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# FORMAL GEOMETRIC QUANTIZATION

PAUL-EMILE PARADAN

ABSTRACT. Let  $K$  be a compact Lie group acting in an Hamiltonian way on a symplectic manifold  $(M, \Omega)$  which is prequantized by a Kostant-Souriau line bundle. We suppose here that the moment map  $\Phi$  is *proper* so that the reduced space  $M_\mu := \Phi^{-1}(K \cdot \mu)/K$  are compact for all  $\mu$ . Following Weitsman [33], we define the “formal geometric quantization” of  $M$  as

$$\mathcal{Q}_K^{-\infty}(M) := \sum_{\mu \in \hat{K}} \mathcal{Q}(M_\mu) V_\mu^K.$$

The aim of this article is to study the functorial properties of the assignment  $M \rightarrow \mathcal{Q}_K^{-\infty}(M)$ .

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## 1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let  $M$  be an Hamiltonian  $K$ -manifold with symplectic form  $\Omega$  and moment map  $\Phi : M \rightarrow \mathfrak{k}^*$ . We assume the existence of a  $K$ -equivariant line bundle  $L$  on  $M$  with connection of curvature equal to  $-i\omega$ . In other words  $M$  is pre-quantizable in the sense of [20] and we call  $L$  the Kostant-Souriau line bundle.

In the process of quantization one tries to associate a unitary representation of  $K$  to the data  $(M, \Omega, \Phi, L)$ . When  $M$  is **compact** one associates to these data a virtual representation  $\mathcal{Q}_K(M) \in R(K)$  of  $K$  defined as the equivariant index of a Dolbeault-Dirac operator :  $\mathcal{Q}_K(M)$  is the geometric quantization of  $M$ .

This quantization process satisfies the following functorial properties :

**[P1]** When  $N$  and  $M$  are respectively pre-quantized compact Hamiltonian  $K_1$  and  $K_2$  manifolds, the product  $M \times N$  is a pre-quantized compact Hamiltonian  $K_1 \times K_2$ -manifold, and we have

$$(1.1) \quad \mathcal{Q}_{K_1 \times K_2}(M \times N) = \mathcal{Q}_{K_1}(M) \otimes \mathcal{Q}_{K_2}(N) \quad \text{in} \quad R(K_1 \times K_2) \simeq R(K_1) \otimes R(K_2).$$

**[P2]** If  $H \subset K$  is a connected Lie subgroup, then the restriction of  $\mathcal{Q}_K(M)$  to  $H$  is equal to  $\mathcal{Q}_H(M)$ .

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Note that [P1] and [P2] gives the following functorial property :

[P3] When  $N$  and  $M$  are pre-quantized compact Hamiltonian  $K$ -manifold, the product  $M \times N$  is a pre-quantized compact Hamiltonian  $K$ -manifold, and we have  $\mathcal{Q}_K(M \times N) = \mathcal{Q}_K(M) \cdot \mathcal{Q}_K(N)$ , where  $\cdot$  denotes the product in  $R(K)$ .

One other fundamental property is the behaviour of the  $K$ -multiplicities of  $\mathcal{Q}_K(M)$  that is known as “quantization commutes with reduction”.

Let  $T$  be a maximal torus of  $K$ ,  $C_K \subset \mathfrak{t}^*$  be a Weyl Chamber,  $\Lambda^* \subset \mathfrak{t}^*$  be the weight lattice, and  $\widehat{K} = \Lambda^* \cap C_K$  be the set of dominant weights. The ring of character  $R(K)$  as a  $\mathbb{Z}$ -basis  $V_\mu^K, \mu \in \widehat{K} : V_\mu^K$  is the irreducible  $K$ -representation with highest weight  $\mu$ .

For any  $\mu \in \widehat{K}$  which is a regular value of  $\Phi$ , the reduced space (or symplectic quotient)  $M_\mu := \Phi^{-1}(K \cdot \mu)/K$  is an orbifold equipped with a symplectic structure  $\Omega_\mu$ . Moreover  $L_\mu := (L|_{\Phi^{-1}(\mu)} \otimes \mathbb{C}_{-\mu})/K_\mu$  is a Kostant line orbundle over  $(M_\mu, \Omega_\mu)$ . The definition of the index of the Dolbeault-Dirac operator carries over to the orbifold case, hence  $\mathcal{Q}(M_\mu) \in \mathbb{Z}$  is defined. In [25], this is extended further to the case of singular symplectic quotients, using partial (or shift) de-singularization. So the integer  $\mathcal{Q}(M_\mu) \in \mathbb{Z}$  is well defined for every  $\mu \in \widehat{K}$  : in particular  $\mathcal{Q}(M_\mu) = 0$  if  $\mu \notin \Phi(M)$ .

The following Theorem was conjectured by Guillemin-Sternberg [16] and is known as “quantization commutes with reduction” [24, 25, 30, 28]. For complete references on the subject the reader should consult [29, 32].

**Theorem 1.1.** (*Meinrenken, Meinrenken-Sjamaar*). *We have the following equality in  $R(K)$*

$$\mathcal{Q}_K(M) = \sum_{\mu \in \widehat{K}} \mathcal{Q}(M_\mu) V_\mu^K .$$

Suppose now that  $M$  is **non-compact** but the moment map  $\Phi : M \rightarrow \mathfrak{t}^*$  is assumed to be **proper** (we will said simply ” $M$  is proper”). In this situation the geometric quantization of  $M$  as an index of an elliptic operator is not well defined. Nevertheless the integers  $\mathcal{Q}(M_\mu), \mu \in \widehat{K}$  are well defined since the symplectic quotient  $M_\mu$  are **compact**.

Following Weitsman [33], we introduce the following

**Definition 1.2.** *The formal quantization of  $(M, \Omega, \Phi)$  is the element of  $R^{-\infty}(K) := \text{hom}_{\mathbb{Z}}(R(K), \mathbb{Z})$  defined by*

$$\mathcal{Q}_K^{-\infty}(M) = \sum_{\mu \in \widehat{K}} \mathcal{Q}(M_\mu) V_\mu^K .$$

A representation  $E$  of  $K$  is admissible if it has finite  $K$ -multiplicities :  $\dim(\text{hom}_K(V_\mu^K, E)) < \infty$  for every  $\mu \in \widehat{K}$ . Here  $R^{-\infty}(K)$  is the Grothendieck group associated to the  $K$ -admissible representations. We have a canonical inclusion  $i : R(K) \hookrightarrow R^{-\infty}(K)$  : to  $V \in R(K)$  we associate the map  $i(V) : R(K) \rightarrow \mathbb{Z}$  defined by  $W \mapsto \dim(\text{hom}_K(V, W))$ . In order to simplify the notation,  $i(V)$  will be written  $V$ . Moreover the tensor product induces a structure of  $R(K)$ -module on  $R^{-\infty}(K)$  since  $E \otimes V$  is an admissible representation when  $V$  and  $E$  are respectively finite dimensional and admissible representation of  $K$ .

It is an easy matter to see that **[P1]** holds for the formal quantization process  $\mathcal{Q}^{-\infty}$ . Let  $N$  and  $M$  be respectively pre-quantized *proper* Hamiltonian  $K_1$  and  $K_2$  manifolds : the product  $M \times N$  is then a pre-quantized *proper* Hamiltonian  $K_1 \times K_2$ -manifold. For the reduced spaces we have  $(M \times N)_{(\mu_1, \mu_2)} \simeq M_{\mu_1} \times N_{\mu_2}$ , for all  $\mu_1 \in \widehat{K}_1, \mu_2 \in \widehat{K}_2$ . It follows then that

$$(1.2) \quad \mathcal{Q}_{K_1 \times K_2}^{-\infty}(M \times N) = \mathcal{Q}_{K_1}^{-\infty}(M) \widehat{\otimes} \mathcal{Q}_{K_2}^{-\infty}(N)$$

in  $R^{-\infty}(K_1 \times K_2) \simeq R^{-\infty}(K_1) \widehat{\otimes} R^{-\infty}(K_2)$ .

The purpose of this article is to show that the functorial property **[P2]** still holds for the formal quantization process  $\mathcal{Q}^{-\infty}$ .

**Theorem 1.3.** *Let  $M$  be a pre-quantized Hamiltonian  $K$ -manifold which is proper. Let  $H \subset K$  be a connected Lie subgroup such that  $M$  is still proper as an Hamiltonian  $H$ -manifold. Then  $\mathcal{Q}_K^{-\infty}(M)$  is  $H$ -admissible and we have the following equality in  $R^{-\infty}(H)$  :*

$$(1.3) \quad \mathcal{Q}_K^{-\infty}(M)|_H = \mathcal{Q}_H^{-\infty}(M).$$

For  $\mu \in \widehat{K}$  and  $\nu \in \widehat{H}$  we denote  $N_\nu^\mu = \dim(\text{hom}_H(V_\nu^H, V_\mu^K|_H))$  the multiplicity of  $V_\nu^H$  in the restriction  $V_\mu^K|_H$ . In the situation of Theorem 1.3, the moment maps relative to the  $K$  and  $H$  actions are  $\Phi_K$  and  $\Phi_H = \text{p} \circ \Phi_K$ , where  $\text{p} : \mathfrak{k}^* \rightarrow \mathfrak{h}^*$  is the canonical projection.

**Corollary 1.4.** *For every  $\nu \in \widehat{H}$ , we have :*

$$(1.4) \quad \mathcal{Q}(M_{\nu, H}) = \sum_{\mu \in \widehat{K}} N_\nu^\mu \mathcal{Q}(M_{\mu, K}).$$

where  $M_{\nu, H} = \Phi_H^{-1}(H \cdot \nu)/H$  and  $M_{\mu, K} = \Phi_K^{-1}(K \cdot \mu)/K$  are respectively the symplectic reductions relative to the  $H$  and  $K$ -actions.

Since  $V_\mu^K$  is equal to the  $K$ -quantization of  $K \cdot \mu$ , the ‘‘quantization commutes with reduction’’ Theorem tells us that  $N_\nu^\mu = \mathcal{Q}((K \cdot \mu)_{\nu, H})$  : in particular  $N_\nu^\mu \neq 0$  implies that  $\nu \in \text{p}(K \cdot \mu) \iff \mu \in K \cdot \text{p}^{-1}(\nu)$ . Finally

$$N_\nu^\mu \mathcal{Q}(M_{\mu, K}) \neq 0 \implies \mu \in K \cdot \text{p}^{-1}(\nu) \quad \text{and} \quad \Phi_K^{-1}(\mu) \neq \emptyset.$$

Theses two conditions imply that we can restrict the sum of RHS of (1.4) to

$$(1.5) \quad \mu \in \widehat{K} \cap \Phi_K(K \cdot \Phi_H^{-1}(\nu))$$

which is *finite* since  $\Phi_H$  is proper.

Theorem 1.3 and (1.2) gives the following extended version of **[P3]**.

**Theorem 1.5.** *Let  $N$  and  $M$  be two pre-quantized Hamiltonian  $K$ -manifold where  $N$  is compact and  $M$  is proper. The product  $M \times N$  is then proper and we have the following equality in  $R^{-\infty}(K)$  :*

$$(1.6) \quad \mathcal{Q}_K^{-\infty}(M \times N) = \mathcal{Q}_K^{-\infty}(M) \cdot \mathcal{Q}_K(N)$$

For  $\mu, \lambda, \theta \in \widehat{K}$  we denote  $C_{\lambda, \theta}^\mu = \dim(\text{hom}_H(V_\mu^K, V_\lambda^K \otimes V_\theta^K))$  the multiplicity of  $V_\mu^K$  in the tensor  $V_\lambda^K \otimes V_\theta^K$ . Since  $V_\lambda^K \otimes V_\theta^K$  is equal to the quantization of the product  $K \cdot \lambda \times K \cdot \theta$ , the ‘‘quantization commutes with reduction’’ Theorem tells us that  $C_{\lambda, \theta}^\mu = \mathcal{Q}((K \cdot \lambda \times K \cdot \theta)_\mu)$  : in particular  $C_{\lambda, \theta}^\mu \neq 0$  implies that (\*)  $\|\lambda\| \leq \|\theta\| + \|\mu\|$ .

**Corollary 1.6.** *In the situation of Theorem 1.5, we have for every  $\mu \in \widehat{K}$  :*

$$(1.7) \quad \mathcal{Q}((M \times N)_\mu) = \sum_{\lambda, \theta \in \widehat{K}} C_{\lambda, \theta}^\mu \mathcal{Q}(M_\lambda) \mathcal{Q}(N_\theta).$$

Since  $N$  is compact,  $\mathcal{Q}(N_\theta) \neq 0$  for a  $(**)$   $\theta \in \{\text{finite set}\}$ . Then  $(*)$  and  $(**)$  show that the sum in the RHS of (1.7) is *finite*.

Weistman [33] studied the formal quantization procedure in the case of  $K = \mathrm{U}(n)$ . Using a method of symplectic cutting [21, 35] he defines for two increasing sequence of positive integers  $r_n, s_n$  a family of cut-spaces  $M^{r_n, s_n}$  which are compact. *Under the hypothesis that the cut spaces  $M^{r_n, s_n}$  are smooth*, he notes that  $\mathcal{Q}_K^{-\infty}(M) = \lim_{n \rightarrow \infty} \mathcal{Q}_K(M^{r_n, s_n})$ , and he was then able to show Theorem 1.5.

Our proof of Theorem 1.3 uses also a technique of symplectic cutting but which is valid for any compact Lie group actions. We have to overpass the difficulties concerning the non-smoothness of the cut-spaces. For this purpose we introduce another method of symplectic cutting that uses the wonderful compactifications of Concini-Procesi [13, 14], and we prove an extension of the “quantization commutes with reduction” Theorem to the *singular* setting.

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## 2. QUANTIZATION COMMUTES WITH REDUCTION

In this section we precise the definition of the quantization of a smooth and compact Hamiltonian manifold. We extend the definition to the case of a *singular* Hamiltonian manifold and we prove a “quantization commutes with reduction” Theorem in the singular setting.

In the Kostant-Souriau framework, an Hamiltonian  $K$ -manifold  $(M, \Omega_M, \Phi_M)$  is pre-quantized if there is an equivariant Hermitian line bundle  $L$  with an invariant Hermitian connection  $\nabla$  such that

$$(2.8) \quad \mathcal{L}(X) - \nabla_{X_M} = i\langle \Phi_M, X \rangle \quad \text{and} \quad \nabla^2 = -i\Omega_M,$$

for every  $X \in \mathfrak{k}$ . Here  $X_M$  is the vector field on  $M$  defined by  $X_M(m) = \frac{d}{dt} e^{-tX} m|_0$ .

$(L, \nabla)$  is also call a Kostant-Souriau line bundle. Remark that the conditions (2.8) implies through the equivariant Bianchi formula the relation

$$(2.9) \quad \iota(X_M)\Omega_M = -d\langle \Phi_M, X \rangle, \quad X \in \mathfrak{k}.$$

We will now recall the notions of geometric quantization.

**2.1. Geometric quantization : the compact and smooth case.** We suppose here that  $(M, \Omega_M, \Phi_M)$  is **compact** and is prequantized by a Hermitian line bundle  $L$ . Choose a  $K$ -invariant almost complex structure  $J$  on  $M$  which is compatible with  $\Omega_M$  in the sense that the symmetric bilinear form  $\Omega_M(\cdot, J\cdot)$  is a Riemannian metric. Let  $\bar{\partial}_L$  be the Dolbeault operator with coefficients in  $L$ , and let  $\bar{\partial}_L^*$  be its (formal) adjoint. The *Dolbeault-Dirac operator* on  $M$  with coefficients in  $L$  is  $D_L = \bar{\partial}_L + \bar{\partial}_L^*$ , considered as an operator from  $\mathcal{A}^{0, \text{even}}(M, L)$  to  $\mathcal{A}^{0, \text{odd}}(M, L)$ .

