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FORMAL GEOMETRIC QUANTIZATION

PAUL-EMILE PARADAN

ABSTRACT. Let K be a compact Lie group acting in an Hamiltonian way on a symplectic manifold (M, Ω) which is prequantized by a Kostant-Souriau line bundle. We suppose here that the moment map Φ is *proper* so that the reduced space $M_\mu := \Phi^{-1}(K \cdot \mu)/K$ are compact for all μ . Following Weitsman [33], we define the “formal geometric quantization” of M as

$$\mathcal{Q}_K^{-\infty}(M) := \sum_{\mu \in \hat{K}} \mathcal{Q}(M_\mu) V_\mu^K.$$

The aim of this article is to study the functorial properties of the assignment $M \rightarrow \mathcal{Q}_K^{-\infty}(M)$.

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1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let M be an Hamiltonian K -manifold with symplectic form Ω and moment map $\Phi : M \rightarrow \mathfrak{k}^*$. We assume the existence of a K -equivariant line bundle L on M with connection of curvature equal to $-i\omega$. In other words M is pre-quantizable in the sense of [20] and we call L the Kostant-Souriau line bundle.

In the process of quantization one tries to associate a unitary representation of K to the data (M, Ω, Φ, L) . When M is **compact** one associates to these data a virtual representation $\mathcal{Q}_K(M) \in R(K)$ of K defined as the equivariant index of a Dolbeault-Dirac operator : $\mathcal{Q}_K(M)$ is the geometric quantization of M .

This quantization process satisfies the following functorial properties :

[P1] When N and M are respectively pre-quantized compact Hamiltonian K_1 and K_2 manifolds, the product $M \times N$ is a pre-quantized compact Hamiltonian $K_1 \times K_2$ -manifold, and we have

$$(1.1) \quad \mathcal{Q}_{K_1 \times K_2}(M \times N) = \mathcal{Q}_{K_1}(M) \otimes \mathcal{Q}_{K_2}(N) \quad \text{in} \quad R(K_1 \times K_2) \simeq R(K_1) \otimes R(K_2).$$

[P2] If $H \subset K$ is a connected Lie subgroup, then the restriction of $\mathcal{Q}_K(M)$ to H is equal to $\mathcal{Q}_H(M)$.

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Note that [P1] and [P2] gives the following functorial property :

[P3] When N and M are pre-quantized compact Hamiltonian K -manifold, the product $M \times N$ is a pre-quantized compact Hamiltonian K -manifold, and we have $\mathcal{Q}_K(M \times N) = \mathcal{Q}_K(M) \cdot \mathcal{Q}_K(N)$, where \cdot denotes the product in $R(K)$.

One other fundamental property is the behaviour of the K -multiplicities of $\mathcal{Q}_K(M)$ that is known as “quantization commutes with reduction”.

Let T be a maximal torus of K , $C_K \subset \mathfrak{t}^*$ be a Weyl Chamber, $\Lambda^* \subset \mathfrak{t}^*$ be the weight lattice, and $\widehat{K} = \Lambda^* \cap C_K$ be the set of dominant weights. The ring of character $R(K)$ as a \mathbb{Z} -basis $V_\mu^K, \mu \in \widehat{K} : V_\mu^K$ is the irreducible K -representation with highest weight μ .

For any $\mu \in \widehat{K}$ which is a regular value of Φ , the reduced space (or symplectic quotient) $M_\mu := \Phi^{-1}(K \cdot \mu)/K$ is an orbifold equipped with a symplectic structure Ω_μ . Moreover $L_\mu := (L|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/K_\mu$ is a Kostant line orbundle over (M_μ, Ω_μ) . The definition of the index of the Dolbeault-Dirac operator carries over to the orbifold case, hence $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is defined. In [25], this is extended further to the case of singular symplectic quotients, using partial (or shift) de-singularization. So the integer $\mathcal{Q}(M_\mu) \in \mathbb{Z}$ is well defined for every $\mu \in \widehat{K}$: in particular $\mathcal{Q}(M_\mu) = 0$ if $\mu \notin \Phi(M)$.

The following Theorem was conjectured by Guillemin-Sternberg [16] and is known as “quantization commutes with reduction” [24, 25, 30, 28]. For complete references on the subject the reader should consult [29, 32].

Theorem 1.1. (*Meinrenken, Meinrenken-Sjamaar*). *We have the following equality in $R(K)$*

$$\mathcal{Q}_K(M) = \sum_{\mu \in \widehat{K}} \mathcal{Q}(M_\mu) V_\mu^K .$$

Suppose now that M is **non-compact** but the moment map $\Phi : M \rightarrow \mathfrak{t}^*$ is assumed to be **proper** (we will said simply ” M is proper”). In this situation the geometric quantization of M as an index of an elliptic operator is not well defined. Nevertheless the integers $\mathcal{Q}(M_\mu), \mu \in \widehat{K}$ are well defined since the symplectic quotient M_μ are **compact**.

Following Weitsman [33], we introduce the following

Definition 1.2. *The formal quantization of (M, Ω, Φ) is the element of $R^{-\infty}(K) := \text{hom}_{\mathbb{Z}}(R(K), \mathbb{Z})$ defined by*

$$\mathcal{Q}_K^{-\infty}(M) = \sum_{\mu \in \widehat{K}} \mathcal{Q}(M_\mu) V_\mu^K .$$

A representation E of K is admissible if it has finite K -multiplicities : $\dim(\text{hom}_K(V_\mu^K, E)) < \infty$ for every $\mu \in \widehat{K}$. Here $R^{-\infty}(K)$ is the Grothendieck group associated to the K -admissible representations. We have a canonical inclusion $i : R(K) \hookrightarrow R^{-\infty}(K)$: to $V \in R(K)$ we associate the map $i(V) : R(K) \rightarrow \mathbb{Z}$ defined by $W \mapsto \dim(\text{hom}_K(V, W))$. In order to simplify the notation, $i(V)$ will be written V . Moreover the tensor product induces a structure of $R(K)$ -module on $R^{-\infty}(K)$ since $E \otimes V$ is an admissible representation when V and E are respectively finite dimensional and admissible representation of K .

It is an easy matter to see that [P1] holds for the formal quantization process $\mathcal{Q}^{-\infty}$. Let N and M be respectively pre-quantized *proper* Hamiltonian K_1 and K_2 manifolds : the product $M \times N$ is then a pre-quantized *proper* Hamiltonian $K_1 \times K_2$ -manifold. For the reduced spaces we have $(M \times N)_{(\mu_1, \mu_2)} \simeq M_{\mu_1} \times N_{\mu_2}$, for all $\mu_1 \in \widehat{K}_1$, $\mu_2 \in \widehat{K}_2$. It follows then that

$$(1.2) \quad \mathcal{Q}_{K_1 \times K_2}^{-\infty}(M \times N) = \mathcal{Q}_{K_1}^{-\infty}(M) \widehat{\otimes} \mathcal{Q}_{K_2}^{-\infty}(N)$$

in $R^{-\infty}(K_1 \times K_2) \simeq R^{-\infty}(K_1) \widehat{\otimes} R^{-\infty}(K_2)$.

The purpose of this article is to show that the functorial property [P2] still holds for the formal quantization process $\mathcal{Q}^{-\infty}$.

Theorem 1.3. *Let M be a pre-quantized Hamiltonian K -manifold which is proper. Let $H \subset K$ be a connected Lie subgroup such that M is still proper as an Hamiltonian H -manifold. Then $\mathcal{Q}_K^{-\infty}(M)$ is H -admissible and we have the following equality in $R^{-\infty}(H)$:*

$$(1.3) \quad \mathcal{Q}_K^{-\infty}(M)|_H = \mathcal{Q}_H^{-\infty}(M).$$

For $\mu \in \widehat{K}$ and $\nu \in \widehat{H}$ we denote $N_\nu^\mu = \dim(\text{hom}_H(V_\nu^H, V_\mu^K|_H))$ the multiplicity of V_ν^H in the restriction $V_\mu^K|_H$. In the situation of Theorem 1.3, the moment maps relative to the K and H actions are Φ_K and $\Phi_H = \text{p} \circ \Phi_K$, where $\text{p} : \mathfrak{k}^* \rightarrow \mathfrak{h}^*$ is the canonical projection.

Corollary 1.4. *For every $\nu \in \widehat{H}$, we have :*

$$(1.4) \quad \mathcal{Q}(M_{\nu, H}) = \sum_{\mu \in \widehat{K}} N_\nu^\mu \mathcal{Q}(M_{\mu, K}).$$

where $M_{\nu, H} = \Phi_H^{-1}(H \cdot \nu)/H$ and $M_{\mu, K} = \Phi_K^{-1}(K \cdot \mu)/K$ are respectively the symplectic reductions relative to the H and K -actions.

Since V_μ^K is equal to the K -quantization of $K \cdot \mu$, the ‘‘quantization commutes with reduction’’ Theorem tells us that $N_\nu^\mu = \mathcal{Q}((K \cdot \mu)_{\nu, H})$: in particular $N_\nu^\mu \neq 0$ implies that $\nu \in \text{p}(K \cdot \mu) \iff \mu \in K \cdot \text{p}^{-1}(\nu)$. Finally

$$N_\nu^\mu \mathcal{Q}(M_{\mu, K}) \neq 0 \implies \mu \in K \cdot \text{p}^{-1}(\nu) \quad \text{and} \quad \Phi_K^{-1}(\mu) \neq \emptyset.$$

Theses two conditions imply that we can restrict the sum of RHS of (1.4) to

$$(1.5) \quad \mu \in \widehat{K} \cap \Phi_K(K \cdot \Phi_H^{-1}(\nu))$$

which is *finite* since Φ_H is proper.

Theorem 1.3 and (1.2) gives the following extended version of [P3].

Theorem 1.5. *Let N and M be two pre-quantized Hamiltonian K -manifold where N is compact and M is proper. The product $M \times N$ is then proper and we have the following equality in $R^{-\infty}(K)$:*

$$(1.6) \quad \mathcal{Q}_K^{-\infty}(M \times N) = \mathcal{Q}_K^{-\infty}(M) \cdot \mathcal{Q}_K(N)$$

For $\mu, \lambda, \theta \in \widehat{K}$ we denote $C_{\lambda, \theta}^\mu = \dim(\text{hom}_H(V_\mu^K, V_\lambda^K \otimes V_\theta^K))$ the multiplicity of V_μ^K in the tensor $V_\lambda^K \otimes V_\theta^K$. Since $V_\lambda^K \otimes V_\theta^K$ is equal to the quantization of the product $K \cdot \lambda \times K \cdot \theta$, the ‘‘quantization commutes with reduction’’ Theorem tells us that $C_{\lambda, \theta}^\mu = \mathcal{Q}((K \cdot \lambda \times K \cdot \theta)_\mu)$: in particular $C_{\lambda, \theta}^\mu \neq 0$ implies that (*) $\|\lambda\| \leq \|\theta\| + \|\mu\|$.

Corollary 1.6. *In the situation of Theorem 1.5, we have for every $\mu \in \widehat{K}$:*

$$(1.7) \quad \mathcal{Q}((M \times N)_\mu) = \sum_{\lambda, \theta \in \widehat{K}} C_{\lambda, \theta}^\mu \mathcal{Q}(M_\lambda) \mathcal{Q}(N_\theta).$$

Since N is compact, $\mathcal{Q}(N_\theta) \neq 0$ for a **(**)** $\theta \in \{\text{finite set}\}$. Then **(*)** and **(**)** show that the sum in the RHS of (1.7) is *finite*.

Weistman [33] studied the formal quantization procedure in the case of $K = \mathrm{U}(n)$. Using a method of symplectic cutting [21, 35] he defines for two increasing sequence of positive integers r_n, s_n a family of cut-spaces M^{r_n, s_n} which are compact. *Under the hypothesis that the cut spaces M^{r_n, s_n} are smooth*, he notes that $\mathcal{Q}_K^{-\infty}(M) = \lim_{n \rightarrow \infty} \mathcal{Q}_K(M^{r_n, s_n})$, and he was then able to show Theorem 1.5.

Our proof of Theorem 1.3 uses also a technique of symplectic cutting but which is valid for any compact Lie group actions. We have to overpass the difficulties concerning the non-smoothness of the cut-spaces. For this purpose we introduce another method of symplectic cutting that uses the wonderful compactifications of Concini-Procesi [13, 14], and we prove an extension of the “quantization commutes with reduction” Theorem to the *singular* setting.

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2. QUANTIZATION COMMUTES WITH REDUCTION

In this section we precise the definition of the quantization of a smooth and compact Hamiltonian manifold. We extend the definition to the case of a *singular* Hamiltonian manifold and we prove a “quantization commutes with reduction” Theorem in the singular setting.

In the Kostant-Souriau framework, an Hamiltonian K -manifold (M, Ω_M, Φ_M) is pre-quantized if there is an equivariant Hermitian line bundle L with an invariant Hermitian connection ∇ such that

$$(2.8) \quad \mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi_M, X \rangle \quad \text{and} \quad \nabla^2 = -i\Omega_M,$$

for every $X \in \mathfrak{k}$. Here X_M is the vector field on M defined by $X_M(m) = \frac{d}{dt} e^{-tX} m|_0$.

(L, ∇) is also call a Kostant-Souriau line bundle. Remark that the conditions (2.8) implies through the equivariant Bianchi formula the relation

$$(2.9) \quad \iota(X_M)\Omega_M = -d\langle \Phi_M, X \rangle, \quad X \in \mathfrak{k}.$$

We will now recall the notions of geometric quantization.

2.1. Geometric quantization : the compact and smooth case. We suppose here that (M, Ω_M, Φ_M) is **compact** and is prequantized by a Hermitian line bundle L . Choose a K -invariant almost complex structure J on M which is compatible with Ω_M in the sense that the symmetric bilinear form $\Omega_M(\cdot, J\cdot)$ is a Riemannian metric. Let $\bar{\partial}_L$ be the Dolbeault operator with coefficients in L , and let $\bar{\partial}_L^*$ be its (formal) adjoint. The *Dolbeault-Dirac operator* on M with coefficients in L is $D_L = \bar{\partial}_L + \bar{\partial}_L^*$, considered as an operator from $\mathcal{A}^{0, \text{even}}(M, L)$ to $\mathcal{A}^{0, \text{odd}}(M, L)$.

