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# On Teichmüller's metric and Thurston's asymmetric metric on Teichmüller space

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## Abstract.

In this chapter, we review some elements of Teichmüller's metric and of Thurston's asymmetric metric on Teichmüller space. One of our objectives is to draw a parallel between these two metrics and to stress on some differences between them. The results that we present on Teichmüller's metric are classical, whereas some of the results on Thurston's asymmetric metric are new. We also discuss some open problems.

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## 1 Introduction

In this chapter,  $\mathfrak{g}$  and  $\mathfrak{n}$  are two nonnegative integers and  $S = S_{\mathfrak{g},\mathfrak{n}}$  is a connected oriented surface obtained from a closed surface of genus  $\mathfrak{g}$  (which we shall denote by  $\hat{S}$ ) by removing  $\mathfrak{n}$  points called the punctures.<sup>1</sup> We assume that the Euler characteristic of  $S$  is negative.  $\mathcal{T} = \mathcal{T}_{\mathfrak{g},\mathfrak{n}}$  denotes the Teichmüller space of  $S$ . This space carries several interesting metrics. In this chapter, we shall consider two of them, namely, Teichmüller's metric and Thurston's asymmetric metric. We shall study these metrics respectively in Sections 2 and 3 below. These two metrics are Finsler metrics, that is, the distance between two points in Teichmüller space can be defined by minimizing the lengths of paths joining these points, and lengths of paths are computed by using a norm defined on the tangent bundle of  $\mathcal{T}$ . In the case of Thurston's asymmetric metric, the norm on the tangent spaces is not symmetric.

Teichmüller space can be defined either as a space of equivalence classes of conformal structures on  $S$  or as a space of equivalence classes of hyperbolic structures on  $S$ . Each of the two metrics considered here is natural from one of these points of view: Teichmüller's metric from the point of view of con-

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<sup>1</sup>Such a surface  $S$  is said to be of *finite type*, in reference to the fact that its fundamental group is of finite type.

formal geometry and Thurston's asymmetric metric from the point of view of hyperbolic geometry. Indeed, Teichmüller's metric is natural as a measure of distances between conformal structures, since it is defined as the logarithm of the least quasi-conformal dilatation of a homeomorphism isotopic to the identity between the conformal structures, whereas Thurston's asymmetric metric is natural from the point of view of measuring distances between hyperbolic structures, since it is defined as the logarithm of the smallest Lipschitz constant of homeomorphisms isotopic to the identity from one hyperbolic structure to the other one. (The order in which we take the hyperbolic surfaces is important in the last definition because the smallest Lipschitz constant in one direction is generally different from the smallest Lipschitz constant in the other direction.) The difference between the conformal and the hyperbolic points of view makes the techniques used in the study of these two metrics on Teichmüller space of different natures: on the one hand, we use complex analysis (quasiconformal mappings, quadratic differentials, extremal length and so on), and, on the other hand, we use two-dimensional hyperbolic geometry. But the problems that we try to solve are formally the same: the global behaviour of geodesic lines (that is, isometric images of  $\mathbb{R}$ ) in Teichmüller space, e.g. the question of whether they are properly embedded or not, the convergence of geodesic rays to a point on Thurston's boundary, the study of visual boundaries, and other related problems. There are also several analogies between the results obtained so far for the two metrics. Finally, there are natural questions about the comparison between the two metrics. These are reasons for which we present the two metrics in parallel.

Let us start with a few words about some analogies and some differences between the general features of the two metrics.

In what follows, we shall recall two descriptions of the Teichmüller distance between two conformal structures. Consider two conformal structures on a surface. On the one hand, the Teichmüller distance is the logarithm of the infimum of quasiconformal dilatations of homeomorphisms isotopic to the identity between the two conformal structures. On the other hand, this distance is the logarithm of the supremum of quotients of extremal lengths of closed curves with respect to these structures. Likewise, there are two descriptions of Thurston's asymmetric distance from one hyperbolic surface to another. On the one hand, this distance is the logarithm of the infimum of Lipschitz constants of homeomorphisms isotopic to the identity between the two hyperbolic surfaces, and on the other hand it is the logarithm of the supremum of quotients of hyperbolic lengths of closed geodesics, with respect to the two hyperbolic surfaces.

Teichmüller proved that given two conformal structures on  $S$ , there exists, in each isotopy class of homeomorphisms, a "best quasiconformal stretch homeomorphism", that is, a homeomorphism for which the infimum of the quasiconformal dilatation in the definition of the Teichmüller distance be-

tween the two conformal structures is attained. Likewise, Thurston proved that given two hyperbolic structures on  $S$ , there exists, in each isotopy class of homeomorphisms, a “best Lipschitz stretch homeomorphism”, that is, a homeomorphism for which the infimum of the Lipschitz constant in the definition of Thurston’s asymmetric distance is attained.

Teichmüller space, equipped with the Teichmüller metric, is a straight G-space in the sense of Busemann (cf. Kravetz [31]). This means that any two distinct points in that space lie on a unique geodesic line. Thurston’s asymmetric metric has a different character: it is not symmetric (as its name indicates). Furthermore, any two distinct points in the space lie on a geodesic line, but this geodesic line is not necessarily unique.

Let us look more closely at the geodesic lines for the two metrics.

Geodesics for Teichmüller’s metric are usually described (since the work of Teichmüller himself) in terms of quadratic differentials. In these notes, we have avoided talking about quadratic differentials, but we have used instead (the equivalent point of view of) pairs of transverse measured foliations. This is not because we do not like quadratic differentials, but it is for the sake of stressing a further analogy between Teichmüller’s metric and Thurston’s asymmetric metric. To describe this analogy, let us be more precise.

To characterize a geodesic for Teichmüller’s metric, we represent each point in Teichmüller space by a conformal structure defined by a pair  $(F_1, F_2)$  of transverse measured foliations on the surface  $S$ . The pair of measured foliations defines a “grid” on the surface, and there is a natural notion of distance measured along the leaves of each of these foliations, coming from the transverse measure of the other foliation. Thus, to a pair of transverse measured foliations, it is easy to associate holomorphic local parameters  $z = x + iy$ , the  $x$ -direction defined by the leaves of the first foliation and the  $y$ -direction by those of the other one. A geodesic line for Teichmüller’s metric is then described as a family of surfaces  $S^t = (e^{-t}F_1, e^tF_2)$ ,  $t \in \mathbb{R}$ . The measured foliations  $F_1$  and  $F_2$  are called respectively the horizontal and vertical foliations associated to the geodesic line  $(S^t)_{t \in \mathbb{R}}$ . This description makes it easy to visualize the holomorphic coordinates of the surface  $S^t$  when this surface varies along a geodesic line. In this way, each conformal structure  $S^t$  is represented by a Euclidean structure with isolated conical singular points on the surface, where the transverse measure of the foliation  $e^{-t}F_1$  (respectively  $e^tF_2$ ) determines the Euclidean length on the leaves of the transverse foliation  $e^tF_2$  (respectively  $e^{-t}F_1$ ).

In the case of Thurston’s asymmetric metric, a geodesic is described using an object which is less symmetrical than a pair of measured foliations. This object is a pair consisting of a complete (not necessarily measured) geodesic lamination, and a measured foliation which is transverse to it. More precisely, Thurston showed that any two points in Teichmüller space can be joined by a geodesic (for his asymmetric metric) made up of a concatenation of pieces

of “stretch lines”, a stretch line being a parametrized family of hyperbolic structures  $S^t = (\mu, e^t F)$ ,  $t \in \mathbb{R}$ , where  $\mu$  is a complete geodesic lamination on  $S^t$  and  $F$  a measured foliation on  $S = S^0$  which is transverse to  $\mu$ . Here also, the pair  $(\mu, e^t F)$  defines a privileged set of directions on the hyperbolic surface  $S^t$ , viz the geodesic directions of the leaves of  $\mu$ , and the perpendicular directions which are the leaves of  $e^t F$  and which are made out of pieces of horocycles, with the transverse measure of  $e^t F$  coinciding on the leaves of  $\mu$  with hyperbolic length. Thus, the measured foliation  $e^t F$  plays here the role of a “vertical foliation”, and the complete lamination  $\mu$  plays the role of a “horizontal foliation”, associated to the stretch line. Varying the parameter  $t$  describes the stretch line.

These descriptions of geodesic lines for the two metrics on Teichmüller space lead naturally to similar questions concerning both metrics, as well as to questions concerning the comparison between them. We now enumerate some of these questions.

- The descriptions of the geodesic rays for Teichmüller’s metric and for Thurston’s asymmetric metric lead to two distinct natural parametrizations of Teichmüller space equipped with actions of the group  $\mathbb{R}_+^*$  of positive reals on the parameter spaces. A first question related to these parametrizations concerns the study of the extension of the parameters by adjoining limit points to the orbits of the  $\mathbb{R}_+^*$ -actions. This amounts to defining boundaries to Teichmüller space, and one natural question is about the dependence of such boundaries on the chosen parametrization in each case. Other questions concern the comparison of these boundaries among themselves and with other geometrically defined boundaries. More precisely, for each of the two metrics, there is a collection of parameter spaces, each of which being a set of equivalence classes of measured foliations. In the case of Teichmüller’s metric, a parameter space is a space of equivalence classes of measured foliations that are transverse to a fixed measured foliation, and in the case of Thurston’s asymmetric metric, a parameter space is a space of equivalence classes of measured foliations that are transverse to a fixed complete geodesic lamination. As such, the two parameter spaces admit natural  $\mathbb{R}_+^*$ -action (induced by the action of  $\mathbb{R}_+^*$  on measures), and the orbits of these actions correspond to geodesic lines in Teichmüller space, for each of these metrics respectively. The questions about the extension of the parameters to the boundary involve the study of the asymptotic behaviour of geodesic rays for each of these metrics. In each case, the  $\mathbb{R}_+^*$ -orbits are properly embedded in the parameter space. In these notes, we present these facts in some detail, as well as results on the following questions:

- *Convergence of geodesic rays*: Some of the convergence results are formulated in terms of a boundary of  $\mathcal{T}$  which Steve Kerckhoff called Teichmüller boundary, others in terms of Thurston’s boundary, and others in terms of the visual boundaries of Teichmüller space. In the early 1980s, Howard Masur

obtained results on the convergence of some particular classes of geodesic rays for Teichmüller’s metric to points on Thurston’s boundary, and Kerckhoff obtained results on the relative behaviour of pairs of Teichmüller geodesic rays. We present these results below. It seems that there were no other significant results of that type until a recent work by Anna Lenzhen, in which she gives a geodesic ray that does not converge to a point on Thurston’s boundary. The question of the description of the behaviour of an arbitrary geodesic ray for Teichmüller’s metric with respect to Thurston’s boundary is still open. Concerning the convergence of geodesic rays for Thurston’s asymmetric metric to a point on Thurston’s boundary, we shall present some recent results.

- *The asymptotic behaviour of “anti-stretch” rays:* An anti-stretch ray is the negative part of a stretch line, oriented in the direction opposite to the one given by the parametrization of the stretch line. Due to the fact that Thurston’s metric is not symmetric, an anti-stretch ray is in general not a stretch ray (even after reparametrization). We note that an anti-stretch ray is (up to reparametrization) a geodesic ray for the asymmetric metric on Teichmüller space which is “dual” to Thurston’s asymmetric metric. Here, the dual  $K^*$  of an asymmetric metric  $K$  is defined by the formula  $K^*(x, y) = K(y, x)$ .

- *The asymptotic behaviour of the length of an arbitrary measured geodesic lamination of compact support under a stretch or an anti-stretch ray:* More precisely, for a given family of hyperbolic surfaces  $(S^t)_{t \in \mathbb{R}}$  parametrized by a stretch line and for any compactly supported measured geodesic lamination  $\alpha$ , we are interested in the existence of the limits  $\lim_{t \rightarrow \infty} l_{S^t}(\alpha)$  and  $\lim_{t \rightarrow -\infty} l_{S^t}(\alpha)$ , and whether these limits are finite or infinite.

We note that in the case of the Teichmüller metric, we have stated some of the results for the restricted case of closed surfaces, because the written sources for these results exist only in that special case (although most of these results are certainly valid in the larger context of surfaces of finite type). For Thurston’s asymmetric metric, we present the results in the case of surfaces with or without punctures.

The results on the Teichmüller metric that we give are classical, but some of those on Thurston’s asymmetric metric are new.

At the end of this chapter, we formulate some open problems which concern especially Thurston’s asymmetric metric.

## 2 Teichmüller’s metric

References for Teichmüller’s metric can be found in Teichmüller’s collected papers [57], as well as in those of Ahlfors and of Bers [2] [12], who rewrote part of the theory. There are several introductory books on the subject of

Teichmüller's theory, for instance the books by Abikoff [1] and by Iwayoshi and Taniguchi [25]. We also recommend the recent book by Hubbard [23].

## 2.1 Measured foliations

We start by recalling a few facts about measured foliations that are used in the theory of deformation of conformal structures which we present below.

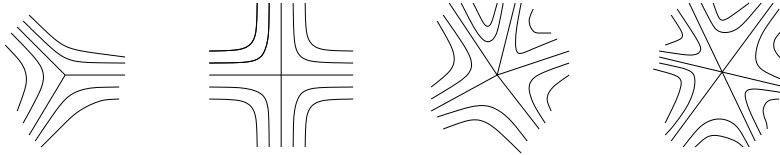


Figure 1. The four pictures represent  $s$ -prong singular points with  $s = 3, 4, 5, 6$  respectively.

**Definition 2.1** (Measured foliation). A *measured foliation* on  $S$  is a foliation with isolated singularities, equipped with a positive measure on each transverse arc that is equivalent to the Lebesgue measure of a closed interval of  $\mathbb{R}$ . (An arc in a surface is, by definition, a homeomorphic image of the interval  $[0, 1]$ ). These measures are invariant by isotopies of the transverse arcs during which each point stays on the same leaf. The isolated singularities are of the type suggested in Figure 1, and we call them  $s$ -prong singularities, where  $s$  can be any integer  $\geq 3$ . We require that at the punctures of  $S$ , the foliation extends as a measured foliation of the unpunctured surface  $\hat{S}$  in such a way that each puncture becomes either a nonsingular point, or an  $s$ -prong singular point with  $s$  being here any integer  $\geq 1$  (see Figure 2). Note that in the case  $s = 2$ , the foliation extends as a nonsingular point at the puncture.

There is an equivalence relation between measured foliations, called *Whitehead-equivalence*. It is generated by the following transformations:

- Homeomorphisms of the surface which are isotopic to the identity sending one foliation to the other and preserving the transverse measures. By abuse of language, we shall sometimes call a homeomorphism isotopic to the identity an *isotopy*. We recall that two homeomorphisms of a surface are isotopic if and only if they are homotopic. This is a result of Baer [6]; see also Mangler [35] and Epstein [18].

- Whitehead moves: These are deformations of the surface that take place in a neighborhood of arcs that join two singular points and whose effect is to collapse such an arc to a point. (Remember that by our definition, an arc is embedded, which implies that after such a collapse, the surface remains a



surface.) An example of a Whitehead move is given in Figure 3. Again, these moves are required to respect the transverse measures. The inverse move of a Whitehead move is also called a Whitehead move.

Note that a singular point involved in a Whitehead move can be at a puncture of  $S$ , and that it is sometimes possible to eliminate a 1-prong singularity at a puncture by using a Whitehead move; see for instance Figure 4.

We let  $\mathcal{MF}(S) = \mathcal{MF}$  denote the set of equivalence classes of measured foliations on  $S$ . An element of  $\mathcal{MF}$  is called a *measured foliation class*.

If  $x$  is a positive real number and if  $F$  is a measured foliation, then  $xF$  denotes the foliation  $F$  (as a topological object), equipped with the transverse measure obtained by multiplying the original transverse measure of  $F$  by the factor  $x$ . This action is compatible with the Whitehead equivalence relation, and it induces an action of  $\mathbb{R}_+^*$  on the set  $\mathcal{MF}$ . The quotient of  $\mathcal{MF}$  by this ac-

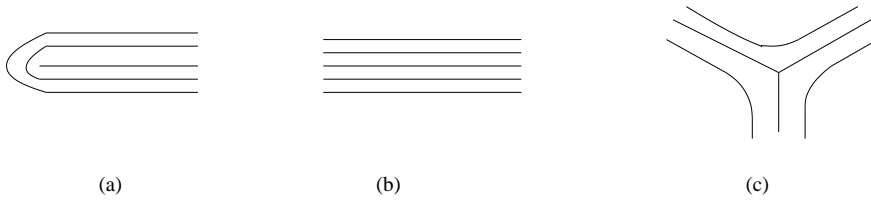


Figure 2. The three pictures represent the local model for a measured foliation at a puncture of  $S$ . The foliation extends in three possible ways: either as a 1-prong singular point (case (a)), or as a regular point (case (b)), or as a singular point with at least 3 prongs (case (c)), that is, like the singular points at interior points of the surface.

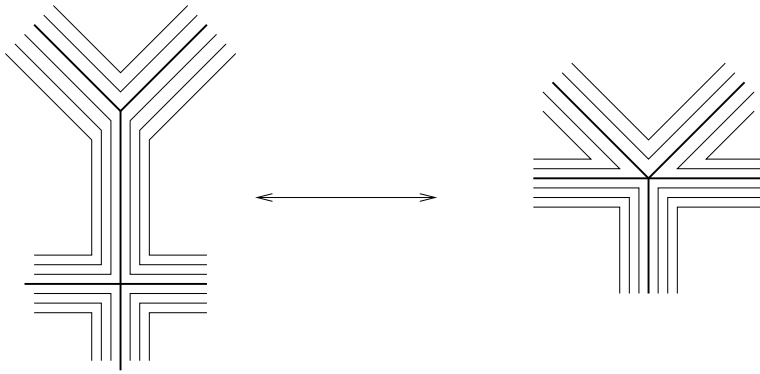


Figure 3. Whitehead move: collapsing or creating an arc joining two singular points.

tion is denoted by  $\mathcal{PMF}(S)$  or  $\mathcal{PMF}$ . An element of  $\mathcal{PMF}$  is called a *projective measured foliation class*.

If  $F$  is a measured foliation, then  $[F]$  will usually denote its equivalence class in  $\mathcal{MF}$ . We shall also sometimes use the same notation,  $[F]$ , for the corresponding element in  $\mathcal{PMF}$ .

Measured foliations already appear in Teichmüller's work as horizontal and vertical line fields associated to holomorphic quadratic differentials on Riemann surfaces, but it was Thurston who initiated their systematic study, and defined the space of equivalence classes of measured foliations, in his paper *On the geometry and dynamics of diffeomorphisms of surfaces* ([64], published several years after it has been written). For a complete presentation of these results in the case of closed surfaces, we refer the reader to [19].

A closed curve in  $S$  will be called *essential* if it is not homotopic to a point or to a puncture, and it is called *simple* if it has no self-intersection, that is, if it is an embedded image of a circle.

Let  $\mathcal{S}$  be the set of homotopy classes of unoriented essential simple closed curves in  $S$ .

There is a natural embedding

$$\mathcal{S} \rightarrow \mathcal{MF} \tag{2.1}$$

defined as follows. For any any element  $\gamma$  in  $\mathcal{S}$ , we take a foliated cylinder  $C$  embedded in  $S$ , whose leaves are simple closed curves that are in the homotopy class  $\gamma$ . This foliated cylinder  $C$  defines a *partial* foliation on  $S$ . Here, the adjective *partial* means that the support of the foliation is a subset of the surface  $S$ . We choose an arc  $c$  that joins the two boundary components of  $C$  and which is transverse to the foliations, and a homeomorphism between this arc and the interval  $[0, 1] \subset \mathbb{R}$ , and we equip the arc  $c$  with the pull-back of the Lebesgue measure of  $[0, 1]$ . We then take the only invariant transverse measure for the foliation on  $C$  that induces the given Lebesgue measure on the

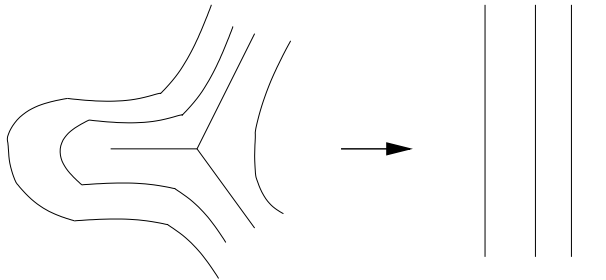


Figure 4. Eliminating a 1-prong singularity at a puncture of  $S$  by applying a Whitehead move.

arc  $c$ . We shall sometimes say that  $C$ , equipped with this measured foliation, is a foliated cylinder of height one with core curve in  $\gamma$ . To get a measured foliation on  $S$ , we collapse the closure of each connected component of  $S \setminus C$  onto a spine. We recall that a spine of a compact surface with boundary is a one-dimensional simplicial complex on which the surface collapses by a homotopy equivalence (see [19] p.90). Here, the spine is chosen so that the singular points of the resulting foliation on  $S$  are of the allowed type. Note that the exposition in [19] works for the case where  $S$  is compact. To deal with the case where  $S$  is noncompact (that is, the case where  $S$  has punctures), we can use the spine of a compact surface with boundary obtained by replacing each puncture by a boundary closed curve and then collapsing that boundary curve to a point. We do this in such a way that the local model of the resulting measured foliation on the original surface  $S$  is of the allowed type, at the interior singularities and at the punctures. The various choices of spines for the complementary components of the cylinder  $C$  differ precisely by Whitehead moves performed on spines. Using this fact, to any element of  $\mathcal{S}$ , we associate a measured foliation on  $S$  which is well-defined up to isotopy and Whitehead moves. It is a fact of the classical theory of measured foliations that the map defined in (2.1) is injective (see [19] p.89).

Note that in the case where the surface  $S$  is the three-punctured sphere, the set  $\mathcal{S}$  is empty. As is well-known, the Teichmüller space in that case consists of a single point, and there is not a lot more to say. Therefore, we discard this case in what follows.

**Definition 2.2** (Measured foliation with one cylinder). A measured foliation (respectively a measured foliation class) which up to a constant factor is obtained from an element of  $\mathcal{S}$  by the map described in (2.1) will be called a *measured foliation* (respectively a *measured foliation class*) with one cylinder.

Equivalently, a measured foliation with one cylinder is a measured foliation such that when extended to the unpunctured surface  $\hat{S}$ , the union of the leaves starting at the singular points, with the singular points included, is a compact graph whose complement is connected. This graph is called the *critical graph* of the foliation.

The *height* of a measured foliation with one cylinder is defined as the total mass of a transverse arc joining the two boundary components of that cylinder.

We need to recall another description of the equivalence relation between measured foliations.

To each measured foliation  $F$ , we associate a map  $i(F, \cdot) : \mathcal{S} \rightarrow \mathbb{R}_+$  defined for each  $\gamma$  in  $\mathcal{S}$  by the formula

$$i(F, \gamma) = \inf_{c \in \gamma} I(F, c) \tag{2.2}$$

where the infimum is taken over all closed curves  $c$  that are in the homotopy class  $\gamma$  and that are made up of a finite concatenation of segments which are either contained in leaves of  $F$  or are transverse to  $F$ , and where  $I(F, c)$  denotes the total mass (with respect to the transverse measure of  $F$ ) of all the sub-segments of  $c$  that are transverse to  $F$ .

In this way, a measured foliation defines an element of  $\mathbb{R}_+^{\mathcal{S}}$ , the set of functions from  $\mathcal{S}$  to  $\mathbb{R}_+$ . Two measured foliations are said to be *measure-equivalent* if they have the same image in  $\mathbb{R}_+^{\mathcal{S}}$ .

The next two theorems summarize some basic results of Thurston that we shall refer to in the rest of the chapter.

**Theorem 2.3** (Thurston, see [19] p. 110). *Two measured foliations are Whitehead-equivalent if and only if they are measure-equivalent.*

Theorem 2.3 gives an embedding of the space  $\mathcal{MF}$  in the function space  $\mathbb{R}_+^{\mathcal{S}}$ , and with this embedding we have the following:

**Theorem 2.4** (Thurston, see [19], p. 117 and 150). *With the topology induced on the space  $\mathcal{MF}$  by its embedding in the function space  $\mathbb{R}_+^{\mathcal{S}}$  equipped with the weak topology,  $\mathcal{MF}$  is homeomorphic to  $\mathbb{R}^{6g-6+2n} \setminus \{0\}$ . The set of elements in  $\mathcal{MF}$  that are of the form  $x\gamma$  with  $x \in \mathbb{R}_+^*$  and  $\gamma \in \mathcal{S}$  (where  $\gamma$  is regarded as an element of  $\mathcal{MF}$  by the embedding defined in (2.1) above) is a dense subset of  $\mathcal{MF}$ . The projectivized space  $\mathcal{PMF}$ , equipped with the quotient topology, is homeomorphic to a  $(6g-7+2n)$ -dimensional sphere  $S^{6g-7+2n}$ , in which the natural image of  $\mathcal{S}$  is dense.*

## 2.2 Conformal structures

We start with a word about atlases.

We shall define a conformal structure and, later on, a hyperbolic structure on  $S$  as an atlas satisfying certain properties. An atlas is a certain collection of local charts satisfying a certain property, and it is possible to form unions of collections of local charts. We shall say that two atlases are *compatible* if their union is an atlas satisfying the required properties. A *maximal* atlas is then a maximal union of compatible atlases.

**Definition 2.5** (Conformal structure). *A conformal structure* (which we shall also call a *conformal atlas*) on  $S$  is a maximal atlas  $\{(U_i, \phi_i)\}_{i \in \mathcal{J}}$  of local charts where for each  $i \in \mathcal{J}$ ,  $U_i$  is an open subset of  $S$  and  $\phi_i$  is a homeomorphism from  $U_i$  onto an open subset of the complex plane  $\mathbb{C}$  satisfying  $\bigcup_{i \in \mathcal{J}} U_i = S$  and such

that for all  $i$  and  $j$  in  $\mathcal{J}$ , the map  $\phi_i \circ \phi_j^{-1}$ , which is called a coordinate change map or transition map, and which is defined on  $\phi_j(U_i \cap U_j)$ , is conformal.

Furthermore, we suppose that each puncture of  $S$  has a neighborhood which is conformally equivalent to a punctured disk in  $\mathbb{C}$ . (Without this condition, the neighborhood of a puncture could also be conformally a cylinder.) We shall sometimes use the word holomorphic instead of conformal. Each pair  $(U_i, \phi_i)$  is called a *holomorphic chart*, and (by abuse of language) the variable  $z \in \phi_i(U_i)$  is called a *holomorphic local coordinate* for the structure. On each domain  $U_i$  of a holomorphic chart, there is a natural orientation induced by the map  $\phi_i$  from the usual orientation of  $\mathbb{C}$ . The transition functions of a conformal atlas, being holomorphic, are orientation preserving. Thus, a conformal structure on  $S$  equips  $S$  with a canonical orientation. We shall always assume that this orientation coincides with the orientation on  $S$  that we started with.

A surface equipped with a conformal structure is also called a *Riemann surface*.

**Example 2.6** (Conformal structure induced by a Riemannian metric). Any Riemannian metric on  $S$  has an underlying conformal structure. This is usually deduced from the existence of the so-called isothermal coordinates, that is, local coordinates  $z$  in which the Riemannian metric can be written as  $ds^2 = \lambda(z)|dz|^2$ , with  $\lambda(z) > 0$ , see [3] p.124–126 and the references therein. Isothermal coordinates have been investigated by Gauss, who proved their existence under some restrictive conditions which have been weakened later on. Conversely, every conformal structure on  $S$  is induced by some Riemannian metric. Furthermore, under our requirement that the Euler characteristic of  $S$  is negative, such a metric can be chosen to be hyperbolic, that is, of constant Gaussian curvature  $-1$ . The existence of such a metric is the “uniformization theorem” for surfaces of negative Euler characteristic, attributed to Klein, Poincaré and Koebe, which is also based on the existence of isothermal parameters, cf. [52] and the references therein. More precisely, the uniformization theorem states that each conformal surface of finite type and of negative Euler characteristic can be realized as a quotient of the hyperbolic plane  $\mathbb{H}^2$  by a discrete group  $\Gamma$  of isometries of  $\mathbb{H}^2$ . The conformal surface is then induced by a hyperbolic metric. The hyperbolic metric is unique up to isotopy, and this makes hyperbolic metrics play a particularly important role in the theory of deformations of conformal structures. We shall use the hyperbolic point of view in Section 3 below.

The connection between a hyperbolic metric and the underlying conformal structure is not easy to handle, and classical Teichmüller theory (that is, the theory based on the techniques developed by Teichmüller) makes little use of hyperbolic geometry. However, there is a class of metrics which is more useful in the conformal theory, because the conformal structures that underly them are convenient to manipulate; these are the singular flat metrics that are defined by pairs of transverse measured foliations on the surface, of which we now recall the definition.

**Example 2.7** (Conformal structure defined by a pair of transverse measured foliations). Let  $F_1$  and  $F_2$  be two transverse measured foliations on  $S$ . Recall from Definition 2.1 that each measured foliation extends to a measured foliation on the closed surface  $\hat{S}$  obtained from  $S$  by filling in the punctures. We require here that at each point of  $\hat{S} \setminus S$ , both  $F_1$  and  $F_2$  extend in the same manner, that is, both of them extend as  $s$ -prong singularities with the same  $s \geq 1$ . The local model for two transverse foliations  $F_1$  and  $F_2$  at a puncture of  $S$  is represented in Figure 6. Such a pair of transverse measured foliations defines a conformal structure on  $S$ , and in fact, a distinguished class of holomorphic parameters, in the following way. We identify the neighborhood of each point on  $S$  which is a nonsingular point of the foliations to a subset of  $\mathbb{C}$  by using a parameter  $z = x + iy \in \mathbb{C}$ , where  $x$  is a variable along the leaves of  $F_1$  and  $y$  is a variable along the leaves of  $F_2$ , with the distance along a leaf of each foliation being measured using the transverse measure of the other foliation. Furthermore, we suppose that the positive orientation of  $x$  followed by the positive orientation of  $y$  coincides with the orientation of the surface  $S$ . The local parameter  $z$  (called a *distinguished* parameter) is defined up to the transformation  $z \mapsto -z$  and up to a translation in  $\mathbb{C}$ . In that way, the leaves of  $F_1$  (respectively  $F_2$ ) are locally defined by the equation  $y = \text{constant}$  (respectively  $x = \text{constant}$ ). The fact that the local parameters  $z = x + iy$  are well-defined at each point up to sign and up to the addition of a complex number implies that the local parameters associated to the various nonsingular points are compatible with each other from the holomorphic point of view, and they define a conformal structure in the complement of the singular points. Now we have to see that this conformal structure extends to the whole surface. Let  $p$  be a singular point of  $F_1$  (or, equivalently,  $F_2$ ) and let  $s$  be the number of prongs at that point. On each small enough disk neighborhood  $V(p)$  of  $p$  in  $S$ , we can define a map  $\phi_p : V(p) \rightarrow \mathbb{C}$  which sends  $p$  to 0 and which at each point of  $V(p) \setminus \{p\}$  coincides with a branch of the map  $z \mapsto z^{2/s}$ ,  $z$  being the distinguished parameter. This map is well-defined up to composition in the range by a rotation of  $\mathbb{C}$  that fixes the origin. The maximal atlas generated by the collection of distinguished local charts at the nonsingular points of  $S$ , together with the maps  $(V(p), \phi_p)$  associated to the various singular points, defines a conformal structure on  $S$ . (We note that the condition we imposed on the way the foliations extend to the punctures of  $S$  ensures that the requirement in Definition 2.5 that each puncture has a neighborhood which is conformally equivalent to a punctured disk in  $\mathbb{C}$  is satisfied.)