**Definition 2.1.** *The geometric quantization of  $(M, \Omega_M, \Phi_M)$  is the element  $\mathcal{Q}_K(M) \in R(K)$  which is defined as the equivariant index of the Dolbeault-Dirac operator  $D_L$ .*

**Remark 2.2.** • *We can define the Dolbeault-Dirac operator  $D_L^J$  for any invariant almost complex structure  $J$ . If  $J_0$  and  $J_1$  are equivariantly homotopic the indices of  $D_L^{J_0}$  and  $D_L^{J_1}$  coincides (see [28]).*

• *Since the set of compatible invariant almost complex structure on  $M$  is path-connected, the element  $\mathcal{Q}_K(M) \in R(K)$  does not depend of the choice of  $J$ .*

**2.2. Geometric quantization : the compact and singular case.** We are interested to defined the geometric quantization of *singular* compact Hamiltonian manifolds : here "singular" means that the manifold is obtain by symplectic reduction.

Let  $(N, \Omega_N)$  be a smooth symplectic manifold equipped with an Hamiltonian action of  $K \times H$  : we denote  $(\Phi_K, \Phi_H) : N \rightarrow \mathfrak{k}^* \oplus \mathfrak{h}^*$  the corresponding moment map. We assume that  $N$  is pre-quantized by a  $K \times H$ -equivariant line bundle  $L$  and we suppose that the map  $\Phi_H$  is **proper**. One wants to define the quantization of the (compact) symplectic quotient

$$N//_0H := \Phi_H^{-1}(0)/H.$$

When 0 is a regular value of  $\Phi_H$ ,  $N//_0H$  is a compact symplectic *orbifold* equipped with an Hamiltonian action of  $K$  : the corresponding moment map is induced by the restriction of  $\Phi_K$  to  $\Phi_H^{-1}(0)$ . The symplectic quotient  $N//_0H$  is pre-quantized by the line orbibundle

$$L_0 := \left( L|_{\Phi_H^{-1}(0)} \right) / H.$$

Definition 2.1 extends to the orbifold case, so one can still defined the quantization of  $N//_0H$  as an element  $\mathcal{Q}_K(N//_0H) \in R(K)$ .

Suppose now that 0 is not a regular value of  $\Phi_H$ . Let  $T_H$  be a maximal torus of  $H$ , and let  $C_H \subset \mathfrak{t}_H^*$  be a weyl chamber. Since  $\Phi_H$  is proper, the convexity theorem says that the image of  $\Phi_H$  intersects  $C_H$  in a closed locally polyhedral convex set, that we denoted  $\Delta_H(M)$  [22].

We consider an element  $a \in \Delta_H(M)$  which is generic and sufficiently closed to  $0 \in \Delta_H(M)$  : we denote  $H_a$  the subgroup of  $H$  which stabilizes  $a$ . When  $a \in \Delta_H(M)$  is generic, one can shows (see [25]) that

$$N//_aH := \Phi_H^{-1}(a)/H_a$$

is a compact  $K$ -Hamiltonian orbifold, and that

$$L_a := \left( L|_{\Phi_H^{-1}(a)} \right) / H_a.$$

is a  $K$ -equivariant line orbibundle over  $N//_aH$  : we can then define like in Definition 2.1 the element  $\mathcal{Q}_K(N//_aH) \in R(K)$  as the equivariant index of the Dolbeault-Dirac operator on  $N//_aH$ .

**Proposition-Definition 2.3.** *The elements  $\mathcal{Q}_K(N//_aH) \in R(K)$  do not depend of the choice of the generic element  $a \in \Delta_H(M)$ , when  $a$  is sufficiently closed to 0. The common value will be taken as the geometric quantization of  $N//_0H$ , and still denoted  $\mathcal{Q}_K(N//_0H)$ .*

PROOF. When  $N$  is compact and  $K = \{e\}$ , the proof can be founded in [25] and in [28]. The  $\mathbf{K}$ -theoretic proof of [28] extends naturally to our case.  $\square$

**2.3. Quantization commutes with reduction : the singular case.** In section 2.2, we have defined the geometric quantization  $\mathcal{Q}_K(N//_0H) \in R(K)$  of a compact symplectic reduced space  $N//_0H$ . We will compute its  $K$ -multiplicities like in Theorem 1.1.

For every  $\mu \in \widehat{K}$ , we consider the coadjoint orbit  $K \cdot \mu \simeq K/K_\mu$  which is pre-quantized by the line bundle  $\mathbb{C}_{[\mu]} \simeq K \times_{K_\mu} \mathbb{C}_\mu$ . We consider the product<sup>1</sup>  $N \times \overline{K \cdot \mu}$  which is an Hamiltonian  $K \times H$  manifold which is pre-quantized by  $K \times H$ -equivariant line bundle  $L \otimes \mathbb{C}_{[\mu]}^{-1}$ . The moment map  $N \times \overline{K \cdot \mu} \rightarrow \mathfrak{k}^* \times \mathfrak{h}^*$ ,  $(n, \xi) \mapsto (\Phi_K(n) - \xi, \Phi_H(n))$  is proper, so the reduced space

$$(N//_0H)_\mu := (N \times \overline{K \cdot \mu})//_{(0,0)} K \times H$$

is compact. Following Proposition 2.3, we can then define its quantization  $\mathcal{Q}((N//_0H)_\mu) \in \mathbb{Z}$ . The main result of this section is the

**Theorem 2.4.** *We have the following equality in  $R(K)$  :*

$$(2.10) \quad \mathcal{Q}_K(N//_0H) = \sum_{\mu \in \widehat{K}} \mathcal{Q}((N//_0H)_\mu) V_\mu^K.$$

PROOF. The proof will occupied the remaining of this section. The starting point is to state another definition of the geometric quantization of a symplectic reduced space which uses the Atiyah-Singer's theory of transversally elliptic operators.

**2.3.1. Transversally elliptic symbols.** Here we give the basic definitions from the theory of transversally elliptic symbols (or operators) defined by Atiyah-Singer in [1]. For an axiomatic treatment of the index morphism see Berline-Vergne [8, 9] and for a short introduction see [28].

Let  $\mathcal{X}$  be a compact  $K_1 \times K_2$ -manifold. Let  $p : T\mathcal{X} \rightarrow \mathcal{X}$  be the projection, and let  $(-, -)_\mathcal{X}$  be a  $K_1 \times K_2$ -invariant Riemannian metric. If  $E^0, E^1$  are  $K_1 \times K_2$ -equivariant vector bundles over  $\mathcal{X}$ , a  $K_1 \times K_2$ -equivariant morphism  $\sigma \in \Gamma(T\mathcal{X}, \text{hom}(p^*E^0, p^*E^1))$  is called a *symbol*. The subset of all  $(x, v) \in T\mathcal{X}$  where  $\sigma(x, v) : E_x^0 \rightarrow E_x^1$  is not invertible is called the *characteristic set* of  $\sigma$ , and is denoted by  $\text{Char}(\sigma)$ .

Let  $T_{K_2}\mathcal{X}$  be the following subset of  $T\mathcal{X}$  :

$$T_{K_2}\mathcal{X} = \{(x, v) \in TM, (v, X_M(x))_M = 0 \text{ for all } X \in \mathfrak{k}_2\}.$$

A symbol  $\sigma$  is *elliptic* if  $\sigma$  is invertible outside a compact subset of  $T\mathcal{X}$  (i.e.  $\text{Char}(\sigma)$  is compact), and is  *$K_2$ -transversally elliptic* if the restriction of  $\sigma$  to  $T_{K_2}\mathcal{X}$  is invertible outside a compact subset of  $T_{K_2}\mathcal{X}$  (i.e.  $\text{Char}(\sigma) \cap T_{K_2}\mathcal{X}$  is compact). An elliptic symbol  $\sigma$  defines an element in the equivariant  $\mathbf{K}$ -theory of  $T\mathcal{X}$  with compact support, which is denoted by  $\mathbf{K}_{K_1 \times K_2}(T\mathcal{X})$ , and the index of  $\sigma$  is a virtual finite dimensional representation of  $K_1 \times K_2$  [3, 4, 5, 6].

A  *$K_2$ -transversally elliptic* symbol  $\sigma$  defines an element of  $\mathbf{K}_{K_1 \times K_2}(T_{K_2}M)$ , and the index of  $\sigma$  is defined as a trace class virtual representation of  $K_1 \times K_2$  (see [1] for the analytic index and [8, 9] for the cohomological one) : in fact  $\text{Index}^\mathcal{X}(\sigma)$  belongs to the tensor product  $R(K_1) \widehat{\otimes} R^{-\infty}(K_2)$ .

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<sup>1</sup> $\overline{K \cdot \mu}$  denotes the coadjoint orbit with the opposite symplectic form.

Remark that any elliptic symbol of  $\mathbf{T}\mathcal{X}$  is  $K_2$ -transversally elliptic, hence we have a restriction map  $\mathbf{K}_{K_1 \times K_2}(\mathbf{T}\mathcal{X}) \rightarrow \mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_2}\mathcal{X})$ , and a commutative diagram

$$(2.11) \quad \begin{array}{ccc} \mathbf{K}_{K_1 \times K_2}(\mathbf{T}\mathcal{X}) & \longrightarrow & \mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_2}\mathcal{X}) \\ \text{Index}^x \downarrow & & \downarrow \text{Index}^x \\ R(K_1) \otimes R(K_2) & \longrightarrow & R(K_1) \widehat{\otimes} R^{-\infty}(K_2) . \end{array}$$

Using the *excision property*, one can easily show that the index map  $\text{Index}^{\mathcal{U}} : \mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_2}\mathcal{U}) \rightarrow R(K_1) \widehat{\otimes} R^{-\infty}(K_2)$  is still defined when  $\mathcal{U}$  is a  $K_1 \times K_2$ -invariant relatively compact open subset of a  $K_1 \times K_2$ -manifold (see [28][section 3.1]).

**2.3.2. Quantization of singular space : second definition.** Let  $(\mathcal{X}, \Omega_{\mathcal{X}})$  be an Hamiltonian  $K_1 \times K_2$ -manifold pre-quantized by a  $K_1 \times K_2$ -equivariant line bundle  $L$ . The moment map  $\Phi_2 : \mathcal{X} \rightarrow \mathfrak{k}_2^*$  relative to the  $K_2$ -action is supposed to be **proper**. Take a compatible  $K_1 \times K_2$ -invariant almost complex structure on  $\mathcal{X}$ . We choose a  $K_1 \times K_2$ -invariant Hermitian metric  $\|v\|^2$  on the tangent bundle  $\mathbf{T}\mathcal{X}$ , and we identify the cotangent bundle  $\mathbf{T}^*\mathcal{X}$  with  $\mathbf{T}\mathcal{X}$ . For  $(x, v) \in \mathbf{T}\mathcal{X}$ , the principal symbol of the Dolbeault-Dirac operator  $\bar{\partial}_L + \bar{\partial}_L^*$  is the clifford multiplication  $\mathbf{c}_{\mathcal{X}}(v)$  on the complex vector bundle  $\Lambda^{\bullet} \mathbf{T}_x \mathcal{X} \otimes L_x$ . It is invertible for  $v \neq 0$ , since  $\mathbf{c}_{\mathcal{X}}(v)^2 = -\|v\|^2$ .

When  $\mathcal{X}$  is compact, the symbol  $\mathbf{c}_{\mathcal{X}}$  is elliptic and then defines an element of the equivariant  $\mathbf{K}$ -group of  $\mathbf{T}\mathcal{X}$ . The topological index of  $\mathbf{c}_{\mathcal{X}} \in \mathbf{K}_{K_1 \times K_2}(\mathbf{T}\mathcal{X})$  is equal to the analytical index of the Dolbeault-Dirac operator  $\bar{\partial}_L + \bar{\partial}_L^*$  :

$$(2.12) \quad \mathcal{Q}_{K_1 \times K_2}(\mathcal{X}) = \text{Index}^{\mathcal{X}}(\mathbf{c}_{\mathcal{X}}) \quad \text{in} \quad R(K_1) \otimes R(K_2).$$

When  $\mathcal{X}$  is not compact the topological index of  $\mathbf{c}_{\mathcal{X}}$  is not defined. In order to give a topological definition of  $\mathcal{Q}_{K_1}(\mathcal{X} //_0 K_2)$ , we will deform the symbol  $\mathbf{c}_{\mathcal{X}}$  as follows. Consider the identification  $\mathfrak{k}_2^* \simeq \mathfrak{k}_2$  defined by a  $K_2$ -invariant scalar product on the Lie algebra  $\mathfrak{k}_2$ . From now on the moment map  $\Phi_2$  will take values in  $\mathfrak{k}_2$ , and we define the vectors field on  $\mathcal{X}$

$$(2.13) \quad \kappa_x = (\Phi_2(x))_M(x), \quad x \in \mathcal{X}.$$

We consider now the symbol

$$\mathbf{c}_{\mathcal{X}}^{\kappa}(v) = \mathbf{c}(v - \kappa_x), \quad v \in \mathbf{T}_x \mathcal{X}.$$

Note that  $\mathbf{c}_{\mathcal{X}}^{\kappa}(v)$  is invertible except if  $v = \kappa_x$ . If furthermore  $v$  belongs to the subset  $\mathbf{T}_{K_2}M$  of tangent vectors orthogonal to the  $K_2$ -orbits, then  $v = 0$  and  $\kappa_x = 0$ . Indeed  $\kappa_x$  is tangent to  $K_2 \cdot x$  while  $v$  is orthogonal.

Since  $\kappa$  is the Hamiltonian vector field of the function  $\|\Phi_2\|^2$ , the set of zeros of  $\kappa$  coincides with the set  $\text{Cr}(\|\Phi_2\|^2)$  of critical points of  $\|\Phi_2\|^2$ .

Let  $\mathcal{U} \subset \mathcal{X}$  be a  $K_1 \times K_2$ -invariant open subset which is *relatively compact*. If the border  $\partial\mathcal{U}$  does not intersect  $\text{Cr}(\|\Phi_2\|^2)$ , then the restriction  $\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}}$  defines a class in  $\mathbf{K}_{K_1 \times K_2}(\mathbf{T}_{K_2}\mathcal{U})$  since

$$\text{Char}(\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}}) \cap \mathbf{T}_{K_2}\mathcal{U} \simeq \text{Cr}(\|\Phi_2\|^2) \cap \mathcal{U}$$

is compact. In this situation the index of  $\mathbf{c}_{\kappa}|_{\mathcal{U}}$  is defined as an element  $\text{Index}^{\mathcal{U}}(\mathbf{c}_{\kappa}|_{\mathcal{U}}) \in R(K_1) \widehat{\otimes} R^{-\infty}(K_2)$ .

**Theorem 2.5.** *The  $K_2$ -invariant part of  $\text{Index}^{\mathcal{U}}(\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}})$  is equal to :*

- $\mathcal{Q}_{K_1}(\mathcal{X} //_0 K_2)$  when  $\Phi_2^{-1}(0) \subset \mathcal{U}$ ,



- 0 in the other case.

PROOF. When  $K_1 = \{e\}$ , the proof is done in [28] (see section 7). This proof works equally well in the general case.

**Remark 2.6.** If  $\mathcal{X}$  is compact we can take  $\mathcal{U} = \mathcal{X}$  in the last Theorem. In this case the symbols  $\mathbf{c}_{\mathcal{X}}^{\kappa}$  and  $\mathbf{c}_{\mathcal{X}}$  defines the same class in  $\mathbf{K}_{K_1 \times K_2}(\mathbb{T}\mathcal{X})$  so they have the same index. Theorem 2.5 corresponds then to the traditional “quantization commutes with reduction” phenomenon :  $[\mathcal{Q}_{K_1 \times K_2}(\mathcal{X})]^{K_2} = \mathcal{Q}_{K_1}(\mathcal{X} //_0 K_2)$ .