Definition 2.1. *The geometric quantization of (M, Ω_M, Φ_M) is the element $\mathcal{Q}_K(M) \in R(K)$ which is defined as the equivariant index of the Dolbeault-Dirac operator D_L .*

Remark 2.2. • *We can define the Dolbeault-Dirac operator D_L^J for any invariant almost complex structure J . If J_0 and J_1 are equivariantly homotopic the indices of $D_L^{J_0}$ and $D_L^{J_1}$ coincides (see [28]).*

• *Since the set of compatible invariant almost complex structure on M is path-connected, the element $\mathcal{Q}_K(M) \in R(K)$ does not depend of the choice of J .*

2.2. Geometric quantization : the compact and singular case. We are interested to defined the geometric quantization of *singular* compact Hamiltonian manifolds : here "singular" means that the manifold is obtain by symplectic reduction.

Let (N, Ω_N) be a smooth symplectic manifold equipped with an Hamiltonian action of $K \times H$: we denote $(\Phi_K, \Phi_H) : N \rightarrow \mathfrak{k}^* \oplus \mathfrak{h}^*$ the corresponding moment map. We assume that N is pre-quantized by a $K \times H$ -equivariant line bundle L and we suppose that the map Φ_H is **proper**. One wants to define the quantization of the (compact) symplectic quotient

$$N //_0 H := \Phi_H^{-1}(0)/H.$$

When 0 is a regular value of Φ_H , $N //_0 H$ is a compact symplectic *orbifold* equipped with an Hamiltonian action of K : the corresponding moment map is induced by the restriction of Φ_K to $\Phi_H^{-1}(0)$. The symplectic quotient $N //_0 H$ is pre-quantized by the line orbundle

$$L_0 := \left(L|_{\Phi_H^{-1}(0)} \right) / H.$$

Definition 2.1 extends to the orbifold case, so one can still defined the quantization of $N //_0 H$ as an element $\mathcal{Q}_K(N //_0 H) \in R(K)$.

Suppose now that 0 is not a regular value of Φ_H . Let T_H be a maximal torus of H , and let $C_H \subset \mathfrak{t}_H^*$ be a weyl chamber. Since Φ_H is proper, the convexity theorem says that the image of Φ_H intersects C_H in a closed locally polyhedral convex set, that we denoted $\Delta_H(M)$ [22].

We consider an element $a \in \Delta_H(M)$ which is generic and sufficiently closed to $0 \in \Delta_H(M)$: we denote H_a the subgroup of H which stabilizes a . When $a \in \Delta_H(M)$ is generic, one can shows (see [25]) that

$$N //_a H := \Phi_H^{-1}(a)/H_a$$

is a compact K -Hamiltonian orbifold, and that

$$L_a := \left(L|_{\Phi_H^{-1}(a)} \right) / H_a.$$

is a K -equivariant line orbundle over $N //_a H$: we can then define like in Definition 2.1 the element $\mathcal{Q}_K(N //_a H) \in R(K)$ as the equivariant index of the Dolbeault-Dirac operator on $N //_a H$.

Proposition-Definition 2.3. *The elements $\mathcal{Q}_K(N //_a H) \in R(K)$ do not depend of the choice of the generic element $a \in \Delta_H(M)$, when a is sufficiently closed to 0. The common value will be taken as the geometric quantization of $N //_0 H$, and still denoted $\mathcal{Q}_K(N //_0 H)$.*

PROOF. When N is compact and $K = \{e\}$, the proof can be founded in [25] and in [28]. The \mathbf{K} -theoretic proof of [28] extends naturally to our case. \square

2.3. Quantization commutes with reduction : the singular case. In section 2.2, we have defined the geometric quantization $\mathcal{Q}_K(N//_0H) \in R(K)$ of a compact symplectic reduced space $N//_0H$. We will compute its K -multiplicities like in Theorem 1.1.

For every $\mu \in \widehat{K}$, we consider the coadjoint orbit $K \cdot \mu \simeq K/K_\mu$ which is pre-quantized by the line bundle $\mathbb{C}_{[\mu]} \simeq K \times_{K_\mu} \mathbb{C}_\mu$. We consider the product¹ $N \times \overline{K \cdot \mu}$ which is an Hamiltonian $K \times H$ manifold which is pre-quantized by $K \times H$ -equivariant line bundle $L \otimes \mathbb{C}_{[\mu]}^{-1}$. The moment map $N \times \overline{K \cdot \mu} \rightarrow \mathfrak{k}^* \times \mathfrak{h}^*$, $(n, \xi) \mapsto (\Phi_K(n) - \xi, \Phi_H(n))$ is proper, so the reduced space

$$(N//_0H)_\mu := (N \times \overline{K \cdot \mu})//_{(0,0)} K \times H$$

is compact. Following Proposition 2.3, we can then define its quantization $\mathcal{Q}((N//_0H)_\mu) \in \mathbb{Z}$. The main result of this section is the

Theorem 2.4. *We have the following equality in $R(K)$:*

$$(2.10) \quad \mathcal{Q}_K(N//_0H) = \sum_{\mu \in \widehat{K}} \mathcal{Q}((N//_0H)_\mu) V_\mu^K.$$

PROOF. The proof will occupied the remaining of this section. The starting point is to state another definition of the geometric quantization of a symplectic reduced space which uses the Atiyah-Singer's theory of transversally elliptic operators.

2.3.1. Transversally elliptic symbols. Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [1]. For an axiomatic treatment of the index morphism see Berline-Vergne [8, 9] and for a short introduction see [28].

Let \mathcal{X} be a compact $K_1 \times K_2$ -manifold. Let $p : T\mathcal{X} \rightarrow \mathcal{X}$ be the projection, and let $(-, -)_\mathcal{X}$ be a $K_1 \times K_2$ -invariant Riemannian metric. If E^0, E^1 are $K_1 \times K_2$ -equivariant vector bundles over \mathcal{X} , a $K_1 \times K_2$ -equivariant morphism $\sigma \in \Gamma(T\mathcal{X}, \text{hom}(p^*E^0, p^*E^1))$ is called a *symbol*. The subset of all $(x, v) \in T\mathcal{X}$ where $\sigma(x, v) : E_x^0 \rightarrow E_x^1$ is not invertible is called the *characteristic set* of σ , and is denoted by $\text{Char}(\sigma)$.

Let $T_{K_2}\mathcal{X}$ be the following subset of $T\mathcal{X}$:

$$T_{K_2}\mathcal{X} = \{(x, v) \in TM, (v, X_M(x))_M = 0 \text{ for all } X \in \mathfrak{k}_2\}.$$

A symbol σ is *elliptic* if σ is invertible outside a compact subset of $T\mathcal{X}$ (i.e. $\text{Char}(\sigma)$ is compact), and is *K_2 -transversally elliptic* if the restriction of σ to $T_{K_2}\mathcal{X}$ is invertible outside a compact subset of $T_{K_2}\mathcal{X}$ (i.e. $\text{Char}(\sigma) \cap T_{K_2}\mathcal{X}$ is compact). An elliptic symbol σ defines an element in the equivariant \mathbf{K} -theory of $T\mathcal{X}$ with compact support, which is denoted by $\mathbf{K}_{K_1 \times K_2}(T\mathcal{X})$, and the index of σ is a virtual finite dimensional representation of $K_1 \times K_2$ [3, 4, 5, 6].

A *K_2 -transversally elliptic* symbol σ defines an element of $\mathbf{K}_{K_1 \times K_2}(T_{K_2}M)$, and the index of σ is defined as a trace class virtual representation of $K_1 \times K_2$ (see [1] for the analytic index and [8, 9] for the cohomological one) : in fact $\text{Index}^\mathcal{X}(\sigma)$ belongs to the tensor product $R(K_1) \widehat{\otimes} R^{-\infty}(K_2)$.

¹ $\overline{K \cdot \mu}$ denotes the coadjoint orbit with the opposite symplectic form.

Remark that any elliptic symbol of $\mathbf{T}\mathcal{X}$ is K_2 -transversally elliptic, hence we have a restriction map $\mathbf{K}_{K_1 \times K_2}(\mathbf{T}\mathcal{X}) \rightarrow \mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_2}\mathcal{X})$, and a commutative diagram

$$(2.11) \quad \begin{array}{ccc} \mathbf{K}_{K_1 \times K_2}(\mathbf{T}\mathcal{X}) & \longrightarrow & \mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_2}\mathcal{X}) \\ \text{Index}^x \downarrow & & \downarrow \text{Index}^x \\ R(K_1) \otimes R(K_2) & \longrightarrow & R(K_1) \widehat{\otimes} R^{-\infty}(K_2) . \end{array}$$

Using the *excision property*, one can easily show that the index map $\text{Index}^{\mathcal{U}} : \mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_2}\mathcal{U}) \rightarrow R(K_1) \widehat{\otimes} R^{-\infty}(K_2)$ is still defined when \mathcal{U} is a $K_1 \times K_2$ -invariant relatively compact open subset of a $K_1 \times K_2$ -manifold (see [28][section 3.1]).

2.3.2. Quantization of singular space : second definition. Let $(\mathcal{X}, \Omega_{\mathcal{X}})$ be an Hamiltonian $K_1 \times K_2$ -manifold pre-quantized by a $K_1 \times K_2$ -equivariant line bundle L . The moment map $\Phi_2 : \mathcal{X} \rightarrow \mathfrak{k}_2^*$ relative to the K_2 -action is supposed to be **proper**. Take a compatible $K_1 \times K_2$ -invariant almost complex structure on \mathcal{X} . We choose a $K_1 \times K_2$ -invariant Hermitian metric $\|v\|^2$ on the tangent bundle $\mathbf{T}\mathcal{X}$, and we identify the cotangent bundle $\mathbf{T}^*\mathcal{X}$ with $\mathbf{T}\mathcal{X}$. For $(x, v) \in \mathbf{T}\mathcal{X}$, the principal symbol of the Dolbeault-Dirac operator $\bar{\partial}_L + \bar{\partial}_L^*$ is the clifford multiplication $\mathbf{c}_{\mathcal{X}}(v)$ on the complex vector bundle $\Lambda^{\bullet} \mathbf{T}_x \mathcal{X} \otimes L_x$. It is invertible for $v \neq 0$, since $\mathbf{c}_{\mathcal{X}}(v)^2 = -\|v\|^2$.

When \mathcal{X} is compact, the symbol $\mathbf{c}_{\mathcal{X}}$ is elliptic and then defines an element of the equivariant \mathbf{K} -group of $\mathbf{T}\mathcal{X}$. The topological index of $\mathbf{c}_{\mathcal{X}} \in \mathbf{K}_{K_1 \times K_2}(\mathbf{T}\mathcal{X})$ is equal to the analytical index of the Dolbeault-Dirac operator $\bar{\partial}_L + \bar{\partial}_L^*$:

$$(2.12) \quad \mathcal{Q}_{K_1 \times K_2}(\mathcal{X}) = \text{Index}^{\mathcal{X}}(\mathbf{c}_{\mathcal{X}}) \quad \text{in} \quad R(K_1) \otimes R(K_2).$$

When \mathcal{X} is not compact the topological index of $\mathbf{c}_{\mathcal{X}}$ is not defined. In order to give a topological definition of $\mathcal{Q}_{K_1}(\mathcal{X} //_0 K_2)$, we will deform the symbol $\mathbf{c}_{\mathcal{X}}$ as follows. Consider the identification $\mathfrak{k}_2^* \simeq \mathfrak{k}_2$ defined by a K_2 -invariant scalar product on the Lie algebra \mathfrak{k}_2 . From now on the moment map Φ_2 will take values in \mathfrak{k}_2 , and we define the vectors field on \mathcal{X}

$$(2.13) \quad \kappa_x = (\Phi_2(x))_M(x), \quad x \in \mathcal{X}.$$

We consider now the symbol

$$\mathbf{c}_{\mathcal{X}}^{\kappa}(v) = \mathbf{c}(v - \kappa_x), \quad v \in \mathbf{T}_x \mathcal{X}.$$

Note that $\mathbf{c}_{\mathcal{X}}^{\kappa}(v)$ is invertible except if $v = \kappa_x$. If furthermore v belongs to the subset $\mathbf{T}_{K_2}M$ of tangent vectors orthogonal to the K_2 -orbits, then $v = 0$ and $\kappa_x = 0$. Indeed κ_x is tangent to $K_2 \cdot x$ while v is orthogonal.

Since κ is the Hamiltonian vector field of the function $\|\Phi_2\|^2$, the set of zeros of κ coincides with the set $\text{Cr}(\|\Phi_2\|^2)$ of critical points of $\|\Phi_2\|^2$.

Let $\mathcal{U} \subset \mathcal{X}$ be a $K_1 \times K_2$ -invariant open subset which is *relatively compact*. If the border $\partial\mathcal{U}$ does not intersect $\text{Cr}(\|\Phi_2\|^2)$, then the restriction $\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}}$ defines a class in $\mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_2}\mathcal{U})$ since

$$\text{Char}(\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}}) \cap \mathbf{T}_{K_2}\mathcal{U} \simeq \text{Cr}(\|\Phi_2\|^2) \cap \mathcal{U}$$

is compact. In this situation the index of $\mathbf{c}_{\kappa}|_{\mathcal{U}}$ is defined as an element $\text{Index}^{\mathcal{U}}(\mathbf{c}_{\kappa}|_{\mathcal{U}}) \in R(K_1) \widehat{\otimes} R^{-\infty}(K_2)$.

Theorem 2.5. *The K_2 -invariant part of $\text{Index}^{\mathcal{U}}(\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}})$ is equal to :*

- $\mathcal{Q}_{K_1}(\mathcal{X} //_0 K_2)$ when $\Phi_2^{-1}(0) \subset \mathcal{U}$,

- 0 in the other case.

PROOF. When $K_1 = \{e\}$, the proof is done in [28] (see section 7). This proof works equally well in the general case.

Remark 2.6. *If \mathcal{X} is compact we can take $\mathcal{U} = \mathcal{X}$ in the last Theorem. In this case the symbols $\mathbf{c}_{\mathcal{X}}^{\kappa}$ and $\mathbf{c}_{\mathcal{X}}$ defines the same class in $\mathbf{K}_{K_1 \times K_2}(\mathbb{T}\mathcal{X})$ so they have the same index. Theorem 2.5 corresponds then to the traditional “quantization commutes with reduction” phenomenon : $[\mathcal{Q}_{K_1 \times K_2}(\mathcal{X})]^{K_2} = \mathcal{Q}_{K_1}(\mathcal{X} //_0 K_2)$.*

From now on we will work with this topological definition for the geometric quantization of the reduced K_1 -Hamiltonian manifold $\mathcal{X} //_0 K_2$ (which is possibly singular): $\mathcal{Q}_{K_1}(\mathcal{X} //_0 K_2) = [\text{Index}^{\mathcal{U}}(\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}})]^{K_2}$ where \mathcal{U} is any relatively compact neighborhood of $\Phi_2^{-1}(0)$ such that $\partial\mathcal{U} \cap \text{Cr}(\|\Phi_2\|^2) = \emptyset$. The functorial properties still holds in this singular setting. In particular :

[P2] If $H \subset K_1$ is a connected lie subgroup, then the restriction of $\mathcal{Q}_{K_1}(\mathcal{X} //_0 K_2)$ to H is equal to $\mathcal{Q}_H(\mathcal{X} //_0 K_2)$.