Since in each small enough neighborhood of each point in  $S$ , the distinguished parameter in  $\mathbb{C}$  is unique up to addition of a complex constant and to multiplication by  $-1$ , such a parameter defines a metric on that neighborhood, which makes the neighborhood isometric to an open subset of  $\mathbb{C}$  equipped with its Euclidean metric. Thus, if  $Z$  denotes the set of singular points of  $F_1$  and  $F_2$ , the surface  $S \setminus Z$  is equipped with a canonical Euclidean metric. The

leaves of the foliations  $F_1$  and  $F_2$  are geodesic with respect to that metric, and the foliations are mutually orthogonal. Such a metric is called a singular flat metric or a Euclidean metric with cone singularities, the singular points being the cone points.

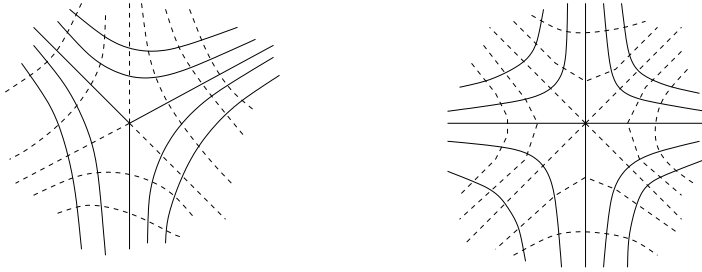


Figure 5. Each picture represents two transverse foliations near a singular point in  $S$ .

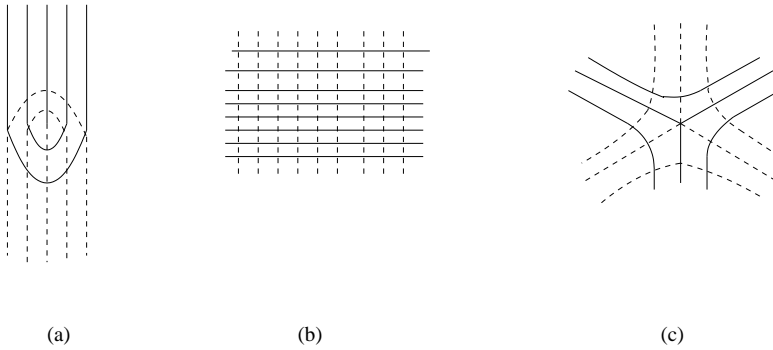


Figure 6. Each picture represents an extension of two transverse foliations at a puncture of  $S$ : in case (a) each foliation extends as a 1-prong singular point, in case (b) it extends as a nonsingular foliation and in case (c) it extends as a 3-prong singular point.

Let  $\text{Hom}(S)$  be the group of orientation-preserving homeomorphisms of  $S$ . We consider the following action of  $\text{Hom}(S)$  on the set of conformal structures on  $S$ : for any maximal atlas  $G = \{(U_i, \phi_i)\}_{i \in \mathcal{J}}$  and for any orientation-preserving homeomorphism  $f : S \rightarrow S$ ,  $f^*G$  is the maximal atlas  $\{(U'_i, \phi'_i)\}_{i \in \mathcal{J}}$  where for each  $i$  in  $\mathcal{J}$ ,  $U'_i = f^{-1}(U_i)$  and  $\phi'_i = \phi_i \circ f|_{U'_i}$ . The maximal atlas  $f^*G$  is then a conformal atlas on  $S$ .

If  $G$  and  $H$  are any two conformal structures on  $S$ , we shall sometimes denote (by abuse of language)  $f : G \rightarrow H$  a map  $f : S \rightarrow S$  in which the domain space (respectively the target space) is the surface  $S$  equipped with the conformal structure  $G$  (respectively  $H$ ).

We shall say that two conformal structures  $G$  and  $H$  are *equivalent* if there exists a homeomorphism  $f : G \rightarrow H$  isotopic to the identity satisfying  $f^*H = G$ .

**Definition 2.8** (Teichmüller space). The *Teichmüller space* of  $S$ , which we shall denote by  $\mathcal{T}_{g,n}$ ,  $\mathcal{T}_{g,n}(S)$ ,  $\mathcal{T}$  or  $\mathcal{T}(S)$ , is the space of equivalence classes of conformal structures on  $S$ .

The subgroup  $\text{Hom}_0(S)$  of  $\text{Hom}(S)$  consisting of the homeomorphisms that are isotopic to the identity is a normal subgroup, and the quotient group  $\mathcal{MCG}(S) = \text{Hom}(S)/\text{Hom}_0(S)$  is called the *mapping class group* of  $S$ . It has a natural action on the Teichmüller space  $\mathcal{T}_{g,n}(S)$ , which is the quotient action of that of  $\text{Hom}(S)$  on the set of maximal conformal atlases defined above.

The set of conformal structures on  $S$  is equipped with a natural topology in which two conformal structures  $G$  and  $H$  are close if we can find two atlases  $\{(U_i, \phi_i)\}_{i \in \mathcal{I}}$  and  $\{(V_j, \psi_j)\}_{j \in \mathcal{J}}$ , where for each  $i \in \mathcal{I}$  (respectively  $j \in \mathcal{J}$ ),  $(U_i, \phi_i)$  (respectively  $(V_j, \psi_j)$ ) is a holomorphic chart for  $G$  (respectively  $H$ ), such that  $\bigcup_{i \in \mathcal{I}} U_i = \bigcup_{j \in \mathcal{J}} V_j = S$  and such that any map of the form  $\phi_i \circ \psi_j^{-1}$  (with the usual convention for its domain of definition) is  $\mathcal{C}^\infty$ -close to the identity. (We are using the fact that a conformal structure on  $S$  defines a canonical  $\mathcal{C}^\infty$ -structure on that surface.) Teichmüller space  $\mathcal{T}_{g,n}(S)$  is equipped with the quotient topology. There are many nice ways of describing that topology, which show that this space is homeomorphic to  $\mathbb{R}^{6g-6+2n}$ . In particular, this topology is induced by the Teichmüller metric which is one of the main subject matters of this chapter (Definition 2.23 below), but we shall also see several other descriptions of this topology.

## 2.3 Moduli and extremal lengths

We shall talk about quadrilaterals and their moduli in Riemann surfaces, but it is natural to start with a few words about triangles, which are simpler objects.

A *topological triangle*  $T$  in a Riemann surface is an embedded closed disk with three distinguished pairwise distinct points on its boundary. Such an object is equipped with a conformal structure, inherited from that of  $S$ . Strictly speaking,  $T$  is equipped with a *conformal structure with boundary*, which is defined as in Definition 2.5 above, except that instead of requiring the range of each homeomorphism  $\phi_i$  to be an open subset of  $\mathbb{C}$ , we require it to be a relatively open subset of the closed half-plane  $\{z \in \mathbb{C} \mid \text{Im}(z) \geq 0\}$ . To



simplify the exposition, we shall suppose that each time we consider a closed disk equipped with a conformal structure and a set of distinguished points on its boundary, these points are taken in a cyclic order which is compatible with the usual orientation on the boundary of the disk that is induced from the orientation of the disk induced from the orientation of the surface. Now any two topological triangles equipped with conformal structures are conformally equivalent, that is, there exists a conformal homeomorphism between the two disks that respects the distinguished points. This follows from the Riemann mapping theorem, which says that any simply connected open subset of the plane bounded by a Jordan curve can be mapped conformally onto the unit disk, that this mapping can be extended to a homeomorphism between the closures of the domains, that the extended homeomorphism is also conformal and that if we choose arbitrarily three distinguished pairwise distinct points in the boundary of the two disks, then the conformal map between the two closed disks can be taken so as to respect the distinguished points, and, finally, that this condition completely determines the map (see [32] Chapter 1).

Thus, topological triangles cannot be used to distinguish different Riemann surfaces, and the next objects of study along that line are topological disks with four distinguished points on their boundary instead of three. These objects are not all conformally equivalent, and they are classified by a conformal invariant called the modulus. In fact, it is sometimes more convenient to deal with closed disks with two disjoint distinguished closed arcs (instead of four distinguished points) on their boundary. The four boundary points of the two arcs will be the distinguished points. Thus, we call a *quadrilateral* in a Riemann surface  $S$  an embedded closed disk with two distinguished disjoint closed arcs in its boundary. We shall call the distinguished arcs the *vertical sides* of the quadrilateral. In the same way as for topological triangles, a quadrilateral in  $S$  is equipped with a conformal structure with boundary induced from that of  $S$ .

**Definition 2.9** (Modulus of a quadrilateral). By the Riemann mapping Theorem, for any quadrilateral  $Q$  in a Riemann surface, there exists a unique positive real number  $\text{Mod}(Q)$  together with a conformal homeomorphism  $\phi$  from  $Q$  to the rectangle  $R$  in the Euclidean plane  $\mathbb{R}^2$  with vertices at  $(0,0)$ ,  $(\text{Mod}(Q),0)$ ,  $(\text{Mod}(Q),1)$  and  $(0,1)$ , such that  $\phi$  sends the vertical sides of  $Q$  to the vertical sides of  $R$  (that is, the sides of length 1, see Figure 7). The value  $\text{Mod}(Q)$  is called the *modulus* of  $Q$  (and of  $R$ ).

**Remarks** 1) Explicit formulae for conformal mappings between quadrilaterals having the same modulus are usually given by means of elliptic integrals (see [32]).

2) Another invariant of quadrilaterals in Riemann surfaces is the cross ratio of the four distinguished points. More precisely, one starts by mapping conformally the quadrilateral on the closed unit disk in the complex plane

(by using the Riemann mapping associated to the interior of the quadrilateral which, as we recalled, extends to the boundary) and then taking the cross ratio of the images of the four distinguished points that we obtain on the boundary of that disk (see [5] p. 343).

3) One can define the modulus of a quadrilateral without using the Riemann mapping theorem (see [32]).

4) After moduli of topological triangles and quadrilaterals, one can study moduli of  $n$ -gons in Riemann surfaces. Here, an  $n$ -gon is defined as a topological disk with  $n$  distinguished points on its boundary. The dimension of the space of moduli (real parameters) of an  $n$ -gon is  $n - 3$ . More generally, Ahlfors and Beurling initiated the study of conformal invariants (or moduli) of arbitrary domains in the plane which are bounded by finitely many closed curves, with a finite number of distinguished points on their boundary and in their interior. The dimension of the moduli space of such a domain is  $3n + 2p + q - 6$ , where  $n$  is the number of boundary components,  $p$  the number of distinguished points in the interior and  $q$  the number of distinguished points on the boundary (see [5] p. 342).

Now back to quadrilaterals.

If  $G$  and  $H$  are Riemann surfaces and if  $f : G \rightarrow H$  is a homeomorphism, then  $f$  transforms any quadrilateral in  $G$  into a quadrilateral in  $H$ . If  $f$  is conformal, it preserves the moduli of quadrilaterals. If  $f : G \rightarrow H$  is a general homeomorphism, then the defect in conformality of  $f$  is measured by a quantity which is called the quasiconformal dilatation of  $f$ , defined as follows.

**Definition 2.10** (Quasiconformal homeomorphism and quasiconformal dilatation). Let  $G$  and  $H$  be two Riemann surfaces and let  $f : G \rightarrow H$  be an

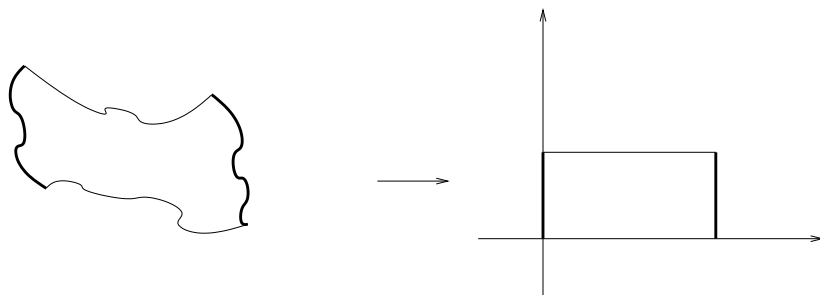


Figure 7. There is a conformal map from the quadrilateral  $Q$  on the left of the picture to the Euclidean rectangle on the right, sending the vertical sides of  $Q$  (which are drawn in bold lines) to the vertical sides of the Euclidean rectangle. The Euclidean rectangle is unique up to isometry, if we take the lengths of its vertical sides to be equal to 1.

orientation-preserving homeomorphism. Then  $f$  is said to be *quasiconformal* if we have

$$K(f) = \sup_Q \frac{\text{Mod}(f(Q))}{\text{Mod}(Q)} < \infty,$$

where the supremum is taken over all quadrilaterals  $Q$  in  $G$ . The value  $K(f)$  is called the *quasiconformal dilatation* of  $f$ . For every  $K \geq K(f)$ ,  $f$  is said to be  $K$ -quasiconformal homeomorphism.

Notice that if  $Q$  is a given quadrilateral and if  $Q'$  is the quadrilateral obtained from  $Q$  by keeping the same topological disk and the same set of distinguished points, but applying to these points an order-one translation of the indices, then we have  $\text{Mod}(Q) = 1/\text{Mod}(Q')$ . With this remark, we can see that the value of  $K(f)$  in Definition 2.10 is always  $\geq 1$ .

Definition 2.10 says in a geometric manner that a map is quasiconformal if it has uniformly bounded distortion. To see that this definition is equivalent to other (more commonly used) analytic definitions of quasiconformal homeomorphisms, we refer the reader to the paper [10] by Bers.

**Remark 2.11.** In the case where  $f$  is a  $\mathcal{C}^1$ -diffeomorphism and where the conformal structures  $G$  and  $H$  are induced by Riemannian metrics, then, for each  $z$  in  $G$ , the differential of  $f$  at  $z$ , being an  $\mathbb{R}$ -linear map, takes a circle centered at the origin in the tangent space  $T_z G$  of  $G$  at  $z$  to an ellipse centered at the origin in the tangent space  $T_{f(z)} H$  of  $H$  at  $f(z)$ . The *(local) quasiconformal dilatation of  $f$  at  $z$* , denoted by  $K_z(f)$ , is defined as the ratio of the major axis to the minor axis of that image ellipse. This ratio does not depend on the choice of the circle in  $T_z G$  centered at the origin that we started with. The quasiconformal dilatation of  $f$  is then equal to

$$K(f) = \sup_{z \in G} K_z(f).$$

Finally, we note that the quasiconformal dilatation of  $f$  at  $z$  is also given by the formula

$$K_z(f) = \frac{\sup\{\|df_z(u)\| \text{ such that } u \in T_z S, \|u\| = 1\}}{\inf\{\|df_z(u)\| \text{ such that } u \in T_z, \|u\| = 1\}}.$$

In this formula, the norm of the tangent vector  $u$  (respectively  $df_z(u)$ ) is measured with respect to the Riemannian metric defining  $G$  (respectively  $H$ ).

The study of quasiconformal dilatations of homeomorphisms between Euclidean rectangles was initiated by H. Grötzsch around 1928, who wrote several papers on that subject. Grötzsch showed that the (real-) affine homeomorphism between two arbitrary Euclidean rectangles realizes the minimum of the quasiconformal dilatation among all homeomorphisms that respect the sides

of these rectangles (see for instance [21]). This result was one of the starting points of the theory of extremal quasiconformal mappings between general Riemann surfaces and in fact the affine map between rectangles is a building block for that general theory. Indeed, it follows from Teichmüller's results (which we shall recall in §2.5 below) that for any two conformal structures  $G$  and  $H$  on  $S$ , one can find two decompositions  $\{R_1, \dots, R_k\}$  and  $\{R_1^*, \dots, R_k^*\}$  of that surface and a homeomorphism  $f : G \rightarrow H$  (which is called the Teichmüller map) such that the following conditions hold:

(1)  $f$  realizes the minimum of the quasi-conformal dilatation in the isotopy class of the identity, and it is the unique map that has this property;

(2) for each  $i = 1, \dots, k$ ,  $R_i$  and  $R_i^*$  are quadrilaterals satisfying  $\bigcup_{i=1}^k R_i =$

$$\bigcup_{i=1}^k R_i^* = S;$$

(3) for each  $i \neq j$ , the interiors of  $R_i$  and  $R_j$  and of  $R_i^*$  and  $R_j^*$  are disjoint;

(4) for all  $i = 1, \dots, k$ , the map  $f$  sends the quadrilateral  $R_i$  to the quadrilateral  $R_i^*$ , and if  $f_i : R_i \rightarrow R_i'$  (respectively  $f_i^* : R_i^* \rightarrow R_i'^*$ ) is the  $G$ - (respectively the  $H$ -) conformal homeomorphism sending  $R_i$  (respectively  $R_i^*$ ) onto a Euclidean rectangle as in Definition 2.9, then, the map  $f_i^* \circ f|_{R_i} \circ f_i^{-1}$  is an affine map;

(5) the quasiconformal dilatations of the affine maps  $f_i^* \circ f|_{R_i} \circ f_i^{-1}$  are equal for all  $i = 1, \dots, k$ .

It is in this sense that the affine map between two rectangles is a basic model for the Teichmüller map between arbitrary Riemann surfaces.

Notice that the local quasiconformal dilatation  $K_z(f)$  of an affine homeomorphism  $f$  between Euclidean rectangles is constant (independent of  $z$ ), and that the local quasiconformal dilatation on each of the quadrilaterals of the decomposition  $\{R_1, \dots, R_k\}$  is the same (independently of the choice of the quadrilateral). This is a basic property of the Teichmüller maps between Riemann surfaces.

**Examples** 1) Consider the surface obtained by doubling a Euclidean rectangle along its open edges (that is, without the vertices). This surface is the four-punctured sphere  $S = S_{0,4}$ , and it is equipped with two transverse measured foliations obtained by gluing the linear vertical (respectively horizontal) foliations of the two rectangles we started with, equipped with the transverse measures that induce Lebesgue measure on the edges of the rectangles. Each of these foliations extends as a one-prong singular point at each of the four punctures. We choose a common  $x$  and  $y$ -coordinate on the sides of the two rectangles. For each  $\lambda > 0$  the real-affine map defined by  $(x, y) \mapsto ((1/\lambda)x, \lambda y)$

is an extremal quasi-conformal map on each rectangle, and the two maps fit together well and define a map on  $S = S_{0,4}$  which is a Teichmüller map.

2) More generally, we can take an arbitrary (simply connected or not) closed subset of the Euclidean plane forming a subsurface with boundary with each boundary curve being made of a finite number of vertical and horizontal segments (see Figure 8 for an example). Doubling this surface, as in Example 1 above, produces a Riemann surface equipped with a canonical product foliation structure (see Definition 2.16), the one induced by the vertical and horizontal measured foliations of the plane. As in the particular case considered above, for each  $\lambda > 0$ , one can easily visualize a Teichmüller map on that surface, as a map induced by an affine transformation of the Euclidean plane. Notice that in all these examples, a singular point of each of the horizontal and vertical measured foliation is either a 1-prong or 3-prong singularity. At a 3-prong singularity, we can close the puncture, that is, we can include in the gluing of the two planar surfaces that we started with the endpoints of the edges that abut on that singular point. However, there are necessarily 1-prong singularities left, and we cannot obtain closed surfaces with this kind of construction. But it is easy to construct closed surfaces equipped with measured foliation pairs (see for instance the examples in the chapter by Herrlich and Schmithüsen in this Handbook [22]).

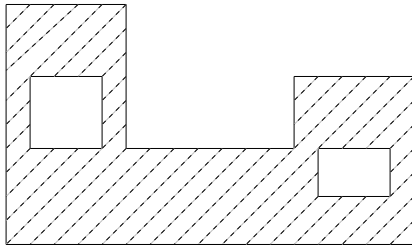


Figure 8. Doubling two such shaded regions along the open edges gives examples of surfaces with punctures equipped with measured foliation pairs

**Definition 2.12** (Modulus of a topological cylinder). Let  $S$  be a Riemann surface and let  $C$  be a *topological cylinder in  $S$* , that is, a surface homeomorphic to  $S^1 \times I$  immersed in  $S$ , with its interior embedded. Such a cylinder is equipped with an induced conformal structure with boundary, and it is conformally equivalent to a Euclidean cylinder  $C^*$  which is unique up to scaling (see [4]). The *modulus* of  $C$  is the height of the Euclidean cylinder  $C^*$  divided by its circumference. It is denoted by  $\text{Mod}_S(C)$  or  $\text{Mod}(C)$ .

**Definition 2.13** (Modulus and extremal length of a homotopy class of curves). Let  $S$  be a Riemann surface and let  $\gamma$  be a homotopy class of essential simple

closed curves in  $S$ . The *modulus* of  $\gamma$ , denoted by  $\text{Mod}_S(\gamma)$  or  $\text{Mod}(\gamma)$ , is the supremum of the moduli of topological cylinders in  $S$  with core curve in the class  $\gamma$ . The *extremal length* of  $\gamma$ , denoted by  $\text{Ext}_S(\gamma)$  or  $\text{Ext}(\gamma)$ , is defined as  $1/\text{Mod}(\gamma)$ .

The notion of extremal length of a family of curves in a Riemann surface was first defined by Beurling and it was developed later on by Beurling and Ahlfors, see [5]. The following result provides examples of Riemann surfaces in which one can see explicitly the maximal modulus cylinder in a given homotopy class.

**Theorem 2.14.** *Let  $S$  be a closed surface equipped with a conformal structure defined by two transverse measured foliations  $F_1$  and  $F_2$  as in Example 2.7. Suppose that  $F_1$  is a foliation with one cylinder, and let  $\gamma$  be the homotopy class of the core curve of that cylinder. Then, the cylinder  $C$  in  $S$  whose interior is the complement of the critical graph of  $F_1$  has the largest modulus among all topological cylinders in  $S$  with core curve in the homotopy class  $\gamma$ . Thus, in this situation, we have  $\text{Mod}(\gamma) = \text{Mod}(C)$  or, equivalently,  $\text{Ext}(\gamma) = 1/\text{Mod}(C)$ .*

Theorem 2.14 is stated (in an equivalent form) by Kerckhoff in [27] (Theorem 3.1) and it is attributed there to J. A. Jenkins and K. Strebel.

There is an extension of this result to the extremal length of an arbitrary measured foliation class instead of a homotopy class of a simple closed curve. This is Theorem 2.21 below and, before stating it, we need to recall Kerckhoff's extension of the notion of extremal length from the set  $\mathcal{S}$  of homotopy classes of simple closed curves to the space  $\mathcal{MF}$  of equivalence classes of measured foliations. (We are using here the inclusion  $\mathcal{S} \subset \mathcal{MF}$  defined in (2.1).)

**Theorem 2.15** (Kerckhoff [27]). *Let  $S$  be a closed surface. The extremal length function defined on the set  $\mathcal{S}$  of homotopy classes of essential simple closed curves in  $S$  extends in a unique way to a continuous function  $\text{ext}_S = \text{ext} : \mathcal{MF} \rightarrow \mathbb{R}_+^*$  satisfying  $\text{ext}(xF) = x^2\text{ext}(F)$  for every  $x$  in  $\mathbb{R}_+^*$  and for every  $F$  in  $\mathcal{MF}$ .*

**Definition 2.16** (Product foliation structure). Let  $\mathcal{Q} = \mathcal{Q}(S)$  be the subset of  $\mathcal{MF} \times \mathcal{MF}$  consisting of pairs of measured foliation classes  $([F_1], [F_2])$  such that  $[F_1]$  and  $[F_2]$  are representable by two foliations  $F_1$  and  $F_2$  that are transverse. We shall call an element of  $\mathcal{Q}$  a *product foliation structure*. The space  $\mathcal{Q}$  is equipped with the topology induced from the weak topology on  $\mathcal{MF} \times \mathcal{MF}$ .

The following lemma will be useful in considerations about product foliation structures.

**Lemma 2.17** (Masur [37]). *Let  $S$  be a closed surface and let  $[F_1]$  and  $[F_2]$  be two measured foliation classes on  $S$  that can be represented by two transverse foliations  $F_1$  and  $F_2$ . Then, the pair  $(F_1, F_2)$  is unique up to isotopy.*

In particular, we cannot perform Whitehead moves on  $F_1$  or on  $F_2$  while keeping the transversality.

**Remark 2.18** (Product foliation structures and quadratic differentials). We must mention that a product foliation structure can be regarded as a Riemann surface equipped with a holomorphic quadratic differential, although we do not make use of this fact in this chapter. We recall that a *holomorphic quadratic differential*  $q$  on a Riemann surface  $S$  is an invariant object that has an expression  $q_i(z)dz^2$  in each holomorphic chart  $(U_i, \phi_i)$ , where  $q_i(z)$  is a holomorphic function of  $z$ , the holomorphic local coordinate in  $\phi_i(U_i)$ . Invariance of  $q$  means that if  $(U_j, \phi_j)$  is another holomorphic chart, if  $w$  is the holomorphic local coordinate in  $\phi_j(U_j)$ , and if a local expression of  $q$  in that chart is  $q_j(w)dw^2$ , then at the overlap between the two charts we have  $q_i(z)dz^2 = q_j(w)dw^2$  or, equivalently,  $q_i(z)(dz/dw)^2 = q_j(w)$ . This invariance property implies in particular that the zeroes of  $q$  and their orders are well-defined. (In other words, the zeroes of the functions  $q_i(z)$  and their orders are independent of the choice of the local chart, see [54] p. 18). Now to each pair  $(F_1, F_2)$  of transverse measured foliations on  $S$  is associated a conformal structure on that surface, as in Example 2.7, with local coordinates  $z = x + iy$  which are well-defined up to sign and up to composition by a translation in  $\mathbb{C}$ . From that we deduce that the local holomorphic differential form  $dz^2$  is the local expression of a well-defined quadratic differential form on  $S$  (that is, the invariance property is satisfied). The *norm* of a quadratic differential is defined as  $\|q\| = \iint |q(z)| dx dy$  (this uses the fact that the area element  $dx dy$  is independent of the choice of the coordinate chart), and the condition that we imposed on the foliations  $F_1$  and  $F_2$  at the punctures of  $S$  (see Definition 2.1) insures that the associated quadratic differential has finite norm. Conversely, to each conformal structure on  $S$  equipped with a holomorphic quadratic differential of finite norm, one can naturally associate a Euclidean metric with cone singularities, and a *horizontal* and a *vertical* foliation defining an element of the space  $\mathcal{Q}(S)$ , these two foliations being orthogonal and their leaves being local geodesics with respect to that Euclidean metric. Let us briefly recall the definitions of these foliations and of the Euclidean structure. Let  $q$  be a nonzero holomorphic quadratic differential on  $S$ , let  $(U_i, \phi_i)$  be a holomorphic chart in  $S$ , and let  $q_i(z)dz^2$  be an expression of  $q$  in that chart. If  $z_0 = \phi_i(x)$  is the image by  $\phi_i$  of a point  $x$  which is a nonzero point of  $q$ , then, since  $q_i(z_0) \neq 0$ , taking if necessary a smaller neighborhood  $U_i$  of  $z_0$ , we may define a branch  $q_i^{\frac{1}{2}}$  of the square root of  $q_i$  on that neighborhood. The integral

$Q_i(z) = \int_{z_0}^z q_i(v)^{\frac{1}{2}} dv$  is then a holomorphic function in  $z$  and it determines a new holomorphic chart for  $S$  at the point  $x$ . The parameter  $w = Q_i(z)$  is called a *distinguished* parameter for  $q$  at the nonzero point  $x$ . In terms of that parameter, the expression of  $q$  is  $dw^2$ . (Remember that the differential  $dw$  is only defined up to sign, but that  $dw^2$  is well-defined.) The distinguished parameter  $w$  is unique up to addition of a complex constant and multiplication by  $-1$ . Therefore, it establishes an isometry between its domain in  $S$  and an open subset of  $\mathbb{C}$  equipped with its Euclidean metric. Thus, if  $Z(q) \subset S$  denotes the set of zeroes of the quadratic differential  $q$ , the surface  $S \setminus Z(q)$  is equipped with a canonical Euclidean structure induced by  $q$ . At a zero of  $q$  of order  $p$ , the Euclidean structure has a cone singularity of cone angle  $(p+2)\pi$ . On  $S \setminus Z(q)$ , the vertical (respectively horizontal) foliation is defined locally by taking a distinguished parameter  $w = u + iv$  and transporting on the domain of this local chart the foliation by vertical (respectively horizontal) lines  $u = \text{constant}$  (respectively  $v = \text{constant}$ ) in  $\mathbb{C}$ . Since the distinguished parameters are unique up to sign and up to translation in  $\mathbb{C}$ , these foliations on the various chart domains match up and give a well-defined vertical (respectively horizontal) foliation on  $S \setminus Z(q)$ . It also follows from this construction that the leaves of these foliations are local geodesics and are orthogonal with respect to the canonical Euclidean structure on  $S \setminus Z(q)$  induced by  $q$ . Of course, this Euclidean structure is the same as the Euclidean structure associated to the pair of transverse measured foliations defined in Example 2.7. For more details on quadratic differentials, we refer the reader to Strebel's book [54].