From now on we will work with this topological definition for the geometric quantization of the reduced  $K_1$ -Hamiltonian manifold  $\mathcal{X} //_0 K_2$  (which is possibly singular):  $\mathcal{Q}_{K_1}(\mathcal{X} //_0 K_2) = [\text{Index}^{\mathcal{U}}(\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{U}})]^{K_2}$  where  $\mathcal{U}$  is any relatively compact neighborhood of  $\Phi_2^{-1}(0)$  such that  $\partial\mathcal{U} \cap \text{Cr}(\|\Phi_2\|^2) = \emptyset$ . The functorial properties still holds in this singular setting. In particular :

**[P2]** If  $H \subset K_1$  is a connected lie subgroup, then the restriction of  $\mathcal{Q}_{K_1}(\mathcal{X} //_0 K_2)$  to  $H$  is equal to  $\mathcal{Q}_H(\mathcal{X} //_0 K_2)$ .

2.3.3. *Proof of theorem 2.4.* We come back in the situation of sections 2.2 and 2.3 .

First we apply Theorem 2.5 to  $\mathcal{X} = N$ ,  $K_1 = K$  and  $K_2 = H$ . (2.10) is trivially true when  $0 \notin \text{Image}(\Phi_H)$ . So we suppose now that  $0 \in \text{Image}(\Phi_H)$ , and we consider an  $K \times H$ -invariant open subset  $\mathcal{U} \subset N$  which is *relatively compact* and such that

$$\Phi_H^{-1}(0) \subset \mathcal{U} \quad \text{and} \quad \partial\mathcal{U} \cap \text{Cr}(\|\Phi_H\|^2) = \emptyset.$$

We have  $\mathcal{Q}_K(N //_0 H) = [\text{Index}^{\mathcal{U}}(\mathbf{c}_N^{\kappa^H}|_{\mathcal{U}})]^H$  and one want to compute its  $K$ - multiplicities  $m_{\mu}, \mu \in \widehat{K}$ . Here  $\kappa^H$  is the vectors field on  $N$  associated to the moment map  $\Phi_H$  (see (2.13)).

Take  $\mu \in \widehat{K}$ . We denote  $\mathbf{c}_{-\mu}$  the principal symbol of the Dolbeault-Dirac operator on  $\overline{K \cdot \mu}$  with values in the line bundle  $\mathbb{C}_{[-\mu]}$  : we have  $\text{Index}^{K \cdot \mu}(\mathbf{c}_{-\mu}) = (V_{\mu}^K)^*$ .

We know then that the multiplicity of  $[\text{Index}^{\mathcal{U}}(\mathbf{c}_N^{\kappa^H}|_{\mathcal{U}})]^H$  relatively to  $V_{\mu}^K$  is equal to

$$(2.14) \quad m_{\mu} := \left[ \text{Index}^{\mathcal{V}}(\mathbf{c}_N^{\kappa^H}|_{\mathcal{U}} \odot \mathbf{c}_{-\mu}) \right]^{K \times H}$$

with  $\mathcal{V} = \mathcal{U} \times K \cdot \mu$ . This identity is due to the fact that we have a ”multiplication”

$$\begin{aligned} \mathbf{K}_{K \times H}(\mathbb{T}H\mathcal{U}) \times \mathbf{K}_K(\mathbb{T}(K \cdot \mu)) &\longrightarrow \mathbf{K}_{K \times H}(\mathbb{T}_{K \times H}(\mathcal{U} \times K \cdot \mu)) \\ (\sigma_1, \sigma_2) &\longmapsto \sigma_1 \odot \sigma_2 . \end{aligned}$$

so that  $\text{Index}^{\mathcal{U} \times K \cdot \mu}(\sigma_1 \odot \sigma_2) = \text{Index}^{\mathcal{U}}(\sigma_1) \cdot \text{Index}^{K \cdot \mu}(\sigma_2)$  in  $R^{-\infty}(K \times H)$ . See [1].

Consider now the case where  $\mathcal{X} = N \times \overline{K \cdot \mu}$ ,  $K_1 = \{e\}$  and  $K_2 = K \times H$ . After Theorem 2.5, we know that

$$(2.15) \quad \mathcal{Q}((N //_0 H)_{\mu}) = \left[ \text{Index}^{\mathcal{V}}(\mathbf{c}_{\mathcal{X}}^{\kappa}|_{\mathcal{V}}) \right]^{K \times H},$$

where  $\kappa$  is the vector field on  $N \times \overline{K \cdot \mu}$  associated to the moment map

$$\begin{aligned} \Phi : N \times \overline{K \cdot \mu} &\longrightarrow \mathfrak{k}^* \times \mathfrak{h}^* \\ (x, \xi) &\longmapsto (\Phi_K(x) - \xi, \Phi_H(n)) \end{aligned}$$

Note that  $\mathcal{V} = \mathcal{U} \times K \cdot \mu$  is a neighborhood of  $\Phi^{-1}(0) \subset (\Phi_H)^{-1}(0)$ .

Our aim now is to prove that the quantities (2.14) and (2.15) are equal.

Since the definition of  $\kappa$  needs the choice of an invariant scalar product on the Lie algebra  $\mathfrak{k} \times \mathfrak{h}$ , we will precise its definition. Let  $\|\cdot\|_K$  and  $\|\cdot\|_H$  be two invariant Euclidean norm respectively on  $\mathfrak{k}$  and  $\mathfrak{h}$ . For any  $r > 0$  we consider on  $\mathfrak{k} \times \mathfrak{h}$  the invariant Euclidean norm  $\|(X, Y)\|_r^2 = r^2 \|X\|_K^2 + \|Y\|_H^2$ .

Let  $\kappa^K$  be the vector field on  $N \times \overline{K \cdot \mu}$  associated to the map  $N \times \overline{K \cdot \mu} \rightarrow \mathfrak{k}^*$ ,  $(x, \xi) \mapsto \Phi_K(x) - \xi$ , and where the identification  $\mathfrak{k} \simeq \mathfrak{k}^*$  is made through the Euclidean norm  $\|\cdot\|_K$  (see (2.13)). For  $(x, \xi) \in N \times \overline{K \cdot \mu}$ , we have the decomposition

$$\kappa^K(x, \xi) = (\kappa_1(x, \xi), \kappa_2(x, \xi)) \in T_x N \times T_\xi K \cdot \mu.$$

Let  $\kappa^H$  be the vector field on  $N \times \overline{K \cdot \mu}$  associated to the map  $N \times \overline{K \cdot \mu} \rightarrow \mathfrak{h}^*$ ,  $(x, \xi) \mapsto \Phi_H(x)$ , and where the identification  $\mathfrak{h} \simeq \mathfrak{h}^*$  is made through the Euclidean norm  $\|\cdot\|_H$ . For  $(x, \xi) \in N \times \overline{K \cdot \mu}$ , we have the decomposition

$$\kappa^H(x, \xi) = (\kappa^H(x), 0) \in T_x N \times T_\xi K \cdot \mu.$$

For any  $r > 0$ , we denote by  $\kappa_r$  the vector field on  $N \times \overline{K \cdot \mu}$  associated to the map  $\Phi$ , and where the identification  $\mathfrak{k} \times \mathfrak{h} \simeq \mathfrak{k}^* \times \mathfrak{h}^*$  is made through the Euclidean norm  $\|\cdot\|_r$ . We have then

$$\begin{aligned} \kappa_r &= \kappa^H + r \kappa^K \\ &= (\kappa^H + r \kappa_1, r \kappa_2) \end{aligned}$$

Now we can precise (2.15). Take an invariant relatively compact neighborhood  $\mathcal{U}$  of  $\Phi_H^{-1}(0)$  such that  $\partial\mathcal{U} \cap \{\text{zeros of } \kappa^H\} = \emptyset$ . With the help of a invariant Riemannian metric on  $\mathcal{X}$  we define

$$\varepsilon_H = \inf_{x \in \partial\mathcal{U}} \|\kappa^H(x)\| > 0 \quad \text{and} \quad \varepsilon_K = \sup_{(x, \xi) \in \partial\mathcal{U} \times K\mu} \|\kappa_1(x, \xi)\|.$$

Note that for any  $0 \leq r < \frac{\varepsilon_H}{\varepsilon_K}$ , we have  $\partial\mathcal{U} \times K \cdot \mu \cap \{\text{zeros of } \kappa^H + r \kappa_1\} = \emptyset$ , and then  $\partial\mathcal{V} \cap \{\text{zeros of } \kappa_r\} = \emptyset$  for the neighborhood  $\mathcal{V} := \mathcal{U} \times K \cdot \mu$  of  $\Phi^{-1}(0)$ . We can then use Theorem 2.5 : for  $0 < r < \frac{\varepsilon_H}{\varepsilon_K}$  we have

$$\mathcal{Q}((N//_0 H)_\mu) = \left[ \text{Index}^\mathcal{V}(\mathbf{c}_{\mathcal{X}}^{\kappa_r} |_{\mathcal{V}}) \right]^{K \times H}$$

We are now close to the end of the proof. Let us compare the symbols  $\mathbf{c}_{\mathcal{X}}^{\kappa_r} |_{\mathcal{V}}$  and  $\mathbf{c}_N^{\kappa^H} |_{\mathcal{U}} \odot \mathbf{c}_{-\mu}$  in  $\mathbf{K}_{K \times H}(T_{K \times H}(\mathcal{U} \times K \cdot \mu))$ . First one sees that the symbols  $\mathbf{c}_{\mathcal{X}}$  is equal to the product  $\mathbf{c}_N \odot \mathbf{c}_{-\mu}$  hence the symbols  $\mathbf{c}_N^{\kappa^H} |_{\mathcal{U}} \odot \mathbf{c}_{-\mu}$  is equal to  $\mathbf{c}_{\mathcal{X}}^{\kappa_r} |_{\mathcal{V}}$  when  $r = 0$ . Since for  $r < \frac{\varepsilon_H}{\varepsilon_K}$  the path  $s \in [0, r] \rightarrow \mathbf{c}_{\mathcal{X}}^{\kappa_s} |_{\mathcal{V}}$  defines an homotopy of  $K \times H$ -transversally elliptic symbols on  $\mathcal{V}$ , we get

$$\text{Index}^\mathcal{V}(\mathbf{c}_{\mathcal{X}}^{\kappa_r} |_{\mathcal{V}}) = \text{Index}^\mathcal{V}(\mathbf{c}_N^{\kappa^H} |_{\mathcal{U}} \odot \mathbf{c}_{-\mu})$$

and then  $m_\mu = \mathcal{Q}((N//_0 H)_\mu)$ .  $\square$

### 3. WONDERFUL COMPACTIFICATIONS AND SYMPLECTIC CUTTING

In this section we use the wonderful compactifications of Concini-Procesi [13, 14] to perform symplectic cutting.

**3.1. Wonderful compactifications : definitions.** We study here the wonderful compactifications from the Hamiltonian point of view.

We consider a compact connected Lie group  $K$  and its complexification  $K_{\mathbb{C}}$ . Let  $T$  be a maximal torus of  $K$ , and let  $W := N(T)/T$  be the Weyl group. Consider  $\mathfrak{t}^*$ , the dual of the Lie algebra of  $T$ , with the lattice  $\Lambda^*$  of real weights. Let  $C_K \subset \mathfrak{t}^*$  be a Weyl chamber and let  $\widehat{K} := \Lambda^* \cap C_K$  be the set dominants weights.

**Definition 3.1.** *A polytope  $P$  in  $\mathfrak{t}^*$  is  $K$ -adapted if :*

- i) *the vertices of  $P$  are regular elements of  $\Lambda^*$ ,*
- ii)  *$P$  is  $W$ -invariant,*
- iii)  *$P$  is Delzant.*

EXAMPLE : When  $K$  as a trivial center, the convex hull of  $W \cdot \mu$  is a  $K$ -adapted polytope for any regular dominant weight  $\mu$ .

**Proposition 3.2.** *There exists  $K$ -adapted polytopes in  $\mathfrak{t}^*$ .*

PROOF. Let us use the dictionary between polytopes and projective fan [27]. Conditions *ii*) and *iii*) of Definition 3.1 means that we are looking after a smooth projective  $W$ -invariant fan  $\mathcal{F}$  in  $\mathfrak{t}$ . Condition *i*) means that each cone of  $\mathcal{F}$  of maximal dimension should not be fixed by any element of  $W \setminus \{\text{Id}\}$ . For a proof of the existence of search fan, see [11, 12]. In particular condition *(\*)* in Proposition 2 of [12] implies *i*).  $\square$

In the rest of this section, we consider a  $K$ -adapted polytope  $P$ . Let  $[P]_1$  be the union of all the closed facet of dimension 1 : we label the elements of  $[P]_1 \cap \widehat{K}$  by  $\{\lambda_1, \dots, \lambda_N\}$ . Since  $P$  is  $K$ -adapted, when  $\lambda_i$  is a vertex of  $P$  there exists  $\alpha_{j_1}, \dots, \alpha_{j_r}$  belonging to  $[P]_1 \cap \Lambda^* = W \cdot \{\lambda_1, \dots, \lambda_N\}$  such that  $\alpha_{j_1} - \lambda_i, \dots, \alpha_{j_r} - \lambda_i$  is a basis of the lattice  $\Lambda^*$ .

Let  $V_{\lambda_i}$  be an irreducible representation of  $K$  with highest weight  $\lambda_i$ : these representations extend canonically to the complexification  $K_{\mathbb{C}}$ . We denote  $\rho : K_{\mathbb{C}} \rightarrow \prod_{i=1}^N GL(V_{\lambda_i})$  the representation of  $K_{\mathbb{C}}$  on  $V := \bigoplus_{i=1}^N V_{\lambda_i}$ . We consider the vector space

$$E = \bigoplus_{i=1}^N \text{End}(V_{\lambda_i})$$

equipped with the action of  $K_{\mathbb{C}} \times K_{\mathbb{C}}$  given by :  $(g_1, g_2) \cdot f = \rho(g_1) \circ f \circ \rho(g_2)^{-1}$ . Let  $\mathbb{P}(E)$  the projective space associated to  $E$  : it is equipped with an algebraic action of the reductive group  $K_{\mathbb{C}} \times K_{\mathbb{C}}$ . We consider the map  $g \mapsto [\rho(g)]$  from  $K_{\mathbb{C}}$  into  $\mathbb{P}(E)$ , that we denote  $\bar{\rho}$ .

**Lemma 3.3.** *The map  $\bar{\rho} : K_{\mathbb{C}} \rightarrow \mathbb{P}(E)$  is an embedding.*

PROOF. Let  $g \in K_{\mathbb{C}}$  such that  $\bar{\rho}(g) = [Id]$  : there exist  $a \in \mathbb{C}^*$  such that  $\rho(g) = aId$ . The Cartan decomposition gives

$$(3.16) \quad \rho(k) = \frac{a}{|a|} Id \quad \text{and} \quad \rho(e^{iX}) = |a| Id$$

for  $g = ke^{iX}$  with  $k \in K$  and  $X \in \mathfrak{k}$ . Since there exist  $Y, Y' \in \mathfrak{t}$  and  $u, u' \in K$  such that  $k = ue^{Y}u^{-1}$  and  $X = u' \cdot Y'$ , (3.16) gives

$$(3.17) \quad \rho(e^Y) = \frac{a}{|a|} Id \quad \text{and} \quad \rho(e^{iY'}) = |a| Id.$$

Since each element of  $[P]_1 \cap \wedge^* = W \cdot \{\lambda_1, \dots, \lambda_N\}$  is a weight for the action of  $T_{\mathbb{C}}$  on  $\bigoplus_{i=1}^N V_{\lambda_i}$ , (3.17) implies that for every  $\alpha \in [P]_1 \cap \wedge^*$  we have

$$(3.18) \quad e^{i\langle \alpha, Y \rangle} = \frac{a}{|a|} \quad \text{and} \quad e^{-\langle \alpha, Y' \rangle} = |a|.$$

Since there exists  $\alpha_{j_0}, \dots, \alpha_{j_r} \in [P]_1 \cap \wedge^*$  such that  $\alpha_{j_1} - \alpha_{j_0}, \dots, \alpha_{j_r} - \alpha_{j_0}$  is a basis of the lattice  $\wedge^*$ , (3.18) implies that  $Y' = 0$  and that  $Y \in \ker(Z \in \mathfrak{t} \rightarrow e^Z)$ . We have proved that  $g = e$ .  $\square$

Let  $T_{\mathbb{C}} \subset K_{\mathbb{C}}$  the complexification of the (compact) torus  $T \subset K$ .