2.3.3. *Proof of theorem 2.4.* We come back in the situation of sections 2.2 and 2.3 .

First we apply Theorem 2.5 to $\mathcal{X} = N$, $K_1 = K$ and $K_2 = H$. (2.10) is trivially true when $0 \notin \text{Image}(\Phi_H)$. So we suppose now that $0 \in \text{Image}(\Phi_H)$, and we consider an $K \times H$ -invariant open subset $\mathcal{U} \subset N$ which is *relatively compact* and such that

$$\Phi_H^{-1}(0) \subset \mathcal{U} \quad \text{and} \quad \partial\mathcal{U} \cap \text{Cr}(\|\Phi_H\|^2) = \emptyset.$$

We have $\mathcal{Q}_K(N //_0 H) = [\text{Index}^{\mathcal{U}}(\mathbf{c}_N^{\kappa^H}|_{\mathcal{U}})]^H$ and one want to compute its K - multiplicities $m_{\mu}, \mu \in \widehat{K}$. Here κ^H is the vectors field on N associated to the moment map Φ_H (see (2.13)).

Take $\mu \in \widehat{K}$. We denote $\mathbf{c}_{-\mu}$ the principal symbol of the Dolbeault-Dirac operator on $\overline{K \cdot \mu}$ with values in the line bundle $\mathbb{C}_{[-\mu]}$: we have $\text{Index}^{K \cdot \mu}(\mathbf{c}_{-\mu}) = (V_{\mu}^K)^*$.

We know then that the multiplicity of $[\text{Index}^{\mathcal{U}}(\mathbf{c}_N^{\kappa^H}|_{\mathcal{U}})]^H$ relatively to V_{μ}^K is equal to

$$(2.14) \quad m_{\mu} := \left[\text{Index}^{\mathcal{V}}(\mathbf{c}_N^{\kappa^H}|_{\mathcal{U}} \odot \mathbf{c}_{-\mu}) \right]^{K \times H}$$

with $\mathcal{V} = \mathcal{U} \times K \cdot \mu$. This identity is due to the fact that we have a ”multiplication”

$$\begin{aligned} \mathbf{K}_{K \times H}(\mathbb{T}H\mathcal{U}) \times \mathbf{K}_K(\mathbb{T}(K \cdot \mu)) &\longrightarrow \mathbf{K}_{K \times H}(\mathbb{T}_{K \times H}(\mathcal{U} \times K \cdot \mu)) \\ (\sigma_1, \sigma_2) &\longmapsto \sigma_1 \odot \sigma_2 . \end{aligned}$$

so that $\text{Index}^{\mathcal{U} \times K \cdot \mu}(\sigma_1 \odot \sigma_2) = \text{Index}^{\mathcal{U}}(\sigma_1) \cdot \text{Index}^{K \cdot \mu}(\sigma_2)$ in $R^{-\infty}(K \times H)$. See [1].

Consider now the case where $\mathcal{X} = N \times \overline{K \cdot \mu}$, $K_1 = \{e\}$ and $K_2 = K \times H$. After Theorem 2.5, we know that

$$(2.15) \quad \mathcal{Q}((N //_0 H)_{\mu}) = \left[\text{Index}^{\mathcal{V}}(\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{V}}) \right]^{K \times H},$$

where κ is the vector field on $N \times \overline{K \cdot \mu}$ associated to the moment map

$$\begin{aligned} \Phi : N \times \overline{K \cdot \mu} &\longrightarrow \mathfrak{k}^* \times \mathfrak{h}^* \\ (x, \xi) &\longmapsto (\Phi_K(x) - \xi, \Phi_H(n)) \end{aligned}$$

Note that $\mathcal{V} = \mathcal{U} \times K \cdot \mu$ is a neighborhood of $\Phi^{-1}(0) \subset (\Phi_H)^{-1}(0)$.

Our aim now is to prove that the quantities (2.14) and (2.15) are equal.

Since the definition of κ needs the choice of an invariant scalar product on the Lie algebra $\mathfrak{k} \times \mathfrak{h}$, we will precise its definition. Let $\|\cdot\|_K$ and $\|\cdot\|_H$ be two invariant Euclidean norm respectively on \mathfrak{k} and \mathfrak{h} . For any $r > 0$ we consider on $\mathfrak{k} \times \mathfrak{h}$ the invariant Euclidean norm $\|(X, Y)\|_r^2 = r^2 \|X\|_K^2 + \|Y\|_H^2$.

Let κ^K be the vector field on $N \times \overline{K \cdot \mu}$ associated to the map $N \times \overline{K \cdot \mu} \rightarrow \mathfrak{k}^*$, $(x, \xi) \mapsto \Phi_K(x) - \xi$, and where the identification $\mathfrak{k} \simeq \mathfrak{k}^*$ is made through the Euclidean norm $\|\cdot\|_K$ (see (2.13)). For $(x, \xi) \in N \times \overline{K \cdot \mu}$, we have the decomposition

$$\kappa^K(x, \xi) = (\kappa_1(x, \xi), \kappa_2(x, \xi)) \in T_x N \times T_\xi K \cdot \mu.$$

Let κ^H be the vector field on $N \times \overline{K \cdot \mu}$ associated to the map $N \times \overline{K \cdot \mu} \rightarrow \mathfrak{h}^*$, $(x, \xi) \mapsto \Phi_H(x)$, and where the identification $\mathfrak{h} \simeq \mathfrak{h}^*$ is made through the Euclidean norm $\|\cdot\|_H$. For $(x, \xi) \in N \times \overline{K \cdot \mu}$, we have the decomposition

$$\kappa^H(x, \xi) = (\kappa^H(x), 0) \in T_x N \times T_\xi K \cdot \mu.$$

For any $r > 0$, we denote by κ_r the vector field on $N \times \overline{K \cdot \mu}$ associated to the map Φ , and where the identification $\mathfrak{k} \times \mathfrak{h} \simeq \mathfrak{k}^* \times \mathfrak{h}^*$ is made through the Euclidean norm $\|\cdot\|_r$. We have then

$$\begin{aligned} \kappa_r &= \kappa^H + r \kappa^K \\ &= (\kappa^H + r \kappa_1, r \kappa_2) \end{aligned}$$

Now we can precise (2.15). Take an invariant relatively compact neighborhood \mathcal{U} of $\Phi_H^{-1}(0)$ such that $\partial \mathcal{U} \cap \{\text{zeros of } \kappa^H\} = \emptyset$. With the help of a invariant Riemannian metric on \mathcal{X} we define

$$\varepsilon_H = \inf_{x \in \partial \mathcal{U}} \|\kappa^H(x)\| > 0 \quad \text{and} \quad \varepsilon_K = \sup_{(x, \xi) \in \partial \mathcal{U} \times K \cdot \mu} \|\kappa_1(x, \xi)\|.$$

Note that for any $0 \leq r < \frac{\varepsilon_H}{\varepsilon_K}$, we have $\partial \mathcal{U} \times K \cdot \mu \cap \{\text{zeros of } \kappa^H + r \kappa_1\} = \emptyset$, and then $\partial \mathcal{V} \cap \{\text{zeros of } \kappa_r\} = \emptyset$ for the neighborhood $\mathcal{V} := \mathcal{U} \times K \cdot \mu$ of $\Phi^{-1}(0)$. We can then use Theorem 2.5 : for $0 < r < \frac{\varepsilon_H}{\varepsilon_K}$ we have

$$\mathcal{Q}((N //_0 H)_\mu) = \left[\text{Index}^{\mathcal{V}}(\mathbf{c}_{\mathcal{X}}^{\kappa_r} |_{\mathcal{V}}) \right]^{K \times H}$$

We are now close to the end of the proof. Let us compare the symbols $\mathbf{c}_{\mathcal{X}}^{\kappa_r} |_{\mathcal{V}}$ and $\mathbf{c}_N^{\kappa^H} |_{\mathcal{U}} \odot \mathbf{c}_{-\mu}$ in $\mathbf{K}_{K \times H}(T_{K \times H}(\mathcal{U} \times K \cdot \mu))$. First one sees that the symbols $\mathbf{c}_{\mathcal{X}}$ is equal to the product $\mathbf{c}_N \odot \mathbf{c}_{-\mu}$ hence the symbols $\mathbf{c}_N^{\kappa^H} |_{\mathcal{U}} \odot \mathbf{c}_{-\mu}$ is equal to $\mathbf{c}_{\mathcal{X}}^{\kappa_r} |_{\mathcal{V}}$ when $r = 0$. Since for $r < \frac{\varepsilon_H}{\varepsilon_K}$ the path $s \in [0, r] \rightarrow \mathbf{c}_{\mathcal{X}}^{\kappa_s} |_{\mathcal{V}}$ defines an homotopy of $K \times H$ -transversally elliptic symbols on \mathcal{V} , we get

$$\text{Index}^{\mathcal{V}}(\mathbf{c}_{\mathcal{X}}^{\kappa_r} |_{\mathcal{V}}) = \text{Index}^{\mathcal{V}}(\mathbf{c}_N^{\kappa^H} |_{\mathcal{U}} \odot \mathbf{c}_{-\mu})$$

and then $m_\mu = \mathcal{Q}((N //_0 H)_\mu)$. \square

3. WONDERFUL COMPACTIFICATIONS AND SYMPLECTIC CUTTING

In this section we use the wonderful compactifications of Concini-Procesi [13, 14] to perform symplectic cutting.

3.1. Wonderful compactifications : definitions. We study here the wonderful compactifications from the Hamiltonian point of view.

We consider a compact connected Lie group K and its complexification $K_{\mathbb{C}}$. Let T be a maximal torus of K , and let $W := N(T)/T$ be the Weyl group. Consider \mathfrak{t}^* , the dual of the Lie algebra of T , with the lattice Λ^* of real weights. Let $C_K \subset \mathfrak{t}^*$ be a Weyl chamber and let $\widehat{K} := \Lambda^* \cap C_K$ be the set dominants weights.

Definition 3.1. *A polytope P in \mathfrak{t}^* is K -adapted if :*

- i) *the vertices of P are regular elements of Λ^* ,*
- ii) *P is W -invariant,*
- iii) *P is Delzant.*

EXAMPLE : When K as a trivial center, the convex hull of $W \cdot \mu$ is a K -adapted polytope for any regular dominant weight μ .

Proposition 3.2. *There exists K -adapted polytopes in \mathfrak{t}^* .*

PROOF. Let us use the dictionary between polytopes and projective fan [27]. Conditions *ii*) and *iii*) of Definition 3.1 means that we are looking after a smooth projective W -invariant fan \mathcal{F} in \mathfrak{t} . Condition *i*) means that each cone of \mathcal{F} of maximal dimension should not be fixed by any element of $W \setminus \{\text{Id}\}$. For a proof of the existence of search fan, see [11, 12]. In particular condition *(*)* in Proposition 2 of [12] implies *i*). \square

In the rest of this section, we consider a K -adapted polytope P . Let $[P]_1$ be the union of all the closed facet of dimension 1 : we label the elements of $[P]_1 \cap \widehat{K}$ by $\{\lambda_1, \dots, \lambda_N\}$. Since P is K -adapted, when λ_i is a vertex of P there exists $\alpha_{j_1}, \dots, \alpha_{j_r}$ belonging to $[P]_1 \cap \Lambda^* = W \cdot \{\lambda_1, \dots, \lambda_N\}$ such that $\alpha_{j_1} - \lambda_i, \dots, \alpha_{j_r} - \lambda_i$ is a basis of the lattice Λ^* .

Let V_{λ_i} be an irreducible representation of K with highest weight λ_i : these representations extend canonically to the complexification $K_{\mathbb{C}}$. We denote $\rho : K_{\mathbb{C}} \rightarrow \prod_{i=1}^N GL(V_{\lambda_i})$ the representation of $K_{\mathbb{C}}$ on $V := \bigoplus_{i=1}^N V_{\lambda_i}$. We consider the vector space

$$E = \bigoplus_{i=1}^N \text{End}(V_{\lambda_i})$$

equipped with the action of $K_{\mathbb{C}} \times K_{\mathbb{C}}$ given by : $(g_1, g_2) \cdot f = \rho(g_1) \circ f \circ \rho(g_2)^{-1}$. Let $\mathbb{P}(E)$ the projective space associated to E : it is equipped with an algebraic action of the reductive group $K_{\mathbb{C}} \times K_{\mathbb{C}}$. We consider the map $g \mapsto [\rho(g)]$ from $K_{\mathbb{C}}$ into $\mathbb{P}(E)$, that we denote $\bar{\rho}$.

Lemma 3.3. *The map $\bar{\rho} : K_{\mathbb{C}} \rightarrow \mathbb{P}(E)$ is an embedding.*

PROOF. Let $g \in K_{\mathbb{C}}$ such that $\bar{\rho}(g) = [Id]$: there exist $a \in \mathbb{C}^*$ such that $\rho(g) = aId$. The Cartan decomposition gives

$$(3.16) \quad \rho(k) = \frac{a}{|a|} Id \quad \text{and} \quad \rho(e^{iX}) = |a| Id$$

for $g = ke^{iX}$ with $k \in K$ and $X \in \mathfrak{k}$. Since there exist $Y, Y' \in \mathfrak{t}$ and $u, u' \in K$ such that $k = ue^{Y}u^{-1}$ and $X = u' \cdot Y'$, (3.16) gives

$$(3.17) \quad \rho(e^Y) = \frac{a}{|a|} Id \quad \text{and} \quad \rho(e^{iY'}) = |a| Id.$$

Since each element of $[P]_1 \cap \wedge^* = W \cdot \{\lambda_1, \dots, \lambda_N\}$ is a weight for the action of $T_{\mathbb{C}}$ on $\bigoplus_{i=1}^N V_{\lambda_i}$, (3.17) implies that for every $\alpha \in [P]_1 \cap \wedge^*$ we have

$$(3.18) \quad e^{i\langle \alpha, Y \rangle} = \frac{a}{|a|} \quad \text{and} \quad e^{-\langle \alpha, Y' \rangle} = |a|.$$

Since there exists $\alpha_{j_0}, \dots, \alpha_{j_r} \in [P]_1 \cap \wedge^*$ such that $\alpha_{j_1} - \alpha_{j_0}, \dots, \alpha_{j_r} - \alpha_{j_0}$ is a basis of the lattice \wedge^* , (3.18) implies that $Y' = 0$ and that $Y \in \ker(Z \in \mathfrak{t} \rightarrow e^Z)$. We have proved that $g = e$. \square

Let $T_{\mathbb{C}} \subset K_{\mathbb{C}}$ the complexification of the (compact) torus $T \subset K$.