Now we can define a map

$$p : \mathcal{Q}(S) \rightarrow \mathcal{T}(S) \tag{2.3}$$

by associating to each product foliation structure  $([F_1], [F_2]) \in \mathcal{Q}(S)$  the equivalence class of conformal structures associated as in Example 2.7 to a pair  $(F_1, F_2)$  representing  $([F_1], [F_2])$ . By Lemma 2.17, the pair  $(F_1, F_2)$  is uniquely defined up to isotopy. Therefore, the corresponding element of Teichmüller space is well-defined. The foliation  $F_1$  (respectively  $F_2$ ) is called the *horizontal* (respectively *vertical*) foliation of  $q$ . We shall say indifferently that the conformal structure is represented by the pair  $(F_1, F_2)$  or by the pair  $([F_1], [F_2])$ .

Jenkins and Strebel studied conformal structures defined by pairs of measured foliations, where the vertical foliation has a special property that we state in the following definition:

**Definition 2.19** (Jenkins-Strebel structure). A *Jenkins-Strebel structure* is a product foliation structure  $([F_1], [F_2])$  such that  $[F_2]$  is the equivalence class of a measured foliation whose leaves are all compact.<sup>2</sup>

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<sup>2</sup>Note that the fact that in this definition we talk about the vertical (rather than the horizontal) foliation is just a matter of convention. In comparing the statements here with



Notice that a measured foliation has all of its leaves compact if and only if the subset of the surface consisting of the union of the leaves that start at singular points is a compact subset of the surface. This subset is then a graph, called the *critical graph* of the foliation. The complement of this graph is a finite union of cylinders foliated by parallel leaves. The core curves of these cylinders are essential and pairwise non-homotopic closed curves. The property for a measured foliation to have all its leaves compact, if it holds for a given measured foliation, holds for any equivalent foliation. A particular class of Jenkins-Strebel structures is the class of product foliation structures whose vertical measured foliations are foliations with one cylinder, that we already considered above (Definition 2.2).

Jenkins studied in [26] (what we now call) Jenkins-Strebel structures as solutions of certain extremal problems concerning maps between Riemann surfaces. It follows from his work and from the work of Strebel in [53] that for any Riemann surface  $S$  and for any measured foliation  $F$  on  $S$  whose leaves are all compact, there exists a unique Jenkins-Strebel structure whose vertical foliation is equivalent to  $F$  and whose underlying conformal structure is the given one on  $S$ . The following more general result was obtained later on by Hubbard and Masur:

**Theorem 2.20** (Hubbard & Masur [24]). *For any Riemann surface  $S$  and for any measured foliation  $F$  on  $S$ , there exists a unique product foliation structure whose underlying conformal structure is the one of  $S$  and whose vertical foliation is equivalent to  $F$ . Moreover, the map  $\psi : \mathcal{MF}(S) \times \mathcal{T}(S) \rightarrow \mathcal{Q}(S)$  that associates to any Riemann surface  $S$  and to any measured foliation class  $[F]$  this uniquely defined element of  $\mathcal{Q}(S)$  is a homeomorphism.*

This implies in particular that the map  $p : \mathcal{Q}(S) \rightarrow \mathcal{T}(S)$  defined in (2.3) is surjective.

Theorem 2.20 was proved by Hubbard and Masur in the case where  $S$  is a closed surface, and another proof was given by Kerckhoff [27]. An adaptation for the case of non-closed surfaces is contained in [20].

Theorem 2.20 gives the following parametrization of Teichmüller space:

Given a measured foliation  $F$  on  $S$ , let  $\mathcal{MF}(F)$  be the space of measured foliation classes that are representable by measured foliations transverse to a measured foliation equivalent to  $F$ . For each  $[F]$  in  $\mathcal{MF}$ , we have a map

$$\psi_{[F]} : \mathcal{T} \rightarrow \mathcal{MF}(F) \tag{2.4}$$

obtained by restricting the map  $\psi$  of Theorem 2.20 to the subset  $\{[F]\} \times \mathcal{T}$  of  $\mathcal{MF} \times \mathcal{T}$ . By Theorem 2.20, the map  $\psi_{[F]}$  is a homeomorphism. We regard this map as a parametrization of Teichmüller space by the subset  $\mathcal{MF}(F)$  of

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statements in the papers of some of the papers that we refer to, one has to be careful about the conventions used in those papers.

measured foliation space. The flowlines of the natural action of  $\mathbb{R}_+^*$  on  $\mathcal{MF}$  correspond by this parametrization to images of geodesic lines in Teichmüller space, for the Teichmüller metric that we recall below.

Using Theorem 2.20, it is possible to compute explicitly the extremal length of some particular measured foliation classes, for Kerckhoff's extension of the extremal length function  $\text{ext} : \mathcal{MF} \rightarrow \mathbb{R}_+$  (Theorem 2.15).

**Theorem 2.21** (Kerckhoff [27] p.34). *Let  $S$  be a closed Riemann surface defined by two transverse measured foliations  $F_1$  and  $F_2$  as in Example 2.7. Then,  $\text{ext}([F_1])$  and  $\text{ext}([F_2])$  are equal to the total area of the singular flat metric associated to the pair  $(F_1, F_2)$ .*

Remember that a homotopy class of essential simple closed curves considered as an element of  $\mathcal{MF}$  is represented by a measured foliation with one cylinder whose height is equal to 1 (see (2.1) above). Therefore, if the foliation  $F_1$  in the statement of Theorem 2.21 is a foliation with one cylinder whose height is  $h$ , then we have  $\text{ext}([F_1]) = h^2 \text{Ext}_S(\gamma)$ , which gives by Theorem 2.14  $\text{ext}([F_1]) = h^2 \frac{\ell}{h} = \ell h$ , where  $\ell$  is the length of a closed leaf of the foliation of the cylinder, the length being computed in the singular flat metric defined by the two transverse foliations  $F_1$  and  $F_2$ . This value  $\ell h$  is the total area of that singular flat metric. Thus, the proof of Theorem 2.21 for  $\text{ext}([F_1])$  follows from Theorem 2.14 which contains that result in the particular case where  $F_1$  is a foliation with one cylinder, and from the continuity of the map  $\psi$  of Theorem 2.20. By symmetry, the same result holds for  $\text{ext}([F_2])$ .

Note that in view of Theorem 2.21 and of the quadratic behaviour in Theorem 2.15, extremal length should be considered as an area rather than as a length.

**Remark 2.22** (Area and intersection number). The *geometric intersection function*  $i : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}_+$  is defined by

$$i(\gamma_1, \gamma_2) = \min \text{Card}\{C_1 \cap C_2\},$$

where the minimum is taken over all simple closed curves  $C_1$  and  $C_2$  in the homotopy classes  $\gamma_1$  and  $\gamma_2$  respectively. Thurston showed that this function  $i$  extends as a continuous function  $i : \mathcal{MF} \times \mathcal{MF} \rightarrow \mathbb{R}_+$ , called the geometric intersection function for measured foliations, which extends the map  $i(F, \gamma)$  defined in (2.2) above. If  $F_1$  and  $F_2$  are two transverse measured foliations, then the intersection  $i([F_1], [F_2])$  is equal to the area of the singular flat metric that this pair defines.

## 2.4 The Teichmüller metric

**Definition 2.23** (The Teichmüller metric). Let  $G$  and  $H$  be two conformal structures on  $S$ . The *Teichmüller distance* between  $G$  and  $H$  is given by

$$d_T(G, H) = \frac{1}{2} \inf_f \log K(f)$$

where the infimum is taken over all quasiconformal homeomorphisms  $f : G \rightarrow H$  that are isotopic to the identity. Since the value  $d_T(G, H)$  remains unchanged if we replace  $G$  or  $H$  by an isotopic conformal structure, the map  $d_T$  induces a map on  $\mathcal{T}_{g,n} \times \mathcal{T}_{g,n}$ . This map is a metric, and it is called the *Teichmüller metric*. We shall denote it by  $d_T$ .

The symmetry of the map  $d_T$  follows from the fact that the inverse of a quasiconformal homeomorphism is a quasiconformal homeomorphism with the same quasiconformal dilatation. The triangle inequality follows from the fact that the composition of a  $K_1$ -quasiconformal homeomorphism with a  $K_2$ -quasiconformal homeomorphism is a  $K_1K_2$ -quasiconformal homeomorphism.

From the way we introduced the quasiconformal dilatation of a homeomorphism, Definition 2.23 gives the Teichmüller distance between two Riemann surfaces as a comparison between moduli of quadrilaterals in these surfaces. The following theorem gives a characterization of the Teichmüller distance as a comparison between extremal lengths.

**Theorem 2.24** (Kerckhoff [27] p. 36). *Let  $S$  be a closed surface and let  $G$  and  $H$  be two conformal structures on  $S$ . The Teichmüller distance between  $G$  and  $H$  is equal to*

$$d_T(G, H) = \frac{1}{2} \log \sup_{\gamma \in \mathcal{S}} \frac{\text{Ext}_H(\gamma)}{\text{Ext}_G(\gamma)}.$$

**Remarks 2.25.** 1) By Kerckhoff's extension of the extremal length function to  $\mathcal{MF}$  (Theorem 2.15) and the fact that the set  $\{x\gamma \mid x \in \mathbb{R}_+^*, \gamma \in \mathcal{S}\}$  is dense in  $\mathcal{MF}$  (Theorem 2.4), the Teichmüller distance is also given by

$$d_T(G, H) = \frac{1}{2} \log \sup_{[F] \in \mathcal{MF}} \frac{\text{ext}_H([F])}{\text{ext}_G([F])}, \quad (2.5)$$

and by

$$d_T(G, H) = \frac{1}{2} \log \sup_{[F] \in \mathcal{PMF}} \frac{\text{ext}_H([F])}{\text{ext}_G([F])}. \quad (2.6)$$

Note that to simplify notation, we are using in 2.6 the same notation,  $[F]$ , to denote the equivalence class of  $F$  in  $\mathcal{MF}$  and that in  $\mathcal{PMF}$ .

In order to see that the last expression is meaningful (even though the value  $\text{ext}_H([F])$  is not defined for  $[F]$  in  $\mathcal{PMF}$ ), notice that the ratio  $\frac{\text{ext}_H([F])}{\text{ext}_G([F])}$  in (2.6) is defined by choosing a representative in  $\mathcal{MF}$  of the element  $[F]$  in  $\mathcal{PMF}$  and that the value of this ratio is independent of the choice that we make.

Formula (2.6) is more useful than Formula (2.5) because the space  $\mathcal{PMF}$  is compact, and therefore the supremum is attained.

2) In particular, we have  $\sup_{[F] \in \mathcal{MF}} \frac{\text{ext}_H([F])}{\text{ext}_G([F])} = \sup_{[F] \in \mathcal{MF}} \frac{\text{ext}_G([F])}{\text{ext}_H([F])}$ , and we note right away that the corresponding quotients that define Thurston's asymmetric metric (see Section 3 below) are not equal.

3) Teichmüller's metric is a Finsler metric.

## 2.5 Teichmüller maps and Teichmüller geodesics

**Definition 2.26** (Teichmüller map). A *Teichmüller map* is the identity map  $f : (S, G) \rightarrow (S, H)$ , where  $G$  and  $H$  are two conformal structures on  $S$  that are defined as follows: there exist two transverse measured foliations  $F_1$  and  $F_2$  and a positive real  $\lambda$ , such that  $G$  is the conformal structure associated to the pair  $(F_1, F_2)$  as in Example 2.7 and  $H$  is the structure associated to the pair  $((1/\lambda)F_1, \lambda F_2)$ .

**Remark 2.27.** In terms of the distinguished local coordinate  $z = x + iy$  associated to the conformal structure determined by the pair  $(F_1, F_2)$ , the homeomorphism  $f$  of Theorem 2.26 is defined by

$$(x, y) \mapsto ((1/\lambda)x, \lambda y). \quad (2.7)$$

The local quasiconformal dilatation  $K_z(f)$  of  $f$  is constant on  $S$ , and it is equal to  $\max(\lambda^2, 1/\lambda^2)$ .

If we start with a topological surface  $S$  equipped with a pair  $(F_1, F_2)$  of transverse measured foliations, and a positive real number  $\lambda$ , then we can define an associated Teichmüller map. This map is the identity map on  $S$  equipped with the conformal structure  $(F_1, F_2)$  on the domain and  $((1/\lambda)F_1, \lambda F_2)$  on the target. In other words, we can consider a Teichmüller map as defining a new conformal atlas on the surface  $S$ , obtained by composing each local chart of the conformal structure defined by  $(F_1, F_2)$  with the real-affine map defined, using the distinguished local coordinates, by Formula (2.7) (see Figure 9).

**Theorem 2.28** (Teichmüller [55], [56]). *For any two conformal structures  $G$  and  $H$  on  $S$ , we can find two transverse measured foliations  $F_1$  and  $F_2$  and a*

positive real  $\lambda$  such that  $(F_1, F_2)$  represents the structure  $G$  and  $((1/\lambda)F_1, \lambda F_2)$  represents the structure  $H$ . Furthermore, the Teichmüller map associated to these conformal structures  $G$  and  $H$  is the unique homeomorphism that has the least quasiconformal dilatation among the homeomorphisms between  $G$  and  $H$  that are isotopic to the identity.

**Remark 2.29.** By Theorem 2.28, the least quasiconformal dilatation depends only on the equivalence classes  $g$  and  $h$  of  $G$  and  $H$  respectively.

**Definition 2.30** (Teichmüller line and Teichmüller ray). A *Teichmüller line* (respectively *Teichmüller ray*) is a map  $\ell : \mathbb{R} \rightarrow \mathcal{T}(S)$  (respectively  $\ell : [0, \infty[ \rightarrow \mathcal{T}(S)$ ) defined by  $t \mapsto \ell(t) = (e^{-t}F_1, e^tF_2)$  where  $F_1$  and  $F_2$  are transverse measured foliations on  $S$ . (Here, as usual, we are identifying each pair of transverse measured foliations with the equivalence class of conformal structures that it determines, as in Example 2.7.) We say that the ray  $\ell$  starts at the (equivalence class of the) conformal structure determined by the pair  $(F_1, F_2)$ . By abuse of language, we shall say that  $F_1$  (respectively the measured foliation class, the projective class of  $F_1$ ) is the *horizontal* measured foliation (respectively measured foliation class, the projective class) of the geodesic line (or ray), and that  $F_2$  is its *vertical* measured foliation (respectively measured foliation class, projective class). (In fact, the pair, and therefore, the foliations are defined up to a constant factor.)

Recall that if  $X$  is a metric space, then a *geodesic* (respectively a *geodesic line*, *geodesic ray*) in  $X$  is a distance-preserving map from a compact interval of  $\mathbb{R}$  (respectively from  $\mathbb{R}$ , from an interval of the form  $[a, \infty)$  with  $a \in \mathbb{R}$ ) into  $X$ . If  $\gamma : [a, b] \rightarrow X$  (respectively  $\gamma : [a, \infty) \rightarrow X$ ) is a geodesic (respectively a geodesic ray) in  $X$ , then we say that  $\gamma$  starts at the point  $\gamma(a) \in X$ , and that  $\gamma$  connects the points  $\gamma(a)$  and  $\gamma(b)$  in  $X$ . A *local geodesic* in  $X$  is a map

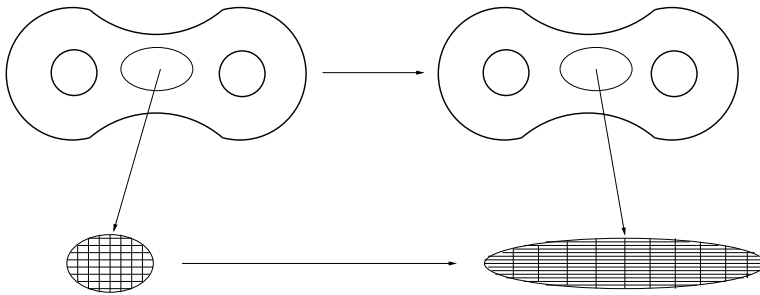


Figure 9. The effect of a Teichmüller map in local coordinates, with  $\lambda > 1$  ( $\lambda$  is the parameter in Definition 2.26).

from an interval  $I$  of  $\mathbb{R}$  into  $X$  such that each point in the interior of  $I$  has a neighborhood such that the restriction of the map to that neighborhood is a geodesic.

We also recall that a metric space  $X$  is said to be *geodesic* if any two points in  $X$  can be connected by a geodesic, and that a metric space is said to be *proper* if its closed balls are compact.

The following is a basic result on the metric structure of Teichmüller space (cf. Kravetz [31]).

**Theorem 2.31** (Teichmüller geodesics). *Teichmüller space  $\mathcal{T}_{g,n}$ , equipped with the Teichmüller metric, is complete, proper and geodesic. Furthermore, each pair of distinct points in  $\mathcal{T}_{g,n}$  is contained in the image of a geodesic line which is unique up to reparametrization. Teichmüller lines are geodesic lines for the Teichmüller metric, that is, for each Teichmüller line  $\ell : \mathbb{R} \rightarrow \mathcal{T}_{g,n}$ , we have, for each  $t_1$  and  $t_2$  in  $\mathbb{R}$ ,  $d_T(\ell(t_1), \ell(t_2)) = |t_2 - t_1|$ . Furthermore, all the bi-infinite geodesic lines for that metric are of this form.*

The following theorem is also well-known. We provide a proof because it follows directly from previously mentioned results.

**Theorem 2.32.** *Let  $S$  be a closed surface, let  $t \in \mathbb{R}$ , let  $F_1$  and  $F_2$  be two transverse measured foliations on  $S$  and let  $G$  and  $H$  be the conformal structures associated respectively to the pair  $(F_1, F_2)$  and  $(e^{-t}F_1, e^tF_2)$ . Then, the Teichmüller distance between  $G$  and  $H$  is equal to  $|t|$ . Furthermore, for  $t > 0$  (respectively  $t < 0$ ), the measured foliation  $F_1$  (respectively  $F_2$ ) realizes the supremum in Formula (2.5) for the Teichmüller distance between the two conformal structures defined by  $G$  and by  $H$ .*

*Proof.* By Remark 2.27, the quasiconformal dilatation of the Teichmüller map between the conformal structures defined by  $(F_1, F_2)$  and  $(e^{-t}F_1, e^tF_2)$  in the domain and target respectively is equal to  $\max(e^{-2t}, e^{2t}) = e^{2|t|}$ . This gives

$$d_T(G, H) = \frac{1}{2} \log e^{2|t|} = |t|.$$

Suppose now that  $t > 0$ . By Theorem 2.21 and Remark 2.22, we have

$$\text{ext}_G(F_1) = i(F_1, F_2)$$

and

$$\text{ext}_H(e^{-t}F_1) = i(e^{-t}F_1, e^tF_2) = i(F_1, F_2).$$

Therefore, we have, by Theorem 2.15

$$\frac{\text{ext}_H(F_1)}{\text{ext}_G(F_1)} = \frac{e^{2t} \text{ext}_H(e^{-t}F_1)}{\text{ext}_G(F_1)} = \frac{e^{2t} i(F_1, F_2)}{i(F_1, F_2)} = e^{2t}.$$

Thus,

$$d_T(G, H) = \frac{1}{2} \log \frac{\text{ext}_H(F_1)}{\text{ext}_G(F_1)} = t.$$

For  $t < 0$ , we can obtain the required result by the same reasoning applied to the Teichmüller map from the structure to the structure  $(e^{-t}F_1, e^tF_2)$  to the structure  $(F_1, F_2)$ .  $\square$

## 2.6 On the asymptotic behaviour of Teichmüller rays.

**Definition 2.33** (Jenkins-Strebel geodesic). A *Jenkins-Strebel geodesic line* (respectively a *Jenkins-Strebel geodesic ray*) is a Teichmüller geodesic line (respectively geodesic ray) defined by  $t \mapsto (e^{-t}F_1, e^tF_2)$  where the vertical measured foliation  $F_2$  has all of its leaves compact.

In other words, a Jenkins-Strebel geodesic ray is a geodesic ray that starts at a Jenkins-Strabel structure (see Definition 2.19).

Recall that if a measured foliation has all its leaves compact, then it consists of a finite union of maximal foliated cylinders with disjoint interiors, each cylinder foliated by homotopic leaves. The homotopy classes of leaves associated to different maximal cylinders are distinct. Thus, to each Jenkins-Strebel geodesic line or ray we can associate a system of homotopy classes of disjoint and pairwise nonhomotopic essential simple closed curves on the surface, which we shall henceforth call the *vertical* system of homotopy classes. In other words, these are the homotopy classes of the nonsingular closed leaves of the associated vertical measured foliation. We shall use the following definition:

**Definition 2.34** (Similar Jenkins-Strebel rays, cf. Masur [36] p.211). Two Jenkins-Strebel geodesic rays are said to be *similar* if the associated vertical system of homotopy classes of closed curves are equal.

**Theorem 2.35** (Masur [36]). *Let  $S$  be a closed surface and let  $r_1 : [0, \infty) \rightarrow \mathcal{T}$  and  $r_2 : [0, \infty) \rightarrow \mathcal{T}$  be two similar Jenkins-Strebel rays starting at the same point. Then there exists  $N > 0$  such that  $d_T(r_1(t), r_2(t)) \leq N$  for all  $t \in [0, \infty)$ .*

In his paper [27], Kerckhoff studied the relative asymptotic behaviour of pairs of Jenkins-Strebel rays that do not necessarily start at the same point. To state a result, we need the following

**Definition 2.36** (Modularly equivalent Jenkins-Strebel rays, cf. Kerckhoff [27] p. 29). Consider two similar Jenkins-Strebel geodesic rays  $r_1 : [0, \infty) \rightarrow \mathcal{T}$  and  $r_2 : [0, \infty) \rightarrow \mathcal{T}$  and let  $\gamma_1, \dots, \gamma_n$  be their associated vertical system of homotopy classes of simple closed curves. For every  $t \in [0, \infty)$ , we consider the

sequences of moduli  $m_1, \dots, m_n$  (respectively  $m'_1, \dots, m'_n$ ) of the homotopy classes  $\gamma_1, \dots, \gamma_n$ , defined with respect to the Riemann surface  $r_1(t)$  (respectively  $r_2(t)$ ). We say that the two rays  $r_1$  and  $r_2$  are *modularly equivalent* if these two sequences of moduli are the same up to a multiplicative constant. (Note that if this condition holds for some  $t > 0$ , then it holds for any  $t > 0$ .)

Unlike the property of similarity between pairs of Jenkins-Strebel rays (Definition 2.34), modular equivalence is a property that involves both the vertical and horizontal foliations of the rays.

**Theorem 2.37** (Kerckhoff, [27] p. 29). *Let  $S$  be a closed surface and let  $r_1 : [0, \infty) \rightarrow \mathcal{T}$  and  $r_2 : [0, \infty) \rightarrow \mathcal{T}$  be two similar Jenkins-Strebel rays (that do not necessarily start at the same point) whose vertical system of homotopy classes of simple closed curves have  $3g - 3$  elements (which is the largest possible number of elements such a system can have). Then, the following are equivalent:*

- (1)  $r_1$  and  $r_2$  are modularly equivalent;
  - (2)  $\lim_{t \rightarrow \infty} \inf_{M = \text{Im}(r_2(t))} d(M, \text{Im}(r_1)) = 0$ .
- (Here,  $\text{Im}$  denotes the image set of a map.)

## 2.7 The Teichmüller boundary and convergent rays

The following definition is contained in Kerckhoff's paper [27].

**Definition 2.38** (The Teichmüller boundary). Let  $M$  be a point of  $\mathcal{T}$ . The *Teichmüller boundary of  $\mathcal{T}$  relative to the basepoint  $M$* , denoted by  $\partial_M \mathcal{T}$ , is the set of geodesic rays (for the Teichmüller metric) starting at  $M$ , equipped with the topology of uniform convergence on compact sets.

Kerckhoff studied a compactification of Teichmüller space obtained by taking a basepoint  $M$  and putting an endpoint at each Teichmüller ray starting at that point. The result is a topological space  $\overline{\mathcal{T}}_M = \mathcal{T} \cup \partial_M \mathcal{T}$  which is homeomorphic to a closed  $(6g - 6 + 2n)$ -dimensional ball, equipped with a natural center (the point  $M$ ) and a ray structure (i.e. an action of  $\mathbb{R}_+^*$ ), in which the closure of the image of each ray starting at  $M$  is the union of that image with a single point in  $\partial_M \mathcal{T}$  (see [27] p. 31). Kerckhoff calls such a compactification of  $\mathcal{T}$  a *Teichmüller compactification* (the name referring to Teichmüller's metric and not only to Teichmüller space), and one of the main objects of the paper [27] is the study of the dependence of such a compactification on the basepoint. For instance, a natural question which Kerckhoff addresses is whether a Teichmüller ray, starting at some point other than the basepoint, has also a unique limit point in  $\partial_M \mathcal{T}$ . He makes the following definition:



**Definition 2.39** (Convergent rays, cf. Kerckhoff [27] p.28). Let  $M$  be a basepoint in Teichmüller space  $\mathcal{T}$  and consider the corresponding Teichmüller boundary  $\partial_M \mathcal{T}$ . Let  $r' : [0, \infty) \rightarrow \mathcal{T}$  be a geodesic ray starting at some other point  $M'$  in  $\mathcal{T}$ , and let  $\overline{\text{Im}(r')}$  be its closure in  $\overline{\mathcal{T}}_M$ . Then,  $r'$  is said to be *convergent* in  $\overline{\mathcal{T}}_M$  if  $\overline{\text{Im}(r')} \setminus \text{Im}(r')$  consists of a single point. In that case,  $\overline{\text{Im}(r')} \setminus \text{Im}(r')$  is the endpoint of some geodesic ray  $r$  starting at  $M$ , and the ray  $r'$  is said to be *convergent* to the ray  $r$ .

For any element  $\gamma$  in  $\mathcal{S}$  and for any  $M$  in  $\mathcal{T}$ , we know, by Theorem 2.20, that there exists a unique Jenkins-Strebel ray  $r_{M,\gamma}$  that starts at  $M$  and whose vertical foliation has one cylinder, with core curve in the homotopy class  $\gamma$ .

**Theorem 2.40** (Kerckhoff [27] p.29). *Let  $S$  be a closed surface. Then, for any homotopy class  $\gamma$  of essential simple closed curves and for any  $M$  and  $M'$  in  $\mathcal{T}$ , the Jenkins-Strebel ray  $r_{M,\gamma}$  is convergent to the Jenkins-Strebel ray  $r_{M',\gamma}$ .*

**Theorem 2.41** (Kerckhoff, [27] p.29). *Let  $S$  be a closed surface and let  $r_1 : [0, \infty) \rightarrow \mathcal{T}$  and  $r_2 : [0, \infty) \rightarrow \mathcal{T}$  be two similar Jenkins-Strebel rays with vertical system of homotopy classes of curves having the largest possible number of elements (that is,  $3g - 3$  elements). Then,  $r_1$  and  $r_2$  are convergent if and only if they are modularly equivalent.*

## 2.8 Convergence to Thurston's boundary

A natural question to consider is whether a Teichmüller ray converges to a point on Thurston's boundary  $\mathcal{PMF}$  of Teichmüller space. We have postponed the definition of Thurston's boundary to Section 3 below (§3.12) since its natural setting is hyperbolic geometry, but we nonetheless state here a result of Masur about convergence of Teichmüller rays to points on that boundary:

**Theorem 2.42** (Limits of Jenkins-Strebel rays, Masur [39]). *Let  $S$  be a closed surface, let  $r : [0, \infty) \rightarrow \mathcal{T}$  be a Jenkins-Strebel ray and let  $\alpha_1, \dots, \alpha_n$  be the associated vertical system of homotopy classes. Let  $F = F(\alpha_1, \dots, \alpha_n)$  be a measured foliation whose nonsingular leaves are all compact, such that the complement of the critical graph of  $F$  consists of  $n$  cylinders whose core curves are in the classes  $\alpha_1, \dots, \alpha_n$  and such that the height of each of these cylinders is equal to one. Then, the ray  $r$  converges to the projective class  $[F]$ , considered as a point in Thurston's boundary of Teichmüller space.*

It follows from Theorem 2.42 that there are pairs of points on Thurston's boundary that are limits of a large family of Teichmüller geodesics. Indeed, let  $F_1$  and  $F_2$  be two transverse measured foliations whose nonsingular leaves

are all compact and such that for each of these foliations, the height of each foliated cylinder in the complement of the critical graph is equal to 1. One can see that such pairs exist from the fact that for each element  $[F]$  in  $\mathcal{MF}$ , the subset  $\mathcal{MF}(F)$  of equivalence classes representable by measured foliations transverse to a representative of  $[F]$  is open, therefore, by the minimality of the action of the mapping class group on  $\mathcal{PMF}$  ([19] p.117), this open set contains a conjugate of the equivalence class of any given measured foliation. By the same argument, for any  $1 \leq m \leq 3g - 3$  and  $1 \leq n \leq 3g - 3$ , we can also choose  $F_1$  (respectively  $F_2$ ) to have its nonsingular closed leaves in  $m$  (respectively  $n$ ) distinct homotopy classes. Let  $c_1, \dots, c_m$  (respectively  $c'_1, \dots, c'_n$ ) be these homotopy classes. Let  $\overline{F_1}$  (respectively  $\overline{F_2}$ ) be now an arbitrary measured foliation having all of its nonsingular leaves compact and with associated homotopy classes  $c_1, \dots, c_m$  (respectively  $c'_1, \dots, c'_n$ ). By Theorem 2.42, the Teichmüller geodesic line  $t \mapsto (e^{-t}\overline{F_1}, e^t\overline{F_2})$  has  $[F_1]$  and  $[F_2]$  as its limit points. Note that if the projective equivalence classes of two such pairs  $(\overline{F_1}, \overline{F_2})$  and  $(\overline{F'_1}, \overline{F'_2})$  are distinct, then, by Teichmüller's uniqueness theorem, the images of the two geodesics  $t \mapsto (e^{-t}\overline{F_1}, e^t\overline{F_2})$  and  $t \mapsto (e^{-t}\overline{F'_1}, e^t\overline{F'_2})$  have at most one common point. Thus, we have a large family of Teichmüller geodesics whose limit points are  $[F_1]$  and  $[F_2]$ .

We recall that a measured foliation  $F$  is said to be *uniquely ergodic* if  $F$ , as a topological foliation, carries a unique transverse measure up to a multiplicative factor. The property of being uniquely ergodic is invariant by Whitehead-equivalence and by homothety. Therefore one can talk about uniquely ergodic projective classes of measured foliations.