**Definition 3.4.** *Let  $\mathcal{X}_P$  be the Zarisky closure of  $\bar{\rho}(K_{\mathbb{C}})$  in  $\mathbb{P}(E)$  and let  $\mathcal{Y}_P \subset \mathcal{X}_P$  be the Zarisky closure of  $\bar{\rho}(T_{\mathbb{C}})$  in  $\mathbb{P}(E)$ .*

Since  $\bar{\rho}(K_{\mathbb{C}}) = K_{\mathbb{C}} \times K_{\mathbb{C}} \cdot [Id]$  and  $\bar{\rho}(T_{\mathbb{C}}) = T_{\mathbb{C}} \times T_{\mathbb{C}} \cdot [Id]$  are orbits of algebraic group actions their Zariski closures coincide with their closures for the Euclidean topology.

**Theorem 3.5.** *The varieties  $\mathcal{X}_P$  and  $\mathcal{Y}_P$  are smooth.*

The proof will be done in the next section

**3.2. Smoothness of  $\mathcal{X}_P$  and  $\mathcal{Y}_P$ .** Let  $E$  be a complex vector space equipped with a linear action of a reductive group  $G$ . Let  $\mathcal{Z} \subset \mathbb{P}(E)$  be a projective variety which is  $G$ -stable. We have the classical fact

**Lemma 3.6.**  $\bullet$   *$\mathcal{Z}$  possess closed  $G$ -orbits.*

$\bullet$   *$\mathcal{Z}$  is smooth if  $\mathcal{Z}$  is smooth near its closed  $G$ -orbits.*

PROOF. Let  $z_0 \in \mathcal{Z}$  and consider the Zariski closure  $\overline{G \cdot z_0} \subset \mathcal{Z}$ . If  $G \cdot z_0$  is not closed, we take  $z_1 \in \overline{G \cdot z_0} \setminus G \cdot z_0$ : we have  $\dim G \cdot z_1 < \dim G \cdot z_0$ . By induction we find a sequence  $z_1, \dots, z_p$  with  $z_{k+1} \in \overline{G \cdot z_k} \setminus G \cdot z_k$  for  $k < p$  and  $G \cdot z_p$  closed. For the second point, we have just to note that if  $\mathcal{Z}$  is singular, the subvariety  $\mathcal{Z}^{sing} \subset \mathcal{Z}$  of singular points is  $G$ -stable and then contains a closed  $G$ -orbits.  $\square$

We are interested here respectively in

- $\bullet$  the  $K_{\mathbb{C}} \times K_{\mathbb{C}}$ -variety  $\mathcal{X}_P \subset \mathbb{P}(E) \subset \mathbb{P}(\text{End}(V))$
- $\bullet$  the  $T_{\mathbb{C}} \times T_{\mathbb{C}}$ -variety  $\mathcal{Y}_P \subset \mathbb{P}(E)$ .

Since the diagonal  $Z_{\mathbb{C}} = \{(t, t) | t \in T_{\mathbb{C}}\}$  stabilizes  $[Id]$ , its action on  $\mathcal{Y}_P$  is trivial. Hence we will restrict ourself to the action of  $T_{\mathbb{C}} \times T_{\mathbb{C}}/Z_{\mathbb{C}} \simeq T_{\mathbb{C}}$  on  $\mathcal{Y}_P$ : for  $t \in T_{\mathbb{C}}$  and  $[y] \in \mathcal{Y}_P$  we take  $t \cdot [y] = [\rho(t) \circ y]$ .

**3.2.1. The case of  $\mathcal{Y}_P$ .** We apply Lemma 3.6 to the  $T_{\mathbb{C}}$ -variety  $\mathcal{Y}_P = \overline{T_{\mathbb{C}} \cdot [Id]}$  in  $\mathbb{P}(E)$ . Let  $\{\alpha_j, j \in J\}$  be the  $T_{\mathbb{C}}$  weights on  $(\bigoplus_{i=1}^N V_{\lambda_i}, \rho)$  counted with their multiplicity. Their exists a orthonormal basis  $\{v_j, j \in J\}$  of  $\bigoplus_{i=1}^N V_{\lambda_i}$  such that  $Id = \sum_{j \in J} v_j \otimes v_j^*$  and

$$(3.19) \quad \rho(e^Z) = \sum_{j \in J} e^{i\langle \alpha_j, Z \rangle} v_j \otimes v_j^*, \quad Z \in \mathfrak{t}_{\mathbb{C}}.$$

So the action of  $e^Z \in T_{\mathbb{C}}$  on  $[Id] \in \mathbb{P}(E)$  is  $e^Z \cdot [Id] = \left[ \sum_{j \in J} e^{i\langle \alpha_j, Z \rangle} v_j \otimes v_j^* \right]$ . We introduce a subset  $J'$  of  $J$  such that for every  $j \in J$  there exists a *unique*  $j' \in J'$  such that  $\alpha_j = \alpha_{j'}$ . So the variety  $\mathcal{Y}_P$  belongs to  $\mathbb{P}(E')$  where  $E' = \bigoplus_{j' \in J'} \mathbb{C} m_{j'}$  with  $m_{j'} = \sum_{j, \alpha_j = \alpha_{j'}} v_j \otimes v_j^*$ . The closed  $T_{\mathbb{C}}$  orbits in  $\mathbb{P}(E')$  are  $[m_{j'}], j' \in J$ .

**Lemma 3.7.**  $[m_{j_o}] \in \mathcal{Y}_P$  if and only if  $\alpha_{j_o}$  is a vertex of the polytope  $P$ .

PROOF. If  $\alpha_{j_o}$  is a vertex of  $P$ , there exists  $X \in \mathfrak{t}$  such that  $\langle \alpha_{j_o}, X \rangle > \langle \alpha_j, X \rangle$  whenever  $\alpha_{j_o} \neq \alpha_j$ . Hence  $e^{-isX} \cdot [Id]$  tends to  $[m_{j_o}]$  when  $s \rightarrow +\infty$ . If  $\alpha_{j_o}$  is not a vertex of  $P$ , there exist  $L \subset J' \setminus \{j_o\}$  such that  $\alpha_{j_o} = \sum_{l \in L} a_l \alpha_l$  with  $0 < a_l < 1$  and  $\sum_l a_l = 1$ . So  $\mathcal{Y}_P$  belongs to the closed subset defined by

$$\left[ \sum_{j' \in J'} \delta_{j'} m_{j'} \right] \in \mathbb{P}(E') \quad \text{such that} \quad \prod_{l \in L} |\delta_l|^{a_l} = |\delta_{j_o}|.$$

Hence  $[m_{j_o}] \notin \mathcal{Y}_P$ .  $\square$

**Remark 3.8.** When  $\alpha_j$  is a vertex of the polytope  $P$ , the multiplicity of  $\alpha_j$  in  $\oplus_{i=1}^N V_{\lambda_i}$  is equal to one, so  $m_j = v_j \otimes v_j^*$ .

Consider now a vertex  $\alpha_{j_o}$  of  $P$  (for  $j_o \in J'$ ). We consider the open subset  $\mathcal{V} \subset \mathbb{P}(E')$  defined by  $[\sum_{j' \in J'} \delta_{j'} m_{j'}] \in \mathcal{V} \Leftrightarrow \delta_{j_o} \neq 0$ , and the diffeomorphism  $\psi : \mathcal{V} \rightarrow \mathbb{C}^{J' \setminus \{j_o\}}$ ,  $[\sum_{j' \in J'} \delta_{j'} m_{j'}] \mapsto (\frac{\delta_{j'}}{\delta_{j_o}})_{j' \neq j_o}$ . The map  $\psi$  realizes an algebraic diffeomorphism between  $\mathcal{Y}_P \cap \mathcal{V}$  and the affine subvariety

$$\mathcal{Z} := \overline{\{(t^{\alpha_{j'} - \alpha_{j_o}})_{j' \neq j_o} \mid t \in T_{\mathbb{C}}\}} \subset \mathbb{C}^{J' \setminus \{j_o\}}.$$

The set of weights  $\alpha_j, j \in J$  contains all the lattice points that belongs to the one dimensional faces of  $P$ . Since the polytope  $P$  is  $K$ -adapted, there exists a subset  $L_{j_o} \subset J'$  such that  $\alpha_l - \alpha_{j_o}, l \in L_{j_o}$  is a  $\mathbb{Z}$ -basis of the group of weights  $\Lambda^*$ . And for every  $j' \neq j_o$  we have

$$(3.20) \quad \alpha_{j'} - \alpha_{j_o} = \sum_{l \in L_{j_o}} n_{j'}^l (\alpha_l - \alpha_{j_o}) \quad \text{with} \quad n_{j'}^l \in \mathbb{N}.$$

We define on  $\mathbb{C}^{L_{j_o}}$  the monomials  $P_{j'}(Z) = \prod_{l \in L_{j_o}} (Z_l)^{n_{j'}^l}$ . Note that  $P_{j'}(Z) = Z_l$  when  $j' = l \in L_{j_o}$ . Now it is not difficult to see that the map

$$\begin{aligned} \mathbb{C}^{L_{j_o}} &\longrightarrow \mathbb{C}^{J' \setminus \{j_o\}} \\ Z &\longmapsto (P_{j'}(Z))_{j' \neq j_o} \end{aligned}$$

realizes an algebraic diffeomorphism between  $\mathbb{C}^{L_{j_o}}$  and  $\mathcal{Z}$ .

Finally we have shown that  $\mathcal{Y}_P$  is smooth near  $[m_{j_o}]$ : hence  $\mathcal{Y}_P$  is a smooth subvariety of  $\mathbb{P}(E)$ . Since  $T_{\mathbb{C}}$  acts on  $\mathcal{Y}_P$  with a dense orbit,  $\mathcal{Y}_P$  is a smooth projective toric variety.

3.2.2. *The case of  $\mathcal{X}_P$ .* Let  $E := \otimes_{i=1}^N \text{End}(V_{\lambda_i})$ . The closed  $K_{\mathbb{C}} \times K_{\mathbb{C}}$ -orbit in  $\mathbb{P}(E)$  are those passing through  $[v_{\lambda_i} \otimes v_{\lambda_i}^*]$  where  $v_{\lambda_i} \in V_{\lambda_i}$  is a highest weight vector (that we take of norm 1 for a  $K$ -invariant hermitian structure).

**Lemma 3.9.**  $[v_{\lambda_i} \otimes v_{\lambda_i}^*] \in \mathcal{X}_P$  if and only if  $\lambda_i$  is a vertex of the polytope  $P$ .

PROOF. If  $\lambda_i$  is a vertex of  $P$ , we have proved in Lemma 3.7 that  $[v_{\lambda_i} \otimes v_{\lambda_i}^*]$  belongs to  $\mathcal{Y}_P$  and so belongs to  $\mathcal{X}_P$ . We prove the converse in Corollary 3.14.  $\square$

For the remaining of this section we consider a vertex  $\lambda_{i_o} \in \widehat{K}$  of the polytope  $P$ . Let  $B^+, B^-$  the Borel subgroups fixing respectively the elements  $[v_{\lambda_{i_o}}] \in \mathbb{P}(V_{\lambda_{i_o}})$  and  $[v_{\lambda_{i_o}}^*] \in \mathbb{P}(V_{\lambda_{i_o}}^*)$ . Consider also the unipotent subgroups  $N^{\pm} \subset B^{\pm}$  fixing respectively the elements  $v_{\lambda_i} \in V_{\lambda_i}$  and  $v_{\lambda_i}^* \in V_{\lambda_i}^*$ .

We consider the open subset  $\mathcal{V}_{\text{End}} \subset \mathbb{P}(E)$  of elements  $[f]$  such that  $\langle v_{\lambda_{i_0}}^*, f(v_{\lambda_{i_0}}) \rangle \neq 0$  :  $\mathcal{V}_{\text{End}}$  is  $B^- \times B^+$  stable. Consider the open subset  $\mathcal{V} \subset \mathbb{P}(V_{\lambda_{i_0}})$  and  $\mathcal{V}^* \subset \mathbb{P}(V_{\lambda_{i_0}}^*)$  defined by :

- $[v] \in \mathcal{V} \Leftrightarrow \langle v_{\lambda_{i_0}}^*, v \rangle \neq 0$  :  $\mathcal{V}$  is  $B^-$  stable,
- $[\xi] \in \mathcal{V}^* \Leftrightarrow \langle \xi, v_{\lambda_{i_0}} \rangle \neq 0$  :  $\mathcal{V}^*$  is  $B^+$  stable.

We consider now the rational maps  $l : \mathbb{P}(E) \dashrightarrow \mathbb{P}(V_{\lambda_{i_0}})$ ,  $f \mapsto f(v_{\lambda_{i_0}})$  and  $r : \mathbb{P}(E) \dashrightarrow \mathbb{P}(V_{\lambda_{i_0}}^*)$ ,  $f \mapsto v_{\lambda_{i_0}}^* \circ f$ . The map  $l$  and  $r$  are defined on  $\mathcal{V}_{\text{End}}$  : they defined respectively  $B^-$ -equivariant map from  $\mathcal{V}_{\text{End}}$  into  $\mathcal{V}$ , and  $B^+$ -equivariant map from  $\mathcal{V}_{\text{End}}$  into  $\mathcal{V}^*$ .

The orbits  $K_{\mathbb{C}} \cdot v_{\lambda_{i_0}} \subset \mathbb{P}(V_{\lambda_{i_0}})$  and  $K_{\mathbb{C}} \cdot v_{\lambda_{i_0}}^* \subset \mathbb{P}(V_{\lambda_{i_0}}^*)$  are closed and we have

$$\begin{aligned} K_{\mathbb{C}} \cdot v_{\lambda_{i_0}} \cap \mathcal{V} &= N^- \cdot v_{\lambda_{i_0}} \simeq N^- \\ K_{\mathbb{C}} \cdot v_{\lambda_{i_0}}^* \cap \mathcal{V}^* &= N^+ \cdot v_{\lambda_{i_0}}^* \simeq N^+. \end{aligned}$$

The rational map  $(l, r) : \mathbb{P}(E) \dashrightarrow \mathbb{P}(V_{\lambda_{i_0}}) \times \mathbb{P}(V_{\lambda_{i_0}}^*)$  induced then a map  $q : \mathcal{V}_{\text{End}} \cap \mathcal{X}_P \rightarrow N^- \times N^+$  which is  $N^- \times N^+$ -equivariant :

$$q((n^-, n^+) \cdot x) = (n^-, n^+) \times q(x)$$

for  $x \in \mathcal{V}_{\text{End}} \cap \mathcal{X}_P$ , and  $n^{\pm} \in N^{\pm}$ .

