \in \mathbb{R}$, one can derive the exact rate of convergence and one proves that, as $n \rightarrow \infty$ :

$$
\begin{equation*}
n^{3 H-1 / 2}\left[\bar{X}_{1}^{(n)}-X_{1}\right] \xrightarrow{\text { Law }} \frac{\alpha}{12} \sigma\left(X_{1}\right) G . \tag{1.14}
\end{equation*}
$$

Here $G$ is a centred Gaussian random variable independent of $X_{1}$, whose variance depends only on $H$. In particular, the upper and lower error bounds are obtained here from a convergence in law.

As was already said, the convergence in (1.11) is somewhat surprising, since there is no analogue for the case of the standard Brownian motion. More precisely, when $H=1 / 2$, it is
proved in (14] that the Euler scheme (1.11) verifies (by denoting $X^{\text {Ito }}$ the solution of (1.10) in the Ito sense), as $n \rightarrow \infty$ :

$$
\begin{equation*}
n^{1 / 2}\left[\widetilde{X}_{1}^{(n)}-X_{1}^{\mathrm{Ito}}\right] \xrightarrow{\mathrm{Law}}-\frac{1}{\sqrt{2}} Y_{1} \int_{0}^{1} \sigma\left(X_{s}^{\mathrm{Ito}}\right) \sigma^{\prime}\left(X_{s}^{\mathrm{Ito}}\right) Y_{s}^{-1} d W_{s} \tag{1.15}
\end{equation*}
$$

Here $W$ is a Brownian motion independent of the Brownian motion $B$ and

$$
Y_{t}=\exp \left(\int_{0}^{t} b^{\prime}\left(X_{s}^{\mathrm{Ito}}\right) d s-\int_{0}^{t} \frac{1}{2} \sigma\left(X_{s}^{\mathrm{Ito}}\right) \sigma^{\prime}\left(X_{s}^{\mathrm{Ito}}\right) d s+\int_{0}^{t} \sigma^{\prime}\left(X_{s}^{\mathrm{Ito}}\right) d B_{s}\right), \quad s \in[0,1]
$$

On the other hand, it can be proved (see Remark 3.8.2) that for the Crank-Nicholson scheme (1.12) we have, as $n \rightarrow \infty$ :

$$
\begin{equation*}
n\left[\bar{X}_{1}^{(n)}-X_{1}^{\mathrm{Str}}\right] \xrightarrow{\mathrm{Law}} \frac{5}{8} \int_{0}^{1}\left(\sigma^{2}\right)^{\prime \prime}\left(X_{s}^{\mathrm{Str}}\right) d W_{s}+\frac{1}{4} \int_{0}^{1} \sigma\left(\sigma^{2}\right)^{\prime \prime \prime}\left(X_{s}^{\mathrm{Str}}\right) d s \tag{1.16}
\end{equation*}
$$

where $X^{\text {Str }}$ denotes the solution of (1.7) in the Stratonovich sense.
In the present paper, we are interested by a better understanding of this phenomenon: which type of convergence allows to derive the upper and lower error bounds for some natural scheme of Milstein's type (see also 22])? To be more precise, let us define, by induction, the family of differential operators $\left(\mathscr{D}^{j}\right)_{j \in \mathbb{N}}$ by:

$$
\begin{equation*}
\mathscr{D}^{0} f=f, \quad \mathscr{D}^{1} f=f^{\prime} \sigma \quad \text { and, for } j \geq 2, \quad \mathscr{D}^{j} f=\mathscr{D}^{1}\left(\mathscr{D}^{j-1} f\right) \tag{1.17}
\end{equation*}
$$

For instance, the first $\mathscr{D}^{j} \sigma$ 's are given by:

$$
\mathscr{D}^{0} \sigma=\sigma, \mathscr{D}^{1} \sigma=\sigma \sigma^{\prime}, \mathscr{D}^{2} \sigma=\sigma \sigma^{\prime 2}+\sigma^{2} \sigma^{\prime \prime}, \mathscr{D}^{3} \sigma=\sigma \sigma^{\prime 3}+4 \sigma^{2} \sigma^{\prime} \sigma^{\prime \prime}+\sigma^{3} \sigma^{\prime \prime \prime}, \text { etc. }
$$

Then, let us consider the following scheme:

$$
\left\{\begin{array}{l}
\widehat{X}_{0}^{(n)}=x  \tag{1.18}\\
\widehat{X}_{(\ell+1) / n}^{(n)}=\widehat{X}_{\ell / n}^{(n)}+\sum_{j=0}^{m} \frac{1}{(j+1)!} \mathscr{D}^{j} \sigma\left(\widehat{X}_{\ell / n}^{(n)}\right) \Delta^{j+1} B_{\ell / n}, \quad \ell \in\{0, \ldots, n-1\},
\end{array}\right.
$$

the integer $m \in \mathbb{N}$ being the size of $\widehat{X}=\left\{\widehat{X}^{(n)}\right\}_{n \in \mathbb{N}^{*}}$. Here, for $j, n \in \mathbb{N}^{*}, \ell \in\{0, \ldots, n-1\}$ and a process $Z=\left(Z_{t}\right)_{t \in[0,1]}$, we set $\Delta^{j} Z_{\ell / n}=\left(Z_{(\ell+1) / n}-Z_{\ell / n}\right)^{j}$. When $j=1$, we denote $\Delta Z_{\ell / n}$ instead of $\Delta^{1} Z_{\ell / n}$ for simplicity. The idea of introducing these schemes will be better explained in $\S 3.1$ below. Let us note that when the size of $\widehat{X}$ is $m=0($ resp. $m=1), \widehat{X}$ is the classical Euler (resp. Milstein) scheme.

The second aim of the present paper is to answer to the following questions: does the sequence $\left\{\widehat{X}_{1}^{(n)}\right\}_{n \in \mathbb{N}^{*}}$ converge? Is the limit $X_{1}^{x}$ given by (1.9) as expected? What is the rate of convergence? Upper and lower error bounds for the approximating scheme (1.18) are obtained from a convergence in law or from a pathwise type convergence?

The paper is organized as follows: the next section contains the results concerning the rates of convergence of weighted power variations for the fractional Brownian motion. In Section 3, after some recalls concerning the definition and main properties of the so-called Newton-Côtes integral and the study of the equation (1.7), we state the result on the rate of convergence of the approximating scheme (1.18). In the last section one gives the proofs of the most technical results.

## 2 Weighted power variations for fractional Brownian motion

In this section we study the rate of convergence of weighted power variations for the process $B$ in order to complete the picture described above. As usually, we start with the standard Brownian motion case and then we proceed with the fractional Brownian motion case. As we shall see, the results and the methods of proofs are somehow different for odd order power variations.

Recall that we denote $\mu_{2 n}\left(n \in \mathbb{N}^{*}\right)$ the $2 n$-moment of a random variable $G \sim \mathscr{N}(0,1)$.
Proposition 2.1 Let $\kappa \in \mathbb{N} \backslash\{0,1\}$ and $g, h \in C_{b}(\mathbb{R})$. Assume that $H=1 / 2$, that is $B$ is the standard Brownian motion.

1. If $\kappa$ is even then, as $n \rightarrow \infty$,

$$
\begin{equation*}
n^{\kappa / 2-1} \sum_{\ell=0}^{n-1} h\left(B_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa} \xrightarrow{\text { Prob }} \mu_{\kappa} \int_{0}^{1} h\left(B_{s}\right) d s . \tag{2.1}
\end{equation*}
$$

2. If $\kappa$ is odd then, as $n \rightarrow \infty$,

$$
\begin{align*}
&\left(B_{1}, n^{\frac{\kappa-1}{2}} \sum_{\ell=0}^{n-1} g\left(B_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa}, n^{\frac{\kappa-1}{2}} \sum_{\ell=0}^{n-1} h\left(B_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa+1}\right) \\
& \xrightarrow{\text { Law }}\left(B_{1}, \mu_{2 \kappa} \int_{0}^{1} g\left(B_{s}\right) d W_{s}, \mu_{\kappa+1} \int_{0}^{1} h\left(B_{s}\right) d s\right) \tag{2.2}
\end{align*}
$$

with $W$ another standard Brownian motion independent of $B$.

## Remark 2.2

We shall see that, for the study of Milstein's type schemes one needs the behaviour of the triplet (2.2) and not only the behaviour of $\sum_{\ell=0}^{n-1} g\left(B_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa}$, with $\kappa$ odd.

Proof. The first part of the proposition is a particular case of the first part of Theorem 2.3 below and which proof will be given in $\S 4$. So, let us admit (2.1) and we prove only (2.2).

