# Existence and uniqueness results for the Gradient Vector Flow and geodesic active contours mixed model

## Laurence Guillot, Maïtine Bergounioux

HAL Id: hal-00130116
https://hal.science/hal-00130116v2

Submitted on 11 Feb 2007

# Existence and uniqueness results for the Gradient Vector Flow and geodesic active contours mixed model

Laurence GUILLOT[*], Maïtine BERGOUNIOUX[*]

February 11, 2007

[*]*UMR 6628-MAPMO, Fédération Denis Poisson, Université d'Orléans, BP. 6759, F-45067 Orléans Cedex 2, laurence.guillot@univ-orleans.fr, maitine.bergounioux@univ-orleans.fr*

## Abstract

This article deals with the so called GVF (Gradient Vector Flow) introduced by C. Xu, J.L. Prince [12, 13]. We give existence and uniqueness results for the front propagation flow for boundary extraction that was initiated by Paragios, Mellina-Gottardo et Ralmesh [9, 10]. The model combines the geodesic active contour flow and the GVF to determine the geometric flow. The motion equation is considered within a level set formulation to result an Hamilton-Jacobi equation.

## 1   Introduction

We consider an image segmentation model : $I$ is a given image and we want to detect boundaries without connexity or convexity assumptions on contours. Therefore we are interested in the Gradient Vector Flow (GVF) as a front propagation flow model. This model builds a class of vector fields derived from the image and has been introduced by Chenyang Xu and Jerry L. Prince in [12]. The GVF can be viewed as an external force for active contour models: it allows to solve problems where classical methods convergence fail to deal with boundary concavities. On the other hand, a new front boundary-based geometric flow for boundary extraction was proposed by Paragios, Mellina-Gottardo and Ralmesh in [9, 10]. In this model the GVF is used to revise the geodesic active contour model of V. Caselles, R. Kimmel, G. Sapiro [2] resulting on a bidirectional geometric flow. The classical parametric active contour model was proposed by D.Terzopoulos, A.Witkin et M.Kass [11]. It derives from an energy

1

functional minimization. Curves are drawn toward the object boundaries under potential external forces action which can be written as the negative gradient of a scalar potential function derived from images. Other forces (as pressure forces) may be added. However, the performance of methods is limited by unstable initialization process and poor convergence when the boundary is concave. Chenyang Xu and Jerry L. Prince [12] set a new external force, dealing with these limitations.

Most of time snakes model provide a local minimum of the functional cost. So, C.Xu et JL.Prince generalized this model from the balance equation at the equilibrium between internal and external forces (Euler equation of the minimization problem). They replace the standard external force $F_{ext}^{(p)} = -\nabla P_{\mathbf{x}}$ (where $P$ is a potential edge detector), by a more general external force.

This new external force field is called the Gradient Vector Flow (GVF). It includes a divergence-free component and a curl-free component [5, 14]. Therefore, this new active contour model cannot be formulated as an energy minimization problem. The external force, denoted $V$ below is introduced via the balance equation that can be written :

$$\alpha \frac{d^2\mathbf{x}}{ds^2} - \beta \frac{d^4\mathbf{x}}{ds^4} + V(\mathbf{x}) = 0 \tag{1.1}$$

where $\mathbf{x} = (x, y) \in \mathbb{R}^2$.

The resulting parametrized curve solving the equation (1.1) is called " GVF-snake ". The final configuration of a GVF-snake satisfies an equilibrium equation which is not a variational problem Euler equation, since $V(x, y)$ is not an irrotational field.

In next section, we introduce the GVF and present the advantages of such a field in view of an active contour model and we present and give a precise definition of the Gradient Vector Flow. In section 3. we present this model which is inspired by the geodesic active contour model combined with the GVF. Finally, the level set strategy leads to an Hamilton-Jacobi equation that has not been studied yet (to our knowledge): in the last section we give existence and uniqueness results.

## 2  The Gradient Vector Flow (GVF)

The GVF $V$ is a 2-dimensional vector field that should minimize the following objective function [12, 9].

$$E(V) := \int_{\Omega} \mu(u_x^2 + u_y^2 + v_x^2 + v_y^2) + f|\nabla f|^2 |V - \nabla f|^2 d\mathbf{x} \tag{2.1}$$

where $\Omega$ is an open, bounded subset of $\mathbb{R}^2$, $u_x, u_y, v_x, v_y$ denote the spatial derivatives (with respect to $x$ and $y$) of the field $V = (u, v)$, $\mu > 0$ and $f : \Omega \to \mathbb{R}$ a continuous edge detector.

There are many choices for $f$: in [13], C.Xu and JL.Prince consider

$$f(x,y) = E_{ext}^{(1)}(x,y) = -|\nabla I(x,y)|^2$$

where $I$ is the (intensity) image or

$$f(x,y) = E_{ext}^{(2)}(x,y) = -|\nabla[G_\sigma * I(x,y)]|^2$$

where $G_\sigma$ is a Gaussian kernel $(G_\sigma(x) = \dfrac{1}{2\pi\sigma^2}e^{\frac{-x^2}{2\sigma^2}}$ )so that the image is filtered. Here we choose detector proposed by R. Deriche and O. Faugeras in [3]

$$f(x,y) = h(|\nabla(G_\sigma * I)(x,y)|^2)$$

where

$$h(b) = 1 - \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{b}{2\sigma^2}} . \tag{2.2}$$

Here $|\cdot|$ denotes the $\mathbb{R}^2$- euclidean norm.
The first term of the functional $E$ is a regularization term whereas the second term is a data-driven component. If $f|\nabla f|^2$ is small, the energy is dominated by the first term and we get a slowly varying field. On the other hand, when $f|\nabla f|^2$ is large, the second term forces $V$ to decrease to $\nabla f$. Therefore $V$ is close to the gradient of the edge map when it is large (this is classical external force for snakes) and does not evolve quickly in homogeneous regions (which increase the snake capture area). The parameter $\mu$ govern the tradeoff between the two integrands of the cost functional and should be set according to the image noise level (more noise increases $\mu$) see [13].

## 2.1 On GVF existence

First we precise the notations : $\Omega$ is an open, bounded subset of $\mathbb{R}^2$ with $\mathcal{C}^\infty$ boundary $\Gamma$: it is the image domain. Let $V = (u,v)$ and $W = (\xi,\chi)$ in $\mathcal{H}(\Omega) := H^1(\Omega) \times H^1(\Omega)$; then $DV(\mathbf{x}) = (\nabla u(\mathbf{x}), \nabla v(\mathbf{x})) \in L^2(\Omega)^4$. The inner product in $L^2(\Omega)^4$ is

$$\langle DV, DW \rangle_2 = \int_\Omega \langle \nabla u, \nabla \xi \rangle_{\mathbb{R}^2} + \langle \nabla v, \nabla \chi \rangle_{\mathbb{R}^2} d\mathbf{x} = \langle \nabla u, \nabla \xi \rangle_{L^2(\Omega)^2} + \langle \nabla v, \nabla \chi \rangle_{L^2(\Omega)^2},$$

and the $L^2(\Omega)^4$-norm of $DV$ is denoted $\|DV\|_2$. The space $\mathcal{H}(\Omega) = H^1(\Omega) \times H^1(\Omega)$ is endowed with the norm

$$\|V\|_{\mathcal{H}(\Omega)}^2 = \|V\|_{L^2(\Omega)^2}^2 + \|DV\|_2^2.$$

A first "definition" of the GVF could be the following :

**Definition 2.1** *The Gradient Vector Flow field is defined as a solution of the following optimization problem*

$$(\mathcal{P}) : \qquad \min\{E(V) \mid V \in \mathcal{H}(\Omega)\}. \tag{2.3}$$

*where the energy functional $E(V)$ is defined by (2.1).* Unfortunately, such a definition is not correct since a minimum is not necessarily attained. The study of $E$ will provide another definition of the GVF as the solution to a decoupled system parabolic partial differential equations.