We can now finish the arguments. The set  $N^- T_{\mathbb{C}} N^+ \subset K_{\mathbb{C}}$  is dense in  $K_{\mathbb{C}}$ , so it is now easy to see that the map

$$\begin{aligned} N^- \times N^+ \times \mathcal{Y}_P \cap \mathcal{V}_{\text{End}} &\longrightarrow \mathcal{X}_P \cap \mathcal{V}_{\text{End}} \\ (n^-, n^+, y) &\longmapsto (n^-, n^+) \cdot y \end{aligned}$$

is a diffeomorphism. We have proved previously that  $\mathcal{Y}_P \cap \mathcal{V}_{\text{End}}$  is a smooth affine variety, hence  $\mathcal{X}_P$  is smooth near the closed orbit  $K_{\mathbb{C}} \times K_{\mathbb{C}} \cdot [v_{\lambda_i} \otimes v_{\lambda_i}^*] \subset \mathcal{X} \cap \mathcal{V}_{\text{End}}$ . Lemma 3.6 tells us then that  $\mathcal{X}_P$  is smooth.

**3.3. Hamiltonian actions.** First consider an Hermitian vector space  $V$ . The Hermitian structure on  $\text{End}(V)$  is  $(A, B) := \text{Tr}(AB^*)$ , hence the associated symplectic structure on  $\text{End}(V)$  is defined by the relation  $\Omega_{\text{End}}(A, B) := -\text{Im}(\text{Tr}(AB^*))$ .

Let  $U(V)$  be the unitary group. Let  $\mathfrak{u}(V)$  be the Lie algebra of  $U(V)$ . We will use the identification  $\epsilon : \mathfrak{u}(V) \simeq \mathfrak{u}(V)^*$ ,  $X \mapsto \epsilon_X$  where  $\epsilon_X(Y) = -\text{Tr}(XY)$ . The action  $U(V) \times U(V)$  on  $\text{End}(V)$  is  $(g, h) \cdot A = gAh^{-1}$ . The moment map relative to this action is

$$\begin{aligned} \text{End}(V) &\longrightarrow \mathfrak{u}(V)^* \times \mathfrak{u}(V)^* \\ A &\longmapsto \frac{-1}{2} (iAA^*, -iA^*A). \end{aligned}$$

We consider now the projective space  $\mathbb{P}(\text{End}(V))$  equipped with the Fubini-Study symplectic form  $\Omega_{\text{FS}}$ . Here the action of  $U(V) \times U(V)$  on  $\mathbb{P}(\text{End}(V))$  is hamiltonian with moment map

$$\begin{aligned} \mathbb{P}(\text{End}(V)) &\longrightarrow \mathfrak{u}(V)^* \times \mathfrak{u}(V)^* \\ [A] &\longmapsto \left( \frac{iAA^*}{\|A\|^2}, \frac{-iA^*A}{\|A\|^2} \right). \end{aligned}$$

where  $\|A\|^2 = \text{Tr}(AA^*)$  (see [26][Section 7]). If  $\rho : K \hookrightarrow U(V)$  is a connected Lie subgroup, we can consider the action of  $K \times K$  on  $\mathbb{P}(\text{End}(V))$ . Let  $\pi_K : \mathfrak{u}(V)^* \rightarrow \mathfrak{k}^*$

be the projection which is dual to the inclusion  $\rho : \mathfrak{k} \hookrightarrow \mathfrak{u}(V)$ . The moment map for the action of  $K \times K$  on  $(\mathbb{P}(\text{End}(V)), \Omega_{\text{FS}})$  is then

$$(3.21) \quad \begin{aligned} \mathbb{P}(\text{End}(V)) &\longrightarrow \mathfrak{k}^* \times \mathfrak{k}^* \\ [A] &\longmapsto \frac{1}{\|A\|^2} (\pi_K(iAA^*), -\pi_K(iA^*A)). \end{aligned}$$

We are interested here respectively in

- the projective variety  $\mathcal{X}_P \subset \mathbb{P}(\text{End}(V))$  with the action of  $K \times K$ ,
- the projective variety  $\mathcal{Y}_P \subset \mathbb{P}(\text{End}(V))$  with the action of  $T \times T$ ,

where  $V = \bigoplus_{i=1}^N V_{\lambda_i}$ . The Fubini-Study two-form restrict into symplectic forms on  $\mathcal{X}_P$  and  $\mathcal{Y}_P$ . The action of  $K \times K$  on  $\mathcal{X}_P$  is Hamiltonian with moment map

$$(3.22) \quad \begin{aligned} \Phi_{K \times K} : \mathcal{X}_P &\longrightarrow \mathfrak{k}^* \times \mathfrak{k}^* \\ [x] &\longmapsto \frac{1}{\|x\|^2} (\pi_K(ixx^*), -\pi_K(ix^*x)). \end{aligned}$$

Since the diagonal  $Z = \{(t, t) | t \in T\}$  acts trivially on  $\mathcal{Y}_P$  we restrict ourself to the action of  $T \times T/Z \simeq T$  on  $\mathcal{Y}_P$ . Let us compute the moment map  $\Phi_T : \mathcal{Y}_P \rightarrow \mathfrak{k}^*$  associated to this action. First we have

$$(3.23) \quad \Phi_T([y]) = \frac{\pi_T(iy^*y)}{\|y\|^2} = \frac{\pi_T(iyy^*)}{\|y\|^2}$$

where  $\pi_T : \mathfrak{u}(V)^* \rightarrow \mathfrak{k}^*$  is the projection which is dual to  $\rho : \mathfrak{t} \rightarrow \mathfrak{u}(V)$ . Since  $\rho(X) = i \sum_{j \in J} \alpha_j(X) v_j \otimes v_j^*$ , a small computation shows that for  $B \in \mathfrak{u}(V) \simeq \mathfrak{u}(V)^*$  we have  $\pi_T(B) = -i \sum_{j \in J} (Bv_j, v_j) \alpha_j$ . Finally for any  $[y] \in \mathcal{Y}_P$  we get

$$\Phi_T([y]) = \sum_{j \in J} \frac{\|yv_j\|^2}{\|y\|^2} \alpha_j.$$

Together with the action on  $T$ , we have also an action of the Weyl group  $W = N(T)/T$  on  $\mathcal{Y}_P$  : for  $\bar{w} \in W$  we take

$$(3.24) \quad \bar{w} \cdot [y] = [\rho(w) \circ y \circ \rho(w)^{-1}], \quad [y] \in \mathcal{Y}_P.$$

This action is well defined since the diagonal  $Z \subset T \times T$  acts trivially on  $\mathcal{Y}_P$ . The set of weights  $\{\alpha_j, j \in J\}$  is stable under the action of  $W$ , hence it is an easy fact to verify that the map  $\Phi_T$  is  $W$ -equivariant.

A dense part of  $\mathcal{Y}_P$  is formed by the elements  $e^Z \cdot [Id] = [\rho(e^Z)]$ . Take  $Z = X + iY \in \mathfrak{t}_{\mathbb{C}}$ . We have  $\Phi_T(e^Z \cdot [Id]) = \psi_T(Y) \in \mathfrak{k}^*$  with

$$(3.25) \quad \psi_T(Y) = \frac{1}{\sum_{j \in J} e^{-2\langle \alpha_j, Y \rangle}} \sum_{j \in J} e^{-2\langle \alpha_j, Y \rangle} \alpha_j.$$

Hence the image of the moment map  $\Phi_T : \mathcal{Y}_P \rightarrow \mathfrak{k}^*$  is equal to the closure of the image of the map  $\psi_T : \mathfrak{t} \rightarrow \mathfrak{k}^*$ .

**Proposition 3.10.** *The map  $\psi_T$  realises a diffeomorphism between  $\mathfrak{t}$  and the interior of the polytope  $P \subset \mathfrak{k}^*$ .*

PROOF. Consider the function  $F_T : \mathfrak{t} \rightarrow \mathbb{R}$ ,  $F_T(Y) = \ln \left( \sum_{j \in J} e^{\langle \alpha_j, Y \rangle} \right)$ , and let  $L_T : \mathfrak{t} \rightarrow \mathfrak{k}^*$  be its Legendre transform :  $L_T(X) = dF_T|_X$ . Note that we have  $L_T(-2Y) = \psi_T(Y)$ .

We see that  $F_T$  is strictly convex. So, it is a classical fact that  $L_T$  realizes a diffeomorphism of  $\mathfrak{t}$  onto its image, and for  $\xi \in \mathfrak{t}^*$  we have

$$\begin{aligned} \xi \in \text{Image}(L_T) &\Leftrightarrow \lim_{Y \rightarrow \infty} F_T(Y) - \langle \xi, Y \rangle = \infty \\ &\Leftrightarrow \lim_{Y \rightarrow \infty} \sum_{j \in J} e^{\langle \alpha_j - \xi, Y \rangle} = \infty. \end{aligned}$$

In order to conclude we need the following

**Lemma 3.11.** *Let  $\{\beta_j, j \in J\}$  be a sequence of elements of  $\mathfrak{t}^*$ , and let  $Q$  be its convex hull. We have*

$$\lim_{Y \rightarrow \infty} \sum_{j \in J} e^{\langle \beta_j, Y \rangle} = \infty \iff 0 \in \text{Interior}(Q)$$

PROOF. First we see that  $0 \notin \text{Interior}(Q)$  if and only there exists  $v \in \mathfrak{t} - \{0\}$  such that  $\langle \beta_j, v \rangle \leq 0$  for all  $j$ : for such vector  $v$ , the map  $t \rightarrow \sum_{j \in J} e^{t \langle \beta_j, v \rangle}$  is bounded. Suppose now that  $\lim_{Y \rightarrow \infty} \sum_{j \in J} e^{\langle \beta_j, Y \rangle} \neq \infty$ . Then there exists a sequence  $(X_k)_k \in \mathfrak{t}$  such that  $\lim_k |X_k| = \infty$  and for all  $j$  the sequence  $(\langle \beta_j, X_k \rangle)_k$  remains bounded. If  $v$  is a limit of a subsequence of  $(\frac{X_k}{|X_k|})_k$  we have then  $\langle \beta_j, v \rangle \leq 0$  for all  $j$ .  $\square$

**Lemma 3.12.** *For  $[y] \in \mathcal{Y}_P$  we have  $\Phi_{K \times K}([y]) = (\Phi_T([y]), -\Phi_T([y]))$ .*

PROOF. It's sufficient to consider the case  $y = \rho(e^Z) = \sum_{j \in J} e^{i \langle \alpha_j, Z \rangle} v_j \otimes v_j^*$ , for  $Z = X + iY \in \mathfrak{t}_{\mathbb{C}}$ . Then  $yy^* = y^*y = \sum_j e^{-2 \langle \alpha_j, Y \rangle} v_j \otimes v_j^* = \rho(e^{2iY})$ . So it remains to prove that  $\pi_K(iyy^*) = \pi_T(iyy^*)$ . We have to check that  $\langle \pi_K(iyy^*), [U, V] \rangle = 0$  for  $U \in \mathfrak{t}$  and  $V \in \mathfrak{k}$ . We have

$$\begin{aligned} \langle \pi_K(iyy^*), [U, V] \rangle &= -i \text{Tr}(yy^* \rho([U, V])) \\ &= -i \text{Tr}(\rho(e^{2iY})[\rho(U), \rho(V)]) \\ &= -i \text{Tr}([\rho(e^{2iY}), \rho(U)]\rho(V)) = 0. \end{aligned}$$

$\square$

**Theorem 3.13.** *We have*

- $\text{Image}(\Phi_T) = P$ ,
- $\text{Image}(\Phi_{K \times K}) = \{(k_1 \cdot \xi, -k_2 \cdot \xi) \mid \xi \in P \text{ and } k_1, k_2 \in K\}$ ,
- $\mathcal{Y}_P \subset \Phi_{K \times K}^{-1}(\mathfrak{t}^* \times \mathfrak{t}^*)$ ,
- $\Phi_{K \times K}^{-1}(\text{interior}(\mathcal{C})) \subset \mathcal{Y}_P$ , where  $\mathcal{C} = C_K \times -C_K$ .

PROOF. The first point follows from Proposition 3.10. Since the map  $(k_1, t, k_2) \mapsto k_1 t k_2$  from  $K \times T_{\mathbb{C}} \times K$  into  $K_{\mathbb{C}}$  is onto, we have

$$(3.26) \quad \mathcal{X}_P = (K \times K) \cdot \mathcal{Y}_P.$$

So if  $[x] \in \mathcal{X}_P$ , there exist  $[y] \in \mathcal{Y}$  and  $k_1, k_2 \in K$  such that  $[x] = (k_1, k_2) \cdot [y]$ , hence

$$(3.27) \quad \begin{aligned} \Phi_{K \times K}([x]) &= (k_1, k_2) \cdot \Phi_{K \times K}([y]) \\ &= (k_1 \cdot \Phi_T([y]), -k_2 \cdot \Phi_T([y])) \end{aligned}$$

The second point is proved. The third point follows also from the identity (3.27) when  $k_1 = k_2 = e$ . Consider now  $[x] = (k_1, k_2) \cdot [y]$  such that  $\Phi_{K \times K}([x])$  belongs to



the interior of the cone  $C_K \times -C_K$ . Then  $k_1 \cdot \Phi_T([y])$  and  $k_2 \cdot \Phi_T([y])$  are regular points of  $C_K$ . This implies that  $k_1, k_2 \in N(T)$  and  $k_2 k_1^{-1} \in T$ . So

$$\begin{aligned} [x] &= (k_1, k_2) \cdot [y] \\ &= (e, k_2 k_1^{-1}) \cdot \left( (k_1, k_1) \cdot [y] \right) \in \mathcal{Y}_P \end{aligned}$$

since  $\mathcal{Y}_P$  is stable under the actions of  $T \times T$  and  $W$ .  $\square$

**Corollary 3.14.** *If  $[v_{\lambda_i} \otimes v_{\lambda_i}^*] \in \mathcal{X}_P$  then  $\lambda_i$  is a vertex of the polytope  $P$ .*

PROOF. Let  $x = v_{\lambda_i} \otimes v_{\lambda_i}^*$ , and suppose that  $[x]$  belongs to  $\mathcal{X}_P$ . In order to show that  $[x] \in \mathcal{Y}_P$ , we compute  $\Phi_{K \times K}([x])$ . We see that  $xx^* = x^*x = x$  and  $\|x\| = 1$  so  $\Phi_{K \times K}([x]) = (\pi_K(ix), -\pi_K(ix))$ . For  $X \in \mathfrak{k}$  we have

$$\begin{aligned} \langle \pi_K(ix), X \rangle &= -i \operatorname{Tr} \left( v_{\lambda_i} \otimes v_{\lambda_i}^* \rho(X) \right) \\ &= -i (\rho(X) v_{\lambda_i}, v_{\lambda_i}) \\ &= \langle \lambda_i, X \rangle. \end{aligned}$$

We have then  $\Phi_{K \times K}([x]) = (\lambda_i, -\lambda_i)$  with  $\lambda_i$  being a regular point of  $C_K$  : hence  $[x] \in \mathcal{Y}_P$ . Now we can conclude with the help of Lemma 3.7. Since  $[v_{\lambda_i} \otimes v_{\lambda_i}^*]$  belongs to  $\mathcal{Y}_P$ , the weight  $\lambda_i$  is a vertex of the polytope  $P$ .  $\square$

**Remark 3.15.** *In this section, Theorem 3.13 was obtain without using the fact that the varieties  $\mathcal{X}_P$  and  $\mathcal{Y}_P$  are smooth. Hence Corollary 3.14 can be used to prove the smoothness of  $\mathcal{X}_P$ .*

**3.4. Symplectic cutting.** Let  $(M, \Omega_M, \Phi_M)$  be an Hamiltonian  $K$ -manifold. At this stage the moment map  $\Phi_M$  is *not assumed to be proper*. We consider also the Hamiltonian  $K \times K$ -manifold  $\mathcal{X}_P$  associated to a  $K$ -adapted polytope  $P$ .

The purpose of this section is to define a *symplectic cutting* of  $M$  which uses  $\mathcal{X}_P$ . The notion of symplectic cutting was introduced by Lerman in [21] in the case of a torus action. Later Woodward [35] extends this procedure to the case of a non-abelian group action (see also [24]). The method of symplectic cutting that we define in this section is different from the one of Woodward.