Definition 3.4. *Let \mathcal{X}_P be the Zarisky closure of $\bar{\rho}(K_{\mathbb{C}})$ in $\mathbb{P}(E)$ and let $\mathcal{Y}_P \subset \mathcal{X}_P$ be the Zarisky closure of $\bar{\rho}(T_{\mathbb{C}})$ in $\mathbb{P}(E)$.*

Since $\bar{\rho}(K_{\mathbb{C}}) = K_{\mathbb{C}} \times K_{\mathbb{C}} \cdot [Id]$ and $\bar{\rho}(T_{\mathbb{C}}) = T_{\mathbb{C}} \times T_{\mathbb{C}} \cdot [Id]$ are orbits of algebraic group actions their Zariski closures coincide with their closures for the Euclidean topology.

Theorem 3.5. *The varieties \mathcal{X}_P and \mathcal{Y}_P are smooth.*

The proof will be done in the next section

3.2. Smoothness of \mathcal{X}_P and \mathcal{Y}_P . Let E be a complex vector space equipped with a linear action of a reductive group G . Let $\mathcal{Z} \subset \mathbb{P}(E)$ be a projective variety which is G -stable. We have the classical fact

Lemma 3.6. \bullet *\mathcal{Z} possess closed G -orbits.*
 \bullet *\mathcal{Z} is smooth if \mathcal{Z} is smooth near its closed G -orbits.*

PROOF. Let $z_0 \in \mathcal{Z}$ and consider the Zariski closure $\overline{G \cdot z_0} \subset \mathcal{Z}$. If $G \cdot z_0$ is not closed, we take $z_1 \in \overline{G \cdot z_0} \setminus G \cdot z_0$: we have $\dim G \cdot z_1 < \dim G \cdot z_0$. By induction we find a sequence z_1, \dots, z_p with $z_{k+1} \in \overline{G \cdot z_k} \setminus G \cdot z_k$ for $k < p$ and $G \cdot z_p$ closed. For the second point, we have just to note that if \mathcal{Z} is singular, the subvariety $\mathcal{Z}^{sing} \subset \mathcal{Z}$ of singular points is G -stable and then contains a closed G -orbits. \square

We are interested here respectively in

- \bullet the $K_{\mathbb{C}} \times K_{\mathbb{C}}$ -variety $\mathcal{X}_P \subset \mathbb{P}(E) \subset \mathbb{P}(\text{End}(V))$
- \bullet the $T_{\mathbb{C}} \times T_{\mathbb{C}}$ -variety $\mathcal{Y}_P \subset \mathbb{P}(E)$.

Since the diagonal $Z_{\mathbb{C}} = \{(t, t) | t \in T_{\mathbb{C}}\}$ stabilizes $[Id]$, its action on \mathcal{Y}_P is trivial. Hence we will restrict ourself to the action of $T_{\mathbb{C}} \times T_{\mathbb{C}} / Z_{\mathbb{C}} \simeq T_{\mathbb{C}}$ on \mathcal{Y}_P : for $t \in T_{\mathbb{C}}$ and $[y] \in \mathcal{Y}_P$ we take $t \cdot [y] = [\rho(t) \circ y]$.

3.2.1. The case of \mathcal{Y}_P . We apply Lemma 3.6 to the $T_{\mathbb{C}}$ -variety $\mathcal{Y}_P = \overline{T_{\mathbb{C}} \cdot [Id]}$ in $\mathbb{P}(E)$. Let $\{\alpha_j, j \in J\}$ be the $T_{\mathbb{C}}$ weights on $(\bigoplus_{i=1}^N V_{\lambda_i}, \rho)$ counted with their multiplicity. Their exists a orthonormal basis $\{v_j, j \in J\}$ of $\bigoplus_{i=1}^N V_{\lambda_i}$ such that $Id = \sum_{j \in J} v_j \otimes v_j^*$ and

$$(3.19) \quad \rho(e^Z) = \sum_{j \in J} e^{i\langle \alpha_j, Z \rangle} v_j \otimes v_j^*, \quad Z \in \mathfrak{t}_{\mathbb{C}}.$$

So the action of $e^Z \in T_{\mathbb{C}}$ on $[Id] \in \mathbb{P}(E)$ is $e^Z \cdot [Id] = \left[\sum_{j \in J} e^{i\langle \alpha_j, Z \rangle} v_j \otimes v_j^* \right]$. We introduce a subset J' of J such that for every $j \in J$ there exists a *unique* $j' \in J'$ such that $\alpha_j = \alpha_{j'}$. So the variety \mathcal{Y}_P belongs to $\mathbb{P}(E')$ where $E' = \bigoplus_{j' \in J'} \mathbb{C} m_{j'}$ with $m_{j'} = \sum_{j, \alpha_j = \alpha_{j'}} v_j \otimes v_j^*$. The closed $T_{\mathbb{C}}$ orbits in $\mathbb{P}(E')$ are $[m_{j'}], j' \in J$.

Lemma 3.7. $[m_{j_o}] \in \mathcal{Y}_P$ if and only if α_{j_o} is a vertex of the polytope P .

PROOF. If α_{j_o} is a vertex of P , there exists $X \in \mathfrak{t}$ such that $\langle \alpha_{j_o}, X \rangle > \langle \alpha_j, X \rangle$ whenever $\alpha_{j_o} \neq \alpha_j$. Hence $e^{-isX} \cdot [Id]$ tends to $[m_{j_o}]$ when $s \rightarrow +\infty$. If α_{j_o} is not a vertex of P , there exist $L \subset J' \setminus \{j_o\}$ such that $\alpha_{j_o} = \sum_{l \in L} a_l \alpha_l$ with $0 < a_l < 1$ and $\sum_l a_l = 1$. So \mathcal{Y}_P belongs to the closed subset defined by

$$\left[\sum_{j' \in J'} \delta_{j'} m_{j'} \right] \in \mathbb{P}(E') \quad \text{such that} \quad \prod_{l \in L} |\delta_l|^{a_l} = |\delta_{j_o}|.$$

Hence $[m_{j_o}] \notin \mathcal{Y}_P$. \square

Remark 3.8. When α_j is a vertex of the polytope P , the multiplicity of α_j in $\oplus_{i=1}^N V_{\lambda_i}$ is equal to one, so $m_j = v_j \otimes v_j^*$.

Consider now a vertex α_{j_o} of P (for $j_o \in J'$). We consider the open subset $\mathcal{V} \subset \mathbb{P}(E')$ defined by $[\sum_{j' \in J'} \delta_{j'} m_{j'}] \in \mathcal{V} \Leftrightarrow \delta_{j_o} \neq 0$, and the diffeomorphism $\psi : \mathcal{V} \rightarrow \mathbb{C}^{J' \setminus \{j_o\}}$, $[\sum_{j' \in J'} \delta_{j'} m_{j'}] \mapsto (\frac{\delta_{j'}}{\delta_{j_o}})_{j' \neq j_o}$. The map ψ realizes an algebraic diffeomorphism between $\mathcal{Y}_P \cap \mathcal{V}$ and the affine subvariety

$$\mathcal{Z} := \overline{\{(t^{\alpha_{j'} - \alpha_{j_o}})_{j' \neq j_o} \mid t \in T_{\mathbb{C}}\}} \subset \mathbb{C}^{J' \setminus \{j_o\}}.$$

The set of weights $\alpha_j, j \in J$ contains all the lattice points that belongs to the one dimensional faces of P . Since the polytope P is K -adapted, there exists a subset $L_{j_o} \subset J'$ such that $\alpha_l - \alpha_{j_o}, l \in L_{j_o}$ is a \mathbb{Z} -basis of the group of weights Λ^* . And for every $j' \neq j_o$ we have

$$(3.20) \quad \alpha_{j'} - \alpha_{j_o} = \sum_{l \in L_{j_o}} n_{j'}^l (\alpha_l - \alpha_{j_o}) \quad \text{with} \quad n_{j'}^l \in \mathbb{N}.$$

We define on $\mathbb{C}^{L_{j_o}}$ the monomials $P_{j'}(Z) = \prod_{l \in L_{j_o}} (Z_l)^{n_{j'}^l}$. Note that $P_{j'}(Z) = Z_l$ when $j' = l \in L_{j_o}$. Now it is not difficult to see that the map

$$\begin{aligned} \mathbb{C}^{L_{j_o}} &\longrightarrow \mathbb{C}^{J' \setminus \{j_o\}} \\ Z &\longmapsto (P_{j'}(Z))_{j' \neq j_o} \end{aligned}$$

realizes an algebraic diffeomorphism between $\mathbb{C}^{L_{j_o}}$ and \mathcal{Z} .

Finally we have shown that \mathcal{Y}_P is smooth near $[m_{j_o}]$: hence \mathcal{Y}_P is a smooth subvariety of $\mathbb{P}(E)$. Since $T_{\mathbb{C}}$ acts on \mathcal{Y}_P with a dense orbit, \mathcal{Y}_P is a smooth projective toric variety.

3.2.2. *The case of \mathcal{X}_P .* Let $E := \otimes_{i=1}^N \text{End}(V_{\lambda_i})$. The closed $K_{\mathbb{C}} \times K_{\mathbb{C}}$ -orbit in $\mathbb{P}(E)$ are those passing through $[v_{\lambda_i} \otimes v_{\lambda_i}^*]$ where $v_{\lambda_i} \in V_{\lambda_i}$ is a highest weight vector (that we take of norm 1 for a K -invariant hermitian structure).

Lemma 3.9. $[v_{\lambda_i} \otimes v_{\lambda_i}^*] \in \mathcal{X}_P$ if and only if λ_i is a vertex of the polytope P .

PROOF. If λ_i is a vertex of P , we have proved in Lemma 3.7 that $[v_{\lambda_i} \otimes v_{\lambda_i}^*]$ belongs to \mathcal{Y}_P and so belongs to \mathcal{X}_P . We prove the converse in Corollary 3.14. \square

For the remaining of this section we consider a vertex $\lambda_{i_o} \in \widehat{K}$ of the polytope P . Let B^+, B^- the Borel subgroups fixing respectively the elements $[v_{\lambda_{i_o}}] \in \mathbb{P}(V_{\lambda_{i_o}})$ and $[v_{\lambda_{i_o}}^*] \in \mathbb{P}(V_{\lambda_{i_o}}^*)$. Consider also the unipotent subgroups $N^{\pm} \subset B^{\pm}$ fixing respectively the elements $v_{\lambda_i} \in V_{\lambda_i}$ and $v_{\lambda_i}^* \in V_{\lambda_i}^*$.

We consider the open subset $\mathcal{V}_{\text{End}} \subset \mathbb{P}(E)$ of elements $[f]$ such that $\langle v_{\lambda_{i_0}}^*, f(v_{\lambda_{i_0}}) \rangle \neq 0$: \mathcal{V}_{End} is $B^- \times B^+$ stable. Consider the open subset $\mathcal{V} \subset \mathbb{P}(V_{\lambda_{i_0}})$ and $\mathcal{V}^* \subset \mathbb{P}(V_{\lambda_{i_0}}^*)$ defined by :

- $[v] \in \mathcal{V} \Leftrightarrow \langle v_{\lambda_{i_0}}^*, v \rangle \neq 0$: \mathcal{V} is B^- stable,
- $[\xi] \in \mathcal{V}^* \Leftrightarrow \langle \xi, v_{\lambda_{i_0}} \rangle \neq 0$: \mathcal{V}^* is B^+ stable.

We consider now the rational maps $l : \mathbb{P}(E) \dashrightarrow \mathbb{P}(V_{\lambda_{i_0}})$, $f \mapsto f(v_{\lambda_{i_0}})$ and $r : \mathbb{P}(E) \dashrightarrow \mathbb{P}(V_{\lambda_{i_0}}^*)$, $f \mapsto v_{\lambda_{i_0}}^* \circ f$. The map l and r are defined on \mathcal{V}_{End} : they defined respectively B^- -equivariant map from \mathcal{V}_{End} into \mathcal{V} , and B^+ -equivariant map from \mathcal{V}_{End} into \mathcal{V}^* .

The orbits $K_{\mathbb{C}} \cdot v_{\lambda_{i_0}} \subset \mathbb{P}(V_{\lambda_{i_0}})$ and $K_{\mathbb{C}} \cdot v_{\lambda_{i_0}}^* \subset \mathbb{P}(V_{\lambda_{i_0}}^*)$ are closed and we have

$$\begin{aligned} K_{\mathbb{C}} \cdot v_{\lambda_{i_0}} \cap \mathcal{V} &= N^- \cdot v_{\lambda_{i_0}} \simeq N^- \\ K_{\mathbb{C}} \cdot v_{\lambda_{i_0}}^* \cap \mathcal{V}^* &= N^+ \cdot v_{\lambda_{i_0}}^* \simeq N^+. \end{aligned}$$

The rational map $(l, r) : \mathbb{P}(E) \dashrightarrow \mathbb{P}(V_{\lambda_{i_0}}) \times \mathbb{P}(V_{\lambda_{i_0}}^*)$ induced then a map $q : \mathcal{V}_{\text{End}} \cap \mathcal{X}_P \rightarrow N^- \times N^+$ which is $N^- \times N^+$ -equivariant :

$$q((n^-, n^+) \cdot x) = (n^-, n^+) \times q(x)$$

for $x \in \mathcal{V}_{\text{End}} \cap \mathcal{X}_P$, and $n^{\pm} \in N^{\pm}$.

We can now finish the arguments. The set $N^- T_{\mathbb{C}} N^+ \subset K_{\mathbb{C}}$ is dense in $K_{\mathbb{C}}$, so it is now easy to see that the map

$$\begin{aligned} N^- \times N^+ \times \mathcal{Y}_P \cap \mathcal{V}_{\text{End}} &\longrightarrow \mathcal{X}_P \cap \mathcal{V}_{\text{End}} \\ (n^-, n^+, y) &\longmapsto (n^-, n^+) \cdot y \end{aligned}$$

is a diffeomorphism. We have proved previously that $\mathcal{Y}_P \cap \mathcal{V}_{\text{End}}$ is a smooth affine variety, hence \mathcal{X}_P is smooth near the closed orbit $K_{\mathbb{C}} \times K_{\mathbb{C}} \cdot [v_{\lambda_i} \otimes v_{\lambda_i}^*] \subset \mathcal{X} \cap \mathcal{V}_{\text{End}}$. Lemma 3.6 tells us then that \mathcal{X}_P is smooth.