From now and in the sequel we assume that the edge function $f$ verifies $(\mathcal{H}_1)$ :

$$(\mathcal{H}_1) : \qquad f \in \mathcal{C}^1(\overline{\Omega}) \text{ and } f \geq 0$$

**Remark 2.1** *Note that if $f$ satisfies $(\mathcal{H}_1)$ then $f \in H^1(\Omega)$ and $f|\nabla f|^2 \in L^\infty(\Omega)$ which are the "minimal" assumptions in a first step.*

**Proposition 2.1** *: Assume $(\mathcal{H}_1)$. The functional $E$ is continuous on $\mathcal{H}(\Omega)$.*
*Proof* - We get

$$E(V) = \mu\|DV\|_2^2 + \int_\Omega f|\nabla f|^2(|V|^2 - 2\langle V, \nabla f\rangle_{\mathbb{R}^2})d\mathbf{x} + \int_\Omega f|\nabla f|^4 d\mathbf{x} = \pi(V,V) + L(V) + C$$

where

$$\pi(X,Y) = \mu\langle DX, DY\rangle_2 + \int_\Omega f|\nabla f|^2\langle X,Y\rangle_{\mathbb{R}^2}d\mathbf{x}, \quad (X,Y) \in (\mathcal{H}(\Omega))^2 \ ,$$

$$L(X) = -2\int_\Omega f|\nabla f|^2\langle X, \nabla f\rangle d\mathbf{x}, \quad X \in \mathcal{H}(\Omega) \text{ and } C = \int_\Omega f|\nabla f|^4 d\mathbf{x} \ .$$

The bilinear form $\pi$ is continuous on $\mathcal{H}(\Omega)$ for the $H^1(\Omega)^2$-norm : let $X$ et $Y$ be in $\mathcal{H}(\Omega)$,

$$
\begin{aligned}
|\pi(X,Y)| &= \left|\mu\langle DX, DY\rangle_2 + \int_\Omega f|\nabla f|^2\langle X,Y\rangle d\mathbf{x}\right| \\
&\leq \mu\|DX\|_2\|DY\|_2 + \||f|\nabla f|^2\|_{L^\infty(\Omega)}\|X\|_{L^2(\Omega)^2}\|Y\|_{L^2(\Omega)^2} \\
&\leq C(\mu, f)\|X\|_{H^1(\Omega)^2}\|Y\|_{H^1(\Omega)^2}
\end{aligned}
$$

where $C(\mu, f)$ is a constant that only depends on data. $L$ is obviously linear and continuous on $\mathcal{H}(\Omega)$. We deduce the continuity of $E$. $\qquad\qquad\square$

**Theorem 2.1** *The functional $E$ is Gâteaux-differentiable on $\mathcal{H}(\Omega)$ and for every $V = (u,v)$ and $W = (\xi, \chi)$ in $\mathcal{H}(\Omega)$ we get*

$$\langle \nabla E(V), W\rangle = 2\int_\Omega \mu\langle DV, DW\rangle + f|\nabla f|^2\langle V - \nabla f, W\rangle d\mathbf{x} \tag{2.4}$$

*Furthermore, if $\mu \in \mathbb{R}^+$ then $E$ is convex on $\mathcal{H}(\Omega)$.*
*Proof* - The Gâteaux -differentiability is clear. In addition for every $V = (u, v)$ and $W = (\xi, \chi)$ in $\mathcal{H}(\Omega)$ we have

$$E(V + W) - E(V) - \langle \nabla E(V), W \rangle = \int_\Omega \mu |DW|^2 + f|\nabla f|^2 |W|^2 d\mathbf{x} \ ,$$

so that, if $\mu \geq 0$, as $f \geq 0$ by $(\mathcal{H}_1)$ $E(V + W) - E(V) - \langle \nabla E(V), W \rangle \geq 0$. So we get the convexity of $E$ on $\mathcal{H}(\Omega)$. $\qquad\square$

**From now we assume $\mu > 0$.**

**Theorem 2.2** *Assume that $f|\nabla f|^2$ does not degenerate on $\Omega$ :*

$$\exists c > 0 \quad such \ as \ f|\nabla f|^2 \geq c \qquad\qquad (2.5)$$

*then $\pi$ is coercive on $\mathcal{H}(\Omega)$. This implies the coercivity and the strict-convexity of $E$ on $\mathcal{H}(\Omega)$ and the problem $(\mathcal{P})$ has a unique solution.*
*Proof* - $\pi$ has been define above and we get

$$\pi(X, X) = \mu \|DX\|_2^2 + \int_\Omega f|\nabla f|^2 |X|_{\mathbb{R}^2}^2 d\mathbf{x} \geq \mu \|DX\|_2^2 + c \int_\Omega |X|_{\mathbb{R}^2}^2 d\mathbf{x} \geq \min(\mu, c) \|X\|_{H^1(\Omega)^2} \ ;$$

therefore $\pi$ is $H^1(\Omega)^2$-coercive on $\mathcal{H}(\Omega)$ and strictly convex. Therefore $E$ is coercive and strictly convex on $\mathcal{H}(\Omega)$ as well. It follows that if (2.5) is verified then $(\mathcal{P})$ admits a unique solution. $\qquad\square$

Let us formally write the Euler-equations of problem $(\mathcal{P})$: assume $V^* = (u^*, v^*)$ is a solution to $(\mathcal{P})$. By convexity, it is a stationary point and $\nabla E(V^*) = 0$. Let $W = (\xi, \chi) \in \mathcal{H}(\Omega)$. Integrating by parts expression (2.4 ) gives

$$\langle \nabla E(V^*), W \rangle$$
$$= -\mu \int_\Omega (\xi \Delta u^* + \chi \Delta v^*) d\mathbf{x} + \int_\Omega f|\nabla f|^2 \langle V^* - \nabla f, W \rangle_{\mathbb{R}^2} d\mathbf{x} + \int_\Gamma \left( \frac{\partial u^*}{\partial \nu} \xi + \frac{\partial v^*}{\partial \nu} \chi \right) d\sigma \quad (2.6)$$

where $\nu(\mathbf{x})$ is the outer unit normal of $\partial\Omega = \Gamma$ at $\mathbf{x}$.
We first suppose that $\xi$ and $\chi$ belong to $\mathcal{D}(\Omega)$ (the space of $C^\infty(\Omega)$ functions with compact support in $\Omega$). Then

$$-\mu \int_\Omega (\xi \Delta u^* + \chi \Delta v^*) d\mathbf{x} + \int_\Omega f|\nabla f|^2 \langle V^* - \nabla f, W \rangle_{\mathbb{R}^2} d\mathbf{x} = 0$$

So

$$\begin{cases} -\mu \Delta u^* + f|\nabla f|^2 (u^* - f_x) = 0 \text{ in } \mathcal{D}'(\Omega) \\ -\mu \Delta v^* + f|\nabla f|^2 (v^* - f_y) = 0 \text{ in } \mathcal{D}'(\Omega) \end{cases}$$

where $f_x$, $f_y$ are spatial derivatives of $f$. With (2.6), we obtain :

$$\langle \begin{pmatrix} \dfrac{\partial u^*}{\partial \nu} \\ \dfrac{\partial v^*}{\partial \nu} \end{pmatrix}, \begin{pmatrix} \xi|_\Gamma \\ \chi|_\Gamma \end{pmatrix} \rangle = 0$$

Finally, if a solution $V^* = (u^*, v^*)$ to problem $(\mathcal{P})$ exists it must verify :

$$\begin{cases} -\mu \Delta u^* + f |\nabla f|^2 (u^* - f_x) = 0 \ \ \text{in } \Omega, \dfrac{\partial u^*}{\partial \nu} = 0 \text{ on } \Gamma \\ -\mu \Delta v^* + f |\nabla f|^2 (v^* - f_y) = 0 \ \ \text{in } \Omega, \dfrac{\partial v^*}{\partial \nu} = 0 \text{ on } \Gamma \ . \end{cases} \tag{2.7}$$

The above equations are equilibrium equations if $V^*$ realizes the minimum of energy $E$. However, we cannot ensure the existence of such a minimum. Indeed, assumption (2.5) is not realistic : the same gray level for image $I$ on a significant area implies $\nabla f = 0$. In this case the functional $E$ is a priori non longer coercive and we do not know if $(\mathcal{P})$ has a solution. So, instead of computing the "exact" GVF, minimizer of the functional $E$, we approach it by a minimizing sequence and we consider it is the equilibrium state of a time evolving vector fields. The stationary problem becomes a **dynamic** one :

$$V^*(\mathbf{x}) = \lim_{t \to +\infty} V(t, \mathbf{x}).$$

A simple way to impose a motion to the vector field is to impose the velocity $\dfrac{\partial V}{\partial t}(\mathbf{x}, t)$ setting :

$$\frac{\partial V}{\partial t}(\mathbf{x}, t) = -\nabla_V E(V(t, \mathbf{x})) \tag{2.8}$$

This leads to parabolic partial differential equations. The gradient vector flow is initialized as the gradient of the edge detector $f$ :

$$V(t = 0, \mathbf{x}) = V_0(\mathbf{x}) = \nabla f(\mathbf{x}) \text{ on } \Omega \tag{2.9}$$

We obtain the following **dynamic formulation** which is the GVF suitable definition :

**Definition 2.2** *The gradient vector flow $V = (u, v)$ is defined by the following decoupled equations respectively verified by each of its coordinates $u$ and $v$ :*