We shall also use the following terminology from [19]: a measured foliation  $F$  is said to be *arational* if it does not contain any closed curve made up of segments connecting singular points. Equivalently, up to performing Whitehead moves on  $F$ , every leaf of  $F$  is dense in the surface.

**Theorem 2.43** (Limits of Teichmüller rays with uniquely ergodic vertical foliations, Masur [39]). *Let  $S$  be a closed surface. Then, any Teichmüller ray whose associated vertical foliation is arational and uniquely ergodic converges in the positive direction to the projective equivalence class of that vertical foliation, considered as an element of Thurston's boundary of Teichmüller space.*

There is a natural Lebesgue measure on the space  $\mathcal{MF}$ , which is provided, for instance, by train track coordinates (see [60] and [49]). By a result which was proved independently by Masur and by Veech (see [38] and [66]), the subspace of  $\mathcal{MF}$  consisting of equivalence classes of uniquely ergodic measured foliations is of full measure with respect to that measure. This measure on  $\mathcal{MF}$  defines a measure class on the set of geodesic rays starting at any given point  $M$  of Teichmüller space. This can be seen by referring to the result of Hubbard and Masur stated in Theorem 2.20 which says that for any element  $M \in \mathcal{T}$  and

for any measured foliation class  $[F_1] \in \mathcal{MF}$ , there exists a unique measured foliation class  $[F_2] \in \mathcal{MF}$  such that the conformal structure defined by the pair  $(F_1, F_2)$  is  $M$ . Thus, the pair  $(F_1, F_2)$  can be regarded as a direction in Teichmüller space, at the point  $M$ , that is, the direction of the ray defined by  $t \mapsto (e^{-t}F_1, e^tF_2)$  starting  $M$ . In this sense, Theorem 2.43 says that for any point  $M$  in Teichmüller space, almost any geodesic starting at  $M$  converges to a point in Thurston's boundary, which is the equivalence class of the vertical foliation of that ray.

In a recent preprint, A. Lenzhen gave an explicit example of a Teichmüller ray that does not have a limit in  $\mathcal{PMF}$ . We state this as

**Theorem 2.44** (Teichmüller rays with no limit in  $\mathcal{PMF}$ , Lenzhen [33]). *On the closed surface of genus 2, there exists a Teichmüller ray that does not converge to any point in Thurston's boundary  $\mathcal{PMF}$ .*

The construction of that ray is based on an arithmetic property of the measured foliations associated to the geodesic ray considered.

The asymptotic behaviour of an arbitrary Teichmüller geodesic ray in terms of the dynamical/topological properties of its vertical foliation is still an open question.

## 2.9 On the visual boundary of the Teichmüller metric

We start by recalling a general definition.

**Definition 2.45** (Asymptotic geodesic rays). Let  $(X, d)$  be a metric space. Two geodesic rays  $r_1 : [a_1, \infty) \rightarrow X$  and  $r_2 : [a_2, \infty) \rightarrow X$  are said to be *asymptotic* if the function defined on  $[0, \infty)$  by  $t \mapsto d(r_1(t + a_1), r_2(t + a_2))$  is bounded.

It is clear that the relation of being asymptotic is an equivalence relation on the set of geodesic rays in a metric space. We shall write  $r_1 \sim r_2$  to say that the two geodesic rays  $r_1$  and  $r_2$  are asymptotic.

**Definition 2.46** (The visual boundary). Let  $X$  be a proper geodesic metric space and let  $x$  be a point in  $X$ . The *visual boundary of  $X$  at  $x$* , denoted by  $\partial_{vis,x}X$ , is the set of equivalence classes (for the relation  $\sim$ ) of geodesic rays starting at  $x$ , equipped with the quotient of the topology of convergence on compact sets.

Note that the visual boundary at a point in Teichmüller space equipped with the Teichmüller metric is a quotient of the Teichmüller boundary relative to that point.

In the paper [42], based on Masur's result stated as Theorem 2.35 above, the following is proved

**Theorem 2.47** (McCarthy-Papadopoulos [42]). *Let  $S$  be a closed surface. For any point  $x$  in the Teichmüller space  $\mathcal{T}(S)$  equipped with the Teichmüller metric, the visual boundary  $\partial_{vis,x}\mathcal{T}(S)$  is not Hausdorff.*

This result was used to give a new proof of the fact due to Masur and Wolf that Teichmüller space equipped with the Teichmüller metric is not Gromov hyperbolic cf. [40].

### 3 Thurston's asymmetric metric

#### 3.1 Hyperbolic structures

**Definition 3.1** (Hyperbolic structure). A *hyperbolic structure* on  $S$  is a maximal atlas  $\{(U_i, \phi_i)\}_{i \in \mathcal{J}}$  where for each  $i \in \mathcal{J}$ ,  $U_i$  is an open subset of  $S$  and  $\phi_i$  is a homeomorphism from  $U_i$  onto an open subset of the hyperbolic plane  $\mathbb{H}^2$ , satisfying  $\bigcup_{i \in \mathcal{J}} U_i = S$  and such that any map of the form  $\phi_i \circ \phi_j^{-1}$  is, on each connected component of  $\phi_j(U_i \cap U_j)$ , the restriction of an orientation-preserving isometry of  $\mathbb{H}^2$ .

A surface equipped with a hyperbolic structure is called a *hyperbolic surface*. It carries a length metric defined as follows. On each chart domain  $U_i$ , we pull-back by the map  $\phi_i$  the metric on  $\phi_i(U_i)$  induced from its inclusion in  $\mathbb{H}^2$ . These metrics on the various sets  $U_i$  give a consistent way of measuring lengths of paths in  $S$ , and the metric we consider on  $S$  is the associated length metric. It is a Riemannian metric of constant curvature  $-1$ . The metric universal covering of  $S$  is a subset of hyperbolic plane  $\mathbb{H}^2$ .

If we regard the various  $\phi_i$ 's as diffeomorphisms (instead of homeomorphisms), then we can pull-back the Riemannian metrics on  $\phi_i(U_i)$  to the various sets  $U_i$ . This is another way of defining the Riemannian metric on  $S$ . In any case, a hyperbolic structure is also regarded as a metric on  $S$ , called a *hyperbolic metric*. We shall only consider hyperbolic metrics that are complete and of finite area. Equivalently, these are the metrics that have the property that each puncture of  $S$  has a neighborhood which is isometric to a *cusp*, that is, the quotient of a subset  $\{x + iy \mid y > a\}$  ( $a > 0$ ) of the upper half-plane model of  $\mathbb{H}^2$  by the map  $z \mapsto z + 1$ . Equivalently, these are the metrics for which the metric universal cover is  $\mathbb{H}^2$  (see [65]).

The group  $\text{Hom}(S)$  of orientation preserving homeomorphisms of  $S$  acts on the set of hyperbolic structures in the same way as it acts on the set of con-

formal structures (cf. Definition 2.8), and, again, we say that two hyperbolic structures are *equivalent* if they are related by a homeomorphism of  $S$  which is isotopic to the identity.

There is a natural one-to-one correspondence between the set of equivalence classes of hyperbolic structures and the set of equivalence classes of conformal structures on  $S$ . This correspondence is given by the quotient of the map which assigns to each hyperbolic structure its underlying conformal structure (Example 2.6 above). The fact that the hyperbolic structure is complete and of finite area implies that the neighborhood of each puncture is conformally a punctured disk (with respect to conformal structure associated to that hyperbolic metric). Using this natural correspondence, one can make the following definition, which is an equivalent form of Definition 2.8 above:

**Definition 3.2** (Teichmüller space). The *Teichmüller space of  $S$* , denoted by  $\mathcal{T}_{g,n}(S)$  or  $\mathcal{T}(S)$ , is the space of equivalence classes of complete finite area hyperbolic structures on  $S$ .

By abuse of language, we shall often call an element of Teichmüller space a “hyperbolic structure” instead of an equivalence class of hyperbolic structures. We shall also use interchangeably the terms “hyperbolic structure” and “hyperbolic metric”.

The topology of  $\mathcal{T}(S)$  can be defined using several metrics. For instance, it is induced by the “quasi-isometry” metric  $d_{qi}$  given by

$$d_{qi}(g, h) = \frac{1}{2} \log \inf \{K\} \quad (3.1)$$

where the infimum is taken over the set of real numbers  $K \geq 1$  such that there exists a  $K$ -quasi-isometry between  $g$  and  $h$ , that is, a map  $f : S \rightarrow S$  satisfying

$$\frac{1}{K} d_g(x, y) \leq d_h(f(x), f(y)) \leq K d_g(x, y)$$

(see Thurston [65] p. 266).

We can also describe the topology of  $\mathcal{T}(S)$  by means of the *length functional*  $l : \mathcal{T}(S) \rightarrow \mathbb{R}_+^{\mathcal{S}}$  defined by

$$g \mapsto l_g(\cdot) : \alpha \mapsto l_g(\alpha), \quad \forall \alpha \in \mathcal{S}, \quad (3.2)$$

with  $l_g(\alpha)$  being the length of the unique  $g$ -geodesic in the homotopy class  $\alpha$ . This map  $l$  is an embedding, and the topology of  $\mathcal{T}(S)$  induced by the metric defined in (3.1) coincides with the one induced on the image of  $l$  by the weak topology on the space  $\mathbb{R}_+^{\mathcal{S}}$ . Thus, a sequence  $(g_n)_{n \geq 0}$  in  $\mathcal{T}(S)$  converges as  $n \rightarrow \infty$  to an element  $g \in \mathcal{T}(S)$  if and only if for every  $\alpha$  in  $\mathcal{S}$ , we have  $l_{g_n}(\alpha) \rightarrow l_g(\alpha)$  as  $n \rightarrow \infty$ . In fact, more is true: there exists a finite set  $\{\alpha_1, \dots, \alpha_k\}$  of homotopy classes of simple closed curves in  $S$  such that an arbitrary sequence  $(g_n)_{n \geq 0}$  in  $\mathcal{T}(S)$  converges to an element  $g$  in that space

if and only if for every  $i = 1, \dots, k$  we have  $l_{g_n}(\alpha_i) \rightarrow l_g(\alpha_i)$  as  $n \rightarrow \infty$ . (For a proof in the case of closed surfaces, see [19].) Finally, let us note that this topology on  $\mathcal{T}(S)$  is the one induced by Teichmüller's metric, and also by Thurston's asymmetric metric, which will be the main object of study in this section (see §3.7 below).

### 3.2 Decompositions by generalized pairs of pants and by ideal triangles

By studying the representation of conformal structures on a surface  $S_{g,n}$  by hyperbolic structures one can easily get an intuition of the fact that its Teichmüller space is homeomorphic to an open ball of dimension  $6g - 6 + 2n$ , a fact that is harder to conceive when one starts with the purely conformal point of view (that is, Definition 2.8). The statement that the space of deformations of equivalence classes of conformal structures on a closed orientable surface of genus  $g \geq 2$  is of complex dimension  $3g - 3$  was made by G. F. B. Riemann (without giving a formal proof) in his paper *Theorie der Abel'schen Functionen*, Crelle J., B. 54 (1857). A homeomorphism  $\mathcal{T}_{g,n} \simeq \mathbb{R}^{6g-6+2n}$ , if one does not use hyperbolic geometry, can be obtained using Teichmüller's results. But when the surface  $S_{g,n}$  is equipped with a hyperbolic structure, it is easy to produce such a homeomorphism using a decomposition of the surface into objects we call "generalized hyperbolic pairs of pants" with geodesic boundary. Another way to realize that homeomorphism, in the case where the set of punctures of  $S$  is not empty, can be achieved by using ideal triangulations. Our aim in this section is to discuss these two sorts of decompositions.

We recall that a *hyperbolic pair of pants* is a sphere with three open disks removed, equipped with a hyperbolic metric for which the three boundary components are geodesic.<sup>3</sup> A *degenerate hyperbolic pair of pants* is a complete finite area hyperbolic surface which is either a sphere with three cusps, or a sphere with two cusps and one open disk removed, or a sphere with one cusp and two open disks removed, and where the boundary components, whenever they exist, are closed geodesics (see Figure 10). There is a precise sense in which any degenerate hyperbolic pair of pants is a limit of a family of hyperbolic pairs of pants. This can be seen from the construction of hyperbolic pairs of pants using right-angled hexagons, which we now briefly recall.

The classification of hyperbolic structures on a pair of pants with geodesic boundary is usually done by decomposing the pair of pants into two isometric right-angled hexagons. The isometry class of a right-angled hexagon is completely determined by the lengths of any three alternating edges. For an

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<sup>3</sup>In talking here about curves in the surface  $S$ , it should be noted that the term *geodesic* is used in the sense of Riemannian geometry, that is, in the sense of (the image of) a locally isometric map, as opposed to the other possible use of that word, denoting an isometric map.

exposition of this fact, we refer the reader to Thurston's book [65] (Exercise 2.4.11, Example 4.6.7 and Figure 4.15).

Consider a right-angled hexagon in hyperbolic space  $\mathbb{H}^2$ . If we make the length of one of its edges tend to zero while keeping fixed the other two which form with the degenerating edge an alternating triple of edges, and keeping all the angles to be right angles, the vertices of the degenerating edge converge to a single point in the boundary of hyperbolic space, and the right-angled hexagon becomes, at the limit, an object we call a *degenerate right-angled hexagon* (see Figure 11 for a picture of such a hexagon). Likewise, we can make (successively) the lengths of two or of three alternating edges tend to zero. We obtain in this manner degenerate right-angled hexagons with one, two or three edges at infinity, called *spikes*. Note that a degenerate right-angled hexagon with three edges at infinity is a hyperbolic ideal triangle (that is, the convex hull, in the hyperbolic plane  $\mathbb{H}^2$ , of three distinct points in the boundary of that plane). A degenerate right-angled hexagon with one, two or three edges at infinity is completely determined up to isometry by the lengths of the edges that make with the one (or the ones) at infinity a triple of alternating edges. Since a hyperbolic pair of pants is obtained by gluing two isometric right-angled hexagons along three alternating edges, it is easy to see that the isometry type of the pair of pants is completely determined by the lengths of its three boundary geodesics. Likewise, a hyperbolic structure on a degenerate pair of pants is completely determined by the lengths of its boundary geodesic curves, a cusp being considered as a boundary curve of length zero. In this sense, the hyperbolic structure is rigid at the cusps. For instance, a degenerate right-angled hexagon with three alternating edges at infinity is a hyperbolic ideal triangle, and, as is well-known, there is a unique hyperbolic ideal triangle up to isometry. In the same way, any two degenerate hyperbolic pairs of pants that are spheres with three cusps are isometric. We shall call a *generalized hyperbolic pair of pants* either a hyperbolic pair of pants in the usual sense, or a degenerate hyperbolic pair of pants with one, two or three cusps.

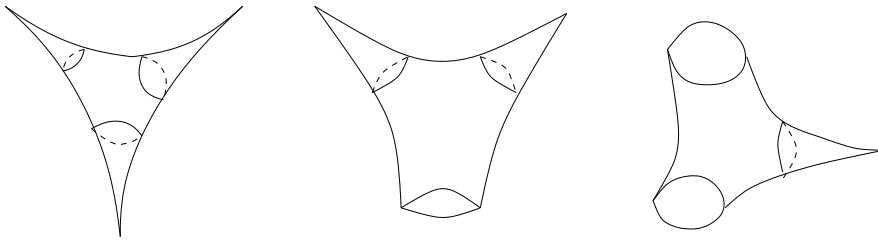


Figure 10. Degenerate hyperbolic pairs of pants.

A closed surface of genus  $g \geq 2$  is decomposable into  $2g - 2$  pairs of pants, by using  $3g - 3$  simple closed curves (see [19]). Using this fact, we can count the number of curves in a generalized pair of pants decomposition of a surface of genus  $g$  with  $n$  cusps by first replacing each cusp (if such a cusp exists) by a boundary closed curve, obtaining a surface with boundary that is homeomorphic to a closed surface of genus  $g$  with  $n$  open disks deleted. Doubling that surface along its boundary curves, we obtain a closed surface of genus  $2g + n - 1$ , which we can decompose into pairs of pants by using  $6g - 6 + 3n$  closed geodesics. By symmetry, we conclude that the number of closed curves needed to decompose the surface  $S_{g,n}$  into generalized pairs of pants is  $3g - 3 + n$ .

Now we can count the parameters for the Teichmüller space of a surface of genus  $g$  with  $n$  punctures. Consider a set of closed geodesics decomposing that surface into generalized hyperbolic pairs of pants. There are  $3g - 3 + n$  such geodesics. There are two parameters associated to each geodesic, one parameter (in  $\mathbb{R}_+^*$ ) being the length of the geodesic itself, and the other parameter (in  $\mathbb{R}$ ) describing the twist along that curve, when we glue together its two sides. (The twist parameter is defined up to the choice of an origin corresponding to the zero twist, and up to an orientation of the boundary component.) This gives two sets of  $3g - 3 + n$  real parameters each. Thus, we have  $6g - 6 + 2n$  parameters, which is an indication of the fact that Teichmüller space is homeomorphic to  $\mathbb{R}^{6g-6+2n}$ . For a geometric and concise proof in the case where  $n = 0$ , we refer the reader to Thurston [65] p.271. There are also proofs of that fact in [1, 8, 19, 25]. These length-twist parameters associated to a (generalized) pair of pants decomposition are called (generalized) *Fenchel-Nielsen parameters*.

Another way of obtaining a homeomorphism  $\mathcal{T}_{g,n} \simeq \mathbb{R}^{6g-6+2n}$  uses a decomposition of  $S_{g,n}$  into ideal triangles, and we now describe it in the case where  $n \geq 1$ . Recall that any hyperbolic ideal triangle has a well-defined center (its center of gravity), and one distinguished point on each of its edges, which is the orthogonal projection on that edge of the center of gravity, or, equivalently, the orthogonal projection on, that edge of the ideal vertex that is opposite to that edge. An *ideal triangulation*  $\lambda$  of  $S = S_{g,n}$  is a decomposition of the associated closed surface  $\hat{S}$  into triangular cells whose vertices are all at the punctures. If  $S$  is equipped with a hyperbolic structure, then one can make the edges of  $\lambda$  to be embedded bi-infinite local geodesics with limit points at the punctures. (The bi-infinite local geodesics are, by definition, embedded images of the bi-infinite geodesic lines in  $\mathbb{H}^2$  realized as the metric universal covering of the surface.) In this way, each face of  $\lambda$  is isometric to the interior of a hyperbolic ideal triangle, and the hyperbolic structure on  $S$  is determined by the gluing maps between the edges of these ideal triangles. In this gluing, there is one parameter associated to each edge of  $\lambda$ , which we can take to be the algebraic distance, called the *shift parameter*, between the two distinguished points on that edge, each distinguished point being asso-



ciated to one of the triangles adjacent to that edge. (Recall that using the orientation on the surface  $S$ , one can define a notion of a left and of a right shift along the edges of  $\mu$ . The sign of the distance between the distinguished points is then defined accordingly.) Thus, the Teichmüller space  $\mathcal{T}_{g,n}$  of  $S_{g,n}$  is seen from that point of view as the set of shift parameters on the edges of an ideal triangulation  $\lambda$  of that surface, and a rough computation of the dimension of  $\mathcal{T}_{g,n}$  can be done in the following way. Let  $E$  denote the number of edges of  $\lambda$  and let  $T$  be the number of its faces (triangles). Since each triangle has 3 boundary edges and since each edge is adjacent to 2 triangles, we have  $E = 3T/2$ . The Euler characteristic of  $S_{g,n}$  is  $\chi = 2 - 2g - n$ . We construct an auxiliary foliation  $F$  on the surface  $\hat{S}$  obtained from  $S_{g,n}$  by filling in the punctures, such that  $F$  is transverse to the edges of the triangulation  $\lambda$ , with a 3-prong singularity in the interior of each triangle, and with a center-type singularity at each puncture. There is an index formula relating the sum of the indices at the singular points of such a foliation to the Euler characteristic  $\chi$  of the surface (see for instance [19] p. 75). The formula says that  $2\chi$  is equal to the sum of the indices of the singular points of the foliation. The index of a 3-prong singularity is  $-1$  and the index of a center-type singularity is  $0$ . Therefore,  $-2\chi$  is equal to the number of 3-prong singularities, which is the number of triangles. In other words, we have  $T = -2\chi$ . Thus, we obtain

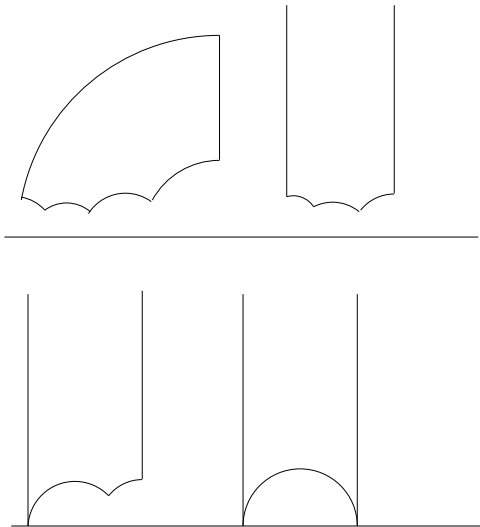


Figure 11. These four generalized polygons represent respectively a right-angled hexagon and three degenerate right-angled hexagons in the upper half-plane model of the hyperbolic plane.

$E = \frac{3}{2}(-2)(2 - 2g - n) = 6g - 6 + 3n$ . Now for the dimension of Teichmüller space, there is one shift parameter associated to each edge and one equation associated to each puncture; this is the equation which ensures that the hyperbolic structure near that puncture is complete, which is equivalent to the fact that around each puncture of  $S_{g,n}$ , there is a foliation by closed leaves made up of pieces of horocycles that are perpendicular to the edges that abut on that puncture, see Definition 3.6 below. Therefore, the number of parameters is  $E - n = 6g - 6 + 3n - n = 6g - 6 + 2n$ , as expected.

### 3.3 Geodesic laminations

**Definition 3.3** (Geodesic lamination). Let  $S$  be a hyperbolic surface and let  $p : \mathbb{H}^2 \simeq \tilde{S} \rightarrow S$  be its metric universal covering. A *geodesic lamination* on  $S$  is a closed subset of  $S$  which is the union of disjoint images of bi-infinite geodesics of  $\mathbb{H}^2$  by the covering map  $p$  such that if  $\ell \subset \tilde{S}$  is such a bi-infinite geodesic, then, either

- (1) the restriction of  $p$  to  $\ell$  is injective, or
- (2)  $p(\ell)$  is a simple closed geodesic on  $S$ .

The decomposition of a geodesic lamination as a union of such images of bi-infinite geodesics of  $\mathbb{H}^2$  is unique, and each such image is called a *leaf* of  $\lambda$ . There are several good references on geodesic laminations. We refer the reader to Thurston's original notes [60], to the book by Penner and Harer [49] or to the book by Casson and Bleiler [15].

By abuse of language, we shall sometimes call the leaves of  $\lambda$  geodesics (rather than local geodesics). We shall talk about *bi-infinite leaves* of  $\lambda$  to denote leaves of kind (1) above, and about *closed leaves* to denote leaves of kind (2).

It is well-known that there is a natural one-to-one correspondence between geodesic laminations associated to any two hyperbolic structures on a given surface (see [34]). This makes it possible to talk about a geodesic lamination on a surface without reference to any particular hyperbolic structure. In fact, we shall sometimes call a geodesic lamination on a surface  $S$  an object which is a geodesic lamination for some hyperbolic structure on  $S$ . Equivalently, a geodesic lamination on  $S$  will be an object which is isotopic to a geodesic lamination for any hyperbolic structure on  $S$ .

A geodesic lamination is said to be *complete* if there is no geodesic lamination that strictly contains it. Equivalently, a geodesic lamination  $\lambda$  is complete if each connected component of  $S \setminus \lambda$ , equipped with its intrinsic metric, is isometric to the interior of a hyperbolic ideal triangle.

An ideal triangulation of a surface  $S = S_{g,n}$  with  $n \geq 1$  is an example of a complete geodesic lamination on  $S$ .

If  $\lambda$  is a geodesic lamination and if  $\mu$  is any complete geodesic lamination containing it, then we shall call  $\mu$  a *completion* of  $\lambda$ .

Any non-complete geodesic lamination can be completed in several ways, and it is interesting to understand these different ways because they will correspond to various ways of constructing geodesics for Thurston's asymmetric metric which we shall study below. It is possible to complete any geodesic lamination  $\lambda$  by inserting a finite number of bi-infinite geodesics in its complement. The surface  $S \setminus \lambda$  has a finite number of connected components, and the metric completion of each such connected component is a hyperbolic surface with geodesic boundary (see [15]). In particular, if  $\lambda$  contains closed leaves, then, in order to obtain a completion of  $\lambda$ , we can add geodesics that spiral around the closed leaves. For instance, in any hyperbolic pair of pants (with geodesic boundary), we can draw three bi-infinite geodesics that spiral around the boundary components so that the complement of these bi-infinite geodesics in the pair of pants is a union of the interiors of two hyperbolic ideal triangles (see Figure 12). In fact, there are 32 distinct ways of completing a hyperbolic pair of pants. If the lamination  $\lambda$  that we started with does not have closed leaves, then we can still complete it by adding a finite number of bi-infinite geodesics that "spiral along bi-infinite leaves of  $\lambda$ ". Spiraling (along closed geodesics or along bi-infinite leaves) can be defined as an operation in the universal covering of  $S$ , that consists in adding in an equivariant manner to the lift of the lamination  $\lambda$  a certain number of geodesic lines that converge in each direction to the limit point of the lift of some (finite or infinite) leaf of  $\lambda$ . To be more precise, let us first suppose that all the leaves of  $\lambda$  are closed. Let  $\alpha$  and  $\beta$  be two closed leaves in  $\lambda$  and let us consider a segment  $c$  having one endpoint on  $\alpha$  and one endpoint on  $\beta$ , and whose interior is disjoint from  $\lambda$ . Starting from the segment  $c$ , we define a continuous family of segments  $(c_t)_{t \geq 0}$  in  $S$ , with  $c_0 = c$ , and such that for any  $t \geq 0$ , the endpoints of  $c_t$  are on  $\alpha$  and on  $\beta$ , and spin around these curves as  $t$  increases to  $\infty$  with constant speed. For each  $t \geq 0$ , we can take the segment  $c_t$  to be a local geodesic, its interior being disjoint from the lamination  $\lambda$ . Let  $p : \tilde{S} \rightarrow S$  be again the metric universal covering of  $S$ , with  $\tilde{S}$  identified with the upper half-plane  $\mathbb{H}^2$ . Consider a continuous family of segments  $(\tilde{c}_t)_{t \geq 0}$  in  $\mathbb{H}^2$ , which lifts the family  $(c_t)_{t \geq 0}$  in  $S$ . For each  $t \geq 0$ , the endpoints of the segment  $\tilde{c}_t$  are on lifts  $\tilde{\alpha}$  and  $\tilde{\beta}$  of  $\alpha$  and  $\beta$ . The family of geodesic segments  $(\tilde{c}_t)_{t \geq 0}$  converges in an obvious sense to the image of a bi-infinite geodesic line  $\tilde{c} : \mathbb{R} \rightarrow \mathbb{H}^2$  whose endpoints are endpoints of  $\tilde{\alpha}$  and  $\tilde{\beta}$ . The distance from the point  $\tilde{c}(t)$  to the geodesic lines  $\tilde{\alpha}$  and  $\tilde{\beta}$  tends to 0 as  $t \rightarrow \infty$  or  $t \rightarrow -\infty$  (and the convergence is of the order of  $e^{-t}$ ). The covering map  $p : \tilde{S} \rightarrow S$  restricted to  $\tilde{c}$  is injective, and the image of  $\tilde{c}$  by this map is a bi-infinite geodesic line in  $S$  which spirals at one end around  $\alpha$  and at the other end around  $\beta$ . We take this geodesic as a new leaf in the completion of  $\lambda$ .

To continue completing the geodesic lamination  $\lambda$  by filling-in the connected components of  $S \setminus \lambda$  with bi-infinite geodesics, we must consider the general case (that is, in the case where leaves of  $\lambda$  are not all closed). Each completed connected component of the surface  $S$  cut along  $\lambda$  is a surface with boundary, with each boundary being either a closed geodesic or a union of bi-infinite geodesics, as in Figure 13. We can decompose each connected component of  $S \setminus \lambda$  into ideal triangles by successively inserting a finite number of bi-infinite geodesics which at each end either spiral along a closed leaf or converge to one of the ends of a bi-infinite leaf or converge to a cusp. The number of leaves that we can add is bounded in terms of the topology of the surface since we eventually decompose the surface into ideal triangles, each of them having a fixed area (equal to  $\pi$ ), and the area of the hyperbolic surface  $S$  is a topological invariant.

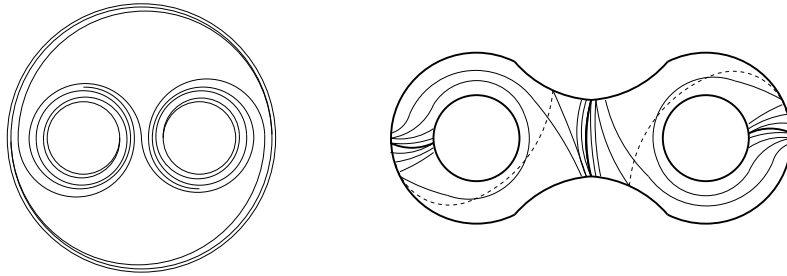


Figure 12. One way of obtaining complete geodesic laminations on a closed surface is to divide that surface into hyperbolic pairs of pants, and then decompose each pair of pants by bi-infinite leaves that spiral around the boundary components, as in the example drawn here. The stump of the complete lamination that we obtain is the union of the closed geodesics defining the pair of pants decomposition.

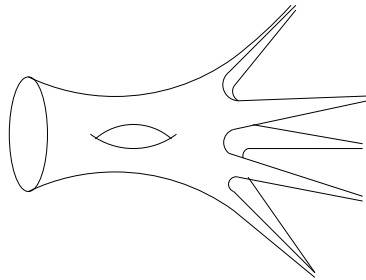


Figure 13. A possible component of the surface  $S$  cut along the lamination  $\mu$ .