3.3. Hamiltonian actions. First consider an Hermitian vector space V . The Hermitian structure on $\text{End}(V)$ is $(A, B) := \text{Tr}(AB^*)$, hence the associated symplectic structure on $\text{End}(V)$ is defined by the relation $\Omega_{\text{End}}(A, B) := -\text{Im}(\text{Tr}(AB^*))$.

Let $U(V)$ be the unitary group. Let $\mathfrak{u}(V)$ be the Lie algebra of $U(V)$. We will use the identification $\epsilon : \mathfrak{u}(V) \simeq \mathfrak{u}(V)^*$, $X \mapsto \epsilon_X$ where $\epsilon_X(Y) = -\text{Tr}(XY)$. The action $U(V) \times U(V)$ on $\text{End}(V)$ is $(g, h) \cdot A = gAh^{-1}$. The moment map relative to this action is

$$\begin{aligned} \text{End}(V) &\longrightarrow \mathfrak{u}(V)^* \times \mathfrak{u}(V)^* \\ A &\longmapsto \frac{-1}{2} (iAA^*, -iA^*A). \end{aligned}$$

We consider now the projective space $\mathbb{P}(\text{End}(V))$ equipped with the Fubini-Study symplectic form Ω_{FS} . Here the action of $U(V) \times U(V)$ on $\mathbb{P}(\text{End}(V))$ is hamiltonian with moment map

$$\begin{aligned} \mathbb{P}(\text{End}(V)) &\longrightarrow \mathfrak{u}(V)^* \times \mathfrak{u}(V)^* \\ [A] &\longmapsto \left(\frac{iAA^*}{\|A\|^2}, \frac{-iA^*A}{\|A\|^2} \right). \end{aligned}$$

where $\|A\|^2 = \text{Tr}(AA^*)$ (see [26][Section 7]). If $\rho : K \hookrightarrow U(V)$ is a connected Lie subgroup, we can consider the action of $K \times K$ on $\mathbb{P}(\text{End}(V))$. Let $\pi_K : \mathfrak{u}(V)^* \rightarrow \mathfrak{k}^*$

be the projection which is dual to the inclusion $\rho : \mathfrak{k} \hookrightarrow \mathfrak{u}(V)$. The moment map for the action of $K \times K$ on $(\mathbb{P}(\text{End}(V)), \Omega_{\text{FS}})$ is then

$$(3.21) \quad \begin{aligned} \mathbb{P}(\text{End}(V)) &\longrightarrow \mathfrak{k}^* \times \mathfrak{k}^* \\ [A] &\longmapsto \frac{1}{\|A\|^2} (\pi_K(iAA^*), -\pi_K(iA^*A)). \end{aligned}$$

We are interested here respectively in

- the projective variety $\mathcal{X}_P \subset \mathbb{P}(\text{End}(V))$ with the action of $K \times K$,
- the projective variety $\mathcal{Y}_P \subset \mathbb{P}(\text{End}(V))$ with the action of $T \times T$,

where $V = \bigoplus_{i=1}^N V_{\lambda_i}$. The Fubini-Study two-form restrict into symplectic forms on \mathcal{X}_P and \mathcal{Y}_P . The action of $K \times K$ on \mathcal{X}_P is Hamiltonian with moment map

$$(3.22) \quad \begin{aligned} \Phi_{K \times K} : \mathcal{X}_P &\longrightarrow \mathfrak{k}^* \times \mathfrak{k}^* \\ [x] &\longmapsto \frac{1}{\|x\|^2} (\pi_K(ixx^*), -\pi_K(ix^*x)). \end{aligned}$$

Since the diagonal $Z = \{(t, t) | t \in T\}$ acts trivially on \mathcal{Y}_P we restrict ourself to the action of $T \times T/Z \simeq T$ on \mathcal{Y}_P . Let us compute the moment map $\Phi_T : \mathcal{Y}_P \rightarrow \mathfrak{k}^*$ associated to this action. First we have

$$(3.23) \quad \Phi_T([y]) = \frac{\pi_T(iy^*y)}{\|y\|^2} = \frac{\pi_T(iyy^*)}{\|y\|^2}$$

where $\pi_T : \mathfrak{u}(V)^* \rightarrow \mathfrak{k}^*$ is the projection which is dual to $\rho : \mathfrak{t} \rightarrow \mathfrak{u}(V)$. Since $\rho(X) = i \sum_{j \in J} \alpha_j(X) v_j \otimes v_j^*$, a small computation shows that for $B \in \mathfrak{u}(V) \simeq \mathfrak{u}(V)^*$ we have $\pi_T(B) = -i \sum_{j \in J} (Bv_j, v_j) \alpha_j$. Finally for any $[y] \in \mathcal{Y}_P$ we get

$$\Phi_T([y]) = \sum_{j \in J} \frac{\|yv_j\|^2}{\|y\|^2} \alpha_j.$$

Together with the action on T , we have also an action of the Weyl group $W = N(T)/T$ on \mathcal{Y}_P : for $\bar{w} \in W$ we take

$$(3.24) \quad \bar{w} \cdot [y] = [\rho(w) \circ y \circ \rho(w)^{-1}], \quad [y] \in \mathcal{Y}_P.$$

This action is well defined since the diagonal $Z \subset T \times T$ acts trivially on \mathcal{Y}_P . The set of weights $\{\alpha_j, j \in J\}$ is stable under the action of W , hence it is an easy fact to verify that the map Φ_T is W -equivariant.

A dense part of \mathcal{Y}_P is formed by the elements $e^Z \cdot [Id] = [\rho(e^Z)]$. Take $Z = X + iY \in \mathfrak{t}_{\mathbb{C}}$. We have $\Phi_T(e^Z \cdot [Id]) = \psi_T(Y) \in \mathfrak{k}^*$ with

$$(3.25) \quad \psi_T(Y) = \frac{1}{\sum_{j \in J} e^{-2\langle \alpha_j, Y \rangle}} \sum_{j \in J} e^{-2\langle \alpha_j, Y \rangle} \alpha_j.$$

Hence the image of the moment map $\Phi_T : \mathcal{Y}_P \rightarrow \mathfrak{k}^*$ is equal to the closure of the image of the map $\psi_T : \mathfrak{t} \rightarrow \mathfrak{k}^*$.

Proposition 3.10. *The map ψ_T realises a diffeomorphism between \mathfrak{t} and the interior of the polytope $P \subset \mathfrak{k}^*$.*

PROOF. Consider the function $F_T : \mathfrak{t} \rightarrow \mathbb{R}$, $F_T(Y) = \ln \left(\sum_{j \in J} e^{\langle \alpha_j, Y \rangle} \right)$, and let $L_T : \mathfrak{t} \rightarrow \mathfrak{k}^*$ be its Legendre transform : $L_T(X) = dF_T|_X$. Note that we have $L_T(-2Y) = \psi_T(Y)$.

We see that F_T is strictly convex. So, it is a classical fact that L_T realizes a diffeomorphism of \mathfrak{t} onto its image, and for $\xi \in \mathfrak{t}^*$ we have

$$\begin{aligned} \xi \in \text{Image}(L_T) &\Leftrightarrow \lim_{Y \rightarrow \infty} F_T(Y) - \langle \xi, Y \rangle = \infty \\ &\Leftrightarrow \lim_{Y \rightarrow \infty} \sum_{j \in J} e^{\langle \alpha_j - \xi, Y \rangle} = \infty. \end{aligned}$$

In order to conclude we need the following

Lemma 3.11. *Let $\{\beta_j, j \in J\}$ be a sequence of elements of \mathfrak{t}^* , and let Q be its convex hull. We have*

$$\lim_{Y \rightarrow \infty} \sum_{j \in J} e^{\langle \beta_j, Y \rangle} = \infty \iff 0 \in \text{Interior}(Q)$$

PROOF. First we see that $0 \notin \text{Interior}(Q)$ if and only there exists $v \in \mathfrak{t} - \{0\}$ such that $\langle \beta_j, v \rangle \leq 0$ for all j : for such vector v , the map $t \rightarrow \sum_{j \in J} e^{t \langle \beta_j, v \rangle}$ is bounded. Suppose now that $\lim_{Y \rightarrow \infty} \sum_{j \in J} e^{\langle \beta_j, Y \rangle} \neq \infty$. Then there exists a sequence $(X_k)_k \in \mathfrak{t}$ such that $\lim_k |X_k| = \infty$ and for all j the sequence $(\langle \beta_j, X_k \rangle)_k$ remains bounded. If v is a limit of a subsequence of $(\frac{X_k}{|X_k|})_k$ we have then $\langle \beta_j, v \rangle \leq 0$ for all j . \square

Lemma 3.12. *For $[y] \in \mathcal{Y}_P$ we have $\Phi_{K \times K}([y]) = (\Phi_T([y]), -\Phi_T([y]))$.*

PROOF. It's sufficient to consider the case $y = \rho(e^Z) = \sum_{j \in J} e^{i \langle \alpha_j, Z \rangle} v_j \otimes v_j^*$, for $Z = X + iY \in \mathfrak{t}_{\mathbb{C}}$. Then $yy^* = y^*y = \sum_j e^{-2 \langle \alpha_j, Y \rangle} v_j \otimes v_j^* = \rho(e^{2iY})$. So it remains to prove that $\pi_K(iyy^*) = \pi_T(iyy^*)$. We have to check that $\langle \pi_K(iyy^*), [U, V] \rangle = 0$ for $U \in \mathfrak{t}$ and $V \in \mathfrak{k}$. We have

$$\begin{aligned} \langle \pi_K(iyy^*), [U, V] \rangle &= -i \text{Tr}(yy^* \rho([U, V])) \\ &= -i \text{Tr}(\rho(e^{2iY})[\rho(U), \rho(V)]) \\ &= -i \text{Tr}([\rho(e^{2iY}), \rho(U)]\rho(V)) = 0. \end{aligned}$$

\square

Theorem 3.13. *We have*

- $\text{Image}(\Phi_T) = P$,
- $\text{Image}(\Phi_{K \times K}) = \{(k_1 \cdot \xi, -k_2 \cdot \xi) \mid \xi \in P \text{ and } k_1, k_2 \in K\}$,
- $\mathcal{Y}_P \subset \Phi_{K \times K}^{-1}(\mathfrak{t}^* \times \mathfrak{t}^*)$,
- $\Phi_{K \times K}^{-1}(\text{interior}(\mathcal{C})) \subset \mathcal{Y}_P$, where $\mathcal{C} = C_K \times -C_K$.

PROOF. The first point follows from Proposition 3.10. Since the map $(k_1, t, k_2) \mapsto k_1 t k_2$ from $K \times T_{\mathbb{C}} \times K$ into $K_{\mathbb{C}}$ is onto, we have

$$(3.26) \quad \mathcal{X}_P = (K \times K) \cdot \mathcal{Y}_P.$$

So if $[x] \in \mathcal{X}_P$, there exist $[y] \in \mathcal{Y}$ and $k_1, k_2 \in K$ such that $[x] = (k_1, k_2) \cdot [y]$, hence

$$(3.27) \quad \begin{aligned} \Phi_{K \times K}([x]) &= (k_1, k_2) \cdot \Phi_{K \times K}([y]) \\ &= (k_1 \cdot \Phi_T([y]), -k_2 \cdot \Phi_T([y])) \end{aligned}$$

The second point is proved. The third point follows also from the identity (3.27) when $k_1 = k_2 = e$. Consider now $[x] = (k_1, k_2) \cdot [y]$ such that $\Phi_{K \times K}([x])$ belongs to

the interior of the cone $C_K \times -C_K$. Then $k_1 \cdot \Phi_T([y])$ and $k_2 \cdot \Phi_T([y])$ are regular points of C_K . This implies that $k_1, k_2 \in N(T)$ and $k_2 k_1^{-1} \in T$. So

$$\begin{aligned} [x] &= (k_1, k_2) \cdot [y] \\ &= (e, k_2 k_1^{-1}) \cdot \left((k_1, k_1) \cdot [y] \right) \in \mathcal{Y}_P \end{aligned}$$

since \mathcal{Y}_P is stable under the actions of $T \times T$ and W . \square

Corollary 3.14. *If $[v_{\lambda_i} \otimes v_{\lambda_i}^*] \in \mathcal{X}_P$ then λ_i is a vertex of the polytope P .*

PROOF. Let $x = v_{\lambda_i} \otimes v_{\lambda_i}^*$, and suppose that $[x]$ belongs to \mathcal{X}_P . In order to show that $[x] \in \mathcal{Y}_P$, we compute $\Phi_{K \times K}([x])$. We see that $xx^* = x^*x = x$ and $\|x\| = 1$ so $\Phi_{K \times K}([x]) = (\pi_K(ix), -\pi_K(ix))$. For $X \in \mathfrak{k}$ we have

$$\begin{aligned} \langle \pi_K(ix), X \rangle &= -i \operatorname{Tr} \left(v_{\lambda_i} \otimes v_{\lambda_i}^* \rho(X) \right) \\ &= -i (\rho(X) v_{\lambda_i}, v_{\lambda_i}) \\ &= \langle \lambda_i, X \rangle. \end{aligned}$$

We have then $\Phi_{K \times K}([x]) = (\lambda_i, -\lambda_i)$ with λ_i being a regular point of C_K : hence $[x] \in \mathcal{Y}_P$. Now we can conclude with the help of Lemma 3.7. Since $[v_{\lambda_i} \otimes v_{\lambda_i}^*]$ belongs to \mathcal{Y}_P , the weight λ_i is a vertex of the polytope P . \square

Remark 3.15. *In this section, Theorem 3.13 was obtain without using the fact that the varieties \mathcal{X}_P and \mathcal{Y}_P are smooth. Hence Corollary 3.14 can be used to prove the smoothness of \mathcal{X}_P .*

3.4. Symplectic cutting. Let (M, Ω_M, Φ_M) be an Hamiltonian K -manifold. At this stage the moment map Φ_M is *not assumed to be proper*. We consider also the Hamiltonian $K \times K$ -manifold \mathcal{X}_P associated to a K -adapted polytope P .

The purpose of this section is to define a *symplectic cutting* of M which uses \mathcal{X}_P . The notion of symplectic cutting was introduced by Lerman in [21] in the case of a torus action. Later Woodward [35] extends this procedure to the case of a non-abelian group action (see also [24]). The method of symplectic cutting that we define in this section is different from the one of Woodward.