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mu \Delta u + f |\nabla f|^2 (u - f_x) = 0 & in \ ]0, +\infty[ \times \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & on \ ]0, +\infty[ \times \Gamma \\ u(0, \cdot) = f_x & in \ \Omega \end{cases} \tag{2.10}$$

$$\begin{cases} \dfrac{\partial v}{\partial t} - \mu \Delta v + f|\nabla f|^2 (v - f_y) = 0 & in \ ]0,+\infty[\times\Omega \\ \dfrac{\partial v}{\partial \nu} = 0 & on \ ]0,+\infty[\times\Gamma \\ v(0,\cdot) = f_y & in \ \Omega \end{cases} \qquad (2.11)$$

## 2.2   GVF properties.

In this subsection, we give regularity properties of solutions to (2.10) and (2.11). Of course, it is sufficient to consider equation (2.10). First, the existence of a unique solution is given by a classical theorem (see for example [7]). The bilinear form $a$ associated to equation (2.10) is

$$a(t;u,v) = \mu \int_\Omega \left( \nabla u(\mathbf{x}).\nabla v(\mathbf{x}) + u(\mathbf{x})v(\mathbf{x}) \right) d\mathbf{x} + \int_\Omega f|\nabla f|^2 u(\mathbf{x})v(\mathbf{x}) d\mathbf{x}$$

where $\mu > 0$ and $f|\nabla f|^2 \in L^\infty(]0,T[\times\Omega)$. It satisfies :

1. $t \mapsto a(t;u,v)$ is measurable $\forall u,v \in H^1(\Omega)$,

2. For almost $t \in [0,T]$ and for all $u,v \in H^1(\Omega)$,

$$|a(t;u,v)| \leq C(\mu,f)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

3. For almost $t \in [0,T]$ and for all $v \in H^1(\Omega)$,

$$a(t;v,v) \geq \mu\|v\|_{H^1(\Omega)}^2 + \|f|\nabla f|^2\|_{L^\infty(\Omega)}|v|_{L^2(\Omega)}^2$$

Note that here $a$ does not depend on $t$. So we may assert that (2.10) has a unique solution $u \in L^2(0,T;H^1(\Omega)) \cap \mathcal{C}([0,T];L^2(\Omega))$ and $\dfrac{\partial u}{\partial t} \in L^2(0,T;H^1(\Omega)')$. Moreover

**Theorem 2.3** *Let $T > 0$, and assume $(\mathcal{H}_1)$. Then the GVF $(u,v)$ (solution of (2.10) and (2.11)) is $\mathcal{C}^1$ on $[0,T] \times \overline{\Omega}$.*
*Proof* - We use a generic regularity result ([7]). We prove the result for the component $u$. Assumption $(\mathcal{H}_1)$ yields that $\nabla f \in L^2(\Omega)^2$ and $f|\nabla f|^2\nabla f \in [\mathcal{C}^1([0,T]\times\overline{\Omega})]^2$. So the solution $u$ of (2.10) belongs to $\mathcal{C}^1([\varepsilon,T]\times\overline{\Omega})$ for all $\varepsilon > 0$. Moreover $\nabla f \in \mathcal{C}^0(\overline{\Omega})$ according to $(\mathcal{H}_1)$ and compatibility conditions are satisfied with respect to boundary and initial data. So we may conclude. $\qquad\qquad\square$

The GVF is built as a spatial diffusion of the edge detector $f$ gradient. This is equivalent to a progressive construction of the gradient vector flow starting from the object boundaries

and moving toward the flat background. In [13], the GVF is normalized to obtain a more efficient propagation. It is denoted $\hat{V}(\mathbf{x}) = (\hat{u}(\mathbf{x}), \hat{v}(\mathbf{x}))$ where

$$\hat{u}(\mathbf{x}) = \frac{u(\mathbf{x})}{\sqrt{u(\mathbf{x})^2 + v(\mathbf{x})^2}}, \qquad \hat{v}(\mathbf{x}) = \frac{v(\mathbf{x})}{\sqrt{u(\mathbf{x})^2 + v(\mathbf{x})^2}}$$

to give the new external force of the geometric flow called **GVF-snake model**. The velocity of the contour $C$ is given by the equation :

$$C_t(s, t) = \alpha \frac{d^2 \mathbf{x}}{ds^2} - \beta \frac{d^4 \mathbf{x}}{ds^4} + \hat{V}(\mathbf{x}) \tag{2.12}$$

C. Xu and J.L. Prince have shown that a such flow is not dependent of initial conditions and deal with concave regions. However, it depends on curve parametrization, cannot manage topological changes, and involves second and fourth order derivatives that are difficult to estimate. The objective of N. Paragios, O. Mellina-Gottardo and V. Ralmesh in [9] is to eliminate these shortcomings by integrating the GVF with the geodesic active contour and implementing it using the level set method [8, 10]. Our aim is to study the Hamilton-Jacobi equation derived from this model.

# 3 Geodesic active contours and GVF

## 3.1 Paragios - Mellina-Gottardo - Ralmesh model

N. Paragios, O. Mellina-Gottardo and V. Ralmesh have defined in [9] a new "front propagation flow for boundary extraction". Their geometric model is inspired by the geodesic active contours ([2]) and directly defined by the contour evolution velocity. It is based on the remark that the Gradient Vector Flow field after the rescaling refers to the direction that has to be followed to locally deform the contour and to reach the closest object boundaries. On the other hand, given the fact that the propagation of a contour often occurs along the normal direction, the propagation will be optimal when $\hat{V}$ and the unit outward normal $\nu$ are colinear. So we choose to project the normalized gradient vector flow onto the outward normal. Then we multiply the velocity by an edge detector function $g$ (different from $f$), which represents the contour information. The contour evolution velocity is then given by the equation :

$$\mathcal{C}_t(\mathbf{x}) = \underbrace{g(|\nabla I_\sigma(\mathbf{x})|^2)}_{\text{boundary}} \underbrace{\langle (\hat{u}, \hat{v})(\mathbf{x}), \nu(\mathbf{x}) \rangle_{\mathbb{R}^2}}_{\text{projection}} \nu(\mathbf{x}) \tag{3.13}$$

where $I_\sigma$ is the filtered image and $g(b) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{b}{2\sigma^2}} = 1 - h(b)$ where $h$ is defined by (2.2). When there is no boundary information ( $|\nabla I_\sigma|^2 \ll 1$), the contour evolution is driven by the inner product between the Normalized Gradient Vector Flow (NGVF) and the normal

direction : it is adapted to deal with concave regions. When the curve reaches the object boundaries neighbourhood ( $|\nabla I_\sigma|^2 \simeq +\infty$ ) then $g \approx 0$ that is, the flow becomes inactive and the equilibrium state is reached.

It is classical to impose a regularity condition on the contour propagation adding a curvature term $\kappa$ and a "balloon force" $H$. The evolution equation becomes

$$\mathcal{C}_t(\mathbf{x}) = g(|\nabla I(\mathbf{x})|^2) \left( \underbrace{-\beta\kappa(\mathbf{x})}_{\text{smoothness}} + \underbrace{(1 - |H(\mathbf{x})|)\langle(\hat{u},\hat{v})(\mathbf{x}), \nu(\mathbf{x})\rangle_{\mathbb{R}^2}}_{\text{boundaries attraction}} + \underbrace{H(\mathbf{x})}_{\text{balloon force}} \right) \nu(\mathbf{x}) \quad (3.14)$$

where $\beta > 0$.

## 3.2   Level set implementation

Problems of topologic changes can be solved using the level set method [8]. The moving 2D-curve is viewed as the zero level set of a 3D surface which equation is $z - \Phi(x, y) = 0$. We denote $\Phi_t$ the partial derivative $\dfrac{\partial\Phi}{\partial t}$ of $\Phi$ towards $t$.