**Definition 3.4** (Measured geodesic lamination). A *measured geodesic lamination*  $\lambda$  on a hyperbolic surface is a geodesic lamination which is equipped with a nonnegative Radon measure on each transverse arc. Furthermore, the measure is invariant under homotopies of the arc respecting the lamination  $\lambda$  and the support of the measure is equal to the intersection of the arc with the support of  $\lambda$ .

To each measured geodesic lamination  $\lambda$  is associated an element of the function space  $\mathbb{R}_+^S$ , by a rule analogous to the one that associates to each measured foliation an element of  $\mathbb{R}_+^S$  (see (2.2) above). On a given hyperbolic surface, two distinct measured geodesic laminations define distinct elements of  $\mathbb{R}_+^S$ , unlike pairs of measured foliations which are more flexible objects. (There is no need to talk about isotopy or Whitehead-equivalence relation between geodesic laminations on the same hyperbolic surface.) The two subsets of  $\mathbb{R}_+^S$  consisting respectively of images of equivalence classes of measured foliations and images of measured geodesic laminations coincide, and this fact can be used to define a one-to-one correspondence between spaces of equivalence classes of measured foliations on a surface  $S$  and spaces of measured geodesic laminations on the surface  $S$  equipped with a hyperbolic structure. Note that there is an alternative and more geometric procedure to associate to each measured foliation class a measured geodesic lamination, which gives the same correspondence between the two sets (see for instance [34]).

We already mentioned that there is a natural one-to-one correspondence between sets of geodesic laminations on the surface  $S$  equipped with two distinct hyperbolic structures. This correspondence leads to a metric-independent definition of a geodesic lamination. In fact, Thurston shows in his Notes [60] that two geodesic laminations which correspond by that natural correspondence are isotopic as topological objects, by a global isotopy of  $S$  that fixes the cusps. Therefore, on a topological surface  $S$ , a geodesic lamination can also be thought of as an object defined up to isotopy.

The *support* of a lamination is the union of its leaves. By abuse of notation, we shall sometimes denote the support of a measured geodesic lamination  $\lambda$  by the same letter  $\lambda$ .

We denote by  $\mathcal{ML}(S)$  the set of measured geodesic laminations on  $S$ , and by  $\mathcal{ML}_0(S) \subset \mathcal{ML}(S)$  the subset consisting of measured geodesic laminations whose support is compact. Note that a compactly supported measured geodesic lamination cannot be complete if the surface has punctures, since by the definition of the transverse measure for a geodesic lamination, the support of the measure is equal to the support of the lamination, and therefore the lamination cannot have leaves which converge to punctures (the closure of such a leaf cannot be compact, and a complete geodesic lamination has necessarily such a leaf).

There is a natural action of the group  $\mathbb{R}_+^*$  of positive reals on the space  $\mathcal{ML}(S)$ , and the quotient of  $\mathcal{ML}(S)$  by this action is the set  $\mathcal{PML}(S)$  of projective classes of measured geodesic laminations, whose elements are called *projective measured geodesic laminations*. The subset of  $\mathcal{PML}(S)$  consisting of the projective classes of compactly supported measured geodesic laminations is denoted by  $\mathcal{PML}_0(S)$ . One can define topologies on the spaces  $\mathcal{ML}(S)$  and  $\mathcal{ML}_0(S)$  by using the inclusion maps of these spaces in the space  $\mathbb{R}_+^{\mathcal{S}}$  and taking the restriction of the weak topology. In this way, the correspondences between spaces of measured geodesic laminations and spaces of equivalence classes of measured foliations become homeomorphisms.

**Definition 3.5** (The stump of a geodesic lamination, [59]). The *stump* of a geodesic lamination  $\mu$  is the support of any maximal (with respect to inclusion) compactly supported measured geodesic sublamination  $\lambda$  of  $\mu$ .

The stump of a geodesic lamination is itself a geodesic lamination, and it is empty if and only if each leaf of  $\mu$  converge at each end towards a cusp of the surface. For instance, if  $\mu$  is complete, then the stump of  $\mu$  is empty if and only if  $\mu$  is an ideal triangulation.

Completing a measured geodesic lamination  $\lambda$  by adding spiraling leaves gives a complete geodesic lamination whose stump is  $\lambda$ .

We shall sometimes consider the stump of a lamination  $\mu$  as being equipped with some transverse measure, and we shall also call this measured lamination the *stump* of  $\mu$ , although the transverse measure it carries is not unique.

We shall use the notion of length of a (compactly supported) measured geodesic lamination and we now recall the definition.

There is a function defined on  $\mathcal{T}(S) \times \mathcal{ML}_0(S)$  which to every hyperbolic structure  $g$  and every compactly supported measured geodesic lamination  $\lambda$  on  $S$  associates a quantity denoted by  $l_g(\lambda)$  and called the *length* of  $\lambda$  with respect to  $g$ . This function is a continuous extension of the notion of length of a simple closed geodesic, the set of simple closed geodesics being considered as a subset of the space of measured geodesic laminations, each simple closed geodesic being equipped with the Dirac measure of mass one. The definition of  $l_g(\lambda)$  can be made by covering the geodesic lamination  $\lambda$  by a finite collection of quadrilaterals  $R_1, \dots, R_n$  with disjoint interiors, such that the intersection of the leaves of  $\lambda$  with each quadrilateral is a union of segments that crosses it from one side to the opposite side. We shall call these two sides the “vertical” sides of  $R_i$ . We choose, for each quadrilateral  $R_i$ , one vertical side, and we call it  $\partial_v R_i$ . The length  $l_g(\lambda)$  is then the sum of the integrals, over all the quadrilaterals  $R_i$ , of the lengths of the geodesic segments of  $\lambda$  that cross them, with respect to the transverse measure  $d\lambda$  of  $\lambda$  induced on the vertical sides  $\partial_v R_i$ . In formulae, if we denote by  $\alpha_x$ ,  $x \in \lambda \cap \partial_v R_i$ , the collection of geodesic sub-segments of  $\lambda$  that traverse the quadrilateral  $R_i$ , we have

$$l_g(\lambda) = \sum_{i=1}^n \int_{\partial_v R_i \cap \lambda} l_g(\alpha_x) d\lambda(x).$$

### 3.4 Horocyclic foliations

We need to consider measured foliations on  $S$  which are slightly different from those we defined in Section 2.1. These foliations naturally appear in the study of hyperbolic surfaces with cusps.

**Definition 3.6** (Measured foliation trivial around the punctures). A measured foliation  $F$  on  $S$  is said to be *trivial around the punctures* if  $F$  is a measured foliation in the sense of Definition 2.1 except for the condition at the punctures, which is replaced by the following: each puncture has a neighborhood on which the induced foliation is a cylinder foliated by homotopic closed leaves, and any segment transverse to  $F$  and converging to a puncture has infinite total mass with respect to the transverse measure of  $F$ .

In what follows, we shall sometimes omit the adjective “trivial around the punctures” when the context is clear. We can pass from a measured foliation trivial around the punctures to a measured foliation in the sense of Definition 2.1 by deleting, for each puncture of  $S$ , the maximal annulus foliated by closed leaves parallel to that puncture, obtaining thus a partial measured foliation on the surface, and then collapsing the complementary components of the support of that partial foliation in order to get a foliation whose support is the entire surface  $S$ . The resulting measured foliation is well-defined up to equivalence, except that it may be empty (or, if one prefers, reduced to a graph). Thus, to each measured foliation which is trivial around the punctures we can naturally associate an object which is either a well-defined element in  $\mathcal{MF}$  or the empty foliation.

For any fixed hyperbolic structure on  $S$ , we next define an object which is “dual” to a complete geodesic lamination and which we call the associated “horocyclic foliation”. We start with a construction in an ideal triangle. Any hyperbolic ideal triangle is equipped with a canonical partial measured foliation which is called the *horocyclic foliation* of the triangle. This partial foliation is characterized by the following three properties:

- (1) the leaves are pieces of horocycles that are perpendicular to the edges;
- (2) the non-foliated region is a triangle bordered by three of these pieces of horocycles (see Figure 14);
- (3) the transverse measure assigned to any arc which is contained in an edge of the ideal triangle coincides with the Lebesgue measure induced from the hyperbolic metric.

Note that the non-foliated triangular region intersects each edge of the ideal triangle at its distinguished point.

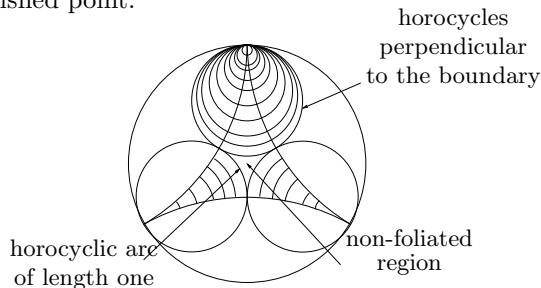


Figure 14. The horocyclic foliation of an ideal triangle.

**Definition 3.7** (Horocyclic measured foliation). Let  $S$  be a hyperbolic surface and let  $\mu$  be a complete geodesic lamination on  $S$ . The completion of each connected component of  $S \setminus \mu$  being an ideal triangle, we can equip it with its horocyclic measured foliation. The union of the horocyclic measured foliations associated to the various ideal triangles fit together smoothly, since they are perpendicular to the edges of the ideal triangles and therefore they form a Lipschitz-line field on the surface. They define a partial measured foliation of  $S$ , which we call the *horocyclic measured foliation associated to  $\mu$  and  $S$* , and which we denote by  $F_\mu(S)$ . Collapsing each nonfoliated triangular piece onto a tripod, we obtain a genuine (i.e. not partial) measured foliation on the surface, which is well-defined up to isotopy, and which we also call the horocyclic measured foliation associated to  $\mu$ .

The fact that the hyperbolic structures that we consider are complete and have finite area is equivalent to the fact that for any complete geodesic lamination  $\mu$ , the associated horocyclic foliation  $F_\mu(S)$  is trivial around the punctures.

The isotopy class of  $F_\mu(S)$  depends only on the isotopy class of the hyperbolic structure and on the complete geodesic lamination  $\mu$  that we started with. Therefore, the equivalence class of  $F_\mu(S)$  is associated to the hyperbolic structure  $S$  viewed as an element of Teichmüller space. As usual,  $[F_\mu(S)] \in \mathcal{MF}$  denotes the equivalence class of the horocyclic foliation  $F_\mu(S)$  after removal of the closed leaves that are parallel to punctures (provided  $F_\mu(S)$  is not the empty foliation).

**Definition 3.8** (Horocyclic measured geodesic lamination). Let  $S$  be a hyperbolic surface and let  $\mu$  be a complete geodesic lamination on  $S$ . The measured geodesic lamination that represents the (partial) measured foliation  $F_\mu(S)$  (after removal of the leaves that are parallel to the punctures, and provided the remaining measured foliation is not the empty foliation) will be called the *horo-*



*cyclic measured geodesic lamination* associated to  $\mu$ , and it will be denoted by  $\lambda_\mu(S)$ .

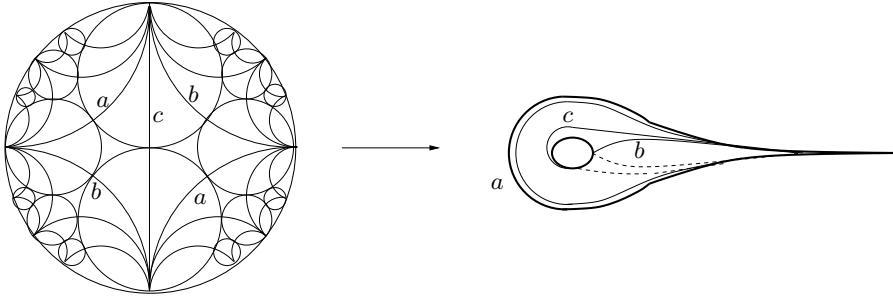


Figure 15. An example of a hyperbolic structure on the punctured torus equipped with a complete geodesic lamination  $\mu$  which is an ideal triangulation. In this case, the associated horocyclic measured geodesic lamination is empty. The gluing of the various ideal triangles in the universal covering is performed in such a way that for each edge of the ideal triangulation  $\mu$ , the two distinguished points corresponding to the two triangles adjacent to it coincide.

The horocyclic measured geodesic lamination  $\lambda_\mu(S)$  is the empty lamination if and only if the horocyclic foliation  $F_\mu(S)$  is a union of cylinders foliated by leaves that are all parallel to punctures. It is easy to construct examples of hyperbolic structures whose associated horocyclic foliation has this property. For instance, we can start, as in Figure 15, with an ideal quadrilateral in  $\mathbb{H}^2$  which is the union of two ideal triangles glued along two edges in such a way that the distinguished points on the common edges agree. We then glue the remaining four edges by pairs, respecting the labels  $a$  and  $b$  in Figure 15, again in such a way that the distinguished points on these edges coincide after the gluing. The resulting hyperbolic surface is a once-punctured torus, and the resulting lamination  $\mu$  on that surface is the union of the images of the edges of the two ideal triangles that we started with.

### 3.5 Thurston's cataclysm coordinates

In this section, we review coordinates for Teichmüller space that were introduced by Thurston, who called them “cataclysm coordinates” and which are essential in the study of Thurston’s asymmetric metric. The parameter space is a space of measured foliations that are transverse to a complete geodesic lamination. We start with the following:

**Definition 3.9** (Measured foliation totally transverse to a lamination). Let  $\mu$  be a geodesic lamination on  $S$ . A measured foliation  $F$  on  $S$  is said to be *totally transverse* to  $\mu$  if it satisfies the following:

- (1)  $F$  is transverse to  $\mu$ ;
- (2)  $F$  is trivial around the punctures.

In this definition, we have tacitly chosen an auxiliary hyperbolic structure on  $S$  in order to talk about geodesic laminations, but recall that we can make a definition which is independent of that choice. In other words, we say that  $\mu$  is a geodesic lamination if  $\mu$  is a geodesic lamination for some hyperbolic structure on  $S$ .

A measured foliation class is said to be *totally transverse* to  $\mu$  if it can be represented by a measured foliation that is totally transverse to  $\mu$ .

For any geodesic lamination  $\mu$  on  $S$ , we denote by  $\mathcal{MF}(\mu) \subset \mathcal{MF}$  the set of measured foliation classes that are totally transverse to  $\mu$ . This subset  $\mathcal{MF}(\mu)$  is open in  $\mathcal{MF}$ .

If  $\mu$  is complete, then, for any choice of a hyperbolic metric  $g$  on  $S$ , the associated horocyclic measured foliation  $F_\mu(g)$  is totally transverse to  $\mu$ . If  $h$  is a hyperbolic metric which is isotopic to  $g$ , then the horocyclic foliation  $F_\mu(h)$  is isotopic to  $F_\mu(g)$ . This enables us to associate to each element  $g$  in  $\mathcal{T}(S)$  a well-defined element in  $\mathcal{MF}(\mu)$ . We denote by  $\phi_\mu : \mathcal{T}(S) \rightarrow \mathcal{MF}(\mu)$  the resulting map.

**Theorem 3.10** (Thurston, [61] §9). *For any complete geodesic lamination  $\mu$  on  $S$ , the map  $\phi_\mu : \mathcal{T}(S) \rightarrow \mathcal{MF}(\mu)$  is a homeomorphism.*

The global coordinates for Teichmüller space that are provided by this map  $\phi_\mu$  are called by Thurston *cataclysm coordinates*.

The proof of Theorem 3.10 is given by Thurston in §4 and §9 of the paper [61]. This proof involves a lot of interesting details which we shall review below. In particular, we shall discuss in length the proof of the surjectivity of the map  $\phi_\mu$ , that is, the fact that for any element  $[F]$  of  $\mathcal{MF}(\mu)$ , there exists a hyperbolic structure  $g \in \mathcal{T}(S)$  such that  $[F] = [F_\mu(g)]$ . This surjectivity is contained in Proposition 9.2 of [61].

Let us first make a few remarks about the proof of the surjectivity of  $\phi_\mu$ . Given an element  $[F] \in \mathcal{MF}(\mu)$ , the transverse measure of a representative  $F$  of  $[F]$  determines a measure on each leaf of  $\mu$ , which we think of as the one-dimensional Lebesgue measure induced on the leaves of  $\mu$  by the hyperbolic metric that we seek. Of course this is not enough to determine the hyperbolic metric; to do this, one needs to define an isometry between each connected component of  $S \setminus \mu$  and a hyperbolic ideal triangle, and then define in a consistent way a gluing of the various ideal triangles among themselves. In the case where  $\mu$  is an ideal triangulation, that is, if every leaf of  $\mu$  is isolated, then the

metric  $g$  can easily be defined by gluing edge-to-edge the ideal triangles (which are finite in number). This gluing is determined by the measured foliation by saying that the algebraic distances between the distinguished points on each edge of  $\mu$  that are associated to the hyperbolic metric and to the measured foliation  $F$  coincide. Here, the distinguished points on an edge of  $\mu$  that are associated to a measured foliation  $F$  are the hitting points of the singular leaves starting at the singular points of  $F$  that are contained in the ideal triangles that are on each side of that edge. But in the case where  $\mu$  has non-isolated leaves, the gluing has to be defined by an infinite process, and the main point of the proof is the convergence of that process.

Let us fix a geodesic lamination  $\mu$  on  $S$ . We start by proving some useful facts about measured foliations totally transverse to  $\mu$ .

**Lemma 3.11.** *Let  $\beta$  be an infinite half-leaf of  $\mu$ . Then, exactly one of the following occurs:*

- (1)  $\beta$  converges to a cusp;
- (2)  $\beta$  has a recurrence point contained in the stump of  $\mu$  (that is, there exists a point  $x$  in the stump of  $\mu$  such that  $\beta$  returns infinitely often in any neighborhood of  $x$ ).

*Proof.* Suppose that  $\beta$  does not converge to a cusp. Then there is a nonempty compactly supported geodesic sub-lamination in its closure. Since each compactly supported geodesic lamination admits a nonzero transverse measure (which may be of smaller support), we obtain a compactly supported measured geodesic sublamination  $\mu_0$  of  $\mu$  in the closure of  $\beta$ . The sub-lamination  $\mu_0$  is contained in the stump of  $\mu$ . The lemma follows then from the fact that each point in a measured geodesic lamination of compact support is a recurrence point.  $\square$

**Lemma 3.12** (infinite measure for infinite half-leaves of  $\mu$ ). *Let  $F$  be a measured foliation totally transverse to  $\mu$ . Then, the  $F$ -transverse measure of any infinite half-leaf of  $\mu$  is infinite.*

*Proof.* We use Lemma 3.11. If  $\beta$  converges to a cusp, then it has infinite transverse measure, by assumption on  $F$  (Definitions 3.9 and 3.6). Suppose now that  $\beta$  does not converge to a cusp and let  $x$  be a recurrence point of  $\beta$  in the stump  $\gamma$  of  $\mu$ . We choose a homeomorphism  $\varphi : U \rightarrow [0, 1]^2$  such that  $U \subset S$  contains  $x$  in its interior, such that the image of  $F|_U$  by  $\varphi$  is made of vertical segments (which we call plaques) defined by the equations  $u = \text{constant}$ , where  $u$  is the coordinate of the first factor of  $[0, 1]^2$ , and such that the connected components of the intersection of  $\mu$  with  $U$  are segments joining the “vertical” sides  $\{u = 0\}$  and  $\{u = 1\}$  of  $[0, 1]^2$ . In particular,

there is a segment of  $\gamma \cap U$  containing  $x$  and crossing the plaques transversely from one vertical side to the other vertical side. Since  $x$  is a recurrence point of  $\beta$ , there are infinitely many segments in  $\beta \cap U$  whose images by  $\varphi$  cross the square  $[0, 1]^2$  from one vertical side to the other one. All these segments have the same transverse measure. Therefore, the transverse measure of  $\beta$  is infinite.  $\square$

In the rest of this section, the geodesic lamination  $\mu$  is complete.

Let  $[F]$  be an element of  $\mathcal{MF}(\mu)$  and let us choose a representative  $F$  of  $[F]$  that is totally transverse to  $\mu$ .

From the transversality of  $F$  to  $\mu$  and by an Euler characteristic argument, we can see that  $F$  has exactly one singular point in each connected component of  $S \setminus \mu$  or on the boundary of that component, and that this singular point is a 3-prong singularity. Up to a small homotopy of  $F$ , we can suppose that all the singular points of  $F$  lie in the interior of the components of  $S \setminus \mu$ .

We first show that  $F$  is homotopic (in a sense that will be made precise) in each ideal triangle to the horocyclic foliation of some hyperbolic structure on  $S$ .

We fix an ideal triangle  $T^*$  in  $\mathbb{H}^2$ , equipped with its horocyclic foliation, modified by collapsing the non-foliated region onto a spine (that is, a tripod). We call  $F_{T^*}$  this measured foliation. We perform the collapse in such a way that it induces the identity map on the boundary of  $T^*$ .

Let  $T$  be a connected component (including its boundary) of  $S \setminus \mu$  equipped with the measured foliation induced by  $F$ . Notice that  $T$  has the structure of a topological ideal triangle. (In fact, it becomes a genuine ideal triangle as soon as we equip the surface  $S$  with a hyperbolic metric, but we do not need to introduce arbitrarily such a metric right now.) We have the following:

**Lemma 3.13** (Equivalence to horocyclic foliation). *There exists a homeomorphism  $\varphi_T : T \rightarrow T^*$  respecting the foliations of these triangles (that is, sending leaves to leaves and preserving the transverse measures) which is unique up to the choice of a one-to-one correspondence between the ideal vertices of  $T$  and those of  $T^*$ , and up to post-composition of  $\varphi_T$  with a homeomorphism of  $T^*$  that preserves its horocyclic foliation and that induces the identity map on the boundary of that triangle.*

*Proof.* Let  $F_T$  denote the measured foliation induced by  $F$  on  $T$ . Choose a one-to-one correspondence between the ideal vertices of the triangles  $T$  and  $T^*$ . There is a homeomorphism  $\varphi_T : T \rightarrow T^*$  preserving the singular points and mapping each leaf of  $F_T$  to a leaf of  $F_{T^*}$ , preserving the transverse measures. To define  $\varphi_T$ , it suffices to start by mapping the three singular leaves of  $F_T$  to the three singular leaves of  $F_{T^*}$  (respecting the correspondence between these leaves induced by the correspondence between the ideal vertices) and then continue mapping homeomorphically each leaf of  $F_T$  onto a leaf of  $F_{T^*}$ .

preserving the transverse measure. This is possible because the transverse measure induced by  $F_T$  on each half-edge of  $T$  is infinite (Lemma 3.12). We thereby get a map  $\varphi_T : T \rightarrow T^*$  which is well-defined up to the deformations mentioned in the statement of the lemma.  $\square$

A *spike* of a hyperbolic ideal triangle of  $\mathbb{H}^2$  is a connected component of the complement of the closure of the unfoliated region of its horocyclic measured foliation (or a connected component of the complement of the singular graph, if the unfoliated region has been collapsed onto a spine). Likewise, given a complete geodesic lamination  $\mu$  on  $S$  together with a totally transverse foliation  $F$ , a *spike* of  $S \setminus \mu$  is a connected component of the complement of the singular leaves of the restriction of  $F$  to a connected component of  $S \setminus \mu$ .

Using Lemma 3.13, we now define a function on each spike of  $S \setminus \mu$ . Let  $q$  be such a spike and let  $T$  be the triangle of  $S \setminus \mu$  that contains it. We choose a homeomorphism  $\varphi_T$  between the triangle  $T$  and a fixed hyperbolic triangle  $T^*$  in  $\mathbb{H}^2$  that preserves the measured foliations of these triangles, as in Lemma 3.13. The *sharpness function*  $f_q : q \rightarrow [0, \infty)$  is defined by

$$f_q(x) = -\log \text{length}(\alpha(\varphi_T(x)))$$

where, for each  $x \in q$ ,  $\alpha(\varphi_T(x))$  denotes the leaf of the horocyclic foliation corresponding to the leaf of  $F_{T^*}$  passing through the point  $\varphi_T(x) \in T^*$ . In other words,  $\alpha(\varphi_T(x))$  is the horocyclic arc that passes through the point of  $T^*$  that is mapped to the point  $\varphi_T(x)$  by the map  $T^* \rightarrow T^*$  that we used to collapse the unfoliated region of that ideal triangle onto a tripod. It is easy to check that the function  $f_q$  does not depend on the choices involved in the definition of  $\varphi_T$ . Since the length of any arc of the horocyclic foliation of an ideal triangle is bounded from above by 1 (which is the length of the horocyclic arc that is on the boundary of the unfoliated region),  $f_q(x)$  is nonnegative for all  $x$  in  $q$ . Furthermore, this function converges to  $\infty$  as the point  $x$  converges to the cusp of  $q$ . (In fact,  $f_q$  is a linear function with respect to the “distance” from  $x$  to the unfoliated region, where this distance is defined by the transverse measure of any transverse arc joining  $x$  to a point in the unfoliated region.)

Thus, associated to the measured foliation  $F$ , we have a family  $\{f_q\}$  of functions, one for each spike  $q$  of  $S \setminus \mu$ . We shall use these functions to construct the hyperbolic structure  $g$  satisfying  $[F] = [F_\mu(g)]$ . Following Thurston, to construct  $g$ , we construct a map from the universal covering of  $S$  to  $\mathbb{H}^2$ . This map will turn out to be the developing map of that structure.

Let us choose a basepoint  $s$  on  $S$  and let  $\pi : \tilde{S} \rightarrow S$  be the topological universal covering of  $S$ . We regard  $\tilde{S}$  as the set of homotopy classes of paths with fixed endpoints  $\alpha : [0, 1] \rightarrow S$  with  $\alpha(0) = s$ ; the projection map  $\pi : \tilde{S} \rightarrow S$  is given by  $[\alpha] \mapsto \alpha(1)$ .

Given such a path  $\alpha$ , we can replace it by a path  $\alpha^*$  which is homotopic to  $\alpha$  by a homotopy with fixed endpoints, which is made of a finite concatenation

of segments, each such segment being contained either in a leaf of  $F$  or in a leaf of  $\mu$ . We call such a path  $\alpha^*$  a *horo-geodesic* path. (Of course, we are using this terminology because we imagine  $S$  as being equipped with a hyperbolic structure  $g$  for which  $\mu$  is geodesic, with  $F$  being the corresponding horocyclic measured foliation; in fact,  $g$  will be the structure that we are seeking.) The construction of  $\alpha^*$  may be done as follows. We start by taking a train track approximation  $\tau$  of  $\mu$ , and for this we can use an auxiliary hyperbolic metric for which  $\mu$  is geodesic, and take the leaves of  $\tau$  so that they are nearly parallel to the geodesic leaves of  $\mu$ , in the metric sense, as in [60] §9.5.

We can cover  $\mu$  by a finite collection of rectangles obtained by thickening the edges of the train track  $\tau$ . The horizontal sides of these rectangles are parallel to the edges of  $\tau$ , and the leaves of  $\mu$  cross each such rectangle from a vertical side to a vertical side. We also take  $\tau$  to be close enough to  $\mu$  in such a way that  $F$  is transverse to  $\tau$ . Moreover, up to taking smaller rectangles, we can assume that the foliation induced by  $F$  on each rectangle consists in segments joining its horizontal sides. Performing a homotopy with fixed endpoints on the path  $\alpha$ , we can replace it by a path whose intersection with each rectangle is a finite number of segments joining its horizontal sides. Finally, replacing each such segment (by using again a homotopy with fixed endpoints) by a concatenation of (at most three) segments in leaves of  $\mu$  and in leaves of  $F$ , we obtain the desired horo-geodesic path  $\alpha^*$ . From this construction, we can assume without loss of generality that  $\alpha^*$  satisfies the following two properties:

- (1) each subsegment of  $\alpha^*$  contained in a leaf of  $F$  or in a leaf of  $\mu$  does not backtrack on that leaf;
- (2) there does not exist any closed immersed disk in  $S$  whose interior is embedded in that surface and whose boundary is the union of a subsegment of  $\alpha^*$  with a segment which is either in a leaf of  $F$  or in a leaf of  $\mu$ .

A horo-geodesic path satisfying these two properties will be called a *minimal horo-geodesic path*.

From now on, we regard  $\tilde{S}$  as the set of homotopy classes with fixed endpoints of minimal horo-geodesic paths  $\alpha^* : [0, 1] \rightarrow S$  satisfying  $\alpha^*(0) = s$ .

We note that in the construction that follows, a change in the auxiliary hyperbolic metric will only affect the developing map of the hyperbolic structure that we are seeking by conjugating it by an orientation-preserving isometry of  $\mathbb{H}^2$ . Precisely, this hyperbolic structure can be seen as the plane  $\mathbb{H}^2$  mod out by a subgroup of the group of isometries of  $\mathbb{H}^2$  acting properly discontinuously on that space and defined up to conjugation, and therefore the choice of the auxiliary hyperbolic metric will not affect the final result.

We now fix a minimal horo-geodesic path  $\alpha^*$ . We shall say that a subpath of  $\alpha^*$  is *maximal* if its image is either contained in a leaf of  $F$  or in a leaf of  $\mu$ , and if this subpath is maximal (with respect to inclusion) for that property. We shall call a maximal subpath of  $\alpha^*$ , if it is contained in a leaf of  $F$  (respectively

of  $\mu$ ), a *maximal vertical* (respectively *horizontal*) subpath. We shall consider the image of  $\alpha^*$  as being equipped with the natural orientation induced from that of the interval  $[0, 1]$ .

