



**HAL**  
open science

# Formulary of geodesics of the projected averaged Kepler Hamiltonian

Jean-Baptiste Caillau

► **To cite this version:**

Jean-Baptiste Caillau. Formulary of geodesics of the projected averaged Kepler Hamiltonian. 2007.  
hal-00130214v2

**HAL Id: hal-00130214**

**<https://hal.science/hal-00130214v2>**

Preprint submitted on 12 Feb 2007 (v2), last revised 15 Feb 2007 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Ecole Nationale Supérieure  
d'Electrotechnique, Electronique  
Informatique, Hydraulique et Télécom  
Institut de Recherche en Informatique de Toulouse

**Formulary of geodesics of the projected  
averaged Kepler Hamiltonian**

J.-B. Caillau

Équipe Algorithmes Parallèles et Optimisation  
ENSEEIHT-IRIT (UMR CNRS 5505)  
2 rue Camichel, F-31071 Toulouse

[www.n7.fr/apo](http://www.n7.fr/apo)

Rapport Technique **RT/APO/07/2**



# Formulary of geodesics of the projected averaged Kepler Hamiltonian\*

J.-B. Caillau<sup>†</sup>

February 2007

## Abstract

This short note gives the quadratures of the geodesics of the averaged Hamiltonian of the controlled Kepler equation (energy criterion) projected on  $\mathbf{S}^2$ . The endpoints of the corresponding cut locus are also deduced, as well as the injectivity radius of the associated Riemannian metric on the 2-sphere.

**Keywords.** Averaged Kepler equation, geodesics on  $\mathbf{S}^2$ , cut locus.

**Classification AMS.** 49K15, 70Q05

The Hamiltonian is

$$H = \frac{1}{2} \left( \frac{p_\theta^2}{G(\varphi)} + p_\varphi^2 \right)$$

with

$$G(\varphi) = \frac{\sin^2 \varphi}{1 - (1 - \mu^2) \sin^2 \varphi}.$$

Herebefore,  $\mu$  is a parameter which is equal to  $1/\sqrt{5}$  in the case of the Kepler equation [1, 2, 3, 4]. The system is clearly integrable, and the geodesics on the level set  $\{H = 1/2\}$  of the corresponding Riemannian metric are as parameterized by  $p_\theta$  (note that  $\theta$  is cyclic) as follows:

$$\varphi = \operatorname{asin} \sqrt{\frac{(1+b) - (1-b) \cos(a(t-t_1))}{2}},$$

and

$$\theta = \operatorname{sign}(p_\theta) \left[ \operatorname{atan} \left( \frac{\tan(a(t-t_1)/2)}{\sqrt{b}} \right) \right]_0^t - (1 - \mu^2) p_\theta t.$$

These two quadratures are valid for  $t$  in  $[t_1, t_1 + T/4]$  with nonnegative  $p_{\varphi_0}$ , and extended on  $[t_1, t_1 + T]$  by using the obvious symmetries of the Hamiltonian (a  $-2t_1$  time translation gives the case when  $p_{\varphi_0}$  is negative). The coordinate

---

\*Travail effectué dans le cadre du Réseau d'Excellence HYCON, contrat numéro FP6-IST-511368.

<sup>†</sup>ENSEEIH-IRIT (UMR CNRS 5505), 2 rue Camichel, F-31071 Toulouse (caillau@n7.fr).

$\varphi$  and the time derivative  $\dot{\theta}$  are  $T$ -periodic and hence clearly extended to the whole real line. The period  $T$ , the time  $t_1$ , and the parameters  $a$  and  $b$  have the following expression ( $\theta_0$  is assumed to be zero by symmetry):

$$\begin{aligned} a &= 2\sqrt{1 + (1 - \mu^2)p_\theta^2}, \\ b &= p_\theta^2 / (1 + (1 - \mu^2)p_\theta^2), \\ t_1 &= (1/a) \left[ -\pi/2 - \operatorname{asin} \left( \frac{2 \sin^2 \varphi_0 - (1 + b)}{1 - b} \right) \right], \\ T &= 4\pi/a. \end{aligned}$$

The cut locus [2] of the Riemannian metric is then easily deduced. For symmetry reasons, it is included in the antipodal parallel  $\{\pi - \varphi_0\}$ , and the coordinates of its left endpoint are

$$\begin{aligned} \theta_l &= \pi(1 - (1 - \mu^2) \sin \varphi_0), \\ \varphi_l &= \pi - \varphi_0. \end{aligned}$$

The time, that is the distance to the cut locus is

$$t_l = \pi \sqrt{1 - (1 - \mu^2) \sin^2 \varphi_0},$$

which gives the injectivity radius of the Riemannian metric  $G(\varphi)d\theta^2 + d\varphi^2$  on  $\mathbf{S}^2$ , that is the infimum of distances from a point to its cut locus (clearly reached on the equator, for  $\varphi_0 = \pi/2$ ),

$$i(\mathbf{S}^2) = \mu\pi.$$

Another consequence of the computations is the existence of a simple closed geodesic which is non-trivial, that is different from the equator  $\{\varphi = \pi/2\}$  or a meridian  $\{\theta = \text{cst}\}$ , for  $p_\theta = 5/4$ .

## References

- [1] B. Bonnard and J.-B. Caillau. A global optimality result with application to orbital transfer. *C. R. Acad. Sci. Paris, Ser. I*, submitted.
- [2] B. Bonnard and J.-B. Caillau. Conjugate and cut loci in averaged orbital transfer. *C. R. Acad. Sci. Paris, Ser. I*, submitted.
- [3] B. Bonnard and J.-B. Caillau. Riemannian metric of the averaged energy minimization problem in orbital transfer with low thrust. *An. I. H. Poincaré*, to appear.
- [4] J.-B. Caillau. Sur la géométrie des transferts orbitaux. Habilitation, Institut National Polytechnique de Toulouse, 2006.