**Theorem 3.1** *The evolution of the 3D-surface $\Phi$ is described by :*

$$\begin{cases} \Phi_t(\mathbf{x}) & = g(|\nabla I_\sigma(\mathbf{x})|^2)\left((\beta\kappa(\mathbf{x}) - H(\mathbf{x}))|\nabla\Phi(\mathbf{x})| \right. \\ & \quad \left. -(1 - |H(\mathbf{x})|)\langle\hat{V}(\mathbf{x}), \nabla\Phi(\mathbf{x})\rangle_{\mathbb{R}^2}\right) & in \; ]0, +\infty[\times\Omega, \\ \dfrac{\partial\Phi}{\partial\nu} & = 0 & on \; ]0, +\infty[\times\Gamma, \\ \Phi(0, \cdot) & = \Phi_o & in \; \Omega \end{cases} \quad (3.15)$$

*where $\Phi_o$ is the **signed distance** defined by :*

$$\Phi_o(\mathbf{x}) = \begin{cases} 0 \; if \; \mathbf{x} \in \Gamma \;, \\ \pm \, d \; the \; distance \; between \; \mathbf{x} \; and \; \Gamma \; ; \end{cases}$$

*the positive sign (resp. negative) is chosen if the point $\mathbf{x}$ is outside (resp. inside) $\Gamma$.*
*Proof* - The curve $\mathcal{C}$ is the zero set level of $\Phi$ : $\Phi(t, \mathcal{C}(\mathbf{x})) = 0$. Deriving formally with respect to $t$ gives for almost every $\mathbf{x} \in \Omega$:

$$\Phi_t(\mathbf{x}) + \langle\nabla\Phi(\mathbf{x}), \mathcal{C}_t(\mathbf{x})\rangle_{\mathbb{R}^2} = 0$$

$$\Phi_t(\mathbf{x}) + \langle\nabla\Phi(\mathbf{x}), g(|\nabla I_\sigma(\mathbf{x})|^2)\left(-\beta\kappa(\mathbf{x}) + (1 - |H(\mathbf{x})|)\langle\hat{V}(\mathbf{x}), \nu(\mathbf{x})\rangle_{\mathbb{R}^2} + H(\mathbf{x})\right)\nu(\mathbf{x})\rangle_{\mathbb{R}^2} = 0$$

where $\nu(\mathbf{x}) = \dfrac{\nabla\Phi(\mathbf{x})}{|\nabla\Phi(\mathbf{x})|}$ is the outward unit normal. A short computation gives

$$\Phi_t(\mathbf{x}) = g(|\nabla I_\sigma(\mathbf{x})|^2)\left(\beta\kappa(\mathbf{x}) - (1 - |H(\mathbf{x})|)\langle\hat{V}(\mathbf{x}), \dfrac{\nabla\Phi(\mathbf{x})}{|\nabla\Phi(\mathbf{x})|}\rangle - H(\mathbf{x})\right)|\nabla\Phi(\mathbf{x})|$$

Finally

$$\Phi_t(\mathbf{x}) = g(|\nabla I_\sigma(\mathbf{x})|^2) \left( \underbrace{(\beta\kappa(\mathbf{x})}_{(a)} - \underbrace{H(\mathbf{x}))}_{(c)}|\nabla\Phi(\mathbf{x})| - (1 - |H(\mathbf{x})|) \underbrace{\langle \hat{V}(\mathbf{x}), \nabla\Phi(\mathbf{x}) \rangle}_{(b)} \right)$$

$\square$

The final flow can be decomposed in

• (a) : a term that provides propagation regularity aims and shrinks the curve toward the object boundaries,

• (b) : a bidirectional flow that moves the curve toward the internal and external objects boundaries,

• (c) : an adaptative balloon force that drives the propagation of the curve when the boundary term becomes inactive [10].

## 3.3 Choice of the edge detector $f$

As mentionned before, we focus on the Gaussian edge detector proposed by R. Deriche et O. Faugeras in [3]:

$$f(x, y) = h(|\nabla(G_\sigma * I)(x, y)|^2) \tag{3.16}$$

where $h$ is defined by (2.2). We must ensure that assumption $(\mathcal{H}_1)$ is satisfied for the edge detector $f$. Therefore, we have to set additional hypothesis on the image intensity function that we have denoted $I$.

$I$ is supposed to have compact support included in $\Omega$ that is image frame : we decide that $I$ is equal 0 out of the "true image". $I \in L^\infty(\Omega) \subset L^2(\Omega) \subset L^1(\Omega)$. On the other hand $I \notin H^1(\Omega)$. Indeed, the image gradient norm becomes infinite at objects contours. Furthermore $I \notin \mathcal{C}^1(\Omega)$. We correct this lack of regularity using a filtering process that makes the filtered image $\mathcal{C}^\infty$ (and of course $\mathcal{C}^1$).

The filtered image is supposed to be $\mathcal{C}^\infty$, with compact support. So, we cannot choose $G_\sigma * I$ since the resulting image has no compact support. So we consider a fixed compact subset $X$ of $\Omega$ and $Y$ a compact subset of $\Omega$ containing $X$.

Let us consider $G_\sigma^X$ a $\mathcal{C}^\infty(\Omega)$ projected of the Gaussian kernel $G_\sigma$ such as :

$$G_\sigma^X(\mathbf{x}) = \begin{cases} G_\sigma(\mathbf{x}) \text{ if } \mathbf{x} \in X \\ 0 \text{ if } \mathbf{x} \notin Y \end{cases}$$

So the regularized (filtered) image $I_\sigma = G_\sigma^X * I$ verifies

$$I_\sigma \in \mathcal{C}_c^\infty(\Omega)$$

and

$$\forall \mathbf{x} \in \Omega, \nabla I_\sigma(\mathbf{x}) = \nabla(G_\sigma^X * I)(\mathbf{x}) = \nabla G_\sigma^X * I(\mathbf{x}).$$

Of course the support of $I_\sigma$, contained in $\overline{\text{supp}(G_\sigma^X) + \text{supp}(I)}$, is not necessarily included in $\Omega$, but even if we must extend the frame $\Omega$ of the image, we consider that the filtered image has a compact support in $\Omega$. More precisely :

**Proposition 3.1** *The function*

$$\begin{cases} \Omega \to \mathbb{R} \\ \mathbf{x} \mapsto \nabla I_\sigma(\mathbf{x}) \end{cases} \tag{3.17}$$

*is $\mathcal{C}^\infty(\Omega)$, with compact support in $\Omega$ and so bounded on $\Omega$.*

$$\nabla I_\sigma \in L^\infty(\Omega) \cap \mathcal{C}_c^\infty(\Omega)$$

From now and in the sequel, we denote $I_\sigma$ by $I$ and we make the following hypothesis on $I$:

$$(\mathcal{H}_I): \quad I \in \mathcal{C}_c^\infty(\Omega) \cap W^{1,\infty}(\Omega)$$

Note that $I$ may be extended by 0 to $\overline{\Omega}$. Now we can give $f$ regularity properties :

**Proposition 3.2** *Assume the image $I$ satisfies $(\mathcal{H}_I)$. The function $f$ defined by :*

$$f : \begin{cases} \overline{\Omega} \to \mathbb{R} \\ \mathbf{x} \mapsto f(\mathbf{x}) = h(|\nabla I(\mathbf{x})|^2) = 1 - \dfrac{1}{\sqrt{2\pi}\sigma} e^{-\frac{|\nabla I(\mathbf{x})|^2}{2\sigma^2}} \end{cases} \tag{3.18}$$

*belongs to $\mathcal{C}^\infty(\overline{\Omega})$.*

*Proof* - It is clear since $I$ is $\mathcal{C}^\infty(\overline{\Omega})$, with compact support in $\Omega$ and $h$ is obvioulsy $\mathcal{C}^\infty$. In addition $f$ is nonnegative and bounded by 1 on $\Omega$. □

Let us denote $\widetilde{g}$ the function

$$\mathbf{x} \in \overline{\Omega} \mapsto g(|\nabla I(\mathbf{x})|^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{|\nabla (I)(\mathbf{x})|^2}{2\sigma^2}} \ ,$$

so that $\widetilde{g} = 1 - f \in \mathcal{C}^\infty(\overline{\Omega})$.

In the sequel, we need the following lemma :

**Lemma 3.1** *The function $\sqrt{\widetilde{g}}$ belongs to $\mathcal{C}^1(\overline{\Omega})$ : it is Lipschitz continuous on $\overline{\Omega}$ with constant $K_1$.*

# 4    Propagation equation study

Equation (3.15) has been obtained quite formally with the level-set method. Now, we give existence and uniqueness of a viscosity solution ([1] for example) . Of course, we assume that the image $I$ verifies $(\mathcal{H}_I)$ so that $f$ satisfies $(\mathcal{H}_1)$. The outward unit normal $\nu$ is a $\mathcal{C}^{1,1}$ Lipschitz vector field.

## 4.1 Viscosity theory framework

Let us precise the notations: for $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \in \mathbb{R}^2$, the matrix $\dfrac{1}{|p|^2} \begin{pmatrix} p_1^2 & p_1 p_2 \\ p_1 p_2 & p_2^2 \end{pmatrix}$ is denoted by $\dfrac{p \otimes p}{|p|^2}$.