Let  $c$  be a maximal vertical subpath of  $\alpha^*$ . The connected components of the complement of  $c$  with respect to  $\mu$  form a linearly ordered set of open intervals  $(a_j, b_j)$ ,  $j \in J$ , each interval  $(a_j, b_j)$  joining opposite edges of a topological ideal triangle  $T_j$  of  $S \setminus \mu$ . To each such interval  $(a_j, b_j)$ , we now associate a parabolic isometry of  $\mathbb{H}^2$ . Recall that to the triangle  $T_j$  containing  $(a_j, b_j)$ , there is an associated homeomorphism  $\varphi_{T_j}$  which is defined up to an isotopy that induces the identity map on the boundary of that triangle and up to a permutation of the ideal vertices, such that  $\varphi_{T_j}$  maps  $T_j$  onto some fixed ideal triangle  $T^*$  of  $\mathbb{H}^2$  equipped with its horocyclic foliation, respecting the foliations and the transverse measures. We also have an associated family of sharpness functions  $\{f_q\}$ , one function for each spike  $q$  of  $S \setminus \mu$ . Let us fix an oriented geodesic line  $\gamma$  in  $\mathbb{H}^2$  which we shall henceforth refer to as the *base geodesic*, and let us choose a basepoint  $x_0$  on (the image of) that geodesic. For each  $j$  in  $J$  and for each choice of a hyperbolic ideal triangle  $T^*$  having the base geodesic  $\gamma$  as one of its edges and situated to the right of  $\gamma$  (we are using an orientation of the hyperbolic plane), we consider the associated orientation-preserving map  $\varphi_{T_j}$  between  $T_j$  and  $T^*$  that sends the edge of  $T_j$  containing  $a_j$  to the base geodesic  $\gamma$ . (The fact that  $\varphi_{T_j}$  preserves orientations implies that the images of the vertices of  $T_j$  are well-defined.) As the ideal vertex of  $T^*$  that is not an endpoint of  $\gamma$  varies on the circle at infinity, the image  $\varphi_{T_j}(a_j)$  varies on the base geodesic  $\gamma$ , and we can manage so that  $\varphi_{T_j}(a_j)$  is the point  $x_0$ . Now let  $P_j$  be the parabolic isometry of  $\mathbb{H}^2$  that fixes the endpoint of the spike of  $T^*$  that contains the segment  $\varphi_{T_j}((a_j, b_j))$  and that sends  $\varphi_{T_j}(a_j)$  to  $\varphi_{T_j}(b_j)$ . Note that the length of the horocyclic arc joining the points  $\varphi_{T_j}(a_j)$  and  $\varphi_{T_j}(b_j)$  equals  $e^{-f_q(a_j)}$ . After associating a parabolic isometry  $P_j$  to each open interval  $(a_j, b_j)$ , we associate to the base subpath  $c$  of  $\alpha^*$  the (possibly infinite) product of isometries  $\prod_{j \in J} P_j$  (see Figure 16). Note that the association of the product of isometries is in the reverse order of the one used for composition of maps. Thus, for instance, if  $\alpha$  crosses the ideal triangles  $T_1, T_2, T_3$  in that order with associated parabolic isometries  $P_1, P_2, P_3$  respectively, then the product will be  $P_1 P_2 P_3$ .

Let  $\text{Isom}_+(\mathbb{H}^2)$  be the group of orientation-preserving isometries of  $\mathbb{H}^2$ . Given a basepoint  $x_0$  in  $\mathbb{H}^2$ , we endow this group with the following complete left-invariant metric

$$\forall A, B \in \text{Isom}_+(\mathbb{H}^2), \quad d_{x_0}(A, B) = \sup_{x \in \mathbb{H}^2} |A(x) - B(x)| e^{-|x_0 - x|},$$

where for every  $x$  and  $y$  in  $\mathbb{H}^2$ ,  $|x - y|$  denotes their hyperbolic distance. It is easy to see that if  $x_1 \in \mathbb{H}^2$  is another basepoint, then we have

$$d_{x_1}(A, B) \leq d_{x_0}(A, B) e^{|x_0 - x_1|}. \quad (3.3)$$

The distance between  $A \in \text{Isom}_+(\mathbb{H}^2)$  and the identity  $I$  is denoted by  $\|A\|_{x_0} = \|A\|$ . Using the left-invariance of the metric  $d_{x_0}$ , it is easy to see that we have  $d_{x_0}(A, B) = \|B^{-1}A\|$  for all  $A$  and  $B$  in  $\text{Isom}_+(\mathbb{H}^2)$ . We need to estimate the norm of a parabolic element.

**Lemma 3.14.** *Let  $P$  be a parabolic element fixing a point  $p$  in  $S_\infty^1$ , the boundary at infinity of  $\mathbb{H}^2$ , and let  $l(0)$  denote the length of the horocyclic arc joining  $x_0$  and  $P(x_0)$  and contained in the horocycle centered at  $p$ . Then,  $\|P\| \leq l(0)$ .*

*Proof.* Let  $\gamma : \mathbb{R} \rightarrow \mathbb{H}^2$  be the geodesic line in  $\mathbb{H}^2$  satisfying  $\gamma(0) = x_0$ , having  $p$  as one of its endpoints and oriented so that  $\lim_{t \rightarrow -\infty} \gamma(t) = p$ . Consider the foliation of  $\mathbb{H}^2$  by the family  $\{H_t : t \in \mathbb{R}\}$  of horocycles centered at  $p$ , where for each  $t$  in  $\mathbb{R}$ ,  $H_t$  denotes the horocycle through  $\gamma(t)$ , and let  $l(t)$  denote the length of the arc in  $H_t$  joining  $\gamma(t)$  to  $P(\gamma(t))$ .

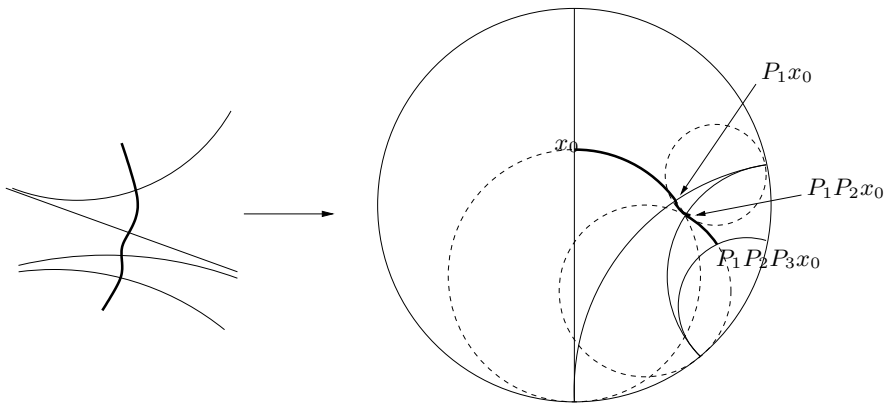


Figure 16.

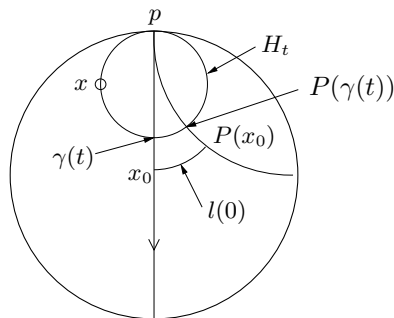


Figure 17. Notation used in Lemma 3.14.



Let  $x \in \mathbb{H}^2$ . This point is on a unique horocycle  $H_t$ ,  $t \in \mathbb{R}$ . We have  $|P(x) - x| = |P(\gamma(t)) - \gamma(t)|$ . Moreover, a geometric argument shows that the projection along horocycles is distance-decreasing, that is,  $|x_0 - x| \geq |x_0 - \gamma(t)| = |t|$ . Therefore,  $\|P\| = \sup_{t \in \mathbb{R}} |P(\gamma(t)) - \gamma(t)| e^{-|t|}$ .

A computation (for instance, using Figure 36 below) gives  $|P(\gamma(t)) - \gamma(t)| = 2 \sinh^{-1}(l(t)/2)$ , where  $l(t)$  denotes the length of the horocyclic arc joining  $\gamma(t)$  and  $P(\gamma(t))$ . Using the fact that  $l(t) = e^{-t}l(0)$  and using the inequality  $\sinh^{-1}(x) \leq x$  for every  $x \geq 0$ , we obtain

$$\begin{aligned} \|P\| &= \sup_{t \in \mathbb{R}} |P(\gamma(t)) - \gamma(t)| e^{-|t|} \\ &= \sup_{t \in \mathbb{R}} 2 \sinh^{-1} \left( \frac{l(t)}{2} \right) e^{-|t|} \\ &\leq l(0) e^{-t-|t|} \leq l(0). \end{aligned}$$

This proves Lemma 3.14.  $\square$

**Proposition 3.15.** *The product of isometries  $\prod_{j \in J} P_j$  that is associated to the maximal horizontal subpath  $c$  of  $\alpha^*$  converges in  $\text{Isom}_+(\mathbb{H}^2)$ .*

*Proof.* In order to show that the product  $\prod_{j \in J} P_j$  converges, we shall use the fact that  $\text{Isom}_+(\mathbb{H}^2)$  is complete. Let  $J_0 \subset J_1 \subset \dots \subset J_n \subset \dots$  be a sequence of finite subsets of  $J$  converging to  $J$ , that is, such that  $\cup_{n \in \mathbb{N}} J_n = J$ . Without loss of generality, we can assume that  $\text{Card}(J_n) = n$ . For each  $n \in \mathbb{N}$ , the finite product  $\prod_{j \in J_{n+1}} P_j$  is obtained from the previous finite product  $\prod_{j \in J_n} P_j$  by inserting one element  $e_n$ . More precisely, we can find elements  $A_n, B_n, e_n \in \text{Isom}_+(\mathbb{H}^2)$  such that  $\prod_{j \in J_n} P_j = A_n B_n$  and  $\prod_{j \in J_{n+1}} P_j = A_n e_n B_n$ . We have

$$\begin{aligned} d_{x_0}(A_n e_n B_n, A_n B_n) &= \|B_n^{-1} e_n B_n\| \\ &= \sup_{x \in \mathbb{H}^2} |B_n^{-1} e_n B_n(x) - x| e^{-|x_0 - x|} \\ &= \sup_{y \in \mathbb{H}^2} |B_n^{-1} e_n(y) - B_n^{-1}(y)| e^{-|x_0 - B_n^{-1}(y)|} \\ &= \sup_{y \in \mathbb{H}^2} |e_n(y) - y| e^{-|B_n(x_0) - y|} \\ &= d_{B_n(x_0)}(e_n, I) \\ &\leq d_{x_0}(e_n, I) e^{|B_n(x_0) - x_0|} \quad (\text{using 3.3}) \\ &\leq \|e_n\| e^{\|B_n\|}. \end{aligned}$$

Let  $\Pi_n$  denote  $\prod_{j \in J_n} P_j$ . We have

$$\begin{aligned} d_{x_0}(\Pi_{n+p}, \Pi_n) &\leq \sum_{k=1}^p d_{x_0}(\Pi_{n+k}, \Pi_{n+k-1}) \\ &\leq \sum_{k=1}^p \|e_{n+k-1}\| e^{\|B_{n+k-1}\|}. \end{aligned}$$

Let  $q$  be a spike of  $S \setminus \mu$ . The intersection of  $q$  with  $\alpha$  is a countable family of horocyclic arcs which, once reordered, have decreasing lengths  $l_n$ ,  $n \in \mathbb{N}$ . There is a positive lower bound  $M(F, c)$ , depending on the compact segment  $c$ , to the set of numbers  $\{l_n - l_{n+1}\}$ . We have  $l_n \leq l_0 e^{-nM(F)}$ . Moreover,  $l(0) \leq 1$ . Therefore, for every spike  $q$ , we have  $l_n \leq e^{-nM(F)}$ . Since there are finitely many spikes in  $S \setminus \mu$ , then, using Lemma using 3.14, we can see that there exists an integer  $m = m(n)$  such that  $\|e_n\| \leq l_m$  where  $l_m$  is the length of the  $m$ -th horocyclic arc of  $q \cap c$ , for every spike  $q$ . Therefore, for all  $\epsilon > 0$ , there exists  $N = N(F, \epsilon) \geq 0$  such that for all  $n \geq N$ , we have  $\|e_n\| \leq \epsilon$ . Thus, the infinite product  $\prod_{j \in J} P_j$  is bounded from above by a convergent geometric series, whence the norm of any finite subproduct is uniformly bounded from above. This shows that the sequence  $\{\Pi_n : n \in \mathbb{N}\}$  is Cauchy, hence converges.  $\square$

Let  $c$  be now a maximal horizontal subsegment of  $\alpha^*$ . We wish to associate to  $c$  a hyperbolic isometry of  $\mathbb{H}^2$ . We do this by induction, and for this, we first consider the isometries that were already associated to the sequence of maximal subsegments of  $\alpha^*$  that precede  $c$ . (In the case where  $c$  is the initial maximal subsegment of  $\alpha^*$ , the preceding isometry is taken to be the identity.) We let  $C$  be the product of all these isometries, this time product being taken in the usual order of composition of maps. In other words, if  $c_1, \dots, c_n$  is the ordered set of maximal subsegments of  $\alpha^*$  that precede  $c$  (that is,  $c_1$  starts at the point  $s$  and the endpoint of  $c_n$  is the starting point of  $c$ ), and if  $C_1, \dots, C_n$  is the ordered set of isometries that we already associated to  $c_1, \dots, c_n$ , then we let  $C = C_n \dots C_1$  (and if  $c$  is the first maximal subsegment of  $\alpha^*$ , we take  $C$  to be the identity). Now if we denote by  $\gamma'$  the image of the base geodesic  $\gamma$  by the isometry  $C$ , then, we associate to  $c$  the hyperbolic isometry  $H \in \text{Isom}_+(\mathbb{H}^2)$  that fixes  $\gamma'$ , whose translation length is equal to the transverse measure of  $c$  with respect to  $F$  and such that the orientation of the action of  $H$  on  $\gamma'$  coincides with the natural orientation associated with the parametrization of  $\alpha^*$ .

Finally, we associate to  $\alpha^*$  the product of the finite sequence of isometries associated to the finite sequence of maximal subsegments of  $\alpha^*$ . (Remember that the product is taken here in the order of composition of maps, as already mentioned.)

**Lemma 3.16.** *The isometry associated to the minimal horogeodesic path  $\alpha^*$  does not depend on the choice of  $\alpha^*$  in its homotopy class with fixed endpoints.*

*Proof.* Any two horogeodesic paths belonging to the same class  $[\alpha]$  are related to each other by a finite number of moves of the type described in Figure 18.



Figure 18. The dotted lines represent leaves of the foliation  $F$ , the horizontal lines are leaves of  $\mu$  and the bold lines represent the maximal subsegments of the path  $\alpha^*$ . (There may be other leaves of  $\mu$  that are inside the rectangles shown and that are not drawn.)

Therefore, it suffices to examine the invariance of the isometry associated to  $\alpha^*$  by such a move. The product of isometries associated to the left-hand rectangle of  $\alpha^*$  in Figure 18 is of the form  $PH$ , where  $H$  is the hyperbolic isometry associated to the maximal horizontal subsegment of  $\alpha^*$  that is drawn in that figure and  $P$  is the product of parabolic isometries that are associated to the maximal vertical segment. Likewise, in the right-hand rectangle, the product is the form  $H'P'$ , where  $H'$  and  $P'$  are defined in an analogous way. From the definition of the products associated to the subsegments of  $\alpha^*$  that are in  $F$ , it easily follows that  $P = P'$ , since these are products of a possibly infinite family of parabolic isometries indexed by the same set and which are all equal. From the invariance of the transverse measure of  $F$ , the translation distances of the two hyperbolic isometries  $H$  and  $H'$  are equal. From this fact and from the way these isometries are defined, it follows that  $H = P^{-1}H'P'$ , that is,  $PH = H'P'$ , which implies the invariance by the move considered. This completes the proof of the lemma.  $\square$

Thus, we have a map  $I_{s,F}$  from  $\tilde{S}$  to  $\text{Isom}_+(\mathbb{H}^2)$ , which associates to each homotopy class of curves with fixed endpoints the isometry associated to a minimal horogeodesic path representative, and which gives a map

$$D_{s,F} : \tilde{S} \longrightarrow \mathbb{H}^2$$

$$[\alpha] \mapsto I_{s,F}([\alpha])x_0.$$

This map is a local isometry. Changes in the choice involved in any of the parameters  $s$  and  $F$  have the effect of conjugating the map  $D_{s,F}$  by an isometry (see [65]).

This map induces a representation of the fundamental group in  $\text{Isom}_+(\mathbb{H}^2)$ , which is free and discrete, since the quotient of  $\mathbb{H}^2$  by the image group  $\Gamma_{[F]}$  is the surface  $S$ . We obtain in this way a hyperbolic structure  $g$  on  $S$  whose developing map is precisely the map  $D_{s,F}$ . The hyperbolic structure  $g$  is complete because each cusp of  $S$  equipped with that structure has a neighborhood

equipped with a foliation by closed leaves that are quotients by  $\Gamma_{[F]}$  of horocycles in  $\mathbb{H}^2$ , which follows from the fact that the measured foliation  $F$  that we started with is trivial around the punctures. This measured foliation is isotopic to the horocyclic measured foliation  $F_\mu(g)$ . This completes the proof of the surjectivity of the map  $\phi_\mu$ . This map is also injective, since it admits an inverse, which is precisely the map which we produced in the proof of the surjectivity. It is clear from the construction of the horocyclic foliation that the map  $\phi_\mu$  is continuous. By invariance of domain, this map is a homeomorphism. This completes the proof of Theorem 3.10.  $\square$

### 3.6 Stretch lines, stretch rays, anti-stretch rays and stretch maps

In the next definition, we use Thurston's cataclysm coordinates  $\phi_\mu$  introduced in §3.5.

**Definition 3.17** (Stretch lines, stretch rays, etc.). Let  $\mu$  be a complete geodesic lamination on  $S$ . For any element  $F$  in  $\mathcal{MF}(\mu)$ , the map from  $\mathbb{R}$  to  $\mathcal{T}$  (respectively from  $[0, \infty)$  to  $\mathcal{T}$ ) defined by  $t \mapsto \phi_\mu^{-1}(e^t F)$  is called a *stretch line directed by  $\mu$  and starting at  $g = \phi_\mu^{-1}(F)$* . A stretch line travelled up backwards is called an anti-stretch line. More precisely, given a stretch line  $t \mapsto \phi_\mu^{-1}(e^t F)$ , the map from  $\mathbb{R}$  to  $\mathcal{T}$  defined by  $t \mapsto \phi_\mu^{-1}(e^{-t} F)$  is called an *anti-stretch line directed by  $\mu$  and starting at  $g = \phi_\mu^{-1}(F)$* . A *stretch ray* (respectively *anti-stretch ray*) directed by  $\mu$  and starting at  $g = \phi_\mu^{-1}(F)$  is the restriction of a stretch line (respectively an anti-stretch line) to the half-line  $\mathbb{R}_+$ . Given a stretch (respectively anti-stretch) ray  $t \mapsto \phi_\mu^{-1}(e^t F)$  that starts at a point  $g$  in  $\mathcal{T}$ , we shall denote by  $g^t$  the hyperbolic surface  $\phi_\mu^{-1}(e^t F)$  (respectively  $\phi_\mu^{-1}(e^{-t} F)$ ). The hyperbolic structure  $g^0$  is the structure  $g$  that we started with. Given a stretch ray starting at  $g$ , then, for each  $t \geq 0$ , the identity map, considered as a map from the surface  $S$  equipped with the hyperbolic structure  $g^0$  to the same surface equipped with the hyperbolic structure  $g^t$ , is called the *stretch map directed by  $\mu$* .

We already mentioned that there exist punctured hyperbolic surfaces  $g$  equipped with complete geodesic laminations for which the associated horocyclic foliation is a union of cylinders that are all parallel to punctures. A stretch ray (or an anti-stretch ray) starting at a hyperbolic structure  $g$  with a horocyclic foliation satisfying this property is a constant ray, that is, we have  $g^t = g$  for all  $t \geq 0$ . For instance, in the example of Figure 15, stretching along  $\mu$  does not change the hyperbolic structure. One way of seeing this is the fact that the relative distances between the distinguished points remain constant (equal to zero). On the other hand, any complete lamination  $\mu$  which is not an ideal triangulation (that is, which is not a finite union of bi-infinite leaves

converging at both ends to punctures) has a nonempty stump. Therefore, in this case, for all  $t \neq 0$ , the hyperbolic structure  $g^t$  is distinct from  $g$  (and for the same reason, for all  $0 \leq t_1 < t_2$ ,  $g^{t_1}$  is distinct from  $g^{t_2}$ ). To see this, it suffices to consider the effect of the stretch map on any nonempty sublamination of  $\mu$  that can be equipped with an invariant measure (for instance, the stump of  $\mu$ ). In the surface  $g^t$ , the length of that measured sub-lamination is multiplied by the factor  $e^t$ , which implies that the structure  $g^t$  is distinct from  $g$ .

### 3.7 Thurston's asymmetric metric

An *asymmetric metric* on a set  $X$  is a map  $L : X \times X \rightarrow \mathbb{R}_+$  that satisfies all the axioms of a metric except the symmetry axiom, and such that the symmetry axiom is not satisfied, i.e. there exist  $x$  and  $y$  in  $X$  such that  $L(x, y) \neq L(y, x)$ . Teichmüller space is equipped with an asymmetric metric that was defined by Thurston. The stretch lines that we considered in Section 3.6 are geodesics for that metric, provided the stump of the complete lamination  $\mu$  is not empty. We now recall the definition of this metric.

Let  $g$  and  $h$  be two hyperbolic structures on  $S$  and let  $\varphi : S \rightarrow S$  be a diffeomorphism of  $S$  which is isotopic to the identity. The *Lipschitz constant*  $\text{Lip}(\varphi)$  of  $\varphi$  is defined as

$$\text{Lip}(\varphi) = \sup_{x \neq y \in S} \frac{d_h(\varphi(x), \varphi(y))}{d_g(x, y)}.$$

The logarithm of the infimum of these Lipschitz constants over all diffeomorphisms  $\varphi$  in the isotopy class of the identity is denoted by

$$L(g, h) = \log \inf_{\varphi \sim \text{Id}_S} \text{Lip}(\varphi). \quad (3.4)$$

It is obvious that  $L$  satisfies the triangle inequality. It is less obvious that  $L(g, h) \geq 0$  for all  $g$  and  $h$  (this uses the fact that any two hyperbolic structures on  $S$  have the same area, see [61] Proposition 2.1).

Making  $g$  and  $h$  vary in their respective homotopy classes does not change  $L(g, h)$  and thus we obtain a function which is well-defined on  $\mathcal{T}(S) \times \mathcal{T}(S)$ , which is Thurston's asymmetric metric. We shall denote it by the same letter:

$$L : \mathcal{T}(S) \times \mathcal{T}(S) \rightarrow \mathbb{R}_+.$$

Thurston showed that the quantity  $L(g, h)$  can also be computed by comparing lengths of homotopic closed geodesics for the metrics  $g$  and  $h$ . More precisely, for any homotopy class  $\alpha$  of essential simple closed curves on  $S$ , we consider the quantity

$$r_{g,h}(\alpha) = \frac{l_h(\alpha)}{l_g(\alpha)}$$

and we set

$$K(g, h) = \log \sup_{\alpha \in \mathcal{S}} r_{g,h}(\alpha). \quad (3.5)$$

Again, it is easy to see that the function  $K$  satisfies the triangle inequality. The fact that we have  $K(g, h) > 0$  for all  $g \neq h$  is not trivial, and it is proved by Thurston in [61] Theorem 3.1.

Since each homeomorphism  $\varphi$  considered in Formula (3.4) is isotopic to the identity, it preserves each homotopy class of simple closed curves in  $S$ . From that observation, it is easy to see that  $K \leq L$ . In his paper [61], Thurston proves the following result:

**Theorem 3.18** (Thurston [61] Theorem 8.5).  $K = L$ .

Note that in the same way as in the definition of the Teichmüller metric (see Remark 2.25 above), by using the extension of the hyperbolic length function to the space of compactly supported measured geodesic laminations, we can replace the supremum in 3.5 by the supremum over the set of compactly supported measured geodesic laminations, that is, we have

$$K(g, h) = \log \sup_{\alpha \in \mathcal{ML}_0} \frac{l_h(\alpha)}{l_g(\alpha)} = \log \sup_{\alpha \in \mathcal{PL}_0} r_{g,h}(\alpha). \quad (3.6)$$

By compactness of the space  $\mathcal{PL}_0(S)$ , the supremum in (3.6) is attained at some measured geodesic lamination.

Along a stretch line directed by a complete geodesic lamination  $\mu$  whose stump  $\gamma$  is not empty, the stretch map between  $g$  and  $g^t$  is  $e^t$ -Lipschitz and  $r_{g,g^t}(\gamma) = e^t$ . This gives  $L(g, g^t) = K(g, g^t) = t$  for all  $t \geq 0$  (and justifies the dilatation factor  $e^t$ ). A measured lamination  $\alpha$  attaining the supremum in (3.6) is generally not unique, even up to a scalar factor. For instance, any nonempty sub-lamination of such a lamination has the same property. There exists a geodesic lamination that is canonically associated to an ordered pair of hyperbolic structures  $(g, h)$ , but this lamination is not necessarily equipped with a transverse measure. Thurston introduced it in the course of proving Theorem 3.18. He first defined the notion of a *maximally stretched geodesic lamination* from a hyperbolic structure  $g$  to a hyperbolic structure  $h$ . This is a (chain-recurrent) geodesic lamination  $\lambda$  for which there exists an  $e^{K(g,h)}$ -Lipschitz map from a neighborhood of  $\lambda$  in the surface  $S$  equipped with the metric  $g$  to a neighborhood of  $\lambda$  in the surface  $S$  equipped with the metric  $h$ . Thurston proved that the union of all maximally stretched geodesic lamina-

tions from  $g$  to  $h$  is a geodesic lamination. This geodesic lamination is denoted by  $\mu(g, h)$ .

**Remarks** 1) For any  $\eta$  in  $\mathcal{ML}_0(S)$ , we can find two hyperbolic structures  $g$  and  $h$  such that the supremum in  $r_{g,h}(\alpha)$  in (3.6) is attained for  $\alpha = \eta$ . This follows from Thurston's construction in [61] §4, taking in that construction a complete geodesic lamination  $\mu$  that contains  $\eta$  and taking  $g$  and  $h$  (in the right order) on a stretch line directed by  $\mu$ .

2) If an element  $\eta$  in  $\mathcal{ML}_0(S)$  realizes the maximum of  $r_{g,h}(\alpha)$  as above, then any other element of  $\mathcal{ML}_0(S)$  which is topologically equal to  $\eta$  (that is, with a possibly different transverse measure) also realizes the maximum. This follows from Thurston's definition of  $\mu(g, h)$  in [61] §8, which is independent of the transverse measure of the lamination (see Theorem 8.1 of [61]).

One can see that the function  $L$  is generally not symmetric by looking at the following example.

**Example 3.19** (The metric  $L$  is not symmetric). In Figure 19, the surface  $S_{2,0}$  is equipped with two hyperbolic structures  $g$  and  $h$  for which the left and right handles are almost isometric for the metrics  $g$  and  $h$ . One can do explicit computations of lengths of closed geodesics, by decomposing the two hyperbolic surfaces into hyperbolic pairs of pants, with the central curve being an element of the decomposition, and with the two other closed curves being the core curves of the two handles represented in Figure 19. Consider the central cylinder that joins the handles. The height of this cylinder, measured in the structures  $g$  and  $h$  respectively, is approximately equal to  $c$  and  $d$ , with  $d > c$ . Applying the formulae that give the distance between two boundary components of a hyperbolic pair of pants in terms of the length of the boundary components (see e.g. [19] p. 151), we obtain  $l_g(\alpha) \simeq Ae^{-c}$  and  $l_h(\alpha) \simeq Ae^{-d}$ , where  $A$  is a constant that depends on the size of the handles. Thus, the smallest Lipschitz constant of a map from  $g$  to  $h$  is approximately equal to  $d/c$ , whereas the smallest Lipschitz constant of a map from  $h$  to  $g$  is bounded below by  $e^{-c}/e^{-d} = e^{d-c}$ . The latter is very large compared to  $d/c$  when  $d$  is large compared to  $c$ . Thus, we obtain examples of hyperbolic structures  $g$  and  $h$  satisfying  $L(g, h) \neq L(h, g)$ .

On the other hand, there are instances of distinct hyperbolic structures  $g$  and  $h$  satisfying  $K(g, h) = K(h, g)$  (hence  $L(g, h) = L(h, g)$ ). An example of such a pair is given in Figure 20.

The notion of a geodesic (respectively of a geodesic line, or a geodesic ray) for an asymmetric metric space can be defined as in the case of a genuine metric, except that one has to be careful about the order of the variables in the case of an asymmetric metric. More precisely, we shall say that a map  $\gamma : I \rightarrow \mathcal{T}(S)$ , (where  $I$  is a closed interval of  $\mathbb{R}$  and where  $\mathcal{T}(S)$  is

equipped with the asymmetric metric  $L$ ) is geodesic if for all  $t \leq t'$  in  $I$  we have  $t' - t = L(\gamma(t), \gamma(t'))$ . Thurston proved the following

**Theorem 3.20** (Stretch lines are geodesics, Thurston [61] Theorem 8.5 and Corollary 4.2). *Let  $\mu$  be a complete geodesic lamination on  $S$ . If the stump of  $\mu$  is not the empty lamination, then any stretch line in  $\mathcal{T}(S)$  directed by  $\mu$  is a geodesic line for Thurston's asymmetric metric. In other words, we have, for every  $0 \leq t \leq t'$ ,*

$$L(\phi_\mu^{-1}(e^t F), \phi_\mu^{-1}(e^{t'} F)) = t' - t.$$

**Theorem 3.21** (Concatenations of stretches, Thurston [61] Theorem 8.5). *Let  $g$  and  $h$  be two hyperbolic structures. Then, we can pass from  $g$  to  $h$  by a geodesic which is a finite concatenation of pieces of stretch lines along complete geodesic laminations  $\mu_1, \dots, \mu_k$ , all of them containing  $\mu(g, h)$ . Furthermore, the number of such pieces needed to go from  $g$  to  $h$  is bounded by a constant which depends only on the topological type of the surface (that is, on the genus and the number of punctures). Such a geodesic path is obtained in the following way. We first choose an arbitrary completion  $\mu_1$  of  $\mu(g, h)$  and we stretch the structure  $g$  along  $\mu_1$  until we reach a first point  $g'$  satisfying  $\mu(g', h) \neq \mu(g, h)$ . We necessarily have  $\mu(g', h) \supset \mu(g, h)$ . Starting now with  $g'$  instead of  $g$  and continuing in the same manner, we reach  $h$  after a finite number of steps.*

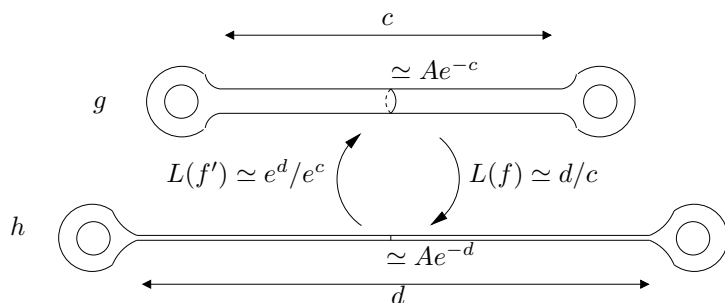


Figure 19. An example (due to Thurston) of hyperbolic structures  $g$  and  $h$  satisfying  $L(g, h) \neq L(h, g)$ .