We shall use the notion of stable convergence for random variables, so let us recall its definition (see for instance, the original paper [27] or the survey work [7]). Let $\mathscr{F} \subset \overline{\mathscr{F}}$ be two $\sigma$-fields. A sequence of $\mathscr{F}$-measurable random variables $\left\{X_{n}\right\}$ converges $\mathscr{F}$-stably to a $\overline{\mathscr{F}}$-measurable random variable $X$ if, for any continuous bounded function $f: \mathbb{R} \rightarrow \mathbb{R}$ and any bounded $\mathscr{F}$-measurable random variable $Z$, we have, as $n \rightarrow \infty$,

$$
\begin{equation*}
\mathrm{E}\left(f\left(X_{n}\right) Z\right) \rightarrow \mathrm{E}(f(X) Z) \tag{2.3}
\end{equation*}
$$

Let $\mathscr{F}^{B}$ be the $\sigma$-field generated by $\left\{B_{t}: 0 \leq t \leq 1\right\}$ and $W$ be a standard Brownian motion independent of $\mathscr{F}^{B}$. We denote $\overline{\mathscr{F}}=\mathscr{F}^{B} \vee \mathscr{F}^{W}$, where $\mathscr{F}^{W}$ is the $\sigma$-field generated by $\left\{W_{t}: 0 \leq t \leq 1\right\}$. On the one hand, it is a consequence of Theorem 1.1 in Jacod (see also [1]) that, as $n \rightarrow \infty$,

$$
\begin{equation*}
X_{n}:=n^{\frac{\kappa-1}{2}} \sum_{\ell=0}^{n-1} g\left(B_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa} \xrightarrow{\mathscr{F}^{B}-\text { stably }} \mu_{2 \kappa} \int_{0}^{1} g\left(B_{s}\right) d W s=: X \tag{2.4}
\end{equation*}
$$

Furthermore, by (2.1), as $n \rightarrow \infty$

$$
\begin{equation*}
Y_{n}:=n^{\frac{\kappa-1}{2}} \sum_{\ell=0}^{n-1} h\left(B_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa+1} \xrightarrow{\text { Prob }} \mu_{\kappa+1} \int_{0}^{1} h\left(B_{s}\right) d s=: Y \tag{2.5}
\end{equation*}
$$

We stress that $Y$ is a $\mathscr{F}^{B}$-measurable random variable. On the other hand, for any $\left(t_{1}, t_{2}, t_{3}\right) \in$ $\mathbb{R}^{3}$, we can write

$$
\begin{aligned}
\mathrm{E}\left[e^{i t_{1} B_{1}+i t_{2} X_{n}+i t_{3} Y_{n}}\right] & -\mathrm{E}\left[e^{i t_{1} B_{1}+i t_{2} X+i t_{3} Y}\right] \\
& =\mathrm{E}\left[\left(e^{i t_{2} X_{n}}-e^{i t_{2} X}\right) e^{i t_{1} B_{1}+i t_{2} Y}\right]+\mathrm{E}\left[\left(e^{i t_{3} Y_{n}}-e^{i t_{3} Y}\right) e^{i t_{1} B_{1}+i t_{2} X_{n}}\right]
\end{aligned}
$$

By using (2.3)-(2.4), the first term on the right hand side of the latter equality tends to zero, while the modulus of the second term tends to zero by (2.5). The proof of (2.2) is done.

The following result concerns the fractional Brownian motion and it is the first main result of this paper:

Theorem 2.3 Let $\kappa \in \mathbb{N} \backslash\{0,1\}$ and assume that $B$ is a fractional Brownian motion with Hurst index $H \in(0,1)$.

1. If $\kappa$ is even and $h \in C_{b}(\mathbb{R})$ then, for any $H \in(0,1)$, as $n \rightarrow \infty$ :

$$
\begin{equation*}
n^{\kappa H-1} \sum_{\ell=0}^{n-1} h\left(B_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa} \xrightarrow{\text { Prob }} \mu_{\kappa} \int_{0}^{1} h\left(B_{s}\right) d s \tag{2.6}
\end{equation*}
$$

2. If $\kappa$ is odd and $h \in C_{b}^{1}(\mathbb{R})$ then, for any $H \in(0,1 / 2)$, as $n \rightarrow \infty$ :

$$
\begin{equation*}
n^{(\kappa+1) H-1} \sum_{\ell=0}^{n-1} h\left(B_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa} \xrightarrow{\text { Prob }}-\frac{\mu_{\kappa+1}}{2} \int_{0}^{1} h^{\prime}\left(B_{s}\right) d s . \tag{2.7}
\end{equation*}
$$

## Remark 2.4

In fact we will prove that the convergence (2.6), and consequently the convergence (2.1), are almost sure.

## 3 Approximating schemes for fractional stochastic differential equations

Here we present an application of the previous results to the approximation schemes for fractional stochastic differential equations.

### 3.1 Newton-Côtes integral and fractional stochastic differential equations

In this paper, we will use, as integral with respect to $B$, the so-called Newton-Côtes integral introduced in [10] and studied further in [23]:

Definition 3.1 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, $X, Y$ be two continuous processes on $[0,1]$ and $N \in \mathbb{N}$. The $N$-order Newton-Côtes integral of $f(Y)$ with respect to $X$ is defined by:

$$
\begin{equation*}
\int_{0}^{t} f\left(Y_{s}\right) d^{\mathrm{NC}, N} X_{s}:=\lim _{\varepsilon \downarrow 0} \operatorname{prob} \frac{1}{\varepsilon} \int_{0}^{t} d s\left(X_{s+\varepsilon}-X_{s}\right) \int_{0}^{1} f\left(Y_{s}+\alpha\left(Y_{s+\varepsilon}-Y_{s}\right)\right) \nu_{N}(d \alpha), t \in[0,1], \tag{3.1}
\end{equation*}
$$

provided the limit exists. Here $\nu_{0}=\delta_{0}, \nu_{1}=\left(\delta_{0}+\delta_{1}\right) / 2$ and, for $N \geq 2$,

$$
\nu_{N}=\sum_{j=0}^{2 N-2}\left(\int_{0}^{1} \prod_{k \neq j} \frac{2(N-1) u-k}{j-k} d u\right) \delta_{j /(2 N-2)},
$$

$\delta_{a}$ being the Dirac measure at point $a$.

## Remark 3.2

1. Note that the 0 - and 1 -order Newton-Côtes integrals are nothing but the forward integral and the symmetric integral in the sense of Russo-Vallois [28], respectively:

$$
\int_{0}^{t} f\left(Y_{s}\right) d^{\mathrm{NC}, 0} X_{s}=\int_{0}^{t} f\left(Y_{s}\right) d^{-} X_{s}=\lim _{\varepsilon \downarrow 0} \operatorname{prob} \frac{1}{\varepsilon} \int_{0}^{t} f\left(Y_{s}\right)\left(X_{s+\varepsilon}-X_{s}\right) d s,
$$

and

$$
\int_{0}^{t} f\left(Y_{s}\right) d^{\mathrm{NC}, 1} X_{s}=\int_{0}^{t} f\left(Y_{s}\right) d^{\circ} X_{s}=\lim _{\varepsilon \downarrow 0} \operatorname{prob} \frac{1}{\varepsilon} \int_{0}^{t} \frac{f\left(Y_{s+\varepsilon}\right)+f\left(Y_{s}\right)}{2}\left(X_{s+\varepsilon}-X_{s}\right) d s
$$

2. Another way to define $\nu_{N}$ is to view it as the unique discrete signed probability carried by $j /(2 N-2)(j=0, \ldots, 2 N-2)$, which coincides with Lebesgue measure on polynomials of degree smaller than $2 N-1$.

The Newton-Côtes integral defined by (3.1) is in fact a special case of so-called $N$-order $\nu$-integrals introduced in [10], p. 789. Moreover, in the same cited paper, p. 795, one proves that the $N$-order Newton-Côtes integral of $f(B)$ with respect to $B$ exists for any $f \in \mathrm{C}^{4 N+1}$ if and only if $H \in(1 /(4 N+2), 1)$. In this case, an Ito's type change of variables formula holds: for any primitive $F$ of $f$, we can write

$$
\begin{equation*}
F\left(B_{t}\right)-F(0)=\int_{0}^{t} f\left(B_{s}\right) d^{\mathrm{NC}, N} B_{s}, \quad t \in[0,1] . \tag{3.2}
\end{equation*}
$$

Moreover, as a consequence of (3.2), let us note that

$$
\int_{0}^{t} f\left(B_{s}\right) d^{\mathrm{NC}, N} B_{s}=\int_{0}^{t} f\left(B_{s}\right) d^{\mathrm{NC}, n} B_{s}=F\left(B_{t}\right)-F(0)
$$

as soon as $f \in \mathrm{C}^{4 N+1}, n<N$ and $H \in(1 /(4 n+2), 1)$. Therefore, for $f$ regular enough, it is possible to define the Newton-Côtes integral without ambiguity by:

$$
\begin{equation*}
\int_{0}^{t} f\left(B_{s}\right) d^{\mathrm{NC}} B_{s}:=\int_{0}^{t} f\left(B_{s}\right) d^{\mathrm{NC}, n} B_{s} \quad \text { if } H \in(1 /(4 n+2), 1) . \tag{3.3}
\end{equation*}
$$

Set $n_{H}:=\inf \{n \geq 1: H>1 /(4 n+2)\}$. Hence, an immediate consequence of (3.2) and (3.3) is that, for any $H \in(0,1)$ and any $f: \mathbb{R} \rightarrow \mathbb{R}$ of class $\mathrm{C}^{4 n_{H}+1}$, the following Ito's type change of variables formula holds:

$$
\begin{equation*}
F\left(B_{t}\right)=F(0)+\int_{0}^{t} f\left(B_{s}\right) d^{\mathrm{NC}} B_{s}, \quad \text { for any primitive } F \text { of } f \tag{3.4}
\end{equation*}
$$

## Remark 3.3

In the sequel we will only use the fact that the Newton-Côtes integral verifies the classical change of variable formula (3.4). Consequently, any other stochastic integral with respect to the fractional Brownian motion, verifying (3.4), could be used in the following.