We choose $\beta \geq 0$ and assume for simplicity that the curvature term $\kappa$ is the standard mean curvature :

$$\kappa(\mathbf{x}) = \operatorname{div}\left(\frac{\nabla\Phi}{|\nabla\Phi|}\right)(\mathbf{x})$$

$$= \left(\frac{\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2}}{|\nabla\Phi|} - |\nabla\Phi|^{-1}\frac{\left(2\frac{\partial\Phi}{\partial x}\frac{\partial\Phi}{\partial y}\frac{\partial^2\Phi}{\partial x\partial y} + \left(\frac{\partial\Phi}{\partial y}\right)^2\frac{\partial^2\Phi}{\partial y^2} + \left(\frac{\partial\Phi}{\partial x}\right)^2\frac{\partial^2\Phi}{\partial x^2}\right)}{|\nabla\Phi|^2}\right)(\mathbf{x}) \qquad (4.19)$$

$$= \frac{\operatorname{Tr}\left(D^2\Phi(\mathbf{x})\right)}{|\nabla\Phi|} - |\nabla\Phi|^{-1}\operatorname{Tr}\left(\frac{\nabla\Phi(\mathbf{x}) \otimes \nabla\Phi(\mathbf{x})}{|\nabla\Phi(\mathbf{x})|^2}D^2\Phi(\mathbf{x})\right)$$

where $D^2\Phi$ is the Hessian matrix of $\Phi$. Equation (3.15) becomes

$$\Phi_t(t,\mathbf{x}) + \widetilde{g}(\mathbf{x})H(\mathbf{x})|\nabla\Phi(\mathbf{x})|$$
$$- \widetilde{g}(\mathbf{x})\beta\left(\operatorname{Tr}\left(D^2\Phi(\mathbf{x})\right) - \operatorname{Tr}\left(\frac{\nabla\Phi(\mathbf{x}) \otimes \nabla\Phi(\mathbf{x})}{|\nabla\Phi(\mathbf{x})|^2}D^2\Phi(\mathbf{x})\right)\right) \qquad (4.20)$$
$$+ \widetilde{g}(\mathbf{x})(1 - |H(\mathbf{x})|)\langle\hat{V}(\mathbf{x}), \nabla\Phi(\mathbf{x})\rangle_{\mathbb{R}^2} = 0$$

So

$$\Phi_t(t,\mathbf{x}) + \widetilde{g}(\mathbf{x})H(\mathbf{x})|\nabla\Phi(\mathbf{x})|$$
$$- \beta\widetilde{g}(\mathbf{x})\operatorname{Tr}\left(\left[I - \frac{\nabla\Phi(\mathbf{x}) \otimes \nabla\Phi(\mathbf{x})}{|\nabla\Phi(\mathbf{x})|^2}\right]D^2\Phi(\mathbf{x})\right) \qquad (4.21)$$
$$+ \widetilde{g}(\mathbf{x})(1 - |H(\mathbf{x})|)\langle\hat{V}(\mathbf{x}), \nabla\Phi(\mathbf{x})\rangle_{\mathbb{R}^2} = 0$$

where $I$ is the identity matrix. Setting

$$A(p) := I - \frac{p \otimes p}{|p|^2}. \qquad (4.22)$$

and $F(\mathbf{x}, p, X) :=$

$$\widetilde{g}(\mathbf{x})H(\mathbf{x})|p| - \beta\widetilde{g}(\mathbf{x})\operatorname{Tr}\left(A(p)X\right) + \widetilde{g}(\mathbf{x})(1 - |H(\mathbf{x})|)\langle\hat{V}(\mathbf{x}), p\rangle_{\mathbb{R}^2} \qquad (4.23)$$

we get

$$\Phi_t + F(\mathbf{x}, \nabla\Phi, D^2\Phi) = 0 . \qquad (4.24)$$

The Hamiltonian $F : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{S}_2 \to \mathbb{R}$, is independent of $t$ and $\Phi$. Here $\mathcal{S}_N$ is $N \times N$ the symmetric matrices space endowed with the classical order.

First, we have to verify we may use the viscosity theory framework.

**Definition 4.1** *Let $\mathcal{F}$ be a function :* $\left\{ \begin{array}{ccc} \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{S}_2 & \to & \mathbb{R} \\ (t, x, p, X) & \mapsto & \mathcal{F}(t, x, p, X) \end{array} \right.$ .

*$\mathcal{F}$ is proper if :*

*- it is a non-decreasing function with respect to the first variable $t$ :*

$$\mathcal{F}(t, x, p, X) \leq \mathcal{F}(s, x, p, X) \; if \; t \leq s. \tag{4.25}$$

*- it is a decreasing function with respect to the last variable $X$*

$$\mathcal{F}(t, x, p, X) \leq \mathcal{F}(t, x, p, Y) \; si \; Y \leq X. \tag{4.26}$$

**Theorem 4.1** *The function $F$ is proper.*

*Proof* - Since $F$ does not explicitly depend on $t$, $F(t, \mathbf{x}, p, X) = F(\mathbf{x}, p, X)$, the condition (4.25) is verified.

Assuming $p \neq 0$. The matrix $A(p)$ defined by (4.22) is semi-definite positive, and can be written as $A = \sigma \sigma^t$. So we have

$$\operatorname{Tr}(AX) = \operatorname{Tr}(\sigma \sigma^t X) = \operatorname{Tr}(\sigma^t X \sigma) = \sum_{i=1}^{2} \sigma_i^t X \sigma_i,$$

where $\sigma_i$ stands for the ith column of $\sigma$.

Let $X, Y \in \mathcal{S}_2$ be given. Let us assume $Y \geq X$, then :

$$\forall i \in \{1, 2\}, \sigma_i^t X \sigma_i \leq \sigma_i^t Y \sigma_i.$$

The constant $\beta$ is positive and the function $g$ as well, so we deduce that :

$$-\beta g(|\nabla I(\mathbf{x})|^2) \operatorname{Tr}(A(p)X) \geq -\beta g(|\nabla I(\mathbf{x})|^2) \operatorname{Tr}(A(p)Y).$$

so $F(\mathbf{x}, p, X) \geq F(\mathbf{x}, p, Y)$.

We conclude that the function $F$ is proper. $\qquad \square$

The general viscosity theory framework is thus well posed.

## 4.2 Ishii and Sato theorem

In what follows we use a theorem by Ishii and Sato [4] that gives existence of viscosity solution to singular degenerate parabolic (Hamilton-Jacobi) equations with nonlinear oblique derivative boundary conditions.

In this subsection we recall this theorem. In the sequel we denote $\rho$ the function defined from $\mathbb{R}^2$ in $\mathbb{R}$ by $\rho(p, q) = \min\left( \dfrac{|p - q|}{\min(|p|, |q|)}, 1 \right)$.

**Theorem 4.2** *([4] Theorem 2.1 p 1079 )*

$H1-$ $\mathcal{F} \in \mathcal{C}([0,T] \times \overline{\Omega} \times \mathbb{R} \times (\mathbb{R}^2 \setminus \{0_{\mathbb{R}_2}\}) \times \mathcal{S}_2)$,

$H2-$ *There exists $\gamma \in \mathbb{R}$ such that for every $(t,x,p,X) \in [0,T] \times \overline{\Omega} \times (\mathbb{R}^2 \setminus \{0_{\mathbb{R}_2}\}) \times \mathcal{S}_2$, the function $\lambda \mapsto \mathcal{F}(t,x,\lambda,p,X) - \gamma\lambda$ is non-decreasing on $\mathbb{R}$.*

$H3-$ *For every $R > 0$, there exists a continuous non-decreasing function $\varphi_R : [0,+\infty[ \to [0,+\infty[$ such that $\varphi_R(0) = 0$ and for every $X,Y \in \mathcal{S}_2$ and $\mu_1, \mu_2 \in [0,+\infty[$ satisfying :*

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \mu_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \mu_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

*then*

$$\begin{aligned} \mathcal{F}(t,x,\lambda,p,X) &- \mathcal{F}(t,y,\lambda,q,-Y) \geq \\ &- \varphi_R \left( \mu_1 (|x-y|^2 + \rho(p,q)^2) + \mu_2 + |p-q| + |x-y|(1 + \max(|p|,|q|)) \right), \end{aligned} \tag{4.27}$$

*$\forall t \in [0,T]$, $x,y \in \overline{\Omega}$, $\lambda \in \mathbb{R}$ such that $|\lambda| \leq R$ and $p,q \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}$.*

$H4-$ *$\mathcal{F}$ is continuous at $(t,x,\lambda,0,0)$ pour tout $(t,x,\lambda) \in [0,T] \times \overline{\Omega} \times \mathbb{R}$ in the following sense:*

$$-\infty < \mathcal{F}_*(t,x,\lambda,0,0) = \mathcal{F}^*(t,x,\lambda,0,0) < +\infty$$

*where $\mathcal{F}^*$ (respectively $\mathcal{F}_*$ ) are the upper (respectively lower) semi-continuous envelopes of $\mathcal{F}$, defined on $[0,T] \times \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \times \mathcal{S}_2$.*

$B1-$ *$B \in \mathcal{C}(\mathbb{R}^2 \times \mathbb{R}^2) \cap \mathcal{C}^{1,1}(\mathbb{R}^2 \times (\mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}))$.*