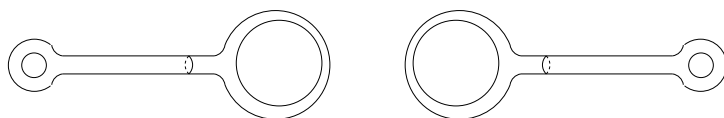


Figure 20. An example of distinct hyperbolic structures  $g$  and  $h$  satisfying  $K(g, h) = K(h, g)$ :  $h$  is the image of  $g$  by an order-two isometry which is not isotopic to the identity.



**Remarks** 1) That the number of pieces in this statement is bounded from above follows from the fact that there is a bound (that depends only on the topological type of the surface) for the length of a strictly increasing (with respect to inclusion) sequence of geodesic laminations.

2) The geodesic path leading from  $g$  to  $h$  is in general not unique, since it depends on the choice of the completion at each step.

3) The bound in 1) does not imply that the number of distinct geodesics from  $g$  to  $h$  is finite, since at each step there are in general infinitely many ways of completing the given geodesic lamination.

4) An anti-stretch ray is (up to reparametrization) a geodesic ray for the asymmetric metric on Teichmüller space which is dual to Thurston's asymmetric metric, that is, the asymmetric metric defined by

$$K^*(g, h) = \log \sup_{\alpha \in \mathcal{S}} \frac{l_g(\alpha)}{l_h(\alpha)} \quad (3.7)$$

for any hyperbolic metrics  $g$  and  $h$ .

Here are some questions related to the example described in Figure 20.

- What does a typical geodesic from  $g$  to  $h$  look like? Is it a segment of a stretch line? is it a concatenation of at least two such segments? is it something else?
- Describe a geodesic from  $g$  to  $h$  that (up to reparametrization) is also a geodesic from  $h$  to  $g$  when traversed in the opposite direction.

We now discuss a few properties of Thurston's asymmetric metric  $L$  with respect to sequences in  $\mathcal{T}(S)$  that tend to infinity. Then, we shall discuss the topologies that this asymmetric metric induces on  $\mathcal{T}(S)$ . Here, as usual, we say that a sequence  $(g_n)$  in  $\mathcal{T}(S)$  tends to infinity (and write  $g_n \rightarrow \infty$ ) if for any compact subset  $\mathcal{K}$  of  $\mathcal{T}(S)$ , we have  $g_n \in \mathcal{T}(S) \setminus \mathcal{K}$  for all  $n$  large enough.

It is known that Teichmüller's metric  $d_T$  is proper (that is, closed balls are compact), which implies that a sequence of points  $(g_n)$  in  $\mathcal{T}$  tends to infinity if and only if for every  $g$  in  $\mathcal{T}$  (or, equivalently, for some  $g$  in  $\mathcal{T}$ ), we have  $d_T(g, g_n) \rightarrow \infty$  as  $n \rightarrow \infty$ . In [48], we prove the following analogous result for the asymmetric metric  $L$ .

**Proposition 3.22.** *For any sequence  $(g_n)$  in  $\mathcal{T}$  and for any  $h \in \mathcal{T}$ , we have the following equivalences:*

$$g_n \rightarrow \infty \iff L(h, g_n) \rightarrow \infty \iff L(g_n, h) \rightarrow \infty.$$

We deduce the following

**Corollary 3.23.** *Let  $t \mapsto g^t$  be a stretch ray starting at a hyperbolic structure  $g$  and directed by a complete geodesic lamination  $\mu$ , such that either the stump*

of  $\mu$  is not empty (or, equivalently,  $\mu$  is not an ideal triangulation) or the horocyclic measured lamination  $\lambda_\mu(g)$  is not empty. Then, the map  $t \mapsto g^t$  is proper. (In other words, as  $t$  tends to infinity, the point  $g^t$  tends to infinity in Teichmüller space.)

*Proof.* If the stump of  $\mu$  is not empty, then we have, by Theorem 3.20,  $L(g, g^t) = t$ , which, by Proposition 3.22, implies that  $g^t$  tends to infinity. If the horocyclic measured lamination  $\lambda_\mu(g)$  is not empty, then, as  $t \rightarrow \infty$ , we have  $l_{g^t}(\lambda_\mu(g)) \rightarrow 0$ . Indeed, by Proposition 3.3 and Lemma 3.9 of [47], there exists a constant  $C > 0$  that depends only on the topological type of the surface  $S$  such that  $l_{g^t}(\lambda_\mu(g^t)) \leq C$ . (In fact, by Lemma 3.9 of [47], we can take  $C = -\chi$  where  $\chi$  is the Euler characteristic of  $S$ .) Note that the setting in [47] is that of closed surfaces, but the arguments work for general surfaces of finite type. We have  $\lambda_\mu(g^t) = e^t(\lambda_\mu(g))$ , therefore  $l_{g^t}(\lambda_\mu(g)) \leq Ce^{-t}$ , which tends to zero as  $t$  tends to infinity. This proves the claim. Now if  $l_{g^t}(\lambda_\mu(g)) \rightarrow 0$ , then  $r_{g^t, g}(\lambda_\mu(g)) = \frac{l_g(\lambda_\mu(g))}{l_{g^t}(\lambda_\mu(g))} \rightarrow \infty$ , which gives  $L(g^t, g) \rightarrow \infty$ , which, again by Proposition 3.22, implies that  $g^t$  tends to infinity.  $\square$

In the same paper [48], we prove the following

**Proposition 3.24.** *For any sequence  $(g_n)$  in  $\mathcal{T}(S)$  and for any element  $g$  in  $\mathcal{T}(S)$ , we have the following equivalences:*

$$g_n \rightarrow g \iff L(g, g_n) \rightarrow 0 \iff L(g_n, g) \rightarrow 0.$$

Associated to the asymmetric metric  $L$ , for each  $g$  in  $\mathcal{T}$  and for each  $R > 0$ , there are two different notions of closed balls, which we call *left closed ball* centered at  $g \in \mathcal{T}(S)$  of radius  $R$  and the *right closed ball* centered at  $g \in \mathcal{T}(S)$  of radius  $R$ , and which we denote respectively by  ${}_gB(R)$  and  $B_g(R)$ . These are defined respectively by

$${}_gB(R) = \{h \in \mathcal{T}(S) \mid L(g, h) \leq R\} \tag{3.8}$$

and

$$B_g(R) = \{h \in \mathcal{T}(S) \mid L(h, g) \leq R\}. \tag{3.9}$$

In [48], we prove the following

**Proposition 3.25.** *For any  $g$  in  $\mathcal{T}(S)$  and for any  $R > 0$ , the closed balls  ${}_gB(R)$  and  $B_g(R)$ , centered at  $g \in \mathcal{T}(S)$  and of radius  $R$ , are compact for the usual topology.*

One can similarly define left (respectively right) open balls in  $\mathcal{J}$ , by taking the same definitions in (3.8) (respectively (3.9)) above, except that the large inequality is replaced by a strict inequality.

Since the metric on Teichmüller space that we are studying here is asymmetric, it is natural to consider two topologies on that space, the one generated by the collection of right open balls, and the one generated by left open balls. We shall call the first of these topologies the topology associated to Thurston's asymmetric metric  $K$  and the second one the topology associated to its dual asymmetric  $K^*$ . It is easy to see that a sequence  $(g_n)$  of points in Teichmüller space converges to a point  $g$  in that space for the topology generated by the left (respectively right) open balls if and only if we have  $K(g, g_n) \rightarrow 0$  as  $n \rightarrow \infty$  (respectively  $K^*(g, g_n) \rightarrow 0$ ) (that is, we have the same convergence criteria than for genuine metrics). Proposition 3.24 is equivalent to the following

**Corollary 3.26.** *The topologies on Teichmüller space generated by Thurston's asymmetric metric and by its dual asymmetric metric coincide with the usual topology of that space.*

From Propositions 3.24 and 3.25 and from Corollary 3.26, we obtain the following

**Proposition 3.27.** *Teichmüller space equipped with Thurston's asymmetric metric is proper. More precisely, left and right closed balls are compact for the topology generated by the asymmetric metric.*

Equivalently, we have the following

**Proposition 3.28.** *For every point  $g$  in Teichmüller space, the functions  $h \mapsto L(g, h)$  and  $h \mapsto L(h, g)$  are proper.*

Herbert Busemann developed a theory of spaces  $(X, \delta)$  satisfying all the axioms of a metric space except the symmetry axiom, and satisfying the following additional axiom:

$$\delta(x, x_n) \rightarrow 0 \iff \delta(x_n, x) \rightarrow 0 \text{ for any } x \text{ and for any sequence } (x_n) \text{ in } X.$$

For such spaces, there is a well-defined topology on  $X$ , the one associated to the bona fide metric  $\max\{\delta(x, y), \delta(y, x)\}$ . Proposition 3.24 insures that Thurston's asymmetric metric fits into that theory, and that this topology is the same as the one generated by the collection of left (or right) open balls. Given such a space  $(X, \delta)$ , Busemann introduced the following notion for such spaces (see [14] Chapter 1):  $(X, \delta)$  is *complete* if and only if for every sequence  $(x_n)_{n \geq 0}$  in  $X$  satisfying  $\delta(x_n, x_{n+m}) \rightarrow 0$  as  $n$  and  $m \rightarrow \infty$ , the sequence converges to a point in  $X$ .

In that setting, the following generalization of a classical theorem of Hopf and Rinow holds (see [14] Theorem 8, p. 4). If a locally compact space  $(X, \delta)$

satisfies Busemann's axioms stated above, and if this "generalized metric"  $\delta$  is intrinsic in the sense that for any  $x$  and  $y$  in  $X$ , the value  $\delta(x, y)$  is equal to the infimum of the lengths of all curves joining  $x$  and  $y$ , then the following two properties are equivalent:

- (1) left closed balls in  $X$  are compact;
- (2)  $X$  is complete.

Here, the length of a curve is defined in a similar way as in the case of a genuine metric space. We do not need to enter here into the details of this definition because, by a result of Thurston (see [61] §6) that we already mentioned, Thurston's asymmetric metric is geodesic (that is, any two points can be joined by a geodesic), and a generalized geodesic metric in this sense is intrinsic (see [14] p. 3).

Thus, we obtain the following

**Corollary 3.29.** *Thurston's asymmetric metric is complete.*

In the next three sections, we study explicit examples of stretch lines that are simple enough to make computations.

### 3.8 Example I of a stretch line: the four-punctured sphere

In this section,  $S = S_{0,4}$  is the four-punctured sphere. We consider the simple case of a hyperbolic structure on  $S$  that is obtained by gluing two ideal quadrilaterals that can be exchanged by an order-two isometry of  $S$  that fixes pointwise each of the four edges of the quadrilaterals. Let us call such a hyperbolic structure a *symmetric* hyperbolic structure on the four-punctured sphere. Figure 21 represents the surface  $S$  as a union of two isometric quadrilaterals, with vertices  $A, B, C, D$ . Each quadrilateral is equipped with a diagonal in such a way that the two diagonals join the same cusps of the surface  $S$ , say  $D$  and  $B$ , as represented in Figure 21.

The union of the edges of the quadrilaterals (that are glued by pairs in the surface), together with the two diagonals that join  $D$  and  $B$ , constitute an ideal triangulation of  $S$  which we denote by  $\mu$ . In the case considered of a symmetric structure, the fact that the hyperbolic structure is complete is equivalent to the fact that for each of the edges  $AB, BC, CD$  and  $DA$ , the two distinguished points (in the sense defined in §3.2) corresponding to the two ideal triangles that are adjacent to that edge coincide. Furthermore, since the two quadrilaterals we started with are isometric, the algebraic distance  $d$  between the distinguished points on each of the two diagonals joining the vertices  $D$  and  $B$  coincide. As a matter of fact, the set of symmetric hyperbolic

structures on the four-punctured sphere is parametrized by  $\mathbb{R}$ , since such a hyperbolic structure is completely determined by the algebraic distance  $d$ .

The horocyclic foliation associated to a symmetric hyperbolic structure  $g$  together with the ideal triangulation  $\mu$  is a foliation which has one cylinder whose height is equal to  $|d|$  (which could be equal to 0) and four other cylinders forming neighborhoods of cusps. In other words, the associated horocyclic measured geodesic lamination  $\lambda_\mu(g)$  is a simple closed geodesic  $\alpha$  equipped with a Dirac transverse measure of mass  $|d|$  (and it is the empty foliation if  $d = 0$ ). Stretching the hyperbolic structure  $g$  in the direction  $\mu$  always produces a symmetric hyperbolic structure. If  $t \mapsto g^t$ ,  $t \in \mathbb{R}$ , is the stretch line starting at  $g$  and directed by  $\mu$ , then, provided  $d \neq 0$ , we have the following:

- (1)  $l_{g^t}(\alpha) \rightarrow 0$  as  $t \rightarrow \infty$ ;
- (2) The distance between the distinguished points on the diagonal  $BD$  for the hyperbolic structure  $g^t$  equals  $e^t$  times that distance for the hyperbolic structure  $g = g^0$ ;
- (3) If  $\nu$  is any compactly supported measured geodesic lamination on  $S$  with  $\nu \neq \alpha$ , then, we have  $i(\nu, \alpha) \neq 0$  and  $l_{g^t}(\nu) \rightarrow \infty$  as  $t \rightarrow \infty$ ;
- (4)  $\lim_{t \rightarrow \infty} g^t = [\alpha]$  as a point on Thurston's boundary  $\mathcal{PMF}_0$ ;
- (5) As  $t \rightarrow -\infty$ ,  $g^t$  does not go to infinity but converges to the hyperbolic structure that is obtained by the completely symmetric gluing of the four ideal triangles that we started with, that is, the gluing where the distances between the pairs of distinguished points on the six edges of  $\mu$  are all equal to zero.

### 3.9 Example II of a stretch line: the punctured torus

In this section, we consider the surface  $S = S_{1,1}$ , that is, the punctured torus, equipped with a hyperbolic metric  $g$  satisfying some properties which we now state. Let  $\mu$  be a complete geodesic lamination on  $S$  whose leaves consist in a

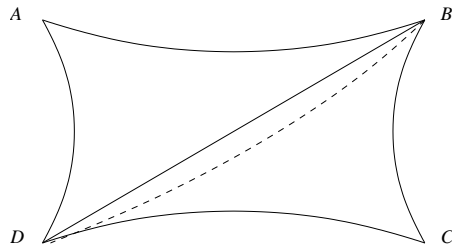


Figure 21. The four-punctured sphere with its ideal triangulation  $\mu$ .

simple closed geodesic  $\gamma$ , together with a bi-infinite geodesic  $\delta$  converging at both ends to the cusp, and in two other bi-infinite geodesics, each having one end converging to the cusp and the other end spiraling around  $\gamma$ , in such a way that the spiraling of the two geodesics takes place in the same direction, as in Figure 22. The surface  $S \setminus \mu$  has two connected components, each of them, with its intrinsic metric, being the interior of a hyperbolic ideal triangle. We choose the metric  $g$  so that its associated horocyclic lamination  $\lambda_\mu(g)$  is a closed geodesic  $\alpha$  satisfying  $i(\alpha, \gamma) = 1$ , as represented in Figure 23.

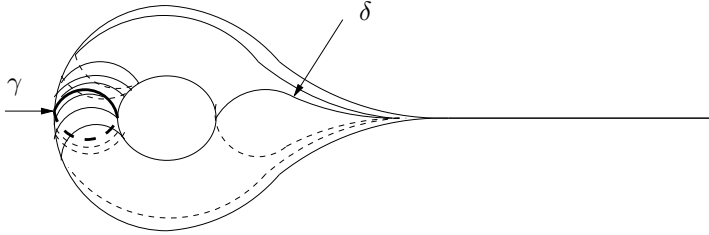


Figure 22. The complete geodesic lamination  $\mu$  is the union of the closed geodesic  $\gamma$ , the bi-infinite leaf  $\delta$  and two other bi-infinite geodesics that spiral around  $\gamma$  at one end and that converge to the cusp of  $S$  at their other end.

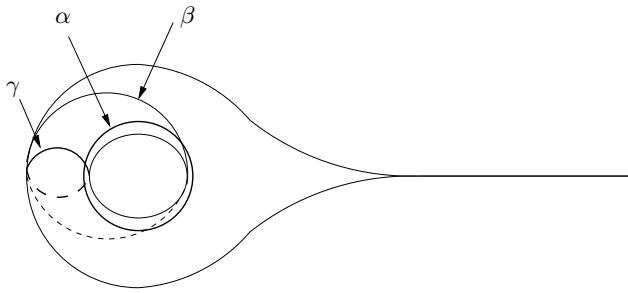


Figure 23.  $\alpha$  is the horocyclic measured geodesic lamination associated to  $g$  and  $\mu$ . We are interested in the behaviour of the lengths of the three curves  $\alpha$ ,  $\beta$  and  $\gamma$  along the stretch and the anti-stretch ray directed by  $\mu$  and starting at  $g$ .

In Figure 23 are represented the stump  $\gamma$  of  $\mu$ , the closed geodesic  $\alpha$  and the simple closed geodesic  $\beta$  obtained from  $\alpha$  by a left Dehn twist along  $\gamma$ .

We shall study the behaviour of the lengths of these particular closed geodesics under the stretch and anti-stretch rays directed by  $\mu$  and starting at  $g = g^0$ . The behaviour of the lengths of these geodesics will permit us to understand the change in geometry of the surface  $g$  under the stretch map.

The closed geodesics  $\alpha$  and  $\beta$  satisfy  $\text{Card}(\gamma \cap \beta) = \text{Card}(\alpha \cap \gamma) = 1$ . The reason why we consider these geodesics is that eventually we want to get an idea of the behaviour of the lengths of an arbitrary simple closed geodesic (and more generally of an arbitrary compactly supported measured geodesic lamination) on  $S$  under the stretch (respectively the anti-stretch) ray  $g^t$  as  $t \rightarrow \infty$ , and we expect this behaviour to depend on the intersection pattern of that lamination with the stump of  $\mu$  and with the horocyclic measured geodesic lamination  $\lambda_\mu(g)$ . The general results are given in Theorems 3.31 and 3.32 below, and the results that we present in the example considered here are illustrations of those general results. What makes things work easily in this example is that we can determine the exact positions of these geodesics for the structure  $g^t$  as  $t$  varies.

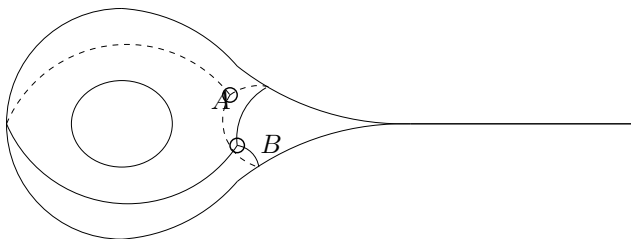


Figure 24. The critical graph of the foliation  $F_\mu(g)$

The horocyclic foliation  $F_\mu(g)$  is the union of a cylinder  $C$  with core curve  $\alpha$  and of another cylinder foliated by leaves that are parallel to the puncture of  $S$ , the two cylinders being glued along the critical graph of  $F_\mu(g)$ . This critical graph is represented in Figure 24. Figure 25 represents the cylinder  $C$  to which are attached the nonfoliated parts of  $F_\mu(g)$ . In that figure, the pairs of regions with the same name are identified in the surface  $S$ . In the same figure, we have drawn in bold lines the segments induced by  $\gamma$  and  $\delta$  on the cylinder  $C$ . This cylinder has a symmetry which is probably more apparent if we cut the cylinder along the arc induced on it by  $\delta$ . After this cutting, we obtain Figure 26, that is, a large quadrilateral  $R$  which is the union of two smaller isometric quadrilaterals  $R_1$  and  $R_2$ . (The internal face of the cylinder  $C$  of Figure 25 is the quadrilateral  $R$  of Figure 26.) The quadrilateral  $R$  has an order-two symmetry of center  $w$ , where  $w$  is the midpoint of the segment induced by  $\gamma$  on the cylinder  $C$ . Let  $s$  denote the midpoint of the arc induced by  $\delta$  on that cylinder.

By the uniqueness of the closed geodesic in each homotopy class of closed curves on  $S$ , the closed geodesic  $\alpha$  is preserved by this order-two symmetry of  $R$ . Therefore this geodesic passes by the points  $s$  and  $w$ . The same holds for the closed geodesic  $\beta$ , which passes by  $w$  (see Figure 26).

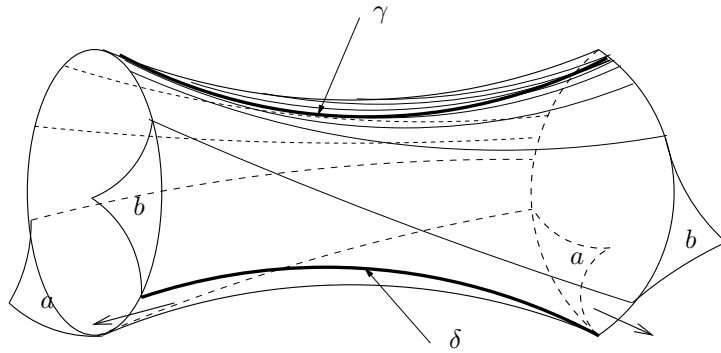


Figure 25. The cylinder  $C$  and on its boundary the nonfoliated regions of  $F_\mu(g)$ . These nonfoliated regions are labeled  $a$  and  $b$ , these letters corresponding to the points  $A$  and  $B$  respectively on the critical graph of Figure 24. The two arrows indicate that the corresponding spikes converge to the cusp.

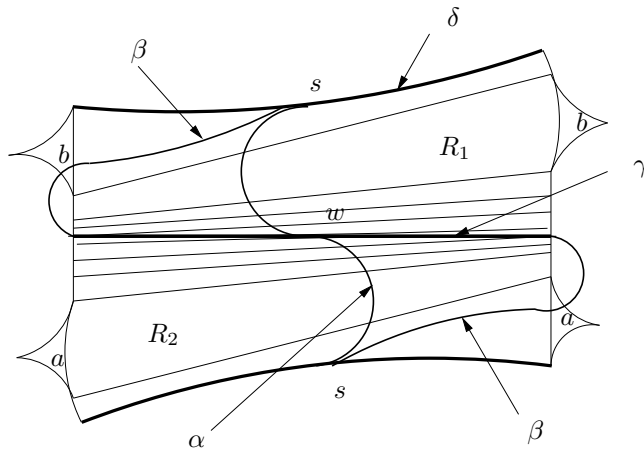


Figure 26. The quadrilateral obtained by cutting the cylinder  $C$  along  $\delta$ .



All these results on the relative positions of  $\alpha$  and  $\beta$  remain true when the metric  $g$  is replaced by the stretched (respectively anti-stretched) metric  $g^t$ .

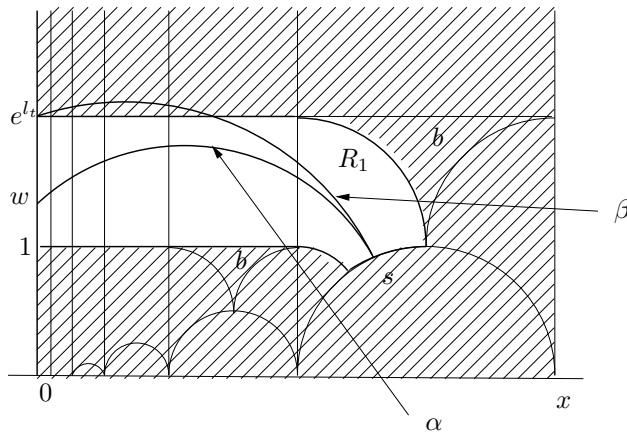


Figure 27. The quadrilateral  $R_1$  in the upper half-plane model of  $\mathbb{H}^2$  is the non-shaded region.

To do computations, we can use Figure 27 in which the quadrilateral  $R_1$  is represented in the upper half-plane model of hyperbolic space. In that figure,  $l_t$  denotes the length of  $\gamma$  for the structure  $g^t$ . However, it is possible to determine the asymptotic behaviour of the lengths of the closed curves  $\alpha$ ,  $\beta$  and  $\gamma$  along the stretch (respectively anti-stretch) ray without computations. It suffices for that to determine the behaviour of the lengths of the sides of  $R_1$  along these rays. We refer to Figure 26, in which the common length of the horizontal sides of  $R$  is equal to the length of  $\gamma$ , and the height of  $R$  is defined to be the length of the leaf of  $F_{g^t}(\mu)$  that is equidistant from the boundaries of  $C$ . This last quantity is an upper bound for the length of  $\alpha$ . Now we observe the following facts:

- (1) The distances between the non-foliated regions grow linearly with  $e^t$  along the stretch ray, and they decrease to zero along the anti-stretch ray;
- (2) In each spike, the lengths of the horocyclic arcs that are not on the boundary of a non-foliated region and that are leaves of the horocyclic foliation decrease to zero along the stretch ray and grow to 1 along the anti-stretch ray.

Using these facts, it is easy to see that along a stretch ray, the height of  $R$  tends to zero, whereas the common length of the horizontal sides of  $R$  tends to infinity. Along an anti-stretch line, the height of  $R$  tends to infinity whereas the length of a horizontal side tends to zero. Now since we know the exact

positions of the various curves, we obtain the following, along the stretch ray  $g^t$ :

- (1)  $\lim_{t \rightarrow \infty} l_{g^t}(\gamma) = \infty$ ;
- (2)  $\lim_{t \rightarrow \infty} l_{g^t}(\alpha) = 0$ ;
- (3)  $\lim_{t \rightarrow \infty} l_{g^t}(\beta) = \infty$ .

Likewise, along the anti-stretch ray  $g^t$ , we have

- (1)  $\lim_{t \rightarrow \infty} l_{g^t}(\gamma) = 0$ ;
- (2)  $\lim_{t \rightarrow \infty} l_{g^t}(\alpha) = \infty$ ;
- (3)  $\lim_{t \rightarrow \infty} l_{g^t}(\beta) = \infty$ .

This example is rather limited in scope because the surface considered is not large enough so as to contain closed curves that are disjoint from the stump, or from the horocyclic measured lamination, or from both. This will be possible in the example which we consider next.

### 3.10 Example III of a stretch line: the closed surface of genus 2

In this example,  $S = S_{2,0}$  is a closed surface of genus 2 equipped with a complete geodesic lamination  $\mu$  whose stump consists of two nonseparating closed geodesics  $\gamma_1$  and  $\gamma_2$ , represented in Figure 28, with the other leaves of  $\mu$  spiraling around  $\gamma_1 \cup \gamma_2$  in the same direction, as represented in Figure 29. We choose a hyperbolic structure  $g$  on  $S$  whose associated horocyclic measured geodesic lamination  $\lambda_\mu(g)$  consists of a simple closed geodesic  $\alpha$ , satisfying  $\text{Card}\{\gamma_1 \cap \alpha\} = \text{Card}\{\gamma_2 \cap \alpha\} = 1$ . The topological pattern made by the curves  $\gamma_1$ ,  $\gamma_2$  and  $\alpha$  is represented in Figure 28.

As in the preceding examples, we study the behaviour of the lengths of some particular simple closed geodesics under a stretch or an anti-stretch ray directed by  $\mu$  and starting at  $g$ . The closed geodesics considered here are  $\gamma_1$  and  $\gamma_2$  (chosen because they are contained in the stump of  $\mu$ ), the geodesic  $\alpha$ , and the two geodesics  $\beta_1$  and  $\beta_2$  that are represented in Figure 30, satisfying  $(\gamma_1 \cup \gamma_2) \cap \beta_1 \neq \emptyset$ ,  $(\gamma_1 \cup \gamma_2) \cap \beta_2 = \emptyset$  and  $\alpha \cap \beta_1 = \alpha \cap \beta_2 = \emptyset$ . Again, the explicit computations that we do in this example are illustrations of Theorems 3.31 and 3.32 below.

The horocyclic foliation  $F_\mu(g)$  is a foliation with one cylinder with core curve in the class of  $\alpha$ . In Figure 31 we have represented the critical graph of  $F_\mu(g)$  (after collapsing the non-foliated regions). Let  $C$  denote the cylinder obtained by cutting the surface along the critical graph of  $F_\mu(g)$ . This cylinder is represented in Figure 33 (together with the non-foliated regions on its

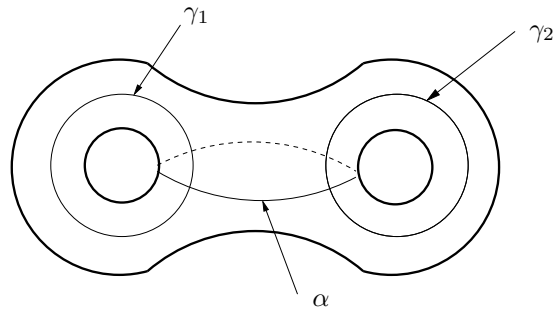


Figure 28. The union  $\gamma_1 \cup \gamma_2$  represents the stump of the complete geodesic lamination  $\mu$ , and  $\alpha$  represents the horocyclic measured geodesic lamination  $\lambda_\mu(g)$  of the surface  $S = S_{2,0}$ .

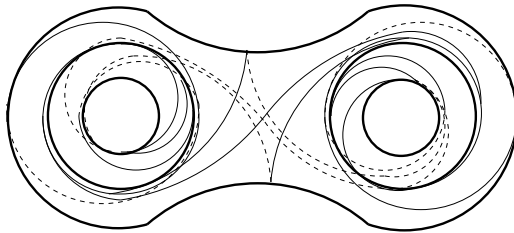


Figure 29. The geodesic lamination  $\mu$ , with stump  $\gamma_1 \cup \gamma_2$ . There are six leaves of  $\mu$  which are not contained in the stump. Each of these leaves is a bi-infinite geodesic that spirals around  $\gamma_1$  or  $\gamma_2$ . The spiraling around each curve  $\gamma_1$  and  $\gamma_2$  is in the same direction.

boundary, as in the preceding example). Each connected component of the intersection of the leaves of  $\mu$  with  $C$  is a segment that joins the two boundary components of that cylinder, as shown in Figure 33. From the definition of the transverse measure of  $F_\mu(g)$ , it follows that all these connected components have the same length. In particular, we have  $l_g(\gamma_1) = l_g(\gamma_2)$ .