All along this paper we will keep an ellipticity assumption and also regularity for the function $\sigma$. More precisely, we suppose that
$(\mathscr{E}) \quad \inf _{\mathbb{R}}|\sigma|>0 \quad$ and $\sigma \in \mathrm{C}^{\infty}(\mathbb{R})$ is bounded together with its derivatives.
Under hypothesis $(\mathscr{E})$, the flow $\phi$ associated to $\sigma$, given by (1.8), is well-defined and verifies the semigroup property:

$$
\begin{equation*}
\forall x, y, z \in \mathbb{R}, \quad \phi(\phi(x, y), z)=\phi(x, y+z) . \tag{3.5}
\end{equation*}
$$

Note that the process $X^{x}$ given by (1.9) verifies:

$$
\begin{equation*}
X_{t}^{x}=x+\int_{0}^{t} \sigma\left(X_{s}^{x}\right) d^{\mathrm{NC}} B_{s}, \quad t \in[0,1] \tag{3.6}
\end{equation*}
$$

as we can see immediately, by applying (3.4).

## Remark 3.4

In (23] (see also [24), one studies a notion of solution for (3.6) and also the existence and the uniqueness of solution. Note however that, in the present work, we will only use the fact that there exists a natural solution to (3.6) given by (1.9).

The following result explains the definition (1.18). By using (1.17), the process $X^{x}$ defined by (1.9) can be expanded as follows:

Lemma 3.5 For any $m \in \mathbb{N}, n \in \mathbb{N}^{*}$ and $\ell \in\{0, \ldots, n-1\}$, we have

$$
\begin{align*}
& X_{(\ell+1) / n}^{x}=X_{\ell / n}^{x}+\sum_{j=0}^{m} \frac{1}{(j+1)!} \mathscr{D}^{j} \sigma\left(X_{\ell / n}^{x}\right) \Delta^{j+1} B_{\ell / n} \\
& \quad+\int_{\ell / n}^{(\ell+1) / n} d^{\mathrm{NC}} B_{t_{1}} \int_{\ell / n}^{t_{1}} d^{\mathrm{NC}} B_{t_{2}} \ldots \int_{\ell / n}^{t_{m}} d^{\mathrm{NC}} B_{t_{m+1}} \int_{\ell / n}^{t_{m+1}} \mathscr{D}^{m+1} \sigma\left(X_{t_{m+2}}^{x}\right) d^{\mathrm{NC}} B_{t_{m+2}} . \tag{3.7}
\end{align*}
$$

Proof. We proceed by induction on $m$. By applying (3.6), and then using (1.9) and (3.4), we can write:

$$
\begin{aligned}
X_{(\ell+1) / n}^{x}=X_{\ell / n}^{x}+\sigma\left(X_{\ell / n}^{x}\right) \Delta & B_{\ell / n}+\int_{\ell / n}^{(\ell+1) / n}\left(\sigma\left(X_{t_{1}}^{x}\right)-\sigma\left(X_{\ell / n}^{x}\right)\right) d^{\mathrm{NC}} B_{t_{1}} \\
& =X_{\ell / n}^{x}+\sigma\left(X_{\ell / n}^{x}\right) \Delta B_{\ell / n}+\int_{\ell / n}^{(\ell+1) / n} d^{\mathrm{NC}} B_{t_{1}} \int_{\ell / n}^{t_{1}} \sigma \sigma^{\prime}\left(X_{t_{2}}^{x}\right) d^{\mathrm{NC}} B_{t_{2}}
\end{aligned}
$$

which is exactly (3.7) for $m=0$.
Let us assume that (3.7) is true for $m \in \mathbb{N}$. Then we can write

$$
\begin{align*}
X_{\ell+1) / n}^{x}= & X_{\ell / n}^{x}+\sum_{j=0}^{m} \frac{1}{(j+1)!} \mathscr{D}^{j} \sigma\left(X_{\ell / n}^{x}\right) \Delta^{j+1} B_{\ell / n} \\
& +\mathscr{D}^{m+1} \sigma\left(X_{\ell / n}^{x}\right) \int_{\ell / n}^{(\ell+1) / n} d^{\mathrm{NC}} B_{t_{1}} \ldots \int_{\ell / n}^{t_{m}} d^{\mathrm{NC}} B_{t_{m+1}} \int_{\ell / n}^{t_{m+1}} d^{\mathrm{NC}} B_{t_{m+2}} \\
& +\int_{\ell / n}^{(\ell+1) / n} d^{\mathrm{NC}} B_{t_{1}} \ldots \int_{\ell / n}^{t_{m}} d^{\mathrm{NC}} B_{t_{m+1}} \int_{\ell / n}^{t_{m+1}}\left(\mathscr{D}^{m+1} \sigma\left(X_{t_{m+2}}^{x}\right)-\mathscr{D}^{m+1} \sigma\left(X_{\ell / n}^{x}\right)\right) d^{\mathrm{NC}} B_{t_{m+2}} \tag{3.8}
\end{align*}
$$

On one hand, using (3.4) repeatedly, it is easy to compute that

$$
\int_{\ell / n}^{(\ell+1) / n} d^{\mathrm{NC}} B_{t_{1}} \ldots \int_{\ell / n}^{t_{m}} d^{\mathrm{NC}} B_{t_{m+1}} \int_{\ell / n}^{t_{m+1}} d^{\mathrm{NC}} B_{t_{m+2}}=\frac{1}{(m+2)!} \Delta^{m+2} B_{\ell / n}
$$

On the other hand, using (1.9) and again (3.4), we can write

$$
\mathscr{D}^{m+1} \sigma\left(X_{t_{m+2}}^{x}\right)-\mathscr{D}^{m+1} \sigma\left(X_{\ell / n}^{x}\right)=\int_{\ell / n}^{t_{m+2}} \sigma\left(\mathscr{D}^{m+1} \sigma\right)^{\prime}\left(X_{t_{m+3}}^{x}\right) d^{\mathrm{NC}} B_{t_{m+3}}
$$

Finally, putting these latter two equalities in (3.8) and noting that $\sigma\left(\mathscr{D}^{m+1} \sigma\right)^{\prime}=\mathscr{D}^{m+2} \sigma$ by definition, we obtain that (3.7) is true for $m+1$.

## Remark 3.6

Clearly (1.18) is the natural scheme constructed from (3.7), by considering the ( $m+2$ )-order iterated integral in the right hand side of (3.7) as a remainder.

### 3.2 Rate of convergence of natural schemes

For $m \in \mathbb{N}$, let us introduce the functions $f_{m}, g_{m}, h_{m}: \mathbb{R} \rightarrow \mathbb{R}$ given by:

$$
f_{m}= \begin{cases}-\frac{2^{-(m+2) / 2}}{((m+2) / 2)!} \mathscr{D}^{m+1} \sigma / \sigma & \text { if } m \text { is even }  \tag{3.9}\\ \frac{2^{-(m+5) / 2}}{((m+3) / 2)!}\left[(m+3) \sigma^{\prime} \mathscr{D}^{m+1} \sigma / \sigma+(m+1) \mathscr{D}^{m+2} \sigma / \sigma\right] & \text { if } m \text { is odd }\end{cases}
$$

$$
\begin{gather*}
g_{m}=\frac{2^{-(m+3) / 2}}{((m+3) / 2)!}\left[(m+3) \sigma^{\prime} \mathscr{D}^{m+1} \sigma / \sigma+\mathscr{D}^{m+2} \sigma / \sigma\right]  \tag{3.10}\\
h_{m}=\frac{(2 m+4)!}{2^{m+2}((m+2)!)^{2}} \mathscr{D}^{m+1} \sigma / \sigma \tag{3.11}
\end{gather*}
$$

For instance, the first $f_{m}$ 's are

$$
f_{0}=-\frac{\sigma^{\prime}}{2}, \quad f_{1}=\frac{1}{8}\left(3 \sigma^{3}+6 \sigma \sigma^{\prime} \sigma^{\prime \prime}+\sigma^{2} \sigma^{\prime \prime \prime}\right), \quad \text { etc. }
$$

Our second main result of this paper contains the answer to the questions in the introduction and can be stated as follows:

Theorem 3.7 Assume that hypothesis ( $\mathscr{E}$ ) holds and let $m \in \mathbb{N}$. Then, for any $H \in$ $(1 /(m+2), 1)$, the sequence $\left\{\widehat{X}_{1}^{(n)}\right\}$ defined by (1.18) converges, as $n \rightarrow \infty$, almost surely toward $X_{1}^{x}=\phi\left(x, B_{1}\right)$. Moreover, as $n \rightarrow \infty$ :

- when $m$ is even and $H \in(1 /(m+2), 1)$,

$$
\begin{equation*}
n^{(m+2) H-1}\left[\widehat{X}_{1}^{(n)}-X_{1}^{x}\right] \xrightarrow{\text { Prob }} \sigma\left(X_{1}^{x}\right) \int_{0}^{1} f_{m}\left(X_{s}^{x}\right) d s \tag{3.12}
\end{equation*}
$$

- when $m$ is odd and $H \in(1 /(m+2), 1 / 2)$,

$$
\begin{equation*}
n^{(m+3) H-1}\left[\widehat{X}_{1}^{(n)}-X_{1}^{x}\right] \xrightarrow{\text { Prob }} \sigma\left(X_{1}^{x}\right) \int_{0}^{1} f_{m}\left(X_{s}^{x}\right) d s \tag{3.13}
\end{equation*}
$$

- when $m$ is odd and $H=1 / 2$,

$$
\begin{equation*}
n^{(m+1) / 2}\left[\widehat{X}_{1}^{(n)}-X_{1}^{x}\right] \xrightarrow{\text { Law }} \sigma\left(X_{1}^{x}\right)\left(\int_{0}^{1} h_{m}\left(X_{s}^{x}\right) d W_{s}+\int_{0}^{1} g_{m}\left(X_{s}^{x}\right) d s\right) \tag{3.14}
\end{equation*}
$$

with $W$ a Brownian motion independent of $X^{x}$.