$B2-$ *Pour tout $x \in \mathbb{R}^2$, the function $p \mapsto B(x,p)$ 1-positively homogeneous with respect to $p$, i.e., $B(x,\lambda p) = \lambda B(x,p), \forall \lambda \geq 0, p \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}$.*

$B3-$ *There exists a positive constant $\theta$ such that $\langle \nu(z), D_p B(z,p) \rangle \geq \theta$ for every $z \in \partial\Omega$ and $p \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}$.*
*Here $\nu(z)$ is the unit outer normal vector of $\Omega$ at $z \in \partial\Omega$.*

*Assume [H1, H2, H3, H4, B1, B2, B3] are satisfied and consider the following problem*

$$(\mathcal{S}) \begin{cases} \phi_t + \mathcal{F}(t,x,\phi,\nabla\phi,D^2\phi) = 0, & \text{in } \mathcal{Q} :=]0,T[\times\Omega \\ B(x,\nabla\phi) = 0 & \text{on } ]0,T[\times\partial\Omega \end{cases} \tag{4.28}$$

• *Let $\phi \in USC([0,T[\times\overline{\Omega})$ and $\psi \in LSC([0,T[\times\overline{\Omega})$ be, respectively, viscosity sub and super-solutions of (4.28). If $\phi(0,x) \leq \psi(0,x)$ for $x \in \overline{\Omega}$, then $\phi \leq \psi$ sur $]0,T[\times\overline{\Omega}$.*
• *For every function $g \in \mathcal{C}(\overline{\Omega})$ there exists a unique viscosity solution $\phi \in \mathcal{C}([0,T[\times\overline{\Omega})$ of (4.28) such that $\phi(0,x) = g(x)$ on $\overline{\Omega}$.*

Here $USC([0,T[\times\overline{\Omega})$ (respectively $LSC([0,T[\times\overline{\Omega}))$ denote the set of upper semicontinuous (respectively lower semicontinuous) functions.

## 4.3   Existence and uniqueness of the solution of the evolution problem

In the sequel we assume that the balloon force $H$ verifies $(\mathcal{H}_2)$ :

$$(\mathcal{H}_2) : \qquad H \text{ is Lipschitz continuous on } \overline{\Omega}.$$

In order to use Ishii-Sato theorem we have to verify every hypothesis. The Hamiltonian $\mathcal{F}(t, x, r, p, X) = F(\mathbf{x}, p, X)$ is defined by (4.23):

$$F(\mathbf{x}, p, X) = \widetilde{g}(\mathbf{x})H(\mathbf{x})|p| - \beta\widetilde{g}(\mathbf{x})\operatorname{Tr}(A(p)X) + \widetilde{g}(\mathbf{x})(1 - |H(\mathbf{x})|)\langle \hat{V}(\mathbf{x}), p\rangle \ . \qquad (4.29)$$

Let us define the symmetric, semi-definite positive matrix $A(\mathbf{x}, p)$ :

$$A(\mathbf{x}, p) = \beta\,\widetilde{g}(\mathbf{x})(I - \frac{p \otimes p}{|p|^2}),$$

so that

$$F(\mathbf{x}, p, X) = -\operatorname{Tr}(A(\mathbf{x}, p)X) + \widetilde{g}(\mathbf{x})H(\mathbf{x})|p| + \widetilde{g}(\mathbf{x})(1 - |H(\mathbf{x})|)\langle \hat{V}(\mathbf{x}), p\rangle \ . \qquad (4.30)$$

In this case $F$ does not depend neither on $t$ nor $\lambda$.
We choose a Neumann-type boundary condition :

$$\langle \nabla\phi(\mathbf{x}), \nu(\mathbf{x})\rangle_{\mathbb{R}^2} = \frac{\partial\Phi}{\partial\nu}(\mathbf{x}) = 0 \text{ on } \Gamma = \partial\Omega$$

[H1]   Function $F \in \mathcal{C}(\overline{\Omega} \times (\mathbb{R}^2 \setminus \{0_{\mathbb{R}_2}\}) \times \mathcal{S}_2)$. $p = 0$ is a singular point.
Theorem 2.3 yields that the gradient vector flow $V$ is continuous on $[0, T] \times \overline{\Omega}$.

In addition we have assumed that the balloon function $H$ is continuous on $\overline{\Omega}$ .
Therefore, the Hamiltonian $F$ is continuous on $\overline{\Omega} \times \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\} \times \mathcal{S}_2$.

[H2]   Let us show there exists a constant $\gamma \in \mathbb{R}$ such that for each $(x, p, X) \in \overline{\Omega} \times (\mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}) \times \mathcal{S}_2$, the function $\lambda \mapsto F(x, p, X) - \gamma\lambda$ is non-decreasing on $\mathbb{R}$. Since $F$ does not explicitly depend on $\lambda$, any negative constant $\gamma$ is suitable.

[H3]   As $F$ does not depend on $t$ and $r$, we have to find a continuous increasing function $\varphi : [0, +\infty[ \to [0, +\infty[$ satisfying $\varphi(0) = 0$ such that if $X, Y \in \mathcal{S}_2$ and $\mu_1, \mu_2 \in [0, +\infty[$ satisfy :

$$\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \mu_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \mu_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \qquad (4.31)$$

then $\forall \mathbf{x}, \mathbf{y} \in \overline{\Omega}$, and $p, q \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}$

$$F(\mathbf{x}, p, X) - F(\mathbf{y}, q, -Y)$$

$$\geq -\varphi\left(\mu_1(|\mathbf{x} - \mathbf{y}|^2 + \rho(p, q)^2) + \mu_2 + |p - q| + |\mathbf{x} - \mathbf{y}|(1 + \max(|p|, |q|))\right),$$

Let use the following lemma [6] :

**Lemma 4.1** *If $p, q \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}$, then :*

$$\left| \frac{p}{|p|} - \frac{q}{|q|} \right| \leq \frac{|p - q|}{\min(|p|, |q|)} := \rho(p, q).$$

Given $X, Y \in \mathcal{S}_2$ and $\mu_1, \mu_2 \in [0, +\infty[$ verifying (4.31). Let be $r, s \in \mathbb{R}^2$, so we have

$$\langle Xr, r \rangle + \langle Ys, s \rangle \leq \mu_1 |r - s|^2 + \mu_2(|r|^2 + |s|^2)$$

Let $\mathbf{x}, \mathbf{y} \in \overline{\Omega}$ and $p, q \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}$. Following C. Le Guyader [6], we split $F$ in three terms and verify [H3] for each term.
$F(\mathbf{x}, p, X) - F(\mathbf{y}, q, -Y) =$

$$- \underbrace{(\mathrm{Tr}\,(A(\mathbf{x}, p)X) + \mathrm{Tr}\,(A(\mathbf{y}, q)Y))}_{(a)}$$

$$+ \underbrace{\widetilde{g}(\mathbf{x})H(\mathbf{x})|p| - \widetilde{g}(\mathbf{y})H(\mathbf{y})|q|}_{(b)} \tag{4.32}$$

$$+ \underbrace{\widetilde{g}(\mathbf{x})(1 - |H(\mathbf{x})|)\langle \hat{V}(\mathbf{x}), p \rangle_{\mathbb{R}^2} - \widetilde{g}(\mathbf{y})(1 - |H(\mathbf{y})|)\langle \hat{V}(\mathbf{y}), q \rangle_{\mathbb{R}^2}}_{(c)} \,.$$

- **(a) estimate.-** As $A(\mathbf{x}, p) = \sigma(\mathbf{x}, p)\sigma^t(\mathbf{x}, p)$,

$$\mathrm{Tr}\,(A(\mathbf{x}, p)X) + \mathrm{Tr}\,(A(\mathbf{y}, q)Y) \leq$$

$$\mu_1 \mathrm{Tr}\,\left((\sigma(\mathbf{x}, p) - \sigma(\mathbf{y}, q))(\sigma(\mathbf{x}, p) - \sigma(\mathbf{y}, q))^t\right) + \mu_2\left(\mathrm{Tr}\,(\sigma(\mathbf{x}, p)\sigma^t(\mathbf{y}, p)) + \mathrm{Tr}\,(\sigma(\mathbf{y}, q)\sigma^t(\mathbf{y}, q))\right)$$

$$\leq \mu_1 \mathrm{Tr}\,\underbrace{\left((\sigma(\mathbf{x}, p) - \sigma(\mathbf{y}, q))(\sigma(\mathbf{x}, p) - \sigma(\mathbf{y}, q))^t\right)}_{(a1)} + 2\mu_2\beta\delta$$

where $\delta$ is an upper bound of $\widetilde{g}$ on $\overline{\Omega}$ (for example $\frac{1}{\sqrt{2\pi\sigma}}$). Expression (a1) verifies :