We have two actions of  $K$  on  $\mathcal{X}_P$  : the action from the left (resp. right) , denoted  $\cdot_l$  (resp.  $\cdot_r$ ), with moment map  $\Phi_l : \mathcal{X}_P \rightarrow \mathfrak{k}^*$  (resp.  $\Phi_r$ ). We consider now the product of  $M \times \mathcal{X}_P$  with

- the action  $k \cdot_1 (m, x) = (k \cdot m, k \cdot_r x)$  : the corresponding moment map is  $\Phi_1(m, x) = \Phi_M(m) + \Phi_r(x)$ ,
- the action  $k \cdot_2 (m, x) = (m, k \cdot_l x)$  : the corresponding moment map is  $\Phi_2(m, x) = \Phi_l(x)$ .

**Definition 3.16.** *We denote  $M_P$  the symplectic reduction at 0 of  $M \times \mathcal{X}_P$  for the action  $\cdot_1$  :  $M_P := (\Phi_1)^{-1}(0)/K$ .*

Note that  $M_P$  is compact when  $\Phi_M$  is proper. The action  $\cdot_2$  on  $M \times \mathcal{X}_P$  induces an action of  $K$  on  $M_P$ . The moment map  $\Phi_2$  induces an equivariant map  $\Phi_{M_P} : M_P \rightarrow \mathfrak{k}^*$ . Let  $\mathcal{Z} \subset (\Phi_1)^{-1}(0)$  be the set of points where  $(K, \cdot_1)$  as a trivial stabilizer.

**Definition 3.17.** *We denote  $M'_P$  the quotient  $\mathcal{Z}/K \subset M_P$ .*

$M'_P$  is an open subset of smooth points of  $M_P$  which is invariant under the  $K$ -action. The symplectic structure of  $M \times \mathcal{X}_P$  induces a canonical symplectic structure on  $M'_P$  that we denote  $\Omega_{M'_P}$ . The action of  $K$  on  $(M'_P, \Omega_{M'_P})$  is Hamiltonian with moment map equal to the restriction of  $\Phi_{M_P} : M_P \rightarrow \mathfrak{k}^*$  to  $M'_P$ .

We start with the easy

**Lemma 3.18.** *The image of  $\Phi_{M_P} : M_P \rightarrow \mathfrak{k}^*$  is equal to the intersection of the image of  $\Phi_M : M \rightarrow \mathfrak{k}^*$  with  $K \cdot P$ .*

Let  $\mathcal{U}_P = K \cdot \text{Interior}(P) \subset K \cdot P$ . We will show now that the open and dense subset  $(\Phi_{M_P})^{-1}(\mathcal{U}_P)$  of  $M_P$  belongs to  $M'_P$ . Afterwards we will prove that  $\Phi_{M_P}^{-1}(\mathcal{U}_P)$  is quasi-symplectomorphic to the open subset  $\Phi_M^{-1}(\mathcal{U}_P)$  of  $M$ .

We consider the open and dense subset of  $\mathcal{X}_P$  which is equal to the open orbit  $\bar{\rho}(K_{\mathbb{C}})$ . From Lemma 3.3, we know that

$$(3.28) \quad \begin{aligned} \Theta : K \times \mathfrak{k} &\longrightarrow \bar{\rho}(K_{\mathbb{C}}) \\ (k, X) &\longmapsto [\rho(ke^{iX})] \end{aligned}$$

is a diffeomorphism. Through  $\Theta$ , the action of  $K \times K$  on  $K \times \mathfrak{k}$  is  $k \cdot_l(a, X) = (ka, X)$  for the action "from the left" and  $k \cdot_r(a, X) = (ak^{-1}, k \cdot X)$  for the action "from the right".

We consider now the map  $\psi_K : \mathfrak{k} \rightarrow \mathfrak{k}^*$  defined by  $\psi_K(X) = \Phi_l([\rho(e^{iX})])$ . In other words,

$$\psi_K(X) = \frac{\pi_K(i\rho(e^{i2X}))}{\text{Tr}(\rho(e^{i2X}))}.$$

Consider the function  $F_K : \mathfrak{k} \rightarrow \mathbb{R}$ ,  $F_K(X) = \ln(\text{Tr}(\rho(e^{-iX})))$ . Let  $L_K : \mathfrak{k} \rightarrow \mathfrak{k}^*$  be its Legendre transform.

**Proposition 3.19.** • We have  $\psi_K(X) = L_K(-2X)$ , for  $X \in \mathfrak{k}$ ,

- The function  $F_K$  is strictly convex,
- The map  $\psi_K$  realizes an equivariant diffeomorphism between  $\mathfrak{k}$  and  $\mathcal{U}_P$ .
- The image of  $\Phi_l : \mathcal{X}_P \rightarrow \mathfrak{k}^*$  is equal to the closure of  $\mathcal{U}_P$ ,
- $\Phi_l^{-1}(\mathcal{U}_P) = \bar{\rho}(K_{\mathbb{C}})$ .

PROOF. For  $X, Y \in \mathfrak{k}$  we consider the function  $\tau(s) = F_K(X + sY)$ . Since  $F_K$  is  $K$ -invariant we can restrict our computation to  $X \in \mathfrak{t}$ . We will use the decomposition of  $Y \in \mathfrak{k}$  relatively to the  $T$ -weights on  $\mathfrak{k}_{\mathbb{C}} : Y = \sum_{\alpha} Y_{\alpha}$  where  $\text{ad}(Z)Y_{\alpha} = i\alpha(Z)Y_{\alpha}$  for any  $Z \in \mathfrak{t}$ , and  $Y_0 \in \mathfrak{t}$ . We have

$$\begin{aligned} \tau'(s) &= \frac{-i}{\text{Tr}(\rho(e^{-iX_s}))} \text{Tr} \left( \rho(e^{-iX_s}) \rho \left( \frac{e^{i\text{ad}(X_s)} - 1}{i\text{ad}(X_s)} Y \right) \right) \\ &= \frac{-i}{\text{Tr}(\rho(e^{-iX_s}))} \text{Tr} (\rho(e^{-iX_s}) \rho(Y)) \\ &= \frac{1}{\text{Tr}(\rho(e^{-iX_s}))} \langle \pi_K(i\rho(e^{-iX_s})), Y \rangle \end{aligned}$$

where  $X_s = X + sY$ . Since by definition  $\tau'(0) = \langle L_K(X), Y \rangle$ , the first point is proved. For the second derivative we have

$$\begin{aligned} \tau''(0) &= - \left( \frac{\text{Tr}(\rho(e^{-iX}) \rho(iY))}{\text{Tr}(\rho(e^{-iX}))} \right)^2 + \frac{\text{Tr} \left( \rho(e^{-iX}) \rho \left( \frac{e^{i\text{ad}(X)} - 1}{i\text{ad}(X)} iY \right) \rho(iY) \right)}{\text{Tr}(\rho(e^{-iX}))} \\ &= R_1 + R_2 \end{aligned}$$

where

$$\begin{aligned} R_1 &= \frac{\mathrm{Tr}(\rho(e^{-iX})\rho(iY_0)\rho(iY_0))}{\mathrm{Tr}(\rho(e^{-iX}))} - \left( \frac{\mathrm{Tr}(\rho(e^{-iX})\rho(iY_0))}{\mathrm{Tr}(\rho(e^{-iX}))} \right)^2 \\ &= \frac{\sum_j e^{-\langle \alpha_j, X \rangle} \langle \alpha_j, Y_0 \rangle^2}{\sum_j e^{-\langle \alpha_j, X \rangle}} - \left( \frac{\sum_j e^{-\langle \alpha_j, X \rangle} \langle \alpha_j, Y_0 \rangle}{\sum_j e^{-\langle \alpha_j, X \rangle}} \right)^2 \end{aligned}$$

and

$$\begin{aligned} R_2 &= \frac{1}{\mathrm{Tr}(\rho(e^{-iX}))} \sum_{\alpha \neq 0, \beta \neq 0} \frac{e^{-\langle \alpha, X \rangle} - 1}{-\langle \alpha, X \rangle} \mathrm{Tr}(\rho(e^{-iX})\rho(iY_\alpha)\rho(iY_\beta)) \\ &= \frac{1}{\mathrm{Tr}(\rho(e^{-iX}))} \sum_{\alpha \neq 0, j} \frac{e^{-\langle \alpha, X \rangle} - 1}{-\langle \alpha, X \rangle} e^{-\langle \alpha_j, X \rangle} \|\rho(Y_\alpha)v_j\|^2. \end{aligned}$$

It is now easy to see that  $R_1$  and  $R_2$  are positive and that  $R_1 + R_2 > 0$  if  $Y \neq 0$ . We have proved that  $F_K$  is strictly convex. So, its Legendre transform  $L_K$  realizes a diffeomorphism of  $\mathfrak{k}$  onto its image. Using the first point we know that  $\psi_K$  realizes a diffeomorphism of  $\mathfrak{k}$  onto its image. The map  $\psi_K$  is equivariant and coincides with  $\psi_T$  on  $\mathfrak{t}$ . We have proved in Proposition 3.10 that the image of  $\psi_T$  is equal to the interior of  $P$ , hence the image of  $\psi_K$  is  $\mathcal{U}_P$ .

For the last two points we first remark that

$$(3.29) \quad \Phi_l([\rho(ke^{iX})]) = k \cdot \psi_K(X)$$

hence the image of  $\Phi_l$  is the closure of  $\mathcal{U}_P$ . If we use the fact that  $\psi_K$  is a diffeomorphism from  $\mathfrak{k}$  onto  $\mathcal{U}_P$ , (3.29) shows that  $\Phi_l^{-1}(K \cdot \xi) \cap \bar{\rho}(K_{\mathbb{C}})$  is a non empty and closed subset of  $\Phi_l^{-1}(K \cdot \xi)$  for any  $\xi \in \mathcal{U}_P$  (in fact it is a  $K \times K$ -orbit). On the other hand  $\Phi_l^{-1}(K \cdot \xi) \cap (\mathcal{X}_P \setminus \bar{\rho}(K_{\mathbb{C}}))$  is also a closed subset of  $\Phi_l^{-1}(K \cdot \xi)$  since  $\bar{\rho}(K_{\mathbb{C}})$  is open in  $\mathcal{X}_P$ . Since  $\Phi_l^{-1}(K \cdot \xi)$  is connected the second subset is empty : in other words  $\Phi_l^{-1}(K \cdot \xi) \subset \bar{\rho}(K_{\mathbb{C}})$ .  $\square$

We introduce now the equivariant diffeomorphism

$$(3.30) \quad \begin{aligned} \Upsilon : K \times \mathcal{U}_P &\longrightarrow \bar{\rho}(K_{\mathbb{C}}) \\ (k, \xi) &\longmapsto \Theta(k, \psi_K^{-1}(\xi)). \end{aligned}$$

We now look at  $K \times \mathcal{U}_P$  equipped with the symplectic structure  $\Upsilon^*(\Omega_{\mathcal{X}_P})$ , and the Hamiltonian action of  $K \times K$  : the moment maps satisfy

$$(3.31) \quad \Upsilon^*(\Phi_l)(k, \xi) = k \cdot \xi \quad \text{and} \quad \Upsilon^*(\Phi_r)(k, \xi) = -\xi.$$

**Proposition 3.20.** *We have*

$$\Upsilon^*(\Omega_{\mathcal{X}_P}) = d\lambda + d\eta$$

where  $\lambda$  is the Liouville 1-form on  $K \times \mathfrak{k}^* \simeq T^*K$  and  $\eta$  is an invariant 1-form on  $\mathcal{U}_P \subset \mathfrak{k}^*$  which is killed by the vectors tangent to the  $K$ -orbits.

PROOF. Let  $E_1, \dots, E_r$  be a basis of  $\mathfrak{k}$ , with dual basis  $\xi^1, \dots, \xi^r$ . Let  $\omega^i$  the 1-form on  $K$ , invariant by left translation and equal to  $\xi^i$  at the identity. The Liouville 1-form is  $\lambda = -\sum_i \omega^i \otimes E_i$ . For  $X \in \mathfrak{k}$  we denote  $X_l(k, \xi) = \frac{d}{dt}|_0 e^{-tX} \cdot_l(k, \xi)$  and  $X_r(k, \xi) = \frac{d}{dt}|_0 e^{-tX} \cdot_r(k, \xi)$  the vectors fields generated by the action of  $K \times K$ . Since  $\iota(X_l)d\lambda = -d\langle \Phi_l, X \rangle$  and  $\iota(X_r)d\lambda = -d\langle \Phi_r, X \rangle$ , the closed invariant 2-form  $\beta = \Upsilon^*(\Omega_{\mathcal{X}_P}) - d\lambda$  is  $K \times K$  invariant and is killed by the vectors tangent to the

orbits : (\*)  $\iota(X_l)\beta = \iota(X_r)\beta = 0$  for all  $X \in \mathfrak{k}$ . We have  $\beta = \beta_2 + \beta_1 + \beta_0$  where  $\beta_2 = \sum_{i,j} a_{ij}(\xi)\omega^i \wedge \omega^j$ ,  $\beta_1 = \sum_{i,j} b_{ij}(\xi)\omega^i \wedge dE_j$ , and  $\beta_0$  is an invariant 2-form on  $\mathcal{U}_P$ . The equalities (\*) gives  $\iota(X_l)\beta_2 = \iota(X_l)\beta_1 = 0$  which implies that  $\beta_2 = \beta_1 = 0$ . So  $\beta = \beta_0$  is a closed invariant 2-form on  $\mathcal{U}_P$  which is killed by the vectors tangent to the  $K$ -orbits. Since  $\mathcal{U}_P$  admits a retraction to  $\{0\}$ ,  $\beta = d\eta$  where  $\eta$  is an invariant 1-form on  $\mathcal{U}_P$  which is killed by the vectors tangent to the  $K$ -orbits.  $\square$

If  $(m, x) \in M \times \mathcal{X}_P$  belongs to  $\Phi_1^{-1}(0)$ , we denote  $[m, x]$  the corresponding element in  $M_P$ . By definition we have  $\Phi_{M_P}([m, x]) = \Phi_l(x)$  for  $[m, x] \in M_P$ , hence the image of  $\Phi_{M_P}$  is included in the closure of  $\mathcal{U}_P$ . We see also that  $[m, x] \in \Phi_{M_P}^{-1}(\mathcal{U}_P)$  if and only if  $x \in \Phi_l^{-1}(\mathcal{U}_P) = \bar{\rho}(K_{\mathbb{C}})$ . Since  $(K, \cdot_r)$  acts freely on  $\bar{\rho}(K_{\mathbb{C}})$ , we see that  $(K, \cdot_1)$  acts freely on  $\Phi_{M_P}^{-1}(\mathcal{U}_P)$  : the open and dense set  $\Phi_{M_P}^{-1}(\mathcal{U}_P) \subset M_P$  is then contained in  $M'_P$ .

Now, we can state our main result which compares the open invariant subsets  $\Phi_M^{-1}(\mathcal{U}_P) \subset M$  and  $\Phi_{M_P}^{-1}(\mathcal{U}_P) \subset M_P$  equipped respectively with the symplectic structures  $\Omega_M$  and  $\Omega_{M'_P}$ .

**Theorem 3.21.**  $\Phi_{M_P}^{-1}(\mathcal{U}_P)$  is an open and dense subset of smooth points in  $M_P$ . There exist an equivariant diffeomorphism  $\Psi : \Phi_M^{-1}(\mathcal{U}_P) \rightarrow \Phi_{M_P}^{-1}(\mathcal{U}_P)$  such that

$$\Psi^*(\Omega_{M'_P}) = \Omega_M + d\Phi_M^*\eta.$$

Here  $\eta$  is an invariant 1-form on  $\mathcal{U}_P$  which is killed by the vectors tangent to the  $K$ -orbits. Moreover the path  $\Omega^t = \Omega_M + td\Phi_M^*\eta$ , defines an homotopy of symplectic 2-forms between  $\Omega_M$  and  $\Psi^*(\Omega_{M'_P})$ .