## Remark 3.8

1. When $m=0$ and $H>1 / 2$, one recovers the convergence (1.11). One can also compare it with (1.15) when $m=0$ and $H=1 / 2$.
2. With the same method used to obtain (3.14), one could prove (1.16). Details are left to the reader.
3. In fact we shall prove that, in (3.12), the convergence is almost sure. Also the convergence in (3.13) is certainly almost sure, but the method of proof we have used here does not allows to obtain it. Thus it remains an open question.
4. According to Theorem $\sqrt[3.7]{ }$, the scheme $\widehat{X}$ of size $m=2 \kappa-1$ has the same rate of convergence than the scheme $\widehat{X}$ of size $m=2 \kappa$, namely $n^{(2 \kappa+2) H-1}$. Thus, it is a priori better to use odd-size schemes.
5. With the same method (see also Theorems 2 and 4 in 20]), one could also derive the exact rate of convergence for the global error on the whole interval $[0,1]$. More precisely, as analogue of (3.12), as $n \rightarrow \infty$, one would have

$$
n^{(m+2) H-1} \max _{\ell=0, \ldots, n}\left|\widehat{X}_{\ell / n}^{(n)}-X_{\ell / n}^{x}\right| \xrightarrow{\text { Prob }} \sup _{t \in[0,1]}\left|\sigma\left(X_{t}^{x}\right) \int_{0}^{t} f_{m}\left(X_{s}^{x}\right) d s\right|
$$

while as analogue of (3.13), as $n \rightarrow \infty$, one would have

$$
n^{(m+3) H-1} \max _{\ell=0, \ldots, n}\left|\widehat{X}_{\ell / n}^{(n)}-X_{\ell / n}^{x}\right| \xrightarrow{\text { Prob }} \sup _{t \in[0,1]}\left|\sigma\left(X_{t}^{x}\right) \int_{0}^{t} f_{m}\left(X_{s}^{x}\right) d s\right| .
$$

Notice that, under $(\mathscr{E})$, the convergences ( $(\sqrt[3.12)]{ })$ and (3.13) give the right lower error bound when

$$
P\left(\int_{0}^{1} f_{m}\left(X_{s}^{x}\right) d s \neq 0\right)>0
$$

Due to (1.9) and the fact that $B_{t}$ has a Gaussian density for any $\left.\left.t \in\right] 0,1\right]$, it is easy to see that it is equivalent to have $f_{m} \not \equiv 0$. Indeed, if $\int_{0}^{1} f\left(B_{s}\right) d s=0$ almost surely for a certain $f \in \mathrm{C}_{b}^{1}(\mathbb{R}, \mathbb{R})$, then $0=D_{u} \int_{0}^{1} f\left(B_{s}\right) d s=\int_{u}^{1} f^{\prime}\left(B_{s}\right) d s$, for any $u \in[0,1]$ (here $D$ denotes the Malliavin derivative with respect to $B$ ). We deduce that $f^{\prime}\left(B_{u}\right)=0$, for any $u \in[0,1]$, and, since the support of the law of $B_{1}$ (for instance) is $\mathbb{R}$, we obtain $f^{\prime} \equiv 0$. The conclusion follows easily.

Solving $f_{m} \equiv 0$ seems, in general, complicated. Nevertheless, when $m \in\{0,1\}$, we can state:

## Proposition 3.9

Assume ( $\mathscr{E})$ and let $f_{m}(m \in \mathbb{N})$ be given by (3.5).

1. The function $f_{0}=-\sigma^{\prime} / 2$ is identically zero if and only if $\sigma$ is a constant function.
2. Assume that $\sigma$ verifies the following additional hypothesis:

$$
\exists j_{0}=-\infty<j_{1}<\ldots<j_{r-1}<j_{r}=\infty \quad \text { s.t. } \quad \sigma_{\mid\left(j_{i}, j_{i+1}\right)}^{\prime} \equiv 0 \quad \text { or } \quad \sigma_{\left.\mid j_{i}, j_{i+1}\right)}^{\prime} \neq 0 .
$$

Then the function $f_{1}=\left(3 \sigma^{\prime 3}+6 \sigma \sigma^{\prime} \sigma^{\prime \prime}+\sigma^{2} \sigma^{\prime \prime \prime}\right) / 8$ is identically zero if and only if $\sigma$ is a constant function.

## Remark 3.10

1. When $\sigma(x)=\sigma$ is constant, we have $\widehat{X}_{1}^{(n)}=X_{1}^{x}=x+\sigma B_{1}$. Consequently, the study of the rate of convergence in the case where $\sigma$ is a constant function is not interesting.
2. A corollary of Theorem 3.7 and Proposition 3.9 is that that upper and lower error bounds come from a convergence in probability, of course except when $\sigma$ is constant. In particular, we never observe a phenomenon of the type (1.14).

Proof. The first point is obvious so we focus on the second one. It is clear that $\sigma(x)=\sigma$ implies $f_{1}=0$. Conversely, assume that $f_{1}=0$. Assume for a moment that $\sigma^{\prime}$ does not vanish on $\left(j_{r-1}, \infty\right)$. Then, on $\left(j_{r-1}, \infty\right)$ we have either $\sigma^{\prime}>0$ or $\sigma^{\prime}<0$. Assume, for instance, that $\sigma^{\prime}>0$, the proof for the other situation being similar. Since $f_{1}=0$, we have $3 \sigma^{\prime}\left(\sigma^{2}\right)^{\prime \prime}=-\sigma\left(\sigma^{2}\right)^{\prime \prime \prime}$. We deduce that the derivative of $\sigma^{3}\left(\sigma^{2}\right)^{\prime \prime}$ is zero on $I:=\left[1+j_{r-1}, \infty\right)$ and then $\left(\sigma^{2}\right)^{\prime \prime}=\alpha \sigma^{-3}$ on $I$, for some $\alpha \neq 0$. Set $h=\sigma^{2}$; we have $h^{\prime \prime} h^{\prime}=\alpha h^{\prime} h^{-3 / 2}$ on $I$ or,
equivalently, $h^{\prime 2}=-4 \alpha h^{-1 / 2}+\beta$ on $I$, for some $\beta \in \mathbb{R}$. In particular, we have $\beta-4 \alpha y^{-1 / 2}>0$, for any $y \in h(I)$. Let $F$ be defined on $h(I)$ by

$$
F(y)=\int_{h\left(1+j_{r-1}\right)}^{y} \frac{d z}{\sqrt{\beta-4 \alpha z^{-1 / 2}}} .
$$

For all $x \in I$, we have

$$
\begin{equation*}
F\left(\sigma(x)^{2}\right)=F(h(x))=x+\gamma, \text { for some } \gamma \in \mathbb{R} . \tag{3.15}
\end{equation*}
$$

The function $\sigma^{2}$ being bounded, we necessarily have $h(x) \rightarrow(4 \alpha / \beta)^{2}$, as $x \rightarrow \infty$. Then $h^{\prime \prime}(x) \rightarrow \beta^{3} /\left(4^{3} \alpha^{2}\right)$, as $x \rightarrow \infty$, which is a contradiction with the fact that $h=\sigma^{2}$ is bounded. Consequently, $\sigma^{\prime}=0$ on $\left(j_{r-1}, \infty\right)$. With the same reasoning, we have either $\sigma^{\prime}=0$ on $\left(j_{r-2}, j_{r-1}\right)$, or $\sigma$ verifies an analogue of (3.15) on $\left(j_{r-2}, j_{r-1}\right)$. But, due to the fact that $\sigma$ is assumed $\mathrm{C}^{\infty}$ and that we already obtained that $\sigma^{\prime}=0$ on $\left(j_{r-1}, \infty\right)$, we necessarily have $\sigma^{\prime}=0$ on $\left(j_{r-2}, j_{r-1}\right)$. Finally, by an induction argument, we have that $\sigma^{\prime}=0$ on $\mathbb{R}$, that is $\sigma$ is constant.

## 4 Proofs of main results

## Proof of Theorem 2.3.

Proof of (2.6). When $h \equiv 1$, it is a classical result (see (13] when $\kappa=2$ ). More precisely, if one fixes $t \in[0,1]$, then, as $n \rightarrow \infty$,

$$
\begin{equation*}
n^{\kappa H-1} \sum_{\ell=0}^{[n t]-1} \Delta^{\kappa} B_{\ell / n}=n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbb{1}_{[0, t]}(\ell / n) \Delta^{\kappa} B_{\ell / n} \longrightarrow \mu_{\kappa} t, \quad \text { a.s. } \tag{4.1}
\end{equation*}
$$

By (4.1) we deduce that, almost surely, (4.1) holds for any $t \in[0,1] \cap \mathbb{Q}$. In other words, there exists $\Omega^{*}$ with $\mathrm{P}\left(\Omega^{*}\right)=1$ verifying that, for every $\omega \in \Omega^{*}$ and $t \in[0,1] \cap \mathbb{Q}$, (4.1) holds.