$$\mathrm{Tr}\,\left((\sigma(\mathbf{x}, p) - \sigma(\mathbf{y}, q))(\sigma(\mathbf{x}, p) - \sigma(\mathbf{y}, q))^t\right)$$
$$= \mathrm{Tr}\,\left(\sigma(\mathbf{x}, p)\sigma^t(\mathbf{x}, p) - \sigma(\mathbf{x}, p)\sigma^t(\mathbf{y}, q) - \sigma(\mathbf{y}, q)\sigma^t(\mathbf{x}, p) + \sigma(\mathbf{y}, q)\sigma^t(\mathbf{y}, q)\right)$$
$$= \mathrm{Tr}\,\left(A(\mathbf{x}, p) - \sigma(\mathbf{x}, p)\sigma^t(\mathbf{y}, q) - \sigma(\mathbf{y}, q)\sigma^t(\mathbf{x}, p) + A(\mathbf{y}, q)\right)$$
$$= \beta\widetilde{g}(\mathbf{x}) + 2\beta\sqrt{\widetilde{g}(\mathbf{x})}\sqrt{\widetilde{g}(\mathbf{y})}\left(-2 + \frac{1}{|p||q|}(p_1 q_1 + p_2 q_2)\right) + \beta\widetilde{g}(\mathbf{y})$$

and with

$$\left(-2 + \frac{1}{|p||q|}(p_1 q_1 + p_2 q_2)\right) \leq -\frac{1}{|p||q|}(p_1 q_1 + p_2 q_2)$$

we get :

$$\text{Tr}\left((\sigma(\mathbf{x},p)-\sigma(\mathbf{y},q))(\sigma(\mathbf{x},p)-\sigma(\mathbf{y},q))^t\right)$$
$$\leq \beta\widetilde{g}(\mathbf{x}) - 2\beta\sqrt{\widetilde{g}(\mathbf{x})}\sqrt{\widetilde{g}(\mathbf{y})}\frac{1}{|p||q|}(p_1q_1+p_2q_2) + \beta\widetilde{g}(\mathbf{y})$$
$$= \left|\sqrt{\beta\widetilde{g}(\mathbf{x})}\frac{p}{|p|} - \sqrt{\beta\widetilde{g}(\mathbf{y})}\frac{q}{|q|}\right|^2$$

We finally obtain

$$\text{Tr}\left(A(\mathbf{x},p)X\right) + \text{Tr}\left(A(\mathbf{y},q)Y\right) \leq \mu_1\left|\sqrt{\beta\widetilde{g}(\mathbf{x})}\frac{p}{|p|} - \sqrt{\beta\widetilde{g}(\mathbf{y})}\frac{q}{|q|}\right|^2 + 2\mu_2\beta\delta \qquad (4.33)$$

Moreover we have the relation :

$$\left|\sqrt{\beta\widetilde{g}(\mathbf{x})}\frac{p}{|p|} - \sqrt{\beta\widetilde{g}(\mathbf{y})}\frac{q}{|q|}\right|^2 \leq \left|\left(\sqrt{\beta\widetilde{g}(\mathbf{x})} - \sqrt{\beta\widetilde{g}(\mathbf{y})}\right)\frac{p}{|p|} + \sqrt{\beta\widetilde{g}(\mathbf{y})}\left(\frac{p}{|p|} - \frac{q}{|q|}\right)\right|^2$$
$$\leq 2\beta\left(\sqrt{\widetilde{g}(\mathbf{x})} - \sqrt{\widetilde{g}(\mathbf{y})}\right)^2 + 2\beta\widetilde{g}(\mathbf{y})\left|\frac{p}{|p|} - \frac{q}{|q|}\right|^2.$$

So we get by lemma 3.1 :

$$\left|\sqrt{\beta\widetilde{g}(\mathbf{x})}\frac{p}{|p|} - \sqrt{\beta\widetilde{g}(\mathbf{y})}\frac{q}{|q|}\right|^2 = \left|\sqrt{\beta\widetilde{g}(\mathbf{x})}\frac{p}{|p|} - \sqrt{\beta\widetilde{g}(\mathbf{y})}\frac{p}{|p|} + \sqrt{\beta\widetilde{g}(\mathbf{y})}\left(\frac{p}{|p|} - \frac{q}{|q|}\right)\right|^2$$
$$\leq 2\beta K_1^2|\mathbf{x}-\mathbf{y}|^2 + 2\beta\delta\left|\frac{p}{|p|} - \frac{q}{|q|}\right|^2.$$

Thus we conclude with lemma 4.1.

$$\text{Tr}\left(A(\mathbf{x},p)X\right) + \text{Tr}\left(A(\mathbf{y},q)Y\right) \leq \mu_1\left(2\beta K_1^2|\mathbf{x}-\mathbf{y}|^2 + 2\beta\delta 4\rho(p,q)^2\right) + 2\mu_2\beta\delta \qquad (4.34)$$

• **(b) estimate.-** We have assumed $H$ to be Lipschitz continuous on $\overline{\Omega}$ so $\mathbf{x} \in \overline{\Omega} \mapsto \widetilde{g}(\mathbf{x})H(\mathbf{x})$ is Lipschitz continuous and bounded as well. Let be $\mathbf{x}, \mathbf{y} \in \overline{\Omega}$

$$|\widetilde{g}(\mathbf{x})H(\mathbf{x})|p| - \widetilde{g}(\mathbf{y})H(\mathbf{y})|q|| \leq |(\widetilde{g}(\mathbf{x})H(\mathbf{x}) - \widetilde{g}(\mathbf{y})H(\mathbf{y}))|p|| + |\widetilde{g}(\mathbf{y})H(\mathbf{y})||p| - |q|||$$
$$\leq K_2|\mathbf{x}-\mathbf{y}|\max(|p|,|q|) + \theta||p| - |q||$$
$$\leq K_2|\mathbf{x}-\mathbf{y}|\max(|p|,|q|) + \theta|p - q|$$

where $K_2$ is the Lipschitz-constant of the function $\mathbf{x} \in \overline{\Omega} \mapsto \widetilde{g}(\mathbf{x})H(\mathbf{x})$ and $\theta$ is a bound of $\mathbf{x} \in \overline{\Omega} \mapsto |\widetilde{g}(\mathbf{x})H(\mathbf{x})|$.

• **(c) estimate.-** We have assumed $(\mathcal{H}_2)$ so $\mathbf{x} \in \overline{\Omega} \mapsto \widetilde{g}(\mathbf{x})(1 - |H(\mathbf{x})|)\hat{V}(\mathbf{x})$ is Lipschitz continuous and bounded.

$$\left| \widetilde{g}(\mathbf{x})(1 - |H(\mathbf{x})|)\langle \hat{V}(\mathbf{x}), p \rangle_{\mathbb{R}^2} - \widetilde{g}(\mathbf{y})(1 - |H(\mathbf{y})|)\langle \hat{V}(\mathbf{y}), q \rangle_{\mathbb{R}^2} \right|$$

$$\leq \left| \langle \widetilde{g}(\mathbf{x})(1 - |H(\mathbf{x})|)\hat{V}(\mathbf{x}) - \widetilde{g}(\mathbf{y})(1 - |H(\mathbf{y})|)\hat{V}(\mathbf{y}), p \rangle_{\mathbb{R}^2} \right|$$

$$+ \left| \langle \widetilde{g}(\mathbf{y})(1 - |H(\mathbf{y})|)\hat{V}(\mathbf{y}), p - q \rangle_{\mathbb{R}^2} \right|$$

$$\leq K_3 |\mathbf{x} - \mathbf{y}| \max(|p|, |q|) + \zeta |p - q|$$

where $K_3$ is the Lipschitz-constant of the function $\mathbf{x} \in \overline{\Omega} \mapsto \widetilde{g}(\mathbf{x})(1 - |H(\mathbf{x})|)\hat{V}(\mathbf{x})$ and $\zeta$ is a bound of $\mathbf{x} \in \overline{\Omega} \mapsto |\widetilde{g}(\mathbf{x})(1 - |H(\mathbf{x})|)\hat{V}(\mathbf{x})|$.
Finally relation (4.32) gives

$$
\begin{aligned}
-(F(\mathbf{x}, p, X) - F(\mathbf{y}, q, -Y)) \quad &\leq \mu_1 \left( 2\beta K_1^2 |\mathbf{x} - \mathbf{y}|^2 + 8\beta\delta\rho(p, q)^2 \right) + 2\mu_2\beta\delta \\
&+ (K_2 |\mathbf{x} - \mathbf{y}| \max(|p|, |q|) + \theta |p - q|) \\
&+ (K_3 |\mathbf{x} - \mathbf{y}| \max(|p|, |q|) + \zeta |p - q|).
\end{aligned}
$$

$$
\begin{aligned}
&\leq \max(2\beta K_1^2, 8\beta\delta, K_2 + K_3, \theta + \zeta)[(\mu_1(|\mathbf{x} - \mathbf{y}|^2 + \rho(p, q)^2) \\
&+ \mu_2) + \max(|p|, |q|)|\mathbf{x} - \mathbf{y}| + |p - q|]
\end{aligned}
$$

$$
\begin{aligned}
&\leq \max(2\beta K_1^2, 8\beta\delta, K_2 + K_3, \theta + \zeta)\left( \mu_1(|\mathbf{x} - \mathbf{y}|^2 + \rho(p, q)^2) \right. \\
&\left. + \mu_2 + (1 + \max(|p|, |q|))|\mathbf{x} - \mathbf{y}| + |p - q| \right) .
\end{aligned}
$$