**Remark 3.22.** The map  $\Psi$  will be call a quasi-symplectomorphism.

PROOF. Consider the immersion

$$\begin{aligned} \psi : \Phi_M^{-1}(\mathcal{U}_P) &\longrightarrow M \times \mathcal{X}_P \\ m &\longmapsto (m, \Upsilon(e, \Phi_M(m))). \end{aligned}$$

We have  $\Phi_1(\psi(m)) = \Phi_M(m) + \Upsilon^*\Phi_r(e, \Phi_M(m)) = 0$ , and  $\Phi_2(\psi(m)) = \Upsilon^*\Phi_l(e, \Phi_M(m)) = \Phi_M(m) \in \mathcal{U}_P$  (see (3.31)). Hence for all  $m \in \Phi_M^{-1}(\mathcal{U}_P)$ , we have  $\psi(m) \in \Phi_1^{-1}(0)$ , and its class  $[\psi(m)] \in M_P$  belongs to  $\Phi_{M_P}^{-1}(\mathcal{U}_P)$ .

We denote  $\Psi : \Phi_M^{-1}(\mathcal{U}_P) \rightarrow \Phi_{M_P}^{-1}(\mathcal{U}_P)$  the map  $m \mapsto [\psi(m)]$ . Let us show that it defines a diffeomorphism. If  $\Psi(m) = \Psi(m')$ , there exists  $k \in K$  such that

$$\begin{aligned} (m, \Upsilon(e, \Phi_M(m))) &= k \cdot_1 (m', \Upsilon(e, \Phi_M(m'))) \\ &= (k \cdot m', k \cdot_r \Upsilon(e, \Phi_M(m'))) \\ &= (k \cdot m', \Upsilon(k^{-1}, k \cdot \Phi_M(m'))). \end{aligned}$$

Since  $\Upsilon$  is a diffeomorphism, we must have  $k = e$  and  $m = m'$  : the map  $\Psi$  is one to one. Consider now  $(m, x) \in \Phi_1^{-1}(0)$  such that  $\Phi_{M_P}([m, x]) = \Phi_l(x) \in \mathcal{U}_P$  : then  $x \in \Phi_l^{-1}(\mathcal{U}_P) = \bar{\rho}(K_{\mathbb{C}}) = \text{Image}(\Upsilon)$ . We have  $x = \Upsilon(k, \xi)$  where  $\xi = -\Phi_r(x) = \Phi_M(m)$ . Finally

$$\begin{aligned} (m, x) &= (m, \Upsilon(k, \Phi_M(m))) \\ &= k^{-1} \cdot_1 (k \cdot m, \Upsilon(e, k \cdot \Phi_M(m))) \\ &= k^{-1} \cdot_1 \psi(k \cdot m). \end{aligned}$$

We have proved that  $\Psi$  is onto.

In order to show that  $\Psi$  is a submersion we must show that for  $m \in \Phi_M^{-1}(\mathcal{U}_P)$

$$\text{Image}(\mathbb{T}_m\psi) \oplus \mathbb{T}_{\psi(m)}(K \cdot_1 \psi(m)) = \mathbb{T}_{\psi(m)}\Phi_1^{-1}(0).$$

Here  $\mathbb{T}_m\psi : \mathbb{T}_mM \rightarrow \mathbb{T}_{\psi(m)}(M \times \mathcal{X}_P)$  is the tangent map, and  $\mathbb{T}_{\psi(m)}(K \cdot_1 \psi(m))$  denotes the tangent space at  $\psi(m)$  of the  $(K, \cdot_1)$ -orbit. We have  $\dim(\text{Image}(\mathbb{T}_m\psi)) + \dim(\mathbb{T}_{\psi(m)}(K \cdot_1 \psi(m))) = \dim(\mathbb{T}_{\psi(m)}\Phi_1^{-1}(0))$  so it is sufficient to prove that

$$\text{Image}(\mathbb{T}_m\psi) \cap \mathbb{T}_{\psi(m)}(K \cdot_1 \psi(m)) = \{0\}.$$

Consider  $(v, w) \in \text{Image}(\mathbb{T}_m\psi) \cap \mathbb{T}_{\psi(m)}(K \cdot_1 \psi(m))$ . There exists  $X \in \mathfrak{k}$  such  $(v, w) = \frac{d}{dt}|_0 e^{tX} \cdot_1 \psi(m)$  :

$$v = \frac{d}{dt}|_0 e^{tX} \cdot m \quad \text{and} \quad w = \frac{d}{dt}|_0 e^{tX} \cdot_r \Upsilon(e, \Phi_M(m))$$

In the other hand since  $(v, w) \in \text{Image}(\mathbb{T}_m\psi)$ , we have

$$w = \frac{d}{dt}|_0 \Upsilon(e, \Phi_M(e^{tX} \cdot m))$$

Since  $e^{tX} \cdot_r \Upsilon(e, \Phi_M(m)) = \Upsilon(e^{-tX}, \Phi_M(e^{tX} \cdot m))$  we obtain that

$$\frac{d}{dt}|_0 \Upsilon(e^{-tX}, \Phi_M(e^{tX} \cdot m)) = \frac{d}{dt}|_0 \Upsilon(e, \Phi_M(e^{tX} \cdot m))$$

or in other words  $\frac{d}{dt}|_0 \Upsilon(e^{-tX}, \Phi_M(m)) = 0$ . Since  $\Upsilon$  is a diffeomorphism we have  $X = 0$ , and then  $(v, w) = 0$ .

We can now compute the pull-back by  $\Psi$  of the symplectic form  $\Omega_{M'_P}$ . We have

$$\begin{aligned} \Psi^*(\Omega_{M'_P}) &= \psi^*(\Omega_M + \Omega_{\mathcal{X}_P}) \\ &= \Omega_M + \Phi_M^* \Upsilon^*(\Omega_{\mathcal{X}_P}) \\ &= \Omega_M + d\Phi_M^* \eta. \end{aligned}$$

It remains to prove that for every  $t \in [0, 1]$ , the 2-form  $\Omega^t = \Omega_M + td\Phi_M^* \eta$  is non-degenerate. Take  $t \neq 0$ ,  $m \in \Phi_M^{-1}(\mathcal{U}_P)$  and suppose that the contraction of  $\Omega^t|_m$  by  $v \in \mathbb{T}_mM$  is equal to 0. For every  $X \in \mathfrak{k}$  we have

$$\begin{aligned} 0 &= \Omega^t(X_M(m), v) \\ &= -\iota(v)d\langle \Phi_M, X \rangle|_m + t\iota(v)\iota(X_M)d\Phi_M^* \eta|_m \\ &= -\iota(v)d\langle \Phi_M, X \rangle|_m \end{aligned}$$

since  $\iota(X_M)d\Phi_M^* \eta = d\Phi_M^*(\iota(X_{\mathfrak{k}^*})\eta) = 0$ . Thus we have  $\mathbb{T}_m\Phi_M(v) = 0$ , and then  $\iota(v)d\Phi_M^* \eta = 0$ . Finally we have that  $0 = \iota(v)\Omega^t|_m = \iota(v)\Omega_M|_m$ . But  $\Omega_M$  is non-degenerate, so  $v = 0$ .  $\square$ .

**3.5. Formal quantization : second definition.** We suppose here that the Hamiltonian  $K$ -manifold  $(M, \Omega_M, \Phi_M)$  is *proper* and admits a Kostant-Souriau line bundle  $L$ . Now we consider the complex  $K \times K$ -submanifold  $\mathcal{X}_P$  of  $\mathbb{P}(E)$ . Since  $\mathcal{O}(-1)$  is a  $K \times K$ -equivariant Kostant-Souriau line bundle on the projective space  $\mathbb{P}(E)$  the restriction

$$(3.32) \quad L_P = \mathcal{O}(-1)|_{\mathcal{X}_P}$$

is a Kostant-Souriau line bundle on  $\mathcal{X}_P$ . Hence  $L \boxtimes L_P$  is a Kostant line bundle on the product  $M \times \mathcal{X}_P$ . In section 2.2 we have defined the quantization  $\mathcal{Q}_K(M_P)$  of the (singular) reduced space  $M_P := (M \times \mathcal{X}_P) //_0 (K, \cdot_1)$ .

NOTATION:  $O_K(r)$  will be any element  $\sum_{\mu \in \widehat{K}} m_\mu V_\mu^K$  of  $R^{-\infty}(K)$  where  $m_\mu = 0$  if  $\|\mu\| < r$ . The limit  $\lim_{r \rightarrow +\infty} O_K(r) = 0$  defines the notion of convergence in  $R^{-\infty}(K)$ .

**Proposition 3.23.** *Let  $\varepsilon_P > 0$  be the radius of the biggest ball center at  $0 \in \mathfrak{t}^*$  which is contains in the polytope  $P$ . We have*

$$(3.33) \quad \mathcal{Q}_K(M_P) = \sum_{\|\mu\| < \varepsilon_P} \mathcal{Q}(M_\mu) V_\mu^K + O_K(\varepsilon_P).$$

PROOF. Theorem 2.4 - ‘‘Quantization commutes with reduction in the singular setting’’ - tells us that  $\mathcal{Q}_K(M_P) = \sum_{\mu \in \widehat{K}} \mathcal{Q}((M_P)_\mu) V_\mu^K$  where  $(M_P)_\mu$  is the symplectic reduction

$$(M_P \times \overline{K_2 \cdot \mu}) //_0 K_2 \cong (M \times \mathcal{X}_P \times \overline{K_2 \cdot \mu}) //_{(0,0)} K_2 \times K_1.$$

Recall what the  $K_1, K_2$ -action are:  $k \cdot_1 (m, x, \xi) = (km, k \cdot_r x, \xi)$  and  $k \cdot_2 (m, x, \xi) = (m, k \cdot_l x, k\xi)$  for  $(m, x, \xi) \in M \times \mathcal{X}_P \times \overline{K_2 \cdot \mu}$  and  $k \in K$ .

Since the image of  $\Phi_{M_P}$  is equal to the intersection of  $K \cdot P = \overline{\mathcal{U}_P}$  with the image of  $\Phi_M$ , we have

$$(3.34) \quad \mathcal{Q}((M_P)_\mu) = 0 \quad \text{if} \quad \mu \notin P \cap \text{Image}(\Phi_M).$$

We will now exploit Theorem 3.21 to show that  $\mathcal{Q}((M_P)_\mu) = \mathcal{Q}(M_\mu)$  if  $\mu$  belongs to the interior of  $P$ .

There exists a quasi-symplectomorphism  $\Psi$  between the open subset  $\Phi_M^{-1}(\mathcal{U}_P)$  of  $M$  and the open and dense subset  $\Phi_{M_P}^{-1}(\mathcal{U}_P)$  of  $M_P$ . Moreover one can see easily that the restriction of the Kostant line bundle  $L_P \rightarrow \mathcal{X}_P$  to the open subset  $\bar{\rho}(K_{\mathbb{C}})$  is *trivial*. If  $L_{M_P}$  is the Kostant line bundle on  $M_P$  induced by  $L \boxtimes L_P$ , we have that the pull-back of the restriction  $L_{M_P}|_{\Phi_{M_P}^{-1}(\mathcal{U}_P)}$  by  $\Psi$  is equivariantly diffeomorphic to the restriction of  $L$  to  $\Phi_M^{-1}(\mathcal{U}_P)$ .

Take now  $\mu \in \widehat{K}$  that belongs to the interior of the polytope  $P$ . The element  $\mathcal{Q}((M_P)_\mu) \in \mathbb{Z}$  is given by the index of a transversally elliptic symbol defined in a (small) neighborhood of  $\Phi_{M_P}^{-1}(\mu) \subset M_P$ . This symbol is defined through two auxiliary data: the Kostant line bundle  $L_{M_P}$  and a compatible almost complex structure  $J$  which defined in a neighborhood of  $\Phi_{M_P}^{-1}(\mu)$ . If we pull back everything by  $\Psi$ , we get a transversally elliptic symbol living in a (small) neighborhood of  $\Phi_M^{-1}(\mu) \subset M$  which is defined by the Kostant line bundle  $L$  and an almost complex structure  $J_1$  compatible with the symplectic structure  $\Omega_1 := \Omega_M + d\Phi_M^* \eta$ . But since  $\Omega_t = \Omega_M + td\Phi_M^* \eta$  defines an homotopy of symplectic structures, any almost complex structure compatible with  $\Omega_M$  is homotopic to  $J_1$ . We have then shown that  $\mathcal{Q}(M_\mu) = \mathcal{Q}((M_P)_\mu)$  for any  $\mu$  belonging to the interior of  $P$ . So we have

$$\mathcal{Q}_K(M_P) = \sum_{\mu \in \text{Interior}(P)} \mathcal{Q}(M_\mu) V_\mu^K + \sum_{\nu \in \partial P} \mathcal{Q}((M_P)_\nu) V_\nu^K.$$

Since for  $\nu \in \partial P$  we have  $\|\nu\| \geq \varepsilon_P$ , the last equality proves (3.33).  $\square$

We work now with the dilated polytope  $nP$ , for any integer  $n \geq 1$ . The polytope  $nP$  is still  $K$ -adapted, so one can consider the reduced spaced  $M_{nP}$  and Proposition 3.23 gives that

$$(3.35) \quad \mathcal{Q}_K(M_{nP}) = \sum_{\|\mu\| < n\varepsilon_P} \mathcal{Q}(M_\mu) V_\mu^K + O_K(n\varepsilon_P).$$

for any integer  $n \geq 1$ . We can summarize the result of this section in the following

**Proposition 3.24.** *Let  $(M, \Omega_M)$  be a pre-quantized Hamiltonian  $K$ -manifold, with a proper moment map  $\Phi_M$ .*

- For any integer  $n \geq 1$ , the (singular) compact Hamiltonian manifold  $M_{nP}$  contains as an open and dense subset, the open subset  $\Phi_M^{-1}(n\mathcal{U}_P)$  of  $M$ .
- We have  $\mathcal{Q}_K^{-\infty}(M) = \lim_{n \rightarrow \infty} \mathcal{Q}_K(M_{nP})$ .

#### 4. FUNCTORIAL PROPERTIES : PROOF OF THEOREM 1.3

This section is devoted to the proof of Theorem 1.3. We will use in a crucial way the characterisation of  $\mathcal{Q}_K^{-\infty}$  given in Proposition 3.24.

Let  $H \subset K$  be a connected Lie subgroup. Here we consider a pre-quantized Hamiltonian  $K$ -manifold  $M$  which is *proper* as an Hamiltonian  $H$ -manifold. We want to compare  $\mathcal{Q}_K^{-\infty}(M)$  and  $\mathcal{Q}_H^{-\infty}(M)$ . For  $\mu \in \widehat{K}$  and  $\nu \in \widehat{H}$  we denote  $N_\nu^\mu$  the multiplicity of  $V_\nu^H$  in the restriction  $V_\mu^K|_H$ . We have seen in the introduction that  $N_\nu^\mu \mathcal{Q}(M_{\mu,K}) \neq 0$  only for the  $\mu$  belonging to *finite* subset  $\widehat{K} \cap \Phi_K(K \cdot \Phi_H^{-1}(\nu))$ . Then  $\mathcal{Q}_K^{-\infty}(M)$  is  $H$ -admissible and we have the following equality in  $R^{-\infty}(H)$  :

$$(4.36) \quad \mathcal{Q}_K^{-\infty}(M)|_H = \sum_{\nu \in \widehat{H}} m_\nu V_\nu^H$$

with  $m_\nu = \sum_\mu N_\nu^\mu \mathcal{Q}(M_{\mu,K})$ . We will now prove that  $\mathcal{Q}_K^{-\infty}(M)|_H = \mathcal{Q}_H^{-\infty}(M)$ .