We have two actions of K on \mathcal{X}_P : the action from the left (resp. right) , denoted \cdot_l (resp. \cdot_r), with moment map $\Phi_l : \mathcal{X}_P \rightarrow \mathfrak{k}^*$ (resp. Φ_r). We consider now the product of $M \times \mathcal{X}_P$ with

- the action $k \cdot_1 (m, x) = (k \cdot m, k \cdot_r x)$: the corresponding moment map is $\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)$,
- the action $k \cdot_2 (m, x) = (m, k \cdot_l x)$: the corresponding moment map is $\Phi_2(m, x) = \Phi_l(x)$.

Definition 3.16. *We denote M_P the symplectic reduction at 0 of $M \times \mathcal{X}_P$ for the action \cdot_1 : $M_P := (\Phi_1)^{-1}(0)/K$.*

Note that M_P is compact when Φ_M is proper. The action \cdot_2 on $M \times \mathcal{X}_P$ induces an action of K on M_P . The moment map Φ_2 induces an equivariant map $\Phi_{M_P} : M_P \rightarrow \mathfrak{k}^*$. Let $\mathcal{Z} \subset (\Phi_1)^{-1}(0)$ be the set of points where (K, \cdot_1) as a trivial stabilizer.

Definition 3.17. *We denote M'_P the quotient $\mathcal{Z}/K \subset M_P$.*

M'_P is an open subset of smooth points of M_P which is invariant under the K -action. The symplectic structure of $M \times \mathcal{X}_P$ induces a canonical symplectic structure on M'_P that we denote $\Omega_{M'_P}$. The action of K on $(M'_P, \Omega_{M'_P})$ is Hamiltonian with moment map equal to the restriction of $\Phi_{M_P} : M_P \rightarrow \mathfrak{k}^*$ to M'_P .

We start with the easy

Lemma 3.18. *The image of $\Phi_{M_P} : M_P \rightarrow \mathfrak{k}^*$ is equal to the intersection of the image of $\Phi_M : M \rightarrow \mathfrak{k}^*$ with $K \cdot P$.*

Let $\mathcal{U}_P = K \cdot \text{Interior}(P) \subset K \cdot P$. We will show now that the open and dense subset $(\Phi_{M_P})^{-1}(\mathcal{U}_P)$ of M_P belongs to M'_P . Afterwards we will prove that $\Phi_{M_P}^{-1}(\mathcal{U}_P)$ is quasi-symplectomorphic to the open subset $\Phi_M^{-1}(\mathcal{U}_P)$ of M .

We consider the open and dense subset of \mathcal{X}_P which is equal to the open orbit $\bar{\rho}(K_{\mathbb{C}})$. From Lemma 3.3, we know that

$$(3.28) \quad \begin{aligned} \Theta : K \times \mathfrak{k} &\longrightarrow \bar{\rho}(K_{\mathbb{C}}) \\ (k, X) &\longmapsto [\rho(ke^{iX})] \end{aligned}$$

is a diffeomorphism. Through Θ , the action of $K \times K$ on $K \times \mathfrak{k}$ is $k \cdot_l(a, X) = (ka, X)$ for the action "from the left" and $k \cdot_r(a, X) = (ak^{-1}, k \cdot X)$ for the action "from the right".

We consider now the map $\psi_K : \mathfrak{k} \rightarrow \mathfrak{k}^*$ defined by $\psi_K(X) = \Phi_l([\rho(e^{iX})])$. In other words,

$$\psi_K(X) = \frac{\pi_K(i\rho(e^{i2X}))}{\text{Tr}(\rho(e^{i2X}))}.$$

Consider the function $F_K : \mathfrak{k} \rightarrow \mathbb{R}$, $F_K(X) = \ln(\text{Tr}(\rho(e^{-iX})))$. Let $L_K : \mathfrak{k} \rightarrow \mathfrak{k}^*$ be its Legendre transform.

Proposition 3.19. • We have $\psi_K(X) = L_K(-2X)$, for $X \in \mathfrak{k}$,

- The function F_K is strictly convex,
- The map ψ_K realizes an equivariant diffeomorphism between \mathfrak{k} and \mathcal{U}_P .
- The image of $\Phi_l : \mathcal{X}_P \rightarrow \mathfrak{k}^*$ is equal to the closure of \mathcal{U}_P ,
- $\Phi_l^{-1}(\mathcal{U}_P) = \bar{\rho}(K_{\mathbb{C}})$.

PROOF. For $X, Y \in \mathfrak{k}$ we consider the function $\tau(s) = F_K(X + sY)$. Since F_K is K -invariant we can restrict our computation to $X \in \mathfrak{t}$. We will use the decomposition of $Y \in \mathfrak{k}$ relatively to the T -weights on $\mathfrak{k}_{\mathbb{C}} : Y = \sum_{\alpha} Y_{\alpha}$ where $\text{ad}(Z)Y_{\alpha} = i\alpha(Z)Y_{\alpha}$ for any $Z \in \mathfrak{t}$, and $Y_0 \in \mathfrak{t}$. We have

$$\begin{aligned} \tau'(s) &= \frac{-i}{\text{Tr}(\rho(e^{-iX_s}))} \text{Tr} \left(\rho(e^{-iX_s}) \rho \left(\frac{e^{i\text{ad}(X_s)} - 1}{i\text{ad}(X_s)} Y \right) \right) \\ &= \frac{-i}{\text{Tr}(\rho(e^{-iX_s}))} \text{Tr} (\rho(e^{-iX_s}) \rho(Y)) \\ &= \frac{1}{\text{Tr}(\rho(e^{-iX_s}))} \langle \pi_K(i\rho(e^{-iX_s})), Y \rangle \end{aligned}$$

where $X_s = X + sY$. Since by definition $\tau'(0) = \langle L_K(X), Y \rangle$, the first point is proved. For the second derivative we have

$$\begin{aligned} \tau''(0) &= - \left(\frac{\text{Tr}(\rho(e^{-iX}) \rho(iY))}{\text{Tr}(\rho(e^{-iX}))} \right)^2 + \frac{\text{Tr} \left(\rho(e^{-iX}) \rho \left(\frac{e^{i\text{ad}(X)} - 1}{i\text{ad}(X)} iY \right) \rho(iY) \right)}{\text{Tr}(\rho(e^{-iX}))} \\ &= R_1 + R_2 \end{aligned}$$

where

$$\begin{aligned} R_1 &= \frac{\operatorname{Tr}(\rho(e^{-iX})\rho(iY_0)\rho(iY_0))}{\operatorname{Tr}(\rho(e^{-iX}))} - \left(\frac{\operatorname{Tr}(\rho(e^{-iX})\rho(iY_0))}{\operatorname{Tr}(\rho(e^{-iX}))} \right)^2 \\ &= \frac{\sum_j e^{-\langle \alpha_j, X \rangle} \langle \alpha_j, Y_0 \rangle^2}{\sum_j e^{-\langle \alpha_j, X \rangle}} - \left(\frac{\sum_j e^{-\langle \alpha_j, X \rangle} \langle \alpha_j, Y_0 \rangle}{\sum_j e^{-\langle \alpha_j, X \rangle}} \right)^2 \end{aligned}$$

and

$$\begin{aligned} R_2 &= \frac{1}{\operatorname{Tr}(\rho(e^{-iX}))} \sum_{\alpha \neq 0, \beta \neq 0} \frac{e^{-\langle \alpha, X \rangle} - 1}{-\langle \alpha, X \rangle} \operatorname{Tr}(\rho(e^{-iX})\rho(iY_\alpha)\rho(iY_\beta)) \\ &= \frac{1}{\operatorname{Tr}(\rho(e^{-iX}))} \sum_{\alpha \neq 0, j} \frac{e^{-\langle \alpha, X \rangle} - 1}{-\langle \alpha, X \rangle} e^{-\langle \alpha_j, X \rangle} \|\rho(Y_\alpha)v_j\|^2. \end{aligned}$$

It is now easy to see that R_1 and R_2 are positive and that $R_1 + R_2 > 0$ if $Y \neq 0$. We have proved that F_K is strictly convex. So, its Legendre transform L_K realizes a diffeomorphism of \mathfrak{k} onto its image. Using the first point we know that ψ_K realizes a diffeomorphism of \mathfrak{k} onto its image. The map ψ_K is equivariant and coincides with ψ_T on \mathfrak{t} . We have proved in Proposition 3.10 that the image of ψ_T is equal to the interior of P , hence the image of ψ_K is \mathcal{U}_P .

For the last two points we first remark that

$$(3.29) \quad \Phi_l([\rho(ke^{iX})]) = k \cdot \psi_K(X)$$

hence the image of Φ_l is the closure of \mathcal{U}_P . If we use the fact that ψ_K is a diffeomorphism from \mathfrak{k} onto \mathcal{U}_P , (3.29) shows that $\Phi_l^{-1}(K \cdot \xi) \cap \bar{\rho}(K_{\mathbb{C}})$ is a non empty and closed subset of $\Phi_l^{-1}(K \cdot \xi)$ for any $\xi \in \mathcal{U}_P$ (in fact it is a $K \times K$ -orbit). On the other hand $\Phi_l^{-1}(K \cdot \xi) \cap (\mathcal{X}_P \setminus \bar{\rho}(K_{\mathbb{C}}))$ is also a closed subset of $\Phi_l^{-1}(K \cdot \xi)$ since $\bar{\rho}(K_{\mathbb{C}})$ is open in \mathcal{X}_P . Since $\Phi_l^{-1}(K \cdot \xi)$ is connected the second subset is empty : in other words $\Phi_l^{-1}(K \cdot \xi) \subset \bar{\rho}(K_{\mathbb{C}})$. \square

We introduce now the equivariant diffeomorphism

$$(3.30) \quad \begin{aligned} \Upsilon : K \times \mathcal{U}_P &\longrightarrow \bar{\rho}(K_{\mathbb{C}}) \\ (k, \xi) &\longmapsto \Theta(k, \psi_K^{-1}(\xi)). \end{aligned}$$

We now look at $K \times \mathcal{U}_P$ equipped with the symplectic structure $\Upsilon^*(\Omega_{\mathcal{X}_P})$, and the Hamiltonian action of $K \times K$: the moment maps satisfy

$$(3.31) \quad \Upsilon^*(\Phi_l)(k, \xi) = k \cdot \xi \quad \text{and} \quad \Upsilon^*(\Phi_r)(k, \xi) = -\xi.$$

Proposition 3.20. *We have*

$$\Upsilon^*(\Omega_{\mathcal{X}_P}) = d\lambda + d\eta$$

where λ is the Liouville 1-form on $K \times \mathfrak{k}^* \simeq T^*K$ and η is an invariant 1-form on $\mathcal{U}_P \subset \mathfrak{k}^*$ which is killed by the vectors tangent to the K -orbits.

PROOF. Let E_1, \dots, E_r be a basis of \mathfrak{k} , with dual basis ξ^1, \dots, ξ^r . Let ω^i the 1-form on K , invariant by left translation and equal to ξ^i at the identity. The Liouville 1-form is $\lambda = -\sum_i \omega^i \otimes E_i$. For $X \in \mathfrak{k}$ we denote $X_l(k, \xi) = \frac{d}{dt}|_0 e^{-tX} \cdot_l(k, \xi)$ and $X_r(k, \xi) = \frac{d}{dt}|_0 e^{-tX} \cdot_r(k, \xi)$ the vectors fields generated by the action of $K \times K$. Since $\iota(X_l)d\lambda = -d\langle \Phi_l, X \rangle$ and $\iota(X_r)d\lambda = -d\langle \Phi_r, X \rangle$, the closed invariant 2-form $\beta = \Upsilon^*(\Omega_{\mathcal{X}_P}) - d\lambda$ is $K \times K$ invariant and is killed by the vectors tangent to the

orbits : (*) $\iota(X_l)\beta = \iota(X_r)\beta = 0$ for all $X \in \mathfrak{k}$. We have $\beta = \beta_2 + \beta_1 + \beta_0$ where $\beta_2 = \sum_{i,j} a_{ij}(\xi)\omega^i \wedge \omega^j$, $\beta_1 = \sum_{i,j} b_{ij}(\xi)\omega^i \wedge dE_j$, and β_0 is an invariant 2-form on \mathcal{U}_P . The equalities (*) gives $\iota(X_l)\beta_2 = \iota(X_l)\beta_1 = 0$ which implies that $\beta_2 = \beta_1 = 0$. So $\beta = \beta_0$ is a closed invariant 2-form on \mathcal{U}_P which is killed by the vectors tangent to the K -orbits. Since \mathcal{U}_P admits a retraction to $\{0\}$, $\beta = d\eta$ where η is an invariant 1-form on \mathcal{U}_P which is killed by the vectors tangent to the K -orbits. \square

If $(m, x) \in M \times \mathcal{X}_P$ belongs to $\Phi_1^{-1}(0)$, we denote $[m, x]$ the corresponding element in M_P . By definition we have $\Phi_{M_P}([m, x]) = \Phi_l(x)$ for $[m, x] \in M_P$, hence the image of Φ_{M_P} is included in the closure of \mathcal{U}_P . We see also that $[m, x] \in \Phi_{M_P}^{-1}(\mathcal{U}_P)$ if and only if $x \in \Phi_l^{-1}(\mathcal{U}_P) = \bar{\rho}(K_{\mathbb{C}})$. Since (K, \cdot_r) acts freely on $\bar{\rho}(K_{\mathbb{C}})$, we see that (K, \cdot_1) acts freely on $\Phi_{M_P}^{-1}(\mathcal{U}_P)$: the open and dense set $\Phi_{M_P}^{-1}(\mathcal{U}_P) \subset M_P$ is then contained in M'_P .

Now, we can state our main result which compares the open invariant subsets $\Phi_M^{-1}(\mathcal{U}_P) \subset M$ and $\Phi_{M_P}^{-1}(\mathcal{U}_P) \subset M_P$ equipped respectively with the symplectic structures Ω_M and $\Omega_{M'_P}$.

Theorem 3.21. $\Phi_{M_P}^{-1}(\mathcal{U}_P)$ is an open and dense subset of smooth points in M_P . There exist an equivariant diffeomorphism $\Psi : \Phi_M^{-1}(\mathcal{U}_P) \rightarrow \Phi_{M_P}^{-1}(\mathcal{U}_P)$ such that

$$\Psi^*(\Omega_{M'_P}) = \Omega_M + d\Phi_M^*\eta.$$

Here η is an invariant 1-form on \mathcal{U}_P which is killed by the vectors tangent to the K -orbits. Moreover the path $\Omega^t = \Omega_M + td\Phi_M^*\eta$, defines an homotopy of symplectic 2-forms between Ω_M and $\Psi^*(\Omega_{M'_P})$.