Finally

$$F(\mathbf{x}, p, X) - F(\mathbf{y}, q, -Y)$$
$$\geq -\varphi \left( \mu_1(|\mathbf{x} - \mathbf{y}|^2 + \rho(p, q)^2) + \mu_2 + (1 + \max(|p|, |q|))|\mathbf{x} - \mathbf{y}| + |p - q| \right)$$

where the function $\varphi$ is defined by :

$$\varphi(m) = \max(2\beta K_1^2, 8\beta\delta, K_2 + K_3, \theta + \zeta)m$$

Hypothesis $[H3]$ is then verified.

[H4] $F$ is continuous at $(\mathbf{x}, 0, 0)$ for any $\mathbf{x} \in \overline{\Omega}$ because $F_*(\mathbf{x}, 0, 0) = F^*(\mathbf{x}, 0, 0) = 0$.

For the three last hypothesis $B1$, $B2$ and $B3$ on the boundary condition, the proof is the same as in C. Le Guyader [6].

[B1] This hypothesis consists in showing that the function $B$ which defines the boundary condition on $]0, +\infty[\times\partial\Omega$ is $\mathcal{C}(\mathbb{R}^N \times \mathbb{R}^N) \cap \mathcal{C}^{1,1}(\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0_{\mathbb{R}^N}\}))$. We have chosen a Neumann-type condition, by denoting $\nu(\mathbf{x})$ the outward unit normal vector of $\partial\Omega$ at the point $\mathbf{x}$, our boundary condition is written :

$$B(\mathbf{x}, p) = \langle \nu(\mathbf{x}), p \rangle_{\mathbb{R}^2}.$$

The point $[B1]$ will be satisfied if $\nu$ is a $\mathcal{C}^{1,1}$ vector field which is the case if $\Omega$ has a $\mathcal{C}^2$ boundary.

[B2] $B$ is 1-positively homogeneous with respect to $p$ :

$$B(\mathbf{x}, \lambda p) = \langle \nu(\mathbf{x}), \lambda p \rangle_{\mathbb{R}^2} = \lambda B(\mathbf{x}, p), \forall \lambda \geq 0, p \in \mathbb{R}^2 \setminus \{0_{\mathbb{R}^2}\}.$$

[B3] Let $z \in \partial\Omega$.

$$\langle \nu(z), D_p B(z, p) \rangle_{\mathbb{R}^2} = |\nu(z)|^2 = 1.$$

The last condition is verified with $\theta = 1$,

We may now conclude since assumptions [ $H1$, $H2$, $H3$, $H4$, $B1$, $B2$ , $B3$] are satisfied.

**Theorem 4.3** *Assume that the image function $I$ verifies the hypothesis $(\mathcal{H}_I)$ and the ballon force $H$ satisfies $(\mathcal{H}_2)$. Consider the following problem*

$$\begin{cases} \Phi_t(t, \mathbf{x}) - g(|\nabla I(\mathbf{x})|^2) \left( (\beta\kappa(\mathbf{x}) - H(\mathbf{x}))|\nabla\Phi(\mathbf{x})| - (1 - |H(\mathbf{x})|)\langle \hat{V}(\mathbf{x}), \nabla\Phi(\mathbf{x}) \rangle \right) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad in \ ]0, +\infty[\times\Omega, \qquad (4.35) \\ \dfrac{\partial\Phi}{\partial\nu}(\mathbf{x}) = 0 \ \ on \ \ \ ]0, +\infty[\times\partial\Omega \ , \end{cases}$$

• *Let $\Phi \in USC([0, T[\times\overline{\Omega})$ and $\Psi \in LSC([0, T[\times\overline{\Omega})$ be, respectively, viscosity sub and supersolutions of : If $\Phi(0, x) \leq \Psi(0, x)$ for $x \in \overline{\Omega}$, then $\Phi \leq \Psi$ in $]0, T[\times\overline{\Omega}$.*
• *For every $\Phi_o \in \mathcal{C}(\overline{\Omega})$, there is a unique viscosity solution $\Phi \in \mathcal{C}([0, T[\times\overline{\Omega})$ of (4.35) satisfying $\Phi(0, x) = \Phi_o(x)$ on $\Omega$.*
• *Equation (3.15) has a unique viscosity solution $\Phi \in \mathcal{C}([0, T[\times\overline{\Omega})$.*

## 5   Conclusion

We have recalled Gradient Vector Flow model that we have justified and we have given regularity properties. Then we proved existence and uniqueness for viscosity solutions of the Hamilton-Jacobi equation derived from the GVF-geodesic active contour model.

Next step is to perform the numerical realization of this GVF-geodesic active contour process solving the Hamilton-Jacobi equation (4.35). We shall use this method to perform

"tuffeau" tomographic images segmentation. This material has been used during past centuries to build monuments as castles and churches in the Val-de-Loire area. These images allow to get information on the structure of the damaged material : we have to identify different phases as calcite (light grey), silice (dark grey) and porosity (black).

We shall combine the GVF-geodesic model with a region segmentation approach to identify the three constituents of tuffeau. The segmentation of these images is a step of pretreatment which aims at reconstructing the stone porosity domain.

# References

[1] **G. Barles**, *Solutions de viscosité des équations d'Hamilton-Jacobi*, Collection Mathématiques et Applications, Vol. 17, Springer-Verlag, 1994.

[2] **V. Caselles, R. Kimmel, G. Sapiro**, *Geodesic Active Contours*, International Journal of Computer Vision, Vol. 22, 1, 1997, pp. 61-79.

[3] **R. Deriche, O. Faugeras**, *Les EDP en Traitement des Images et Vision par Ordinateur*, Traitement du Signal, 13, 1996.

[4] **H. Ishii, M-H. Sato**, *Nonlinear oblique derivative problems for singular degenerate parabolic equations on a general domain*, Nonlinear Analysis, Vol 57, 2004, pp. 1077-1098.

[5] **S. Kichenassamy, A. Kumar, P. Olver, A. Tannenbaum, A. Yezzi**, *Gradient Flows and Geometric Active Contour Models*, Proc. IEEE Int'l Conf. Computer Vision, vol.1, pp. 67-73, 2001.

[6] **C. Le Guyader**, *Imagerie mathématique : segmentation sous contraintes géométriques*, Thèse, Université de Rouen, Décembre 2004.

[7] **J.L. Lions, E. Magenes**, *Problèmes aux limites non homogènes et applications*, Volume 3, Dunod, 1968.

[8] **S. Osher, Sethian** *Fronts propagating with curvature-dependent velocity : algorithms based on the Hamilton-Jacobi formulation*, Journal of Computational Physics, 79, pp. 12-49, 1988.

[9] **N. Paragios, O. Mellina-Gottardo, V. Ralmesh**, *Gradient Vector Flow Fast Geodesic Active Contours*, Proc. IEEE Int'l Conf. Computer Vision, vol.1, pp. 67-73, 2001.

[10] **N. Paragios, O. Mellina-Gottardo, V. Ralmesh**, *Gradient Vector Flow Fast Geometric Active Contours*, IEEE Transactions on pattern analysis and Machine intelligence, vol. 26, 3 , pp. 67-73, March 2004.

[11] **D. Terzopoulos, A. Witkin, M. Kass**, *Snakes : Active contour models*, International Journal of Computer Vision, Vol. 1, 1988, pp. 321-331.

[12] **C. Xu, J.L. Prince**, *Gradient Vector Flow : A new external force for snakes*, IEEE Proc. Conf. on Comp. Vis. Patt. Recog. (CVPR'97), 1997, pp. 66-71.

[13] **C. Xu, J.L. Prince**, *Snakes, Shapes, and Gradient Vector Flow*, IEEE Transactions on image processing, Vol.7, 3, pp. 359-369, March 1998.

[14] **C. Xu, A. Yezzi, J.L. Prince**, *On the relationship between parametric and geometric active contours*, In Proc. of 34th Asilomar Conference on Signals, Systems, and Computers, pp. 483-489, October 2000.