**Lemma 4.1.** *The restriction  $\mathcal{Q}_K^{-\infty}(M)|_H$  is equal to  $\lim_{n \rightarrow \infty} \mathcal{Q}_K(M_{nP})|_H$ .*

PROOF. Let us denote  $P^\circ$  and  $\partial P$  respectively the interior and the border of the  $K$ -adapted polytope  $P$ . We write

$$\mathcal{Q}_K^{-\infty}(M) = \sum_{\mu \in nP^\circ} \mathcal{Q}(M_{\mu,K}) V_\mu^K + \sum_{\mu \notin nP^\circ} \mathcal{Q}(M_{\mu,K}) V_\mu^K.$$

On the other side

$$\mathcal{Q}_K(M_{nP}) = \sum_{\mu \in nP^\circ} \mathcal{Q}(M_{\mu,K}) V_\mu^K + \sum_{\mu \in n\partial P} \mathcal{Q}((M_{nP})_{\mu,K}) V_\mu^K.$$

So the difference  $D(n) = \mathcal{Q}_K^{-\infty}(M) - \mathcal{Q}_K(M_{nP})$  is equal to

$$D(n) = - \sum_{\mu' \in n\partial P} \mathcal{Q}((M_{nP})_{\mu',K}) V_{\mu'}^K + \sum_{\mu \notin nP^\circ} \mathcal{Q}(M_{\mu,K}) V_\mu^K.$$

We show now that the restriction  $D(n)|_H$  tends to 0 in  $R^{-\infty}(H)$  as  $n$  goes to infinity. For this purpose, we will prove that for any  $c > 0$  there exist  $n_c \in \mathbb{N}$  such that  $D(n)|_H = O_H(c)$  for any  $n \geq n_c$ .

For  $c > 0$  we consider the compact subset of  $\mathfrak{k}^*$  defined by

$$(4.37) \quad \mathcal{K}_c = \Phi_K(K \cdot \Phi_H^{-1}(\xi \in \mathfrak{h}^*, \|\xi\| \leq c)).$$

Let  $n_c \in \mathbb{N}$  such that  $\mathcal{K}_c$  is included in  $K \cdot (n_c P^\circ)$  : hence  $\mathcal{K}_c \subset K \cdot (nP^\circ)$  for any  $n \geq n_c$ . We know that for  $\mu \in \widehat{K}$ , we have  $N_\nu^\mu \mathcal{Q}(M_{\mu,K}) \neq 0$  only for  $\mu \in \Phi_K(K \cdot \Phi_H^{-1}(\nu))$ , and for  $\mu' \in \widehat{K}$ , we have  $N_{\nu'}^{\mu'} \mathcal{Q}((M_{nP})_{\mu',K}) \neq 0$  only for  $\mu' \in nP \cap \Phi_K(K \cdot \Phi_H^{-1}(\nu))$ .

Then if  $n \geq n_c$ , we have

$$N_\nu^\mu \mathcal{Q}(M_{\mu,K}) = N_\nu^{\mu'} \mathcal{Q}((M_{nP})_{\mu',K}) = 0$$

for any  $\nu \in \widehat{H} \cap \{\xi \in \mathfrak{h}^*, \|\xi\| \leq c\}$ ,  $\mu \notin nP^o$  and  $\mu' \in n\partial P$ . It means that  $D(n)|_H = O_H(c)$  for any  $n \geq n_c$ .  $\square$

Since  $\mathcal{Q}_K(M_{nP})|_H = \mathcal{Q}_H(M_{nP})$ , we are no led to the

**Lemma 4.2.** *The limit  $\lim_{n \rightarrow \infty} \mathcal{Q}_H(M_{nP})$  is equal to  $\mathcal{Q}_H^{-\infty}(M)$ .*

PROOF. Theorem 2.4 - ‘‘Quantization commutes with reduction in the singular setting’’ - tells us that  $\mathcal{Q}_H(M_{nP}) = \sum_{\nu \in \widehat{H}} \mathcal{Q}((M_{nP})_{\nu,H}) V_\nu^H$  where  $(M_{nP})_{\nu,H}$  is the symplectic reduction

$$(M_{nP} \times \overline{H \cdot \nu}) //_0 H \cong (M \times \mathcal{X}_{nP} \times \overline{H \cdot \mu}) //_{(0,0)} H \times K.$$

For  $c > 0$  we consider the compact subset of  $\mathcal{K}_c$  defined in (4.37). Let  $n_c \in \mathbb{N}$  such that  $\mathcal{K}_c \subset K \cdot (nP^o)$  for any  $n \geq n_c$ . It implies that

$$\Phi_H^{-1}(\xi \in \mathfrak{h}^*, \|\xi\| \leq c) \subset \Phi_K^{-1}(K \cdot (nP^o))$$

for  $n \geq n_c$ . Since  $M_{nP}$  ‘‘contains’’ as an the open subset  $\Phi_K^{-1}(K \cdot (nP^o))$ , the arguments similar to those used in the proof of Proposition 3.23 show that  $\mathcal{Q}((M_{nP})_{\nu,H}) = \mathcal{Q}(M_{\nu,H})$  for  $\|\nu\| \leq c$  and  $n \geq n_c$ . It means that

$$\mathcal{Q}_H(M_{nP}) = \sum_{\|\nu\| \leq c} \mathcal{Q}(M_{\nu,H}) V_\nu^H + O_H(c) \quad \text{when } n \geq n_c.$$

It follows that  $\lim_{n \rightarrow \infty} \mathcal{Q}_H(M_{nP}) = \sum_{\nu \in \widehat{H}} \mathcal{Q}(M_{\nu,H}) V_\nu^H = \mathcal{Q}_H^{-\infty}(M)$ .  $\square$

## 5. THE CASE OF AN HERMITIAN SPACE

Let  $(E, \mathfrak{h})$  be an Hermitian vector space of dimension  $n$ .

**5.1. The quantization of  $E$ .** Let  $U := U(E)$  be the unitary group with Lie algebra  $\mathfrak{u}$ . We use the isomorphism  $\epsilon : \mathfrak{u} \rightarrow \mathfrak{u}^*$  defined by  $\langle \epsilon(X), Y \rangle = -\text{Tr}(XY) \in \mathbb{R}$ . For  $v, w \in E$ , let  $v \otimes w^* : E \rightarrow E$  be the linear map  $x \mapsto \mathfrak{h}(x, w)v$ .

Let  $E_{\mathbb{R}}$  be the space  $E$  view as a real vector space. Let  $\Omega$  be the imaginary part of  $-\mathfrak{h}$ , and let  $J$  the complex structure on  $E_{\mathbb{R}}$ . Then on  $E_{\mathbb{R}}$ ,  $\Omega$  is a (constant) symplectic structure and  $\Omega(-, J-)$  defines a scalar product. The action of  $U$  on  $(E_{\mathbb{R}}, \Omega)$  is Hamiltonian with moment map  $\Phi : E \rightarrow \mathfrak{u}^*$  defined by  $\langle \Phi(v), X \rangle = \frac{1}{2} \Omega(Xv, v)$ . Through  $\epsilon$ , the moment map  $\Phi$  is defined by

$$(5.38) \quad \Phi(v) = \frac{1}{2i} v \otimes v^*.$$

The pre-quantization data  $(L, \langle -, - \rangle, \nabla)$  on the Hamiltonian  $U$ -manifold  $(E_{\mathbb{R}}, \Omega, \Phi)$  is a trivial line bundle  $L$  with a trivial action of  $U$  equipped with the Hermitian structure  $\langle s, s' \rangle_v = e^{\frac{-\mathfrak{h}(v,v)}{2}} \overline{s} s'$  and the Hermitian connexion  $\nabla = d - i\theta$  where  $\theta$  is the 1-form on  $E$  defined by  $\theta = \frac{1}{2} \Omega(v, dv)$ .

The traditional quantization of the Hamiltonian  $U$ -manifold  $(E_{\mathbb{R}}, \Omega, \Phi)$ , that we denote  $\mathcal{Q}_U^{L^2}(E)$ , is the *Bargman space* of entire holomorphic functions on  $E$  which are  $L^2$  integrable with respect to the Gaussian measure  $e^{\frac{-\mathfrak{h}(v,v)}{2}} \Omega^n$ . The representation  $\mathcal{Q}_U^{L^2}(E)$  of  $U$  is admissible. The irreducible representations of  $U$



that occur in  $\mathcal{Q}_U^{\mathbb{L}^2}(E)$  are the vector subspaces  $S^j(E^*)$  formed by the homogeneous polynomial on  $E$  of degree  $j \geq 0$ .

On the other hand, the moment map  $\Phi$  is proper (see 5.38). Hence we can consider the formal quantization  $\mathcal{Q}_U^{-\infty}(E) \in R^{-\infty}(U)$  of the  $U$ -action on  $E$ .

**Lemma 5.1.** *The two quantizations of  $(E, \Omega, \Phi)$ ,  $\mathcal{Q}_U^{\mathbb{L}^2}(E)$  and  $\mathcal{Q}_U^{-\infty}(E)$  coincide in  $R^{-\infty}(U)$ . In other words, we have*

$$(5.39) \quad \mathcal{Q}_U^{-\infty}(E) = S^\bullet(E^*) := \sum_{j \geq 0} S^j(E^*) \quad \text{in } R^{-\infty}(U).$$

PROOF. Let  $T \subset U$  be a maximal torus with Lie algebra  $\mathfrak{t} \subset \mathfrak{u}$ . There exists an orthonormal basis  $(e_k)_{k=1, \dots, n}$  of  $E$  and characters  $(\chi_k)_{k=1, \dots, n}$  of  $T$  such that  $t \cdot e_k = \chi_k(t)e_k$  for all  $k$ . The family  $(ie_k \otimes e_k^*)_{k=1, \dots, n}$  is then a basis of  $\mathfrak{t}$  such that  $\frac{1}{i}d\chi_l(ie_k \otimes e_k^*) = \delta_{l,k}$ . The set  $\hat{U} \subset \mathfrak{t}^* \subset \mathfrak{u}^*$  of dominant weights is composed, through  $\epsilon$ , by the elements

$$\underline{\lambda} = i \sum_{k=1}^n \lambda_k e_k \otimes e_k^*,$$

where  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$  is a *decreasing* sequence of integer.

The formal quantization  $\mathcal{Q}_U^{-\infty}(E) \in R^{-\infty}(U)$  is defined by

$$\mathcal{Q}_U^{-\infty}(E) = \sum_{\lambda_1 \geq \dots \geq \lambda_n} \mathcal{Q}(E_{\underline{\lambda}}) V_{\underline{\lambda}}$$

where  $E_{\underline{\lambda}} = \Phi^{-1}(U \cdot \underline{\lambda})/U$  is the reduced space and  $V_{\underline{\lambda}}$  is the irreducible representation of  $U$  with highest weight  $\underline{\lambda}$ .

It is now easy to check that

$$E_{\underline{\lambda}} = \begin{cases} \{\text{pt}\} & \text{if } \lambda = (0, \dots, 0, -j) \text{ with } j \geq 0, \\ \emptyset & \text{in the other cases,} \end{cases}$$

and then

$$\mathcal{Q}(E_{\underline{\lambda}}) = \begin{cases} 1 & \text{if } \lambda = (0, \dots, 0, -j) \text{ with } j \geq 0, \\ 0 & \text{in the other cases.} \end{cases}$$

Finally (5.39) follows from the fact that  $V_{(0, \dots, 0, -j)} = S^j(E^*)$ .  $\square$

**5.2. The quantization of  $E$  restricted to a subgroup of  $U$ .** Let  $K \subset U$  be a connected Lie subgroup with Lie algebra  $\mathfrak{k}^*$ . Let  $K_{\mathbb{C}} \subset \text{GL}(E)$  be its complexification. The moment map relative to the  $K$ -action on  $(E_{\mathbb{R}}, \Omega)$  is the map

$$\Phi_K : E \rightarrow \mathfrak{k}^*$$

equal to the composition of  $\Phi$  with the projection  $\mathfrak{u}^* \rightarrow \mathfrak{k}^*$ .

**Lemma 5.2.** *The following conditions are equivalent :*

- (a) *the map  $\Phi_K$  is proper,*
- (b)  *$\Phi_K^{-1}(0) = \{0\}$ ,*
- (c)  *$\{0\}$  is the only closed  $K_{\mathbb{C}}$ -orbit in  $E$ ,*
- (d) *for every  $v \in E$  we have  $0 \in \overline{K_{\mathbb{C}} \cdot v}$ ,*
- (e)  *$S^\bullet(E^*)$  is an admissible representation of  $K$ ,*
- (f) *the  $K$ -invariant polynomials on  $E$  are the constant polynomials.*

PROOF. The equivalence (a)  $\iff$  (b) is due to the fact that  $\Phi_K$  is quadratic.

Let  $\mathcal{O}$  be a  $K_{\mathbb{C}}$ -orbit in  $E$ . Classical results of Geometric Invariant Theory [26, 18] assert that  $\overline{\mathcal{O}} \cap \Phi_K^{-1}(0) \neq \emptyset$  and that  $\mathcal{O}$  is closed if and only if  $\mathcal{O} \cap \Phi_K^{-1}(0) \neq \emptyset$ . Hence (b)  $\iff$  (c)  $\iff$  (d).

After Lemma 5.1 we know that  $\mathcal{Q}_{\mathbb{U}}^{-\infty}(E) = S^{\bullet}(E^*)$ . Since  $\mathcal{Q}_{\mathbb{U}}^{-\infty}(E)$  is  $K$ -admissible when  $\widehat{\Phi}_K$  is proper (see Section 4), we have (a)  $\implies$  (e).

For every  $\mu \in \widehat{K}$ , the  $\mu$ -isotopic component  $[S^{\bullet}(E^*)]_{\mu}$  is a module over  $[S^{\bullet}(E^*)]_0 = [S^{\bullet}(E^*)]^K$ . Hence  $\dim[S^{\bullet}(E^*)]_{\mu} < \infty$  implies that  $[S^{\bullet}(E^*)]^K = \mathbb{C}$ . We have (e)  $\implies$  (f).

Finally (f)  $\implies$  (d) follows from the following fundamental fact. For any  $v, w \in E$  we have  $\overline{K_{\mathbb{C}} \cdot v} \cap \overline{K_{\mathbb{C}} \cdot w} \neq \emptyset$  if and only if  $P(v) = P(w)$  for all  $P \in [S^{\bullet}(E^*)]^K$ .  $\square$

Theorem 1.3 implies the following

**Proposition 5.3.** *Let  $K \subset \mathbb{U}(E)$  be a closed connected subgroup such that  $S^{\bullet}(E^*)$  is an admissible representation of  $K$ . For every  $\mu \in \widehat{K}$ , we have*

$$\dim([S^{\bullet}(E^*)]_{\mu}) = \mathcal{Q}(E_{\mu, K})$$

where  $[S^{\bullet}(E^*)]_{\mu}$  is the  $\mu$ -isotopic component of  $S^{\bullet}(E^*)$  and  $E_{\mu, K}$  is the reduced space  $\Phi_K^{-1}(K \cdot \mu)/K$ .

In the following examples the condition  $\Phi_K^{-1}(0) = \{0\}$  is easy to check.

- 1) the subgroup  $K \subset \mathbb{U}(E)$  contains the center of  $\mathbb{U}(E)$ ,
- 2)  $E = \wedge^2 \mathbb{C}^n$  or  $E = S^2(\mathbb{C}^n)$  and  $K = \mathbb{U}(n) \subset \mathbb{U}(E)$ ,
- 3)  $E = M_{n, k}$  is the vector space of  $n \times k$ -matrices and  $K = \mathbb{U}(n) \times \mathbb{U}(k) \subset \mathbb{U}(E)$ .

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