Now, fix $\omega \in \Omega^{*}$ and $t \in[0,1]$. Let $\left\{t_{m}\right\},\left\{t_{m}^{\prime}\right\} \subset \mathbb{Q} \cap[0,1]$ such that $t_{m} \uparrow t$ and $t_{m}^{\prime} \downarrow t$, as $m \rightarrow \infty$. Then

$$
n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbb{1}_{\left[0, t_{m}\right]}(\ell / n) \Delta^{\kappa} B_{\ell / n} \leq n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbb{1}_{[0, t]}(\ell / n) \Delta^{\kappa} B_{\ell / n} \leq n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbb{1}_{\left[0, t_{m}^{\prime}\right]}(\ell / n) \Delta^{\kappa} B_{\ell / n}
$$

Letting $n \rightarrow \infty$, we obtain

$$
\mu_{\kappa} t_{m} \leq \liminf _{n \rightarrow \infty} n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbb{1}_{[0, t]}(\ell / n) \Delta^{\kappa} B_{\ell / n} \leq \limsup _{n \rightarrow \infty} n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbb{1}_{[0, t]}(\ell / n) \Delta^{\kappa} B_{\ell / n} \leq \mu_{\kappa} t_{m}^{\prime}
$$

Letting $m \rightarrow \infty$, we obtain that, for every $\omega \in \Omega^{*}$ and $t \in[0,1]$, (4.1) holds. Thanks to second Dini's theorem ${ }^{2}$, we obtain that, for every $\omega \in \Omega^{*}$, as $n \rightarrow \infty$ :

$$
\sup _{t \in[0,1]}\left|n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbb{1}_{[0, t]}(\ell / n) \Delta^{\kappa} B_{\ell / n}-\mu_{\kappa} t\right| \longrightarrow 0
$$

The following (deterministic) lemma allows us to finish the proof of (2.6).

[^2]Lemma 4.1 Let $b, g:[0,1] \rightarrow \mathbb{R}$ be two continuous functions and $\kappa$ be an even integer. If

$$
\begin{equation*}
\sup _{t \in[0,1]}\left|n^{\kappa H-1} \sum_{\ell=0}^{n-1} \mathbb{1}_{[0, t]}(\ell / n) \Delta^{\kappa} b_{\ell / n}-\mu_{\kappa} t\right| \longrightarrow 0, \quad \text { as } n \rightarrow \infty \tag{4.2}
\end{equation*}
$$

then

$$
\begin{equation*}
\sup _{t \in[0,1]}\left|n^{\kappa H-1} \sum_{\ell=0}^{n-1} g_{\ell / n} \Delta^{\kappa} b_{\ell / n}-\mu_{\kappa} \int_{0}^{t} g_{s} d s\right| \longrightarrow 0, \quad \text { as } n \rightarrow \infty \tag{4.3}
\end{equation*}
$$

Proof. If $g=\mathbb{1}_{[0, t]}$ then (4.2) and (4.3) are the same. By linearity, (4.3) holds for piecewise functions. An argument of density allows us to finish easily the proof.

Proof of (2.7). This proof is rather technical and we split it in two steps.
First step. We prove that, as $n \rightarrow \infty$,

$$
\begin{equation*}
n^{(\kappa+1) H-1} \sum_{\ell=0}^{n-1}\left(h\left(B_{(\ell+1) / n}\right)+h\left(B_{\ell / n}\right)\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa} \xrightarrow{\text { Prob }} 0 . \tag{4.4}
\end{equation*}
$$

For that purpose, we follow the same strategy as [11]. Set

$$
I_{n}(h):=\sum_{\ell=0}^{n-1}\left(h\left(B_{(\ell+1) / n}\right)+h\left(B_{\ell / n}\right)\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa}
$$

and it suffices to prove that $\lim _{n \rightarrow \infty} n^{(2 \kappa+2) H-2} \mathrm{E}\left[I_{n}(h)^{2}\right]=0$. We shall use the following Gaussian mean-zero random vector

$$
G=\left(G_{1}, G_{2}, G_{3}, G_{4}\right):=\left(B_{k / n}, B_{\ell / n}, B_{(k+1) / n}-B_{k / n}, B_{(\ell+1) / n}-B_{\ell / n}\right)
$$

Clearly,
$I_{n}(h)^{2}=\sum_{k, \ell=0}^{n-1}\left[h\left(B_{(k+1) / n}\right)+h\left(B_{k / n}\right)\right]\left(B_{(k+1) / n}-B_{k / n}\right)^{\kappa}\left[h\left(B_{(\ell+1) / n}\right)+h\left(B_{\ell / n}\right)\right]\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa}$,
hence

$$
\begin{equation*}
\mathrm{E}\left[I_{n}(h)^{2}\right]=\sum_{k, \ell=0}^{n-1} \mathrm{E}\left\{\left[h\left(G_{1}+G_{3}\right)+h\left(G_{1}\right)\right]\left[h\left(G_{2}+G_{4}\right)+h\left(G_{2}\right)\right] G_{3}^{\kappa} G_{4}^{\kappa}\right\} \tag{4.5}
\end{equation*}
$$

For random variables $\xi, \zeta, \phi_{\varepsilon}$, we will denote

$$
\xi \stackrel{(\text { law })}{=} \zeta+o(\varepsilon) \text { as } \varepsilon \downarrow 0, \text { if } \xi \stackrel{(\text { law })}{=} \zeta+\varepsilon \phi_{\varepsilon}, \text { with } \mathrm{E}\left[\sup _{0<\varepsilon<1}\left|\phi_{\varepsilon}\right|^{p}\right]<\infty, \forall p
$$

Then, by (5.46) in 11, p. 1812, we can write, as $n \rightarrow \infty$,

$$
\left\{\begin{array}{l}
G_{3}:=B_{(k+1) / n}-B_{k / n} \stackrel{(\text { law })}{=} N_{1} n^{-H}+Q_{1} n^{-2 H}-\frac{\lambda_{11}}{8} N_{1} n^{-3 H}+o\left(n^{-3 H}\right)  \tag{4.6}\\
G_{4}:=B_{(\ell+1) / n}-B_{\ell / n} \stackrel{(\text { law })}{=} N_{2} n^{-H}+Q_{2} n^{-2 H}-\left(\frac{\lambda_{12}}{4} N_{1}+\frac{\lambda_{22}}{8} N_{2}\right) n^{-3 H}+o\left(n^{-3 H}\right),
\end{array}\right.
$$

where $N_{1}, N_{2}$ are independent standard Gaussian random variables independent also of $G_{1}, G_{2}$ and where

$$
\begin{align*}
& Q_{1}:=-\frac{1}{2}\left(\lambda_{11} G_{1}+\lambda_{12} G_{2}\right), Q_{2}:=-\frac{1}{2}\left(\lambda_{12} G_{1}+\lambda_{22} G_{2}\right)  \tag{4.7}\\
& \left(\begin{array}{ll}
\lambda_{11} & \lambda_{12} \\
\lambda_{12} & \lambda_{22}
\end{array}\right):=\left(\begin{array}{cc}
(k / n)^{2 H} & K(k / n, \ell / n) \\
K(\ell / n, k / n) & (\ell / n)^{2 H}
\end{array}\right)^{-1}=\operatorname{Cov}_{\left(G_{1}, G_{2}\right)}^{-1} \tag{4.8}
\end{align*}
$$

with

$$
K(k / n, \ell / n)=\operatorname{Cov}\left(B_{k / n}, B_{\ell / n}\right)=\frac{1}{2}\left((k / n)^{2 H}+(\ell / n)^{2 H}-(|k-\ell| / n)^{2 H}\right) .
$$

Moreover, by (4.6) and by computations in (11], p. 1810, we can write

$$
\begin{equation*}
G_{3} G_{4}=n^{-2 H}\left(N_{1} N_{2}+S_{n}\right) \tag{4.9}
\end{equation*}
$$

where, as $n \rightarrow \infty$,

$$
\begin{equation*}
S_{n} \stackrel{(\text { law })}{=} n^{-H}\left(N_{1} Q_{2}+N_{2} Q_{1}\right)+n^{-2 H}\left(Q_{1} Q_{2}-\frac{\lambda_{12}}{4} N_{1}^{2}-\frac{\lambda_{11}+\lambda_{22}}{8} N_{1} N_{2}\right)+o\left(n^{-2 H}\right) \tag{4.10}
\end{equation*}
$$

Then,

$$
\begin{equation*}
S_{n}^{2} \stackrel{(\text { law })}{=} n^{-2 H}\left(N_{1}^{2} Q_{2}^{2}+2 N_{1} N_{2} Q_{1} Q_{2}+N_{2}^{2} Q_{1}^{2}\right)+o\left(n^{-2 H}\right) \text { and } S_{n}^{3} \stackrel{(\text { law })}{=} o\left(n^{-2 H}\right) \tag{4.11}
\end{equation*}
$$