Remark 3.22. The map Ψ will be call a quasi-symplectomorphism.

PROOF. Consider the immersion

$$\begin{aligned} \psi : \Phi_M^{-1}(\mathcal{U}_P) &\longrightarrow M \times \mathcal{X}_P \\ m &\longmapsto (m, \Upsilon(e, \Phi_M(m))). \end{aligned}$$

We have $\Phi_1(\psi(m)) = \Phi_M(m) + \Upsilon^*\Phi_r(e, \Phi_M(m)) = 0$, and $\Phi_2(\psi(m)) = \Upsilon^*\Phi_l(e, \Phi_M(m)) = \Phi_M(m) \in \mathcal{U}_P$ (see (3.31)). Hence for all $m \in \Phi_M^{-1}(\mathcal{U}_P)$, we have $\psi(m) \in \Phi_1^{-1}(0)$, and its class $[\psi(m)] \in M_P$ belongs to $\Phi_{M_P}^{-1}(\mathcal{U}_P)$.

We denote $\Psi : \Phi_M^{-1}(\mathcal{U}_P) \rightarrow \Phi_{M_P}^{-1}(\mathcal{U}_P)$ the map $m \mapsto [\psi(m)]$. Let us show that it defines a diffeomorphism. If $\Psi(m) = \Psi(m')$, there exists $k \in K$ such that

$$\begin{aligned} (m, \Upsilon(e, \Phi_M(m))) &= k \cdot_1 (m', \Upsilon(e, \Phi_M(m'))) \\ &= (k \cdot m', k \cdot_r \Upsilon(e, \Phi_M(m'))) \\ &= (k \cdot m', \Upsilon(k^{-1}, k \cdot \Phi_M(m))). \end{aligned}$$

Since Υ is a diffeomorphism, we must have $k = e$ and $m = m'$: the map Ψ is one to one. Consider now $(m, x) \in \Phi_1^{-1}(0)$ such that $\Phi_{M_P}([m, x]) = \Phi_l(x) \in \mathcal{U}_P$: then $x \in \Phi_l^{-1}(\mathcal{U}_P) = \bar{\rho}(K_{\mathbb{C}}) = \text{Image}(\Upsilon)$. We have $x = \Upsilon(k, \xi)$ where $\xi = -\Phi_r(x) = \Phi_M(m)$. Finally

$$\begin{aligned} (m, x) &= (m, \Upsilon(k, \Phi_M(m))) \\ &= k^{-1} \cdot_1 (k \cdot m, \Upsilon(e, k \cdot \Phi_M(m))) \\ &= k^{-1} \cdot_1 \psi(k \cdot m). \end{aligned}$$

We have proved that Ψ is onto.

In order to show that Ψ is a submersion we must show that for $m \in \Phi_M^{-1}(\mathcal{U}_P)$

$$\text{Image}(\mathbb{T}_m\psi) \oplus \mathbb{T}_{\psi(m)}(K \cdot_1 \psi(m)) = \mathbb{T}_{\psi(m)}\Phi_1^{-1}(0).$$

Here $\mathbb{T}_m\psi : \mathbb{T}_mM \rightarrow \mathbb{T}_{\psi(m)}(M \times \mathcal{X}_P)$ is the tangent map, and $\mathbb{T}_{\psi(m)}(K \cdot_1 \psi(m))$ denotes the tangent space at $\psi(m)$ of the (K, \cdot_1) -orbit. We have $\dim(\text{Image}(\mathbb{T}_m\psi)) + \dim(\mathbb{T}_{\psi(m)}(K \cdot_1 \psi(m))) = \dim(\mathbb{T}_{\psi(m)}\Phi_1^{-1}(0))$ so it is sufficient to prove that

$$\text{Image}(\mathbb{T}_m\psi) \cap \mathbb{T}_{\psi(m)}(K \cdot_1 \psi(m)) = \{0\}.$$

Consider $(v, w) \in \text{Image}(\mathbb{T}_m\psi) \cap \mathbb{T}_{\psi(m)}(K \cdot_1 \psi(m))$. There exists $X \in \mathfrak{k}$ such $(v, w) = \frac{d}{dt}|_0 e^{tX} \cdot_1 \psi(m)$:

$$v = \frac{d}{dt}|_0 e^{tX} \cdot m \quad \text{and} \quad w = \frac{d}{dt}|_0 e^{tX} \cdot_r \Upsilon(e, \Phi_M(m))$$

In the other hand since $(v, w) \in \text{Image}(\mathbb{T}_m\psi)$, we have

$$w = \frac{d}{dt}|_0 \Upsilon(e, \Phi_M(e^{tX} \cdot m))$$

Since $e^{tX} \cdot_r \Upsilon(e, \Phi_M(m)) = \Upsilon(e^{-tX}, \Phi_M(e^{tX} \cdot m))$ we obtain that

$$\frac{d}{dt}|_0 \Upsilon(e^{-tX}, \Phi_M(e^{tX} \cdot m)) = \frac{d}{dt}|_0 \Upsilon(e, \Phi_M(e^{tX} \cdot m))$$

or in other words $\frac{d}{dt}|_0 \Upsilon(e^{-tX}, \Phi_M(m)) = 0$. Since Υ is a diffeomorphism we have $X = 0$, and then $(v, w) = 0$.

We can now compute the pull-back by Ψ of the symplectic form $\Omega_{M'_P}$. We have

$$\begin{aligned} \Psi^*(\Omega_{M'_P}) &= \psi^*(\Omega_M + \Omega_{\mathcal{X}_P}) \\ &= \Omega_M + \Phi_M^* \Upsilon^*(\Omega_{\mathcal{X}_P}) \\ &= \Omega_M + d\Phi_M^* \eta. \end{aligned}$$

It remains to prove that for every $t \in [0, 1]$, the 2-form $\Omega^t = \Omega_M + td\Phi_M^* \eta$ is non-degenerate. Take $t \neq 0$, $m \in \Phi_M^{-1}(\mathcal{U}_P)$ and suppose that the contraction of $\Omega^t|_m$ by $v \in \mathbb{T}_mM$ is equal to 0. For every $X \in \mathfrak{k}$ we have

$$\begin{aligned} 0 &= \Omega^t(X_M(m), v) \\ &= -\iota(v)d\langle \Phi_M, X \rangle|_m + t\iota(v)\iota(X_M)d\Phi_M^* \eta|_m \\ &= -\iota(v)d\langle \Phi_M, X \rangle|_m \end{aligned}$$

since $\iota(X_M)d\Phi_M^* \eta = d\Phi_M^*(\iota(X_{\mathfrak{k}^*})\eta) = 0$. Thus we have $\mathbb{T}_m\Phi_M(v) = 0$, and then $\iota(v)d\Phi_M^* \eta = 0$. Finally we have that $0 = \iota(v)\Omega^t|_m = \iota(v)\Omega_M|_m$. But Ω_M is non-degenerate, so $v = 0$. \square .

3.5. Formal quantization : second definition. We suppose here that the Hamiltonian K -manifold (M, Ω_M, Φ_M) is *proper* and admits a Kostant-Souriau line bundle L . Now we consider the complex $K \times K$ -submanifold \mathcal{X}_P of $\mathbb{P}(E)$. Since $\mathcal{O}(-1)$ is a $K \times K$ -equivariant Kostant-Souriau line bundle on the projective space $\mathbb{P}(E)$ the restriction

$$(3.32) \quad L_P = \mathcal{O}(-1)|_{\mathcal{X}_P}$$

is a Kostant-Souriau line bundle on \mathcal{X}_P . Hence $L \boxtimes L_P$ is a Kostant line bundle on the product $M \times \mathcal{X}_P$. In section 2.2 we have defined the quantization $\mathcal{Q}_K(M_P)$ of the (singular) reduced space $M_P := (M \times \mathcal{X}_P) //_0 (K, \cdot_1)$.

NOTATION: $O_K(r)$ will be any element $\sum_{\mu \in \widehat{K}} m_\mu V_\mu^K$ of $R^{-\infty}(K)$ where $m_\mu = 0$ if $\|\mu\| < r$. The limit $\lim_{r \rightarrow +\infty} O_K(r) = 0$ defines the notion of convergence in $R^{-\infty}(K)$.

Proposition 3.23. *Let $\varepsilon_P > 0$ be the radius of the biggest ball center at $0 \in \mathfrak{t}^*$ which is contains in the polytope P . We have*

$$(3.33) \quad \mathcal{Q}_K(M_P) = \sum_{\|\mu\| < \varepsilon_P} \mathcal{Q}(M_\mu) V_\mu^K + O_K(\varepsilon_P).$$

PROOF. Theorem 2.4 - ‘‘Quantization commutes with reduction in the singular setting’’ - tells us that $\mathcal{Q}_K(M_P) = \sum_{\mu \in \widehat{K}} \mathcal{Q}((M_P)_\mu) V_\mu^K$ where $(M_P)_\mu$ is the symplectic reduction

$$(M_P \times \overline{K_2 \cdot \mu}) //_0 K_2 \cong (M \times \mathcal{X}_P \times \overline{K_2 \cdot \mu}) //_{(0,0)} K_2 \times K_1.$$

Recall what the K_1, K_2 -action are: $k \cdot_1 (m, x, \xi) = (km, k \cdot_r x, \xi)$ and $k \cdot_2 (m, x, \xi) = (m, k \cdot_l x, k\xi)$ for $(m, x, \xi) \in M \times \mathcal{X}_P \times \overline{K_2 \cdot \mu}$ and $k \in K$.

Since the image of Φ_{M_P} is equal to the intersection of $K \cdot P = \overline{\mathcal{U}_P}$ with the image of Φ_M , we have

$$(3.34) \quad \mathcal{Q}((M_P)_\mu) = 0 \quad \text{if} \quad \mu \notin P \cap \text{Image}(\Phi_M).$$

We will now exploit Theorem 3.21 to show that $\mathcal{Q}((M_P)_\mu) = \mathcal{Q}(M_\mu)$ if μ belongs to the interior of P .

There exists a quasi-symplectomorphism Ψ between the open subset $\Phi_M^{-1}(\mathcal{U}_P)$ of M and the open and dense subset $\Phi_{M_P}^{-1}(\mathcal{U}_P)$ of M_P . Moreover one can see easily that the restriction of the Kostant line bundle $L_P \rightarrow \mathcal{X}_P$ to the open subset $\bar{\rho}(K_{\mathbb{C}})$ is *trivial*. If L_{M_P} is the Kostant line bundle on M_P induced by $L \boxtimes L_P$, we have that the pull-back of the restriction $L_{M_P}|_{\Phi_{M_P}^{-1}(\mathcal{U}_P)}$ by Ψ is equivariantly diffeomorphic to the restriction of L to $\Phi_M^{-1}(\mathcal{U}_P)$.

Take now $\mu \in \widehat{K}$ that belongs to the interior of the polytope P . The element $\mathcal{Q}((M_P)_\mu) \in \mathbb{Z}$ is given by the index of a transversally elliptic symbol defined in a (small) neighborhood of $\Phi_{M_P}^{-1}(\mu) \subset M_P$. This symbol is defined through two auxiliary data: the Kostant line bundle L_{M_P} and a compatible almost complex structure J which defined in a neighborhood of $\Phi_{M_P}^{-1}(\mu)$. If we pull back everything by Ψ , we get a transversally elliptic symbol living in a (small) neighborhood of $\Phi_M^{-1}(\mu) \subset M$ which is defined by the Kostant line bundle L and an almost complex structure J_1 compatible with the symplectic structure $\Omega_1 := \Omega_M + d\Phi_M^* \eta$. But since $\Omega_t = \Omega_M + td\Phi_M^* \eta$ defines an homotopy of symplectic structures, any almost complex structure compatible with Ω_M is homotopic to J_1 . We have then shown that $\mathcal{Q}(M_\mu) = \mathcal{Q}((M_P)_\mu)$ for any μ belonging to the interior of P . So we have

$$\mathcal{Q}_K(M_P) = \sum_{\mu \in \text{Interior}(P)} \mathcal{Q}(M_\mu) V_\mu^K + \sum_{\nu \in \partial P} \mathcal{Q}((M_P)_\nu) V_\nu^K.$$

Since for $\nu \in \partial P$ we have $\|\nu\| \geq \varepsilon_P$, the last equality proves (3.33). \square

We work now with the dilated polytope nP , for any integer $n \geq 1$. The polytope nP is still K -adapted, so one can consider the reduced spaced M_{nP} and Proposition 3.23 gives that

$$(3.35) \quad \mathcal{Q}_K(M_{nP}) = \sum_{\|\mu\| < n\varepsilon_P} \mathcal{Q}(M_\mu) V_\mu^K + O_K(n\varepsilon_P).$$

for any integer $n \geq 1$. We can summarize the result of this section in the following

Proposition 3.24. *Let (M, Ω_M) be a pre-quantized Hamiltonian K -manifold, with a proper moment map Φ_M .*

- For any integer $n \geq 1$, the (singular) compact Hamiltonian manifold M_{nP} contains as an open and dense subset, the open subset $\Phi_M^{-1}(n\mathcal{U}_P)$ of M .
- We have $\mathcal{Q}_K^{-\infty}(M) = \lim_{n \rightarrow \infty} \mathcal{Q}_K(M_{nP})$.

4. FUNCTORIAL PROPERTIES : PROOF OF THEOREM 1.3

This section is devoted to the proof of Theorem 1.3. We will use in a crucial way the characterisation of $\mathcal{Q}_K^{-\infty}$ given in Proposition 3.24.

Let $H \subset K$ be a connected Lie subgroup. Here we consider a pre-quantized Hamiltonian K -manifold M which is *proper* as an Hamiltonian H -manifold. We want to compare $\mathcal{Q}_K^{-\infty}(M)$ and $\mathcal{Q}_H^{-\infty}(M)$. For $\mu \in \widehat{K}$ and $\nu \in \widehat{H}$ we denote N_ν^μ the multiplicity of V_ν^H in the restriction $V_\mu^K|_H$. We have seen in the introduction that $N_\nu^\mu \mathcal{Q}(M_{\mu,K}) \neq 0$ only for the μ belonging to *finite* subset $\widehat{K} \cap \Phi_K(K \cdot \Phi_H^{-1}(\nu))$. Then $\mathcal{Q}_K^{-\infty}(M)$ is H -admissible and we have the following equality in $R^{-\infty}(H)$:

$$(4.36) \quad \mathcal{Q}_K^{-\infty}(M)|_H = \sum_{\nu \in \widehat{H}} m_\nu V_\nu^H$$

with $m_\nu = \sum_\mu N_\nu^\mu \mathcal{Q}(M_{\mu,K})$. We will now prove that $\mathcal{Q}_K^{-\infty}(M)|_H = \mathcal{Q}_H^{-\infty}(M)$.