We consider the cylinder  $C$  as the union of four quadrilaterals  $R_1, \dots, R_4$ . This decomposition into quadrilaterals is induced by  $\gamma_1$  and  $\gamma_2$  and by two other geodesic segments that join distinguished points on the leaves of  $\mu$ , whose trace on the cylinder  $C$  cut up along  $\gamma_1$  is shown in Figure 34. It is easy to see that the quadrilaterals  $R_1, \dots, R_4$  are congruent. For each of these quadrilaterals, two of the opposite sides are geodesics contained in leaves of  $\mu$ , and the remaining two sides are made out of horocycles contained in leaves of  $F_\mu(g)$ . (Note that each such side is a segment of a horocycle and not just a union of segments of horocycles. This can be seen by examining Figure 35 which represents one of the quadrilaterals drawn in the upper half-plane.) The cylinder  $C$  has an order-two symmetry with respect to the midpoint of

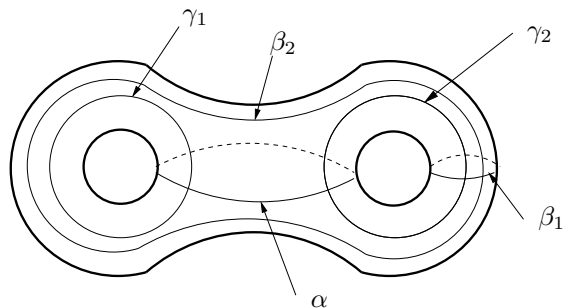


Figure 30. We are interested in the behaviour of the lengths of the five closed geodesics represented here under the stretch line directed by  $\mu$  (of stump  $\gamma_1 \cup \gamma_2$ ) and passing by the hyperbolic structure whose horocyclic measured geodesic lamination is  $\alpha$ .

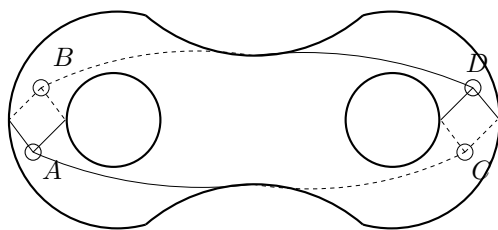


Figure 31. The critical graph of the foliation obtained from  $F_\mu(g)$  by collapsing the non-foliated regions on the points  $A, B, C, D$ .

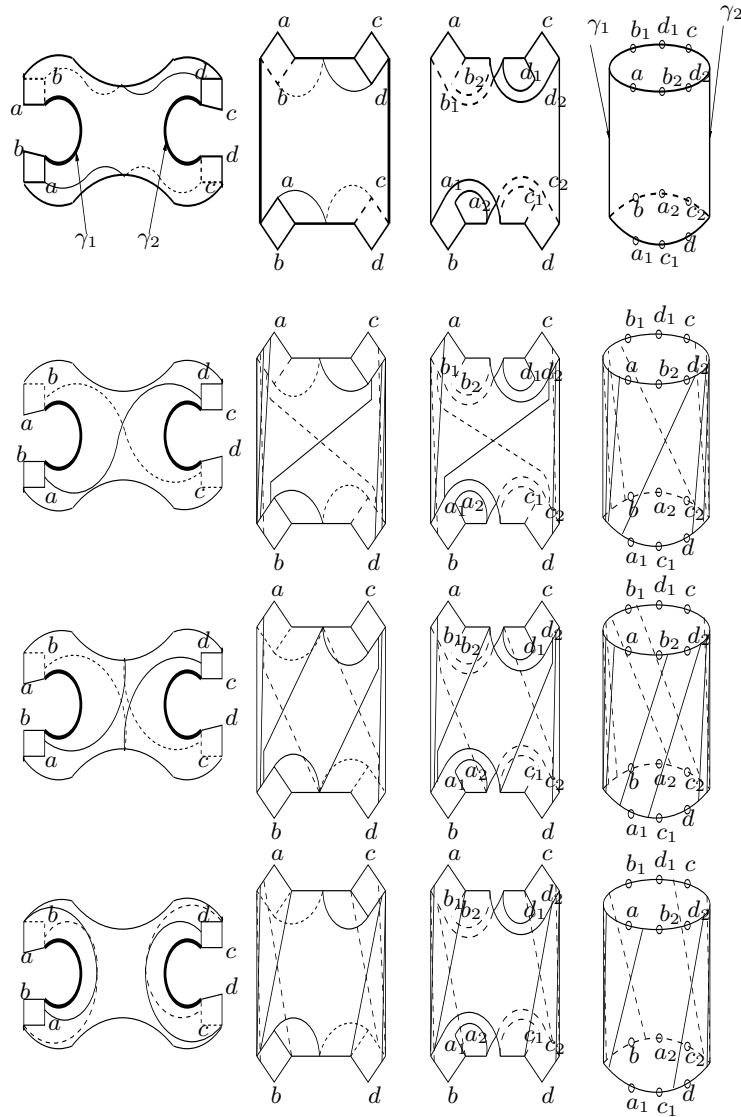


Figure 32. Line 1 represents the passage from the singular graph to the cylinder  $C$  obtained by cutting the surface along the singular graph (see Figure 33). On lines 2, 3 and 4 we have repeated the cutting, showing at each line the two closed curves  $\gamma_1$  and  $\gamma_2$  in bold lines, together with two other leaves of  $\mu$  that spiral along these two closed curves.

each geodesic side of any of the quadrilaterals  $R_i$ . To see this, let us consider the quadrilateral  $R$  obtained by cutting the cylinder  $C$  along the segment  $\gamma_1$ . This quadrilateral is represented in Figure 34, with its induced decomposition into the four quadrilaterals  $R_1, \dots, R_4$ . The four points  $s, w, u, v$  are the intersection points of the leaf of  $F_\mu(g)$  that is equidistant from the boundary of  $C$  (distances being measured with respect to the transverse measure of that foliation) with the geodesic sides of the quadrilaterals  $R_1, \dots, R_4$ . The point  $u$  is the intersection point of that leaf with  $\gamma_2$ . We claim that  $u$  is a center of symmetry for the segments induced on  $R$  by each of the curves  $\alpha, \beta_1$  and  $\beta_2$ . To see this, first of all, note that these segments are situated as shown in Figure 34 because of the combinatorics of the gluing of the boundary components of  $R$ . Now, if we call  $\alpha', \beta'_1, \beta'_2$  the images of the curves  $\alpha, \beta_1, \beta_2$  by the symmetry of center  $u$ , then  $\alpha', \beta'_1, \beta'_2$  are also simple and closed since the interior of  $R$  is symmetric with respect to  $u$ . Since these curves are closed geodesics that are homotopic to the closed geodesics  $\alpha, \beta_1$  and  $\beta_2$  respectively, they coincide with them. In particular,  $\alpha$  passes through the four points  $s, w, u, v$ .

For later use, we compute the value  $x$ , shown in Figure 35, which represents the quadrilateral  $R_1$  in the upper half-plane. This value  $x$  is a Euclidean distance, measured on the boundary of the upper half-plane. It is also equal to the hyperbolic length of the horocyclic edge of the rectangle  $R_1$  lying at height 1, in that same figure. The Euclidean distance  $y$  represented in the same figure is equal to 3, because it is equal to the length of three horocyclic segments that are boundaries of non-foliated triangles in the standard horocyclic foliation of an ideal triangle, and that are at height 1 in the upper half-plane. Therefore, we have the following (see also the caption of Figure 35):

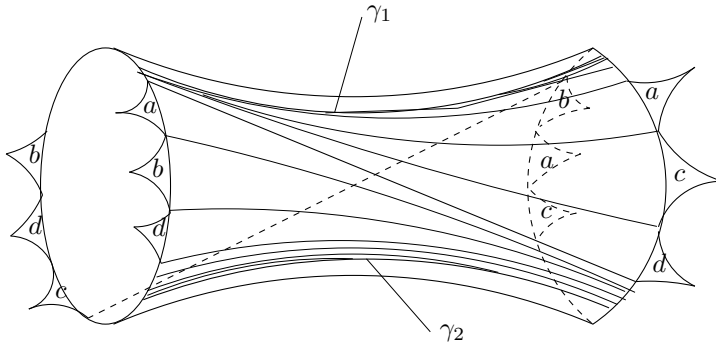


Figure 33. The cylinder  $C$ , together with the non-foliated regions of ideal triangles on its boundary. Non-foliated regions with the same names are identified in the surface  $S$ .

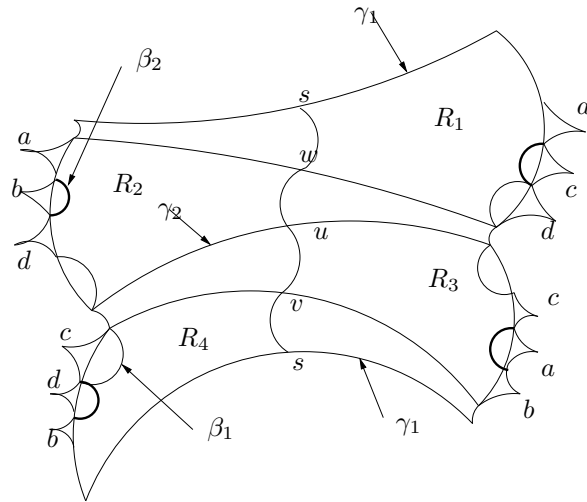


Figure 34. The quadrilateral  $R$  representing the cylinder  $C$  cut along the geodesic  $\gamma_1$ . This quadrilateral is the union of the four quadrilaterals  $R_1, \dots, R_4$  that are drawn. The two horizontal sides of these quadrilaterals that are not labeled are geodesic segments in  $\mu$  that join distinguished points on the edges of that lamination. We have also represented the leaf of  $F_\mu(g)$  that is equidistant from the two boundary components of  $C$ , together with its intersection points  $s, w, u, v$  with the geodesic edges of the quadrilaterals  $R_1, \dots, R_4$ . Finally, we have represented the geodesic segments induced by  $\beta_1$  and  $\beta_2$  on the quadrilateral  $R$ . Each of these segments has two boundary points on the same vertical side of  $R$ . The trace of  $\beta_2$  is drawn in bold lines.

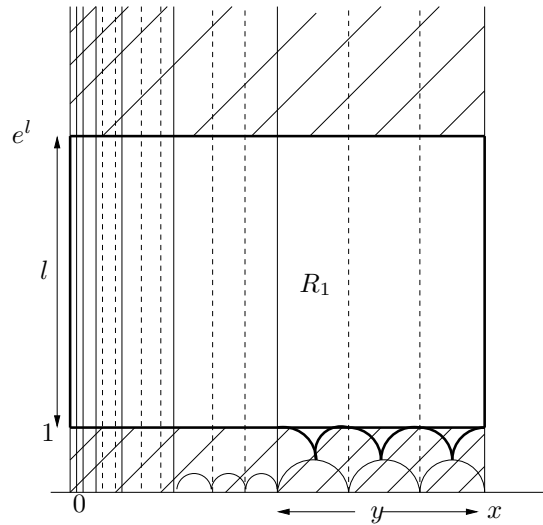


Figure 35. The quadrilateral  $R_1$ , represented in the upper half-plane model of hyperbolic plane, is the region between the two shaded regions. It is bounded by two vertical geodesic segments, both of length  $l = l_g(\gamma_1) = l_g(\gamma_2)$ , and two horizontal horocyclic segments, the lower one having length  $x$  and the upper one having length  $x/h = xe^{-l}$ . The geodesic segment corresponding to  $\gamma_1$  is the segment contained in the vertical ray starting at the origin, joining the point at height 1 to the point at height  $e^l$ . Both the continuous and the dotted vertical lines in the picture are induced by the leaves of  $\mu$  that cross  $R_1$  and that spiral along  $\gamma_1$ . This spiraling is obtained by iterating, under the map  $z \mapsto e^{-l}z$ , the right hand block delimited by the geodesics that are based at the abscissas  $x$  and  $x - y$ . This block is made out of the three ideal triangles suggested by the dotted lines. The (Euclidean) distance  $y$  is equal to 3.



$$x = y(1 + e^{-l} + e^{-2l} + \dots) = \frac{3}{1 - e^{-l}} = \frac{3e^{l/2}}{2 \sinh(l/2)},$$

where  $l = l_g(\gamma_1) = l_g(\gamma_2)$ .

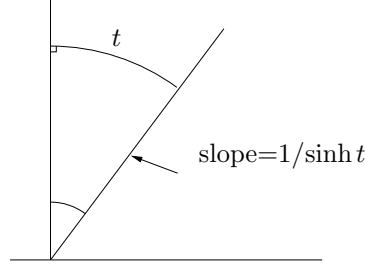


Figure 36. This is a useful ingredient for computing  $l_{g^t}(\alpha)$ ,  $l_{g^t}(\beta_1)$  and  $l_{g^t}(\beta_2)$ . In this figure, the vertical line is a geodesic in the upper half-space model of  $\mathbb{H}^2$ , and the oblique line is a hypercycle (a line consisting of points at hyperbolic distance  $t$  from the geodesic line). The value  $t$  is equal to the (hyperbolic) length of any arc of circle joining these two lines perpendicularly. The slope of the oblique line is then equal to  $1/\sinh t$ .

Now we compute the lengths of the segments labeled  $p$ ,  $q$  and  $r$  in Figure 37. These segments are induced by  $\alpha$ ,  $\beta_1$  and  $\beta_2$  respectively on the quadrilateral represented. We use the formula of Figure 36 giving a relation between the length and the slope.

For the segment  $p$ , the slope is  $\frac{2e^{l/2}}{x}$ , therefore the length is

$$2\sinh^{-1}\left(\frac{x}{2e^{l/2}}\right) = 2\sinh^{-1}\left(\frac{3}{4 \sinh(l/2)}\right).$$

For the segment  $q$ , the slope is  $\frac{2}{x-3}$ , therefore the length is

$$2\sinh^{-1}\left(\frac{x-3}{2}\right) = 2\sinh^{-1}\left(\frac{3e^{-l/2}}{4 \sinh(l/2)}\right).$$

For the segment  $r$ , the slope is 2, therefore

$$2\sinh^{-1}\left(\frac{1}{2}\right) = 2 \log\left(\frac{1 + \sqrt{5}}{2}\right).$$

Now we stretch the structure  $g$  along  $\mu$ . Recall that  $g^t$  is the hyperbolic structure defined by the equality  $F_\mu(g^t) = e^t F_\mu(g)$ . The distances between the ideal triangles of  $S \setminus \mu$  are uniformly stretched. All the remarks that we made about the quadrilaterals  $R_1, \dots, R_4$  with respect to the metric  $g$  are valid for

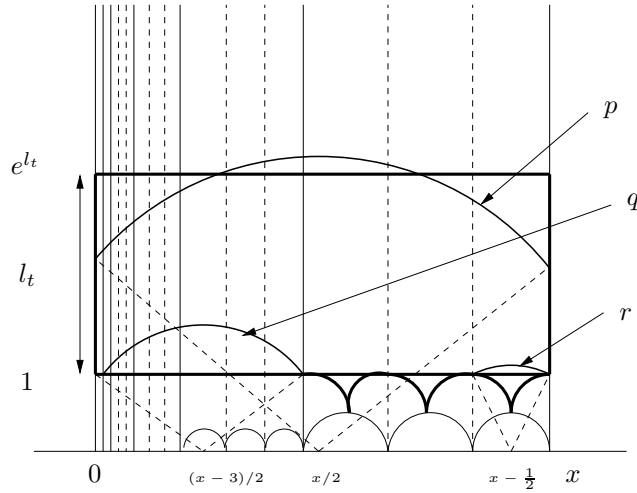


Figure 37. One of the four quadrilaterals of Figure 34, drawn in the upper half-space. The segments labeled  $p, q, r$  are the segments induced by the geodesics  $\alpha, \beta_1$  and  $\beta_2$  respectively. The segment  $p$  is only contained in  $\alpha$ , the segment  $q$  is only contained in  $\beta_2$ , whereas the segments  $r$  is contained in  $\beta_1$  and  $\beta_2$ . The computation of the lengths of these segments uses the the coordinates that are indicated here and the ingredient in Figure 36. The labels  $(x-3)/2, x/2$  and  $x-\frac{1}{2}$  are the abscissas of the corners in dotted lines that are above these labels.

$g^t$ . The lengths of the boundary sides of these quadrilaterals depend on  $t$ , and again, using the invariance of the transverse measure of  $F_\mu(g^t)$ , we have

$$l_{g^t}(\gamma_1) = l_{g^t}(\gamma_2).$$

We set

$$l_t = l_{g^t}(\gamma_1) = l_{g^t}(\gamma_2).$$

Note that  $l_t = e^t l_g(\gamma_1) = e^t l_g(\gamma_2)$ , since  $\gamma_1$  and  $\gamma_2$  are in the stump of  $\mu$ .

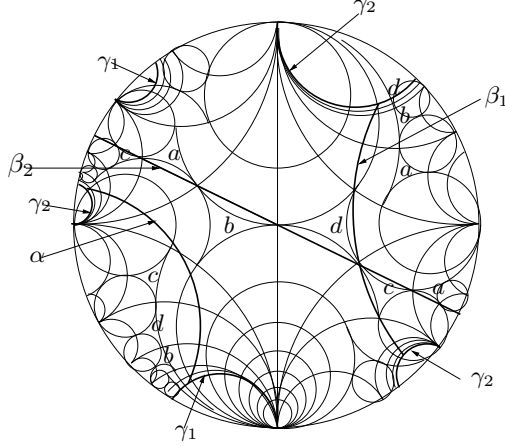


Figure 38. The universal covering of the structure  $g$ . In this picture, we can see the annuli of the horocyclic foliation by examining the non-foliated triangular regions of the ideal triangulation. Lifts of  $\alpha$ ,  $\beta_1$  and  $\beta_2$  are also represented.

The curve  $\alpha$  consists of four segments of the type labeled by  $p$  in Figure 37. The curve  $\beta_1$  is composed of two segments of the type  $r$  and two segments of the type  $q$ , and the curve  $\beta_2$  is composed of four segments of the type  $r$ . From this we deduce the following:

$$l_{g^t}(\alpha) = 8 \sinh^{-1} \left( \frac{3}{4 \sinh(l_t/2)} \right),$$

$$l_{g^t}(\beta_1) = 4 \sinh^{-1} \left( \frac{3e^{-l_t/2}}{4 \sinh(l_t/2)} \right) + 4 \log \left( 1 + \frac{\sqrt{5}}{2} \right)$$

and

$$l_{g^t}(\beta_2) = 8 \log \left( \frac{1 + \sqrt{5}}{2} \right).$$

Summing up, if  $t \mapsto g^t$  is the stretch ray directed by  $\mu$  and starting at  $g$ , we have

- (1)  $\lim_{t \rightarrow \infty} l_{g^t}(\alpha) = 0$ . More precisely, we have  $l_{g^t} \sim_{t \rightarrow \infty} 12e^{-t/2} = 12e^{-e^t/2}$ ;
- (2)  $\{l_{g^t}(\beta_1) \mid t \geq 0\}$  and  $\{l_{g^t}(\beta_2) \mid t \geq 0\}$  are bounded subsets of  $\mathbb{R}_+^*$ . In fact,  $l_{g^t}(\beta_2)$  is constant and  $l_{g^t}(\beta_1)$  decreases towards  $4 \log\left(\frac{1 + \sqrt{5}}{2}\right)$ .

Now if  $t \mapsto g^t$  is the anti-stretch ray directed by  $\mu$  and starting at  $g$ , we have

- (1)  $\lim_{t \rightarrow \infty} l_{g^t}(\alpha) = \infty$ ;
- (2)  $\lim_{t \rightarrow \infty} l_{g^t}(\gamma_i) = 0$  for  $i = 1, 2$ ;
- (3)  $\{l_{g^t}(\beta_2) \mid t \leq 0\}$  is a bounded subset of  $\mathbb{R}_+^*$  (and in fact it is constant);
- (4)  $\lim_{t \rightarrow \infty} l_{g^t}(\beta_1) = \infty$ .

From these computations, one gets the feeling that the following facts hold:

— Changing the orientation of a stretch line seems to interchange the roles of the stump and of the horocyclic measured geodesic lamination.

— The length of any simple closed geodesic with empty intersection with the stump and with the horocyclic measured geodesic lamination is almost constant along a stretch line.

— The length of a simple closed geodesic intersecting the horocyclic lamination (respectively the stump) tends to infinity under the stretch ray (respectively anti-stretch ray).

The results stated in Section 3.11 below (Theorems 3.31 and 3.32) show that this is indeed the case, and not only for lengths of simple closed geodesics, but more generally for lengths of arbitrary measured geodesic laminations.

Figure 39 gives another view on the dynamical behaviour of the stretch line considered in the example that we are studying in this section. In that figure, we have drawn the Poincaré disk realized as the universal covering of the surface  $S$  equipped respectively with hyperbolic structures  $f$ ,  $g$  and  $h$  that belong to a stretch line directed by  $\mu$ , and appearing in that order. The central disk represents the structure  $g$  that we started with, the left-hand disk corresponds to a structure  $f$  that lies before  $g$  on the stretch line (that is, we can obtain  $g$  by stretching  $f$  along the same complete lamination  $\mu$ ) and the right-hand figure represents the structure  $h$  obtained by stretching  $g$  along  $\mu$ .

From this example, we deduce the following result which says that Thurston's asymmetric metric and its dual are not Lipschitz-equivalent.

**Proposition 3.30.** *There does not exist any constant  $C$  satisfying  $K(g, h) \leq CK(h, g)$  for all  $g$  and  $h$  in  $\mathcal{T}$ .*

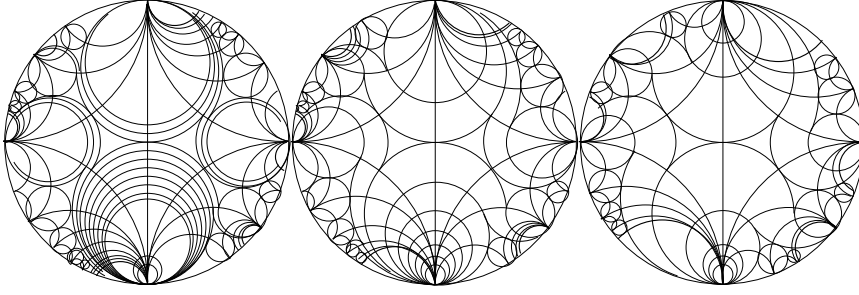


Figure 39. Each of these three pictures represents the universal covering  $\mathbb{H}^2$  of the surface  $S_{2,0}$  studied in Section 3.10 successively equipped with the structures  $g^{-t}$ ,  $g = g^0$  and  $g^t$  for some  $t > 0$ . We have represented a part of the preimage of  $\gamma_1 \cup \gamma_2$  by geodesics in bold lines, as well as the preimage of  $\mu$  that decomposes in a neat way  $\mathbb{H}^2$  into ideal triangles. In each ideal triangle, we have represented the small non-foliated triangular region, which permits the visualization of the horocyclic foliation. In this way, we can see that the width of the cylinder  $C$  increases whereas its circumference decreases.

*Proof.* We consider the stretch line studied above, directed by  $\mu$  and starting at  $g$ , and the closed geodesic  $\alpha$ . Then, we have, for every  $t \geq 0$ ,

$$r_{g^t, g}(\alpha) \sim_{t \rightarrow \infty} \frac{l_g(\alpha)}{12e^{-\frac{e^t}{2}}} = A(g)e^{\frac{e^t}{2}}$$

where  $A(g)$  is a constant that depends on the metric  $g$ . Thus, we obtain  $K(g^t, g) \geq \log r_{g^t, g}(\alpha) \sim e^t$ , whereas  $K(g, g^t) = t$ . This proves the proposition.  $\square$

We can use the explicit formulae that we produced in this example to plot the graphs of the functions  $t \mapsto r_{g, g^t}(\lambda)$  for  $\lambda \in \{\alpha, \beta_1, \beta_2, \gamma_1, \gamma_2\}$ . These graphs give us an idea of the behaviour of the hyperbolic surfaces  $g^t$  as  $t$  varies. It is also interesting to draw the graph of the function  $t \mapsto r_{g, g^t}(\delta)$  where  $\delta$  is the closed geodesic represented in Figure 40, because the intersection pattern of  $\delta$  with  $\alpha$  satisfies  $\delta \cap \alpha \neq \emptyset$  and  $\delta \cap \gamma = \emptyset$ . Figures 41 and 42 represent respectively the intersection of  $\delta$  with the cylinder  $R$  of Figures 34 and 37.

The computations give

$$l_{g^t}(\delta) = \log \left( \frac{e^{lt} + \frac{1}{16}(e^{lt} - 5 - \sqrt{(e^{lt} - 5)^2 + 16e^{lt}})^2}{e^{lt/2} \left(1 + \frac{1}{16}(e^{lt} + 3 - \sqrt{(e^{lt} - 5)^2 + 16e^{lt}})^2\right)} \right)$$

Therefore,

$$\lim_{t \rightarrow \infty} l_{g^t}(\delta) = \infty$$

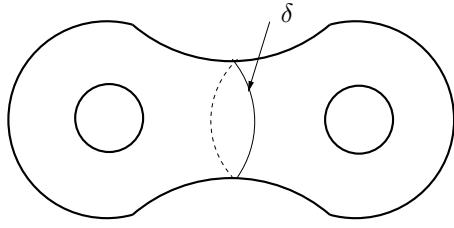


Figure 40.

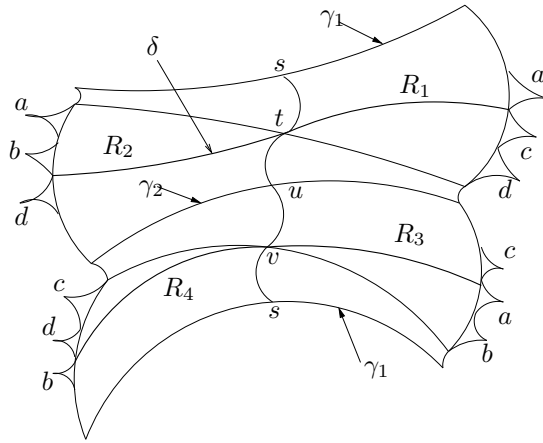


Figure 41.

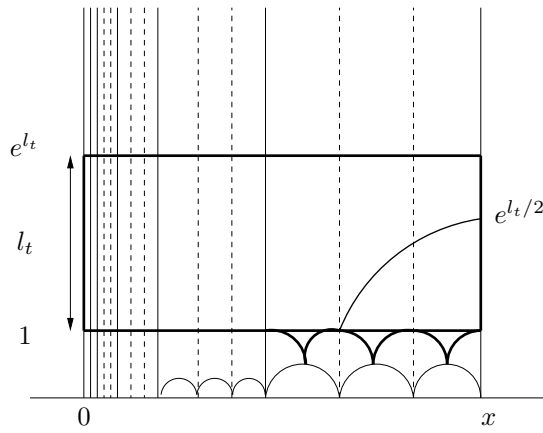


Figure 42.

and

$$\lim_{t \rightarrow -\infty} l_{g^t}(\delta) = 4 \log(1 + \sqrt{2}).$$

We have drawn below the graphs of the functions  $t \mapsto r_{g,g^t}(\lambda)$  for  $\lambda \in \{\alpha, \beta_1, \delta\}$ .

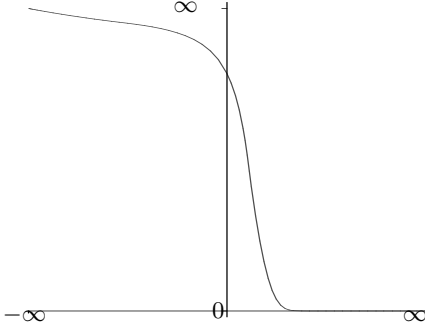


Figure 43.  $t \mapsto l_{g^t}(\alpha)$

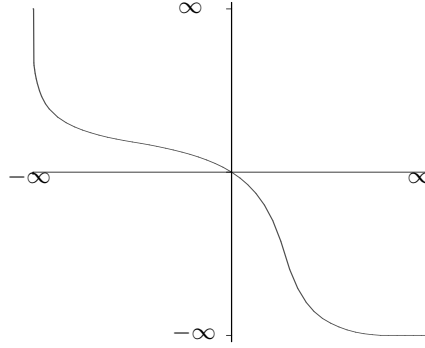


Figure 44.  $t \mapsto \log(r_{g,g^t}(\alpha))$

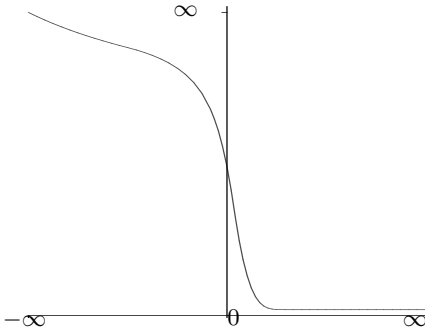


Figure 45.  $t \mapsto l_{g^t}(\beta_1)$

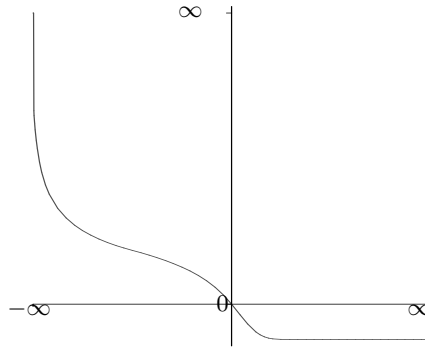
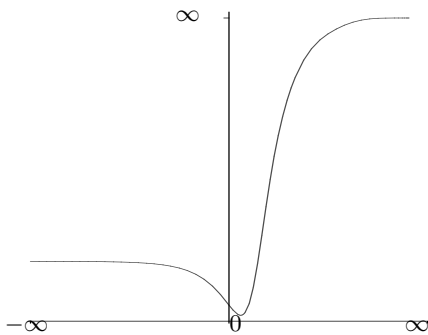