Using (4.9), (4.10) and (4.11) we see that to obtain an expansion of $\left(G_{3} G_{4}\right)^{\kappa}$ up to order $n^{-2(\kappa+1) H}$ it suffices to keep only the first three terms in the binomial expansion of $\left(N_{1} N_{2}+S_{n}\right)^{\kappa}$. Hence, as $n \rightarrow \infty$,

$$
\begin{equation*}
\left(G_{3} G_{4}\right)^{\kappa} \stackrel{(\text { law })}{=} n^{-2 \kappa H}\left(N_{1}^{\kappa} N_{2}^{\kappa}+\kappa N_{1}^{\kappa-1} N_{2}^{\kappa-1} S_{n}+\frac{\kappa(\kappa-1)}{2} N_{1}^{\kappa-2} N_{2}^{\kappa-2} S_{n}^{2}+o\left(n^{-2 H}\right)\right) \tag{4.12}
\end{equation*}
$$

Again by Taylor expansion and using (4.6), as $n \rightarrow \infty$,

$$
\begin{equation*}
h\left(G_{1}+G_{3}\right) \stackrel{(\text { law })}{=} h\left(G_{1}\right)+h^{\prime}\left(G_{1}\right) N_{1} n^{-H}+\left(h^{\prime}\left(G_{1}\right) Q_{1}+\frac{1}{2} h^{\prime \prime}\left(G_{1}\right) N_{1}^{2}\right) n^{-2 H}+o\left(n^{-2 H}\right) \tag{4.13}
\end{equation*}
$$

and similarly

$$
\begin{equation*}
h\left(G_{2}+G_{4}\right) \stackrel{(\text { law })}{=} h\left(G_{2}\right)+h^{\prime}\left(G_{2}\right) N_{2} n^{-H}+\left(h^{\prime}\left(G_{2}\right) Q_{2}+\frac{1}{2} h^{\prime \prime}\left(G_{2}\right) N_{2}^{2}\right) n^{-2 H}+o\left(n^{-2 H}\right) \tag{4.14}
\end{equation*}
$$

Replacing (4.12)-(4.14) in (4.5) we see that we need to compute the following product:

$$
\begin{aligned}
& n^{-2 \kappa H}\left\{2 h\left(G_{1}\right)+h^{\prime}\left(G_{1}\right) N_{1} n^{-H}+\left(h^{\prime}\left(G_{1}\right) Q_{1}+\frac{1}{2} h^{\prime \prime}\left(G_{1}\right) N_{1}^{2}\right) n^{-2 H}+o\left(n^{-2 H}\right)\right\} \\
& \quad \times\left\{2 h\left(G_{2}\right)+h^{\prime}\left(G_{2}\right) N_{2} n^{-H}+\left(h^{\prime}\left(G_{2}\right) Q_{2}+\frac{1}{2} h^{\prime \prime}\left(G_{2}\right) N_{2}^{2}\right) n^{-2 H}+o\left(n^{-2 H}\right)\right\} \\
& \times\left\{N_{1}^{\kappa} N_{2}^{\kappa}+\kappa N_{1}^{\kappa-1} N_{2}^{\kappa-1}\left(N_{1} Q_{2}+N_{2} Q_{1}\right) n^{-H}+\left[\kappa^{2} N_{1}^{\kappa-1} N_{2}^{\kappa-1} Q_{1} Q_{2}-\frac{\lambda_{12}}{4} \kappa N_{1}^{\kappa+1} N_{2}^{\kappa-1}\right.\right. \\
& \\
& \left.\left.-\frac{\lambda_{11}+\lambda_{22}}{8} \kappa N_{1}^{\kappa} N_{2}^{\kappa}+\frac{\kappa(\kappa-1)}{2} N_{1}^{\kappa} N_{2}^{\kappa-2} Q_{2}^{2}+\frac{\kappa(\kappa-1)}{2} N_{1}^{\kappa-2} N_{2}^{\kappa} Q_{1}^{2}\right] n^{-2 H}+o\left(n^{-2 H}\right)\right\}
\end{aligned}
$$

We are looking for the coefficients of powers of $n$ :

$$
n^{-2 \kappa H}: 4 \mathrm{E}\left\{h\left(G_{1}\right) h\left(G_{2}\right) N_{1}^{\kappa} N_{2}^{\kappa}\right\},
$$

and this term vanishes by the independence of $N_{1}, N_{2}$ and ( $G_{1}, G_{2}$ ) ( $\kappa$ being an odd integer);

$$
\begin{aligned}
n^{-(2 \kappa+1) H}: 2 \mathrm{E}\left\{h\left(G_{2}\right) h^{\prime}\left(G_{1}\right) N_{1}^{\kappa+1} N_{2}^{\kappa}+2\right. & 2\left(G_{1}\right) h^{\prime}\left(G_{2}\right) N_{1}^{\kappa} N_{2}^{\kappa+1} \\
& \left.+4 \kappa h\left(G_{1}\right) h\left(G_{2}\right)\left[N_{1}^{\kappa} N_{2}^{\kappa-1} Q_{2}+N_{1}^{\kappa-1} N_{2}^{\kappa} Q_{1}\right]\right\},
\end{aligned}
$$

and again this term vanishes. After similar computations we also obtain the coefficient of

$$
\begin{array}{ll}
n^{-(2 \kappa+2) H}: & \mathrm{E}\left\{4 \kappa^{2} \mu_{\kappa-1}^{2} h\left(G_{1}\right) h\left(G_{2}\right) Q_{1} Q_{2}-\kappa \lambda_{12} \mu_{\kappa+1} \mu_{\kappa-1} h\left(G_{1}\right) h\left(G_{2}\right)\right. \\
\left.\quad+\mu_{\kappa+1}^{2} h^{\prime}\left(G_{1}\right) h^{\prime}\left(G_{2}\right)+2 \kappa \mu_{\kappa+1} \mu_{\kappa-1} h^{\prime}\left(G_{1}\right) h\left(G_{2}\right) Q_{2}+2 \kappa \mu_{\kappa+1} \mu_{\kappa-1} h\left(G_{1}\right) h^{\prime}\left(G_{2}\right) Q_{1}\right\} .
\end{array}
$$

Recall that $\kappa \mu_{\kappa-1}=\mu_{\kappa+1}$. Hence, as $n \rightarrow \infty$,

$$
\mathrm{E}\left[I_{n}(h)^{2}\right]=n^{-(2 \kappa+2) H} \mu_{\kappa+1}^{2} \sum_{k, \ell=0}^{n-1} c_{k, \ell}^{(n)}+o\left(n^{2-(2 \kappa+2) H}\right),
$$

where

$$
\begin{align*}
c_{k, \ell}^{(n)}:=\mathrm{E}\left\{h\left(G_{1}\right) h\left(G_{2}\right)\left(4 Q_{1} Q_{2}-\lambda_{12}\right)\right\} & +\mathrm{E}\left\{h^{\prime}\left(G_{1}\right) h^{\prime}\left(G_{2}\right)\right\} \\
& +\mathrm{E}\left\{2 h^{\prime}\left(G_{1}\right) h\left(G_{2}\right) Q_{2}\right\}+E\left\{2 h\left(G_{1}\right) h^{\prime}\left(G_{2}\right) Q_{1}\right\} . \tag{4.15}
\end{align*}
$$

Again we use some identities in [11], p. 1802. We write $K$ for $K(k / n, \ell / n)$ and $\Delta$ for $(k / n)^{2 H}(\ell / n)^{2 H}-K^{2}$. Hence

$$
\lambda_{11}=\frac{(\ell / n)^{2 H}}{\Delta}, \lambda_{22}=\frac{(k / n)^{2 H}}{\Delta}, \lambda_{12}=-\frac{K}{\Delta} \text { and }\binom{G_{1}}{G_{2}}=\left(\begin{array}{cc}
(k / n)^{H} & 0 \\
\frac{K}{(k / n)^{H}} & \frac{\sqrt{\Delta}}{(k / n)^{H}}
\end{array}\right)\binom{Z_{1}}{Z_{2}}
$$

where $Z_{1}, Z_{2}$ are two independent standard Gaussian random variables. After some algebraic computations

$$
Q_{1}=\frac{-1}{2(k / n)^{H}} Z_{1}+\frac{K}{2(k / n)^{H} \sqrt{\Delta}} Z_{2}, \quad Q_{2}=\frac{-(k / n)^{H}}{2 \sqrt{\Delta}} Z_{2} \quad \text { and } \quad 4 Q_{1} Q_{2}=\frac{Z_{1} Z_{2}}{\sqrt{\Delta}}-\frac{K Z_{2}^{2}}{\Delta} .
$$

