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**Formulary of geodesics of the projected  
averaged Kepler Hamiltonian**

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# Formulary of geodesics of the projected averaged Kepler Hamiltonian\*

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## Abstract

This short note gives the quadratures of the geodesics of the averaged Hamiltonian of the controlled Kepler equation (energy criterion) projected on  $\mathbf{S}^2$ . The endpoints of the corresponding cut locus are also deduced, as well as the injectivity radius of the associated Riemannian metric on the 2-sphere.

**Keywords.** Averaged Kepler equation, geodesics on  $\mathbf{S}^2$ , cut locus.

**Classification AMS.** 49K15, 70Q05

The Hamiltonian is

$$H = \frac{1}{2} \left( \frac{p_\theta^2}{G(\varphi)} + p_\varphi^2 \right)$$

with

$$G(\varphi) = \frac{\sin^2 \varphi}{1 - (1 - \mu^2) \sin^2 \varphi}.$$

Herebefore,  $\mu$  is a parameter which is equal to  $1/\sqrt{5}$  in the case of the Kepler equation [1, 2, 3, 4]. The system is clearly integrable, and the geodesics on the level set  $\{H = 1/2\}$  of the corresponding Riemannian metric are as parameterized by  $p_\theta$  (note that  $\theta$  is cyclic) as follows:

$$\varphi = \operatorname{asin} \sqrt{\frac{(1+b) - (1-b) \cos(a(t-t_1))}{2}},$$

and

$$\theta = \operatorname{sign}(p_\theta) \left[ \operatorname{atan} \left( \frac{\tan(a(t-t_1)/2)}{\sqrt{b}} \right) \right]_0^t - (1 - \mu^2) p_\theta t.$$

These two quadratures are valid for  $t$  in  $[t_1, t_1 + T/4]$  with nonnegative  $p_{\varphi_0}$ , and extended on  $[t_1, t_1 + T]$  by using the obvious symmetries of the Hamiltonian (a  $-2t_1$  time translation gives the case when  $p_{\varphi_0}$  is negative). The coordinate

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$\varphi$  and the time derivative  $\dot{\theta}$  are  $T$ -periodic and hence clearly extended to the whole real line. The period  $T$ , the time  $t_1$ , and the parameters  $a$  and  $b$  have the following expression ( $\theta_0$  is assumed to be zero by symmetry):

$$\begin{aligned} a &= 2\sqrt{1 + (1 - \mu^2)p_\theta^2}, \\ b &= p_\theta^2 / (1 + (1 - \mu^2)p_\theta^2), \\ t_1 &= (1/a) \left[ -\pi/2 - \operatorname{asin} \left( \frac{2 \sin^2 \varphi_0 - (1 + b)}{1 - b} \right) \right], \\ T &= 4\pi/a. \end{aligned}$$

The cut locus [2] of the Riemannian metric is then easily deduced. For symmetry reasons, it is included in the antipodal parallel  $\{\pi - \varphi_0\}$ , and the coordinates of its left endpoint are

$$\begin{aligned} \theta_l &= \pi(1 - (1 - \mu^2) \sin \varphi_0), \\ \varphi_l &= \pi - \varphi_0. \end{aligned}$$

The time, that is the distance to the cut locus is

$$t_l = \pi \sqrt{1 - (1 - \mu^2) \sin^2 \varphi_0},$$

which gives the injectivity radius of the Riemannian metric  $G(\varphi)d\theta^2 + d\varphi^2$  on  $\mathbf{S}^2$ , that is the infimum of distances from a point to its cut locus (clearly reached on the equator, for  $\varphi_0 = \pi/2$ ),

$$i(\mathbf{S}^2) = \mu\pi.$$

As consequence of the computations, any geodesic is—up to a rotation in  $\theta$ —a *pseudo-equator*, that is a geodesic generated by  $p_\varphi = 0$  on  $\{H = 1/2\}$ . In Kepler case, every pseudo-equator starting from a rational initial eccentricity is closed. There exist five simple closed geodesic modulo rotations with respect to  $\theta$ , and the shortest closed geodesics are the meridians whose length is  $2\pi$ .

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