Lemma 4.1. *The restriction $\mathcal{Q}_K^{-\infty}(M)|_H$ is equal to $\lim_{n \rightarrow \infty} \mathcal{Q}_K(M_{nP})|_H$.*

PROOF. Let us denote P° and ∂P respectively the interior and the border of the K -adapted polytope P . We write

$$\mathcal{Q}_K^{-\infty}(M) = \sum_{\mu \in nP^\circ} \mathcal{Q}(M_{\mu,K}) V_\mu^K + \sum_{\mu \notin nP^\circ} \mathcal{Q}(M_{\mu,K}) V_\mu^K.$$

On the other side

$$\mathcal{Q}_K(M_{nP}) = \sum_{\mu \in nP^\circ} \mathcal{Q}(M_{\mu,K}) V_\mu^K + \sum_{\mu \in n\partial P} \mathcal{Q}((M_{nP})_{\mu,K}) V_\mu^K.$$

So the difference $D(n) = \mathcal{Q}_K^{-\infty}(M) - \mathcal{Q}_K(M_{nP})$ is equal to

$$D(n) = - \sum_{\mu' \in n\partial P} \mathcal{Q}((M_{nP})_{\mu',K}) V_{\mu'}^K + \sum_{\mu \notin nP^\circ} \mathcal{Q}(M_{\mu,K}) V_\mu^K.$$

We show now that the restriction $D(n)|_H$ tends to 0 in $R^{-\infty}(H)$ as n goes to infinity. For this purpose, we will prove that for any $c > 0$ there exist $n_c \in \mathbb{N}$ such that $D(n)|_H = O_H(c)$ for any $n \geq n_c$.

For $c > 0$ we consider the compact subset of \mathfrak{k}^* defined by

$$(4.37) \quad \mathcal{K}_c = \Phi_K(K \cdot \Phi_H^{-1}(\xi \in \mathfrak{h}^*, \|\xi\| \leq c)).$$

Let $n_c \in \mathbb{N}$ such that \mathcal{K}_c is included in $K \cdot (n_c P^\circ)$: hence $\mathcal{K}_c \subset K \cdot (nP^\circ)$ for any $n \geq n_c$. We know that for $\mu \in \widehat{K}$, we have $N_\nu^\mu \mathcal{Q}(M_{\mu,K}) \neq 0$ only for $\mu \in \Phi_K(K \cdot \Phi_H^{-1}(\nu))$, and for $\mu' \in \widehat{K}$, we have $N_{\nu'}^{\mu'} \mathcal{Q}((M_{nP})_{\mu',K}) \neq 0$ only for $\mu' \in nP \cap \Phi_K(K \cdot \Phi_H^{-1}(\nu))$.

Then if $n \geq n_c$, we have

$$N_\nu^\mu \mathcal{Q}(M_{\mu,K}) = N_\nu^{\mu'} \mathcal{Q}((M_{nP})_{\mu',K}) = 0$$

for any $\nu \in \widehat{H} \cap \{\xi \in \mathfrak{h}^*, \|\xi\| \leq c\}$, $\mu \notin nP^o$ and $\mu' \in n\partial P$. It means that $D(n)|_H = O_H(c)$ for any $n \geq n_c$. \square

Since $\mathcal{Q}_K(M_{nP})|_H = \mathcal{Q}_H(M_{nP})$, we are no led to the

Lemma 4.2. *The limit $\lim_{n \rightarrow \infty} \mathcal{Q}_H(M_{nP})$ is equal to $\mathcal{Q}_H^{-\infty}(M)$.*

PROOF. Theorem 2.4 - ‘‘Quantization commutes with reduction in the singular setting’’ - tells us that $\mathcal{Q}_H(M_{nP}) = \sum_{\nu \in \widehat{H}} \mathcal{Q}((M_{nP})_{\nu,H}) V_\nu^H$ where $(M_{nP})_{\nu,H}$ is the symplectic reduction

$$(M_{nP} \times \overline{H \cdot \nu}) //_0 H \cong (M \times \mathcal{X}_{nP} \times \overline{H \cdot \mu}) //_{(0,0)} H \times K.$$

For $c > 0$ we consider the compact subset of \mathcal{K}_c defined in (4.37). Let $n_c \in \mathbb{N}$ such that $\mathcal{K}_c \subset K \cdot (nP^o)$ for any $n \geq n_c$. It implies that

$$\Phi_H^{-1}(\xi \in \mathfrak{h}^*, \|\xi\| \leq c) \subset \Phi_K^{-1}(K \cdot (nP^o))$$

for $n \geq n_c$. Since M_{nP} ‘‘contains’’ as an the open subset $\Phi_K^{-1}(K \cdot (nP^o))$, the arguments similar to those used in the proof of Proposition 3.23 show that $\mathcal{Q}((M_{nP})_{\nu,H}) = \mathcal{Q}(M_{\nu,H})$ for $\|\nu\| \leq c$ and $n \geq n_c$. It means that

$$\mathcal{Q}_H(M_{nP}) = \sum_{\|\nu\| \leq c} \mathcal{Q}(M_{\nu,H}) V_\nu^H + O_H(c) \quad \text{when } n \geq n_c.$$

It follows that $\lim_{n \rightarrow \infty} \mathcal{Q}_H(M_{nP}) = \sum_{\nu \in \widehat{H}} \mathcal{Q}(M_{\nu,H}) V_\nu^H = \mathcal{Q}_H^{-\infty}(M)$. \square

5. THE CASE OF AN HERMITIAN SPACE

Let (E, \mathfrak{h}) be an Hermitian vector space of dimension n .

5.1. The quantization of E . Let $U := U(E)$ be the unitary group with Lie algebra \mathfrak{u} . We use the isomorphism $\epsilon : \mathfrak{u} \rightarrow \mathfrak{u}^*$ defined by $\langle \epsilon(X), Y \rangle = -\text{Tr}(XY) \in \mathbb{R}$. For $v, w \in E$, let $v \otimes w^* : E \rightarrow E$ be the linear map $x \mapsto \mathfrak{h}(x, w)v$.

Let $E_{\mathbb{R}}$ be the space E view as a real vector space. Let Ω be the imaginary part of $-\mathfrak{h}$, and let J the complex structure on $E_{\mathbb{R}}$. Then on $E_{\mathbb{R}}$, Ω is a (constant) symplectic structure and $\Omega(-, J-)$ defines a scalar product. The action of U on $(E_{\mathbb{R}}, \Omega)$ is Hamiltonian with moment map $\Phi : E \rightarrow \mathfrak{u}^*$ defined by $\langle \Phi(v), X \rangle = \frac{1}{2} \Omega(Xv, v)$. Through ϵ , the moment map Φ is defined by

$$(5.38) \quad \Phi(v) = \frac{1}{2i} v \otimes v^*.$$

The pre-quantization data $(L, \langle -, - \rangle, \nabla)$ on the Hamiltonian U -manifold $(E_{\mathbb{R}}, \Omega, \Phi)$ is a trivial line bundle L with a trivial action of U equipped with the Hermitian structure $\langle s, s' \rangle_v = e^{\frac{-\mathfrak{h}(v,v)}{2}} \overline{s} s'$ and the Hermitian connexion $\nabla = d - i\theta$ where θ is the 1-form on E defined by $\theta = \frac{1}{2} \Omega(v, dv)$.

The traditional quantization of the Hamiltonian U -manifold $(E_{\mathbb{R}}, \Omega, \Phi)$, that we denote $\mathcal{Q}_U^{L^2}(E)$, is the *Bargman space* of entire holomorphic functions on E which are L^2 integrable with respect to the Gaussian measure $e^{\frac{-\mathfrak{h}(v,v)}{2}} \Omega^n$. The representation $\mathcal{Q}_U^{L^2}(E)$ of U is admissible. The irreducible representations of U

that occur in $\mathcal{Q}_U^{\mathbb{L}^2}(E)$ are the vector subspaces $S^j(E^*)$ formed by the homogeneous polynomial on E of degree $j \geq 0$.

On the other hand, the moment map Φ is proper (see 5.38). Hence we can consider the formal quantization $\mathcal{Q}_U^{-\infty}(E) \in R^{-\infty}(U)$ of the U -action on E .

Lemma 5.1. *The two quantizations of (E, Ω, Φ) , $\mathcal{Q}_U^{\mathbb{L}^2}(E)$ and $\mathcal{Q}_U^{-\infty}(E)$ coincide in $R^{-\infty}(U)$. In other words, we have*

$$(5.39) \quad \mathcal{Q}_U^{-\infty}(E) = S^\bullet(E^*) := \sum_{j \geq 0} S^j(E^*) \quad \text{in } R^{-\infty}(U).$$

PROOF. Let $T \subset U$ be a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{u}$. There exists an orthonormal basis $(e_k)_{k=1, \dots, n}$ of E and characters $(\chi_k)_{k=1, \dots, n}$ of T such that $t \cdot e_k = \chi_k(t)e_k$ for all k . The family $(ie_k \otimes e_k^*)_{k=1, \dots, n}$ is then a basis of \mathfrak{t} such that $\frac{1}{i}d\chi_l(ie_k \otimes e_k^*) = \delta_{l,k}$. The set $\hat{U} \subset \mathfrak{t}^* \subset \mathfrak{u}^*$ of dominant weights is composed, through ϵ , by the elements

$$\underline{\lambda} = i \sum_{k=1}^n \lambda_k e_k \otimes e_k^*,$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a decreasing sequence of integer.

The formal quantization $\mathcal{Q}_U^{-\infty}(E) \in R^{-\infty}(U)$ is defined by

$$\mathcal{Q}_U^{-\infty}(E) = \sum_{\lambda_1 \geq \dots \geq \lambda_n} \mathcal{Q}(E_{\underline{\lambda}}) V_{\underline{\lambda}}$$

where $E_{\underline{\lambda}} = \Phi^{-1}(U \cdot \underline{\lambda})/U$ is the reduced space and $V_{\underline{\lambda}}$ is the irreducible representation of U with highest weight $\underline{\lambda}$.

It is now easy to check that

$$E_{\underline{\lambda}} = \begin{cases} \{\text{pt}\} & \text{if } \lambda = (0, \dots, 0, -j) \text{ with } j \geq 0, \\ \emptyset & \text{in the other cases,} \end{cases}$$

and then

$$\mathcal{Q}(E_{\underline{\lambda}}) = \begin{cases} 1 & \text{if } \lambda = (0, \dots, 0, -j) \text{ with } j \geq 0, \\ 0 & \text{in the other cases.} \end{cases}$$

Finally (5.39) follows from the fact that $V_{(0, \dots, 0, -j)} = S^j(E^*)$. \square

5.2. The quantization of E restricted to a subgroup of U . Let $K \subset U$ be a connected Lie subgroup with Lie algebra \mathfrak{k}^* . Let $K_{\mathbb{C}} \subset \text{GL}(E)$ be its complexification. The moment map relative to the K -action on $(E_{\mathbb{R}}, \Omega)$ is the map

$$\Phi_K : E \rightarrow \mathfrak{k}^*$$

equal to the composition of Φ with the projection $\mathfrak{u}^* \rightarrow \mathfrak{k}^*$.

Lemma 5.2. *The following conditions are equivalent :*

- (a) *the map Φ_K is proper,*
- (b) *$\Phi_K^{-1}(0) = \{0\}$,*
- (c) *$\{0\}$ is the only closed $K_{\mathbb{C}}$ -orbit in E ,*
- (d) *for every $v \in E$ we have $0 \in \overline{K_{\mathbb{C}} \cdot v}$,*
- (e) *$S^\bullet(E^*)$ is an admissible representation of K ,*
- (f) *the K -invariant polynomials on E are the constant polynomials.*

PROOF. The equivalence (a) \iff (b) is due to the fact that Φ_K is quadratic.

Let \mathcal{O} be a $K_{\mathbb{C}}$ -orbit in E . Classical results of Geometric Invariant Theory [26, 18] assert that $\overline{\mathcal{O}} \cap \Phi_K^{-1}(0) \neq \emptyset$ and that \mathcal{O} is closed if and only if $\mathcal{O} \cap \Phi_K^{-1}(0) \neq \emptyset$. Hence (b) \iff (c) \iff (d).

After Lemma 5.1 we know that $\mathcal{Q}_{\mathbb{U}}^{-\infty}(E) = S^{\bullet}(E^*)$. Since $\mathcal{Q}_{\mathbb{U}}^{-\infty}(E)$ is K -admissible when $\widehat{\Phi}_K$ is proper (see Section 4), we have (a) \implies (e).

For every $\mu \in \widehat{K}$, the μ -isotopic component $[S^{\bullet}(E^*)]_{\mu}$ is a module over $[S^{\bullet}(E^*)]_0 = [S^{\bullet}(E^*)]^K$. Hence $\dim[S^{\bullet}(E^*)]_{\mu} < \infty$ implies that $[S^{\bullet}(E^*)]^K = \mathbb{C}$. We have (e) \implies (f).

Finally (f) \implies (d) follows from the following fundamental fact. For any $v, w \in E$ we have $\overline{K_{\mathbb{C}} \cdot v} \cap \overline{K_{\mathbb{C}} \cdot w} \neq \emptyset$ if and only if $P(v) = P(w)$ for all $P \in [S^{\bullet}(E^*)]^K$. \square

Theorem 1.3 implies the following

Proposition 5.3. *Let $K \subset \mathbb{U}(E)$ be a closed connected subgroup such that $S^{\bullet}(E^*)$ is an admissible representation of K . For every $\mu \in \widehat{K}$, we have*

$$\dim([S^{\bullet}(E^*)]_{\mu}) = \mathcal{Q}(E_{\mu, K})$$

where $[S^{\bullet}(E^*)]_{\mu}$ is the μ -isotopic component of $S^{\bullet}(E^*)$ and $E_{\mu, K}$ is the reduced space $\Phi_K^{-1}(K \cdot \mu)/K$.

In the following examples the condition $\Phi_K^{-1}(0) = \{0\}$ is easy to check.

- 1) the subgroup $K \subset \mathbb{U}(E)$ contains the center of $\mathbb{U}(E)$,
- 2) $E = \wedge^2 \mathbb{C}^n$ or $E = S^2(\mathbb{C}^n)$ and $K = \mathbb{U}(n) \subset \mathbb{U}(E)$,
- 3) $E = M_{n, k}$ is the vector space of $n \times k$ -matrices and $K = \mathbb{U}(n) \times \mathbb{U}(k) \subset \mathbb{U}(E)$.

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