Classical integration by parts formula gives (see also (5.28) in [11] , p. 1803):

$$
\begin{gather*}
\mathrm{E}\left\{h^{\prime}\left(G_{1}\right) h^{\prime}\left(G_{2}\right)\right\}=\mathrm{E}\left\{h\left(G_{1}\right) h\left(G_{2}\right)\left[\frac{Z_{1} Z_{2}}{\sqrt{\Delta}}-\frac{K}{\Delta}\left(Z_{2}^{2}-1\right)\right]\right\},  \tag{4.16}\\
\mathrm{E}\left\{h^{\prime}\left(G_{1}\right) h\left(G_{2}\right) Z_{2}\right\}=\frac{1}{(k / n)^{H}} \mathrm{E}\left\{h\left(G_{1}\right) h\left(G_{2}\right) Z_{1} Z_{2}\right\}-\frac{K}{(k / n)^{H} \sqrt{\Delta}} \mathrm{E}\left\{h\left(G_{1}\right) h\left(G_{2}\right)\left(Z_{2}^{2}-1\right)\right\}, \tag{4.17}
\end{gather*}
$$

$$
\begin{equation*}
\mathrm{E}\left\{h\left(G_{1}\right) h^{\prime}\left(G_{2}\right) Z_{1}\right\}=\frac{(k / n)^{H}}{\sqrt{\Delta}} \mathrm{E}\left\{h\left(G_{1}\right) h\left(G_{2}\right) Z_{1} Z_{2}\right\} \tag{4.18}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{E}\left\{h\left(G_{1}\right) h^{\prime}\left(G_{2}\right) Z_{2}\right\}=\frac{(k / n)^{H}}{\sqrt{\Delta}} \mathrm{E}\left\{h\left(G_{1}\right) h\left(G_{2}\right)\left(Z_{2}^{2}-1\right)\right\} . \tag{4.19}
\end{equation*}
$$

Replacing (4.16) -(4.19) in the expectation in (4.15) we get

$$
\begin{gathered}
c_{k, \ell}^{(n)}=\mathrm{E}\left\{h ( G _ { 1 } ) h ( G _ { 2 } ) \left[\frac{Z_{1} Z_{2}}{\sqrt{\Delta}}-\frac{K Z_{2}^{2}}{\Delta}-\frac{K}{\Delta}+\frac{Z_{1} Z_{2}}{\sqrt{\Delta}}-\frac{K}{\Delta}\left(Z_{2}^{2}-1\right)-\frac{(k / n)^{H}}{\sqrt{\Delta}}\left(\frac{1}{(k / n)^{H}} Z_{1} Z_{2}\right.\right.\right. \\
\left.\left.\left.-\frac{K}{(k / n)^{H} \sqrt{\Delta}}\left(Z_{2}^{2}-1\right)\right)-\frac{1}{(k / n)^{H}}\left(\frac{(k / n)^{H}}{\sqrt{\Delta}} Z_{1} Z_{2}+\frac{(k / n)^{H}}{\sqrt{\Delta}}\left(Z_{2}^{2}-1\right)\right)\right]\right\}=0
\end{gathered}
$$

and finally, $\lim _{n \rightarrow \infty} n^{(2 \kappa+2) H-2} \mathrm{E}\left[I_{n}(h)^{2}\right]=0$ which implies (4.4).
Second step. Let us now prove that, as $n \rightarrow \infty$,

$$
\begin{equation*}
n^{(\kappa+1) H-1} \sum_{\ell=0}^{n-1}\left(h\left(B_{(\ell+1) / n}\right)-h\left(B_{\ell / n}\right)\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa} \xrightarrow{\text { Prob }} \mu_{\kappa+1} \int_{0}^{1} h^{\prime}\left(B_{u}\right) d u . \tag{4.20}
\end{equation*}
$$

It is clear that $(2.6)$ is then a consequence of $(4.4)$ and $(4.20)$.
By Taylor expansion, as $n \rightarrow \infty$,

$$
h\left(B_{(\ell+1) / n}\right)-h\left(B_{\ell / n}\right)=h^{\prime}\left(\theta_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right),
$$

with $\theta_{\ell / n}$ between $B_{\ell / n}$ and $B_{(\ell+1) / n}$. Replacing in the previous sum we get

$$
\begin{aligned}
n^{(\kappa+1) H-1} \sum_{\ell=0}^{n-1}\left(h\left(B_{(\ell+1) / n}\right)-h\left(B_{\ell / n}\right)\right)\left(B_{(\ell+1) / n}\right. & \left.-B_{\ell / n}\right)^{\kappa} \\
& =n^{(\kappa+1) H-1} \sum_{\ell=0}^{n-1} h^{\prime}\left(\theta_{\ell / n}\right)\left(B_{(\ell+1) / n}-B_{\ell / n}\right)^{\kappa+1} .
\end{aligned}
$$

By an easy adaptation of (2.6), this quantity tends, as $n \rightarrow \infty$, toward $-\mu_{\kappa+1} \int_{0}^{1} h^{\prime}\left(B_{u}\right) d u$ $(\kappa+1$ being an even integer $)$, and $(4.20)$ is proved. This ends the proof of Theorem 2.3 .

Proof of Theorem 3.7. We assume that $H \in(1 /(m+2), 1)$ and we denote in the following

$$
\Delta^{p}(B)=\max _{k=0, \ldots, n-1}\left|\Delta^{p} B_{k / n}\right|
$$

for $p \in \mathbb{N}^{*}$. Again we split the proof in several steps.

1. General computations. The following lemma can be showed by using the same method as in the proof of Lemma 3.5, but with Lebesgue integral instead of Newton-Côtes integral:

Lemma 4.2 For any $x, y \in \mathbb{R}$, we have, as $y \rightarrow 0$,

$$
\phi(x, y)=x+\sum_{j=0}^{m+2} \frac{1}{(j+1)!} \mathscr{D}^{j} \sigma(x) y^{j+1}+O\left(y^{m+4}\right)
$$

By applying this lemma with $x=\widehat{X}_{k / n}^{(n)}$ and $y=\Delta B_{k / n}$, we obtain

$$
\begin{aligned}
\widehat{X}_{(k+1) / n}^{(n)}=\phi\left(\widehat{X}_{k / n}^{(n)}, \Delta B_{k / n}\right)-\mathscr{D}^{m+1} \sigma\left(\widehat{X}_{k / n}^{(n)}\right) & \frac{\Delta^{m+2} B_{k / n}}{(m+2)!} \\
& \quad-\mathscr{D}^{m+2} \sigma\left(\widehat{X}_{k / n}^{(n)}\right) \frac{\Delta^{m+3} B_{k / n}}{(m+3)!}+O\left(\Delta^{m+4}(B)\right) .
\end{aligned}
$$

By straightforward computations, we get ${ }^{3}$

$$
\begin{equation*}
\widehat{X}_{(k+1) / n}^{(n)}=\phi\left(\widehat{X}_{k / n}^{(n)}, \Delta B_{k / n}+h_{m}\left(\widehat{X}_{k / n}^{(n)}\right) \Delta^{m+2} B_{k / n}+g_{m}\left(\widehat{X}_{k / n}^{(n)}\right) \Delta^{m+3} B_{k / n}+O\left(\Delta^{m+4}(B)\right)\right), \tag{4.21}
\end{equation*}
$$

with $g_{m}$ and $h_{m}$ given respectively by (3.10) and (3.11). By the semi-group property (3.5), we finally obtain that, for any $\ell \in\{0, \ldots, n\}$ :

$$
\begin{equation*}
\widehat{X}_{\ell / n}^{(n)}=\phi\left(x, B_{\ell / n}+\sum_{k=0}^{\ell-1} h_{m}\left(\widehat{X}_{k / n}^{(n)}\right) \Delta^{m+2} B_{k / n}+\sum_{k=0}^{\ell-1} g_{m}\left(\widehat{X}_{k / n}^{(n)}\right) \Delta^{m+3} B_{k / n}+O\left(n \Delta^{m+4}(B)\right)\right) \tag{4.22}
\end{equation*}
$$

In particular, we deduce, as $n \rightarrow \infty$,

$$
\begin{equation*}
\sup _{\ell \in\{0, \ldots, n\}}\left|\widehat{X}_{\ell / n}^{(n)}-X_{\ell / n}^{x}\right|=\sup _{\ell \in\{0, \ldots, n\}}\left|\widehat{X}_{\ell / n}^{(n)}-\phi\left(x, B_{\ell / n}\right)\right|=O\left(n \Delta^{m+2}(B)\right), \tag{4.23}
\end{equation*}
$$

and $\widehat{X}_{1}^{(n)}$ converges almost surely to $X_{1}^{x}$, as $n \rightarrow \infty$, since $H>1 /(2 m+2)$.
2. Proof of (3.12). Assume that $m$ is an even integer. As a consequence of (4.22)-(4.23) and the fact that $H>1 /(m+2)$, we can write

$$
\widehat{X}_{1}^{(n)}=\phi\left(x, B_{1}+\sum_{k=0}^{n-1} h_{m}\left(X_{k / n}^{x}\right) \Delta^{m+2} B_{k / n}+o\left(n \Delta^{m+2}(B)\right)\right)
$$

By using (2.6) with $\kappa=m+2$ and due to the fact that $X_{t}^{x}=\phi\left(x, B_{t}\right)$ and $(\partial \phi / \partial y)(x, y)=$ $\sigma(\phi(x, y)$ ), we finally obtain (3.12).
3. Proof of (3.13) for $H \in(2 /(2 m+3), 1 / 2)$. Assume that $m$ is an odd integer and that $H \in(2 /(2 m+3), 1 / 2)$. Thanks to (4.23), (4.22) can be transformed in

$$
\begin{equation*}
\widehat{X}_{\ell / n}^{(n)}=\phi\left(x, B_{\ell / n}+\sum_{k=0}^{\ell-1} h_{m}\left(\widehat{X}_{k / n}^{(n)}\right) \Delta^{m+2} B_{k / n}+\sum_{k=0}^{\ell-1} g_{m}\left(X_{k / n}^{x}\right) \Delta^{m+3} B_{k / n}+o\left(n \Delta^{m+3}(B)\right)\right) . \tag{4.24}
\end{equation*}
$$

$$
\begin{aligned}
& { }^{3} \text { In fact, we rather obtain } \\
& \qquad \widehat{X}_{(k+1) / n}^{(n)}=\phi\left(\widehat{X}_{k / n}^{(n)}, \Delta B_{k / n}+h_{m}\left(\widehat{X}_{k / n}^{(n)}\right) \Delta^{m+2} B_{k / n}+g_{m}\left(\widehat{X}_{k / n}^{(n)}\right) \Delta^{m+3} B_{k / n}\right)+O\left(\Delta^{m+4}(B)\right),
\end{aligned}
$$

which is not exactly (4.21). But, in order to introduce $O\left(\Delta^{m+4}(B)\right)$ in the argument of $\phi$, we proceed as follows, by using hypothesis ( $\mathscr{E}$ ):

$$
\phi(x, z)+O(\delta)=\phi\left(x, \phi^{-1}(x, \phi(x, z)+O(\delta))\right)=\phi(x, z+O(\delta)) .
$$

Due to the assumptions on $\sigma$, we have

$$
\phi\left(x, y_{2}\right)=\phi\left(x, y_{1}\right)+\sum_{j=1}^{M} \frac{1}{j!} \frac{\partial^{j} \phi}{(\partial y)^{j}}\left(x, y_{1}\right)\left(y_{2}-y_{1}\right)^{j}+O\left(\left(y_{2}-y_{1}\right)^{M+1}\right) .
$$

Thus we get

$$
\begin{array}{r}
\widehat{X}_{k / n}^{(n)}=X_{k / n}^{x}+\sum_{j=1}^{M} \frac{1}{j!} \frac{\partial^{j} \phi}{(\partial y)^{j}}\left(x, B_{k / n}\right)\left(\sum_{k_{1}=0}^{k-1} h_{m}\left(\widehat{X}_{k_{1} / n}^{(n)}\right) \Delta^{m+2} B_{k_{1} / n}+O\left(n \Delta^{m+3}(B)\right)\right)^{j} \\
+O\left(n^{M+1} \Delta^{(m+2)(M+1)}(B)\right) \tag{4.25}
\end{array}
$$

By using (4.25) with $M=1$ and the equality $(\partial \phi / \partial y)(x, y)=\sigma(\phi(x, y))$ once more, we get

$$
\widehat{X}_{k / n}^{(n)}=X_{k / n}^{x}+\sigma\left(X_{k_{1} / n}^{x}\right) \sum_{k_{1}=0}^{k-1} h_{m}\left(\widehat{X}_{k / n}^{(n)}\right) \Delta^{m+2} B_{k_{1} / n}+O\left(n^{2} \Delta^{2 m+4}(B)\right)+O\left(n \Delta^{m+3}(B)\right)
$$

and then

$$
\widehat{X}_{k / n}^{(n)}=X_{k / n}^{x}+\sigma\left(X_{k / n}^{x}\right) \sum_{k_{1}=0}^{k-1} h_{m}\left(X_{k_{1} / n}^{x}\right) \Delta^{m+2} B_{k_{1 / n}}+O\left(n^{2} \Delta^{2 m+4}(B)\right)+O\left(n \Delta^{m+3}(B)\right) .
$$

By inserting the previous equality in (4.24) with $\ell=n$, we obtain

$$
\begin{align*}
& \widehat{X}_{1}^{(n)}=\phi\left(x, B_{1}+\sum_{k=0}^{n-1} h_{m}\left(X_{k / n}^{x}\right) \Delta^{m+2} B_{k / n}+\sum_{k=0}^{n-1} g_{m}\left(X_{k / n}^{x}\right) \Delta^{m+3} B_{k / n}\right. \\
& \left.\quad+\sum_{k=0}^{n-1} \sigma h_{m}^{\prime}\left(X_{k / n}^{x}\right) \Delta^{m+2} B_{k / n} \sum_{j=0}^{k} h_{m}\left(X_{j / n}^{x}\right) \Delta^{m+2} B_{j / n}+o\left(n^{3} \Delta^{3 m+6}(B)\right)+o\left(n \Delta^{m+3}(B)\right)\right) . \tag{4.26}
\end{align*}
$$

But we have due to (2.6) with $\kappa=m+2$, as $n \rightarrow \infty$,

$$
n^{(m+3) H-1} \sum_{k=0}^{n-1} h_{m}\left(X_{k / n}^{x}\right) \Delta^{m+2} B_{k / n} \xrightarrow{\text { Prob }}-\frac{(m+3)!}{2^{(m+5) / 2}((m+3) / 2)!} \int_{0}^{1} \sigma h_{m}^{\prime}\left(X_{s}^{x}\right) d s
$$

and also, due this time to (2.6) with $\kappa=m+3$, as $n \rightarrow \infty$,

$$
n^{(m+3) H-1} \sum_{k=0}^{n-1} g_{m}\left(X_{k / n}^{x}\right) \Delta^{m+3} B_{k / n} \xrightarrow{\text { Prob }} \frac{(m+3)!}{2^{(m+3) / 2}((m+3) / 2)!} \int_{0}^{1} g_{m}\left(X_{s}^{x}\right) d s .
$$

Moreover, since $H>2 /(2 m+3)$ we have, as $n \rightarrow \infty$,

$$
n^{(m+3) H+2} \Delta^{3 m+6}(B) \xrightarrow{\text { Prob }} 0 .
$$

On the other hand, since

$$
\begin{align*}
2 \sum_{k=0}^{n-1} \sigma h_{m}^{\prime}\left(X_{k / n}^{x}\right) \Delta^{m+2} B_{k / n} & \sum_{j=0}^{k} h_{m}\left(X_{j / n}^{x}\right) \Delta^{m+2} B_{j / n}
\end{align*}=\left(\sum_{k=0}^{n-1} \sigma h_{m}^{\prime}\left(X_{k / n}^{x}\right) \Delta^{m+2} B_{k / n}\right) ~ 子\left(\sum_{j=0}^{n-1} h_{m}\left(X_{j / n}^{x}\right) \Delta^{m+2} B_{j / n}\right)-\sum_{k=0}^{n-1} \sigma h_{m} h_{m}^{\prime}\left(X_{k / n}^{x}\right) \Delta^{2 m+4} B_{k / n} .
$$

we deduce, this time thanks to (2.6) with $\kappa=m+2$ and (2.6) with $\kappa=2 m+4$, that, as $n \rightarrow \infty$,

$$
n^{(m+3) H-1} \sum_{k=0}^{n-1} \sigma h_{m}^{\prime}\left(X_{k / n}^{x}\right) \Delta^{m+2} B_{k / n} \sum_{j=0}^{k} h_{m}\left(X_{j / n}^{x}\right) \Delta^{m+2} B_{j / n} \xrightarrow{\text { Prob }} 0 .
$$

Finally, we obtain (3.13) when $H>2 /(2 m+3)$.
4. Proof of (3.13) for $H \in(1 /(m+2), 1 / 2)$. It suffices to use (4.25) with the appropriate $M$ for the considered $H$ and then to proceed as in the previous step. The remaining details are left to the reader.
5. Proof of (3.14). By (4.22)-(4.23), we have

$$
\widehat{X}_{1}^{(n)}=\phi\left(x, B_{1}+\sum_{k=0}^{n-1} h_{m}\left(X_{k / n}^{x}\right) \Delta^{m+2} B_{k / n}+\sum_{k=0}^{n-1} g_{m}\left(X_{k / n}^{x}\right) \Delta^{m+3} B_{k / n}+O\left(n^{2} \Delta^{2 m+4}(B)\right)\right) .
$$

When $m \geq 3$ and since $H=1 / 2$, we have, as $n \rightarrow \infty$ :

$$
n^{(m+1) / 2+2} \Delta^{2 m+4}(B) \xrightarrow{\text { Prob }} 0 .
$$

Hence, for $m \geq 3$, (3.14) is easily obtained by (2.2) and these latter two relations. If $m=1$, we rather use (4.26). Moreover, since $H=1 / 2$, we have, as $n \rightarrow \infty$ :

$$
n^{4} \Delta^{9}(B) \xrightarrow{\text { Prob }} 0 \quad \text { and } \quad o\left(n^{2} \Delta^{4}(B)\right) \xrightarrow{\text { Prob }} 0 .
$$

Finally, using these convergences, (2.1), (2.2) and (4.27), we obtain (3.14) also for $m=1$.
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[^1]:    ${ }^{1}$ More precisely, in (3) one does not prove exactly (1.5) but a quite similar result.

[^2]:    ${ }^{2}$ Second Dini's theorem: Let $\left(f_{n}\right)$ be a sequence of real functions defined on $[0,1]$. We assume, for every $x \in[0,1]$, that $f_{n}(x) \rightarrow f(x)$, as $n \rightarrow \infty$, with $f:[0,1] \rightarrow \mathbb{R}$ a continuous function. Moreover, we assume, for any $n \in \mathbb{N}$, that $f_{n}$ is an increasing function. Then $\sup _{x \in[0,1]}\left|f_{n}(x)-f(x)\right| \rightarrow 0$ as $n \rightarrow \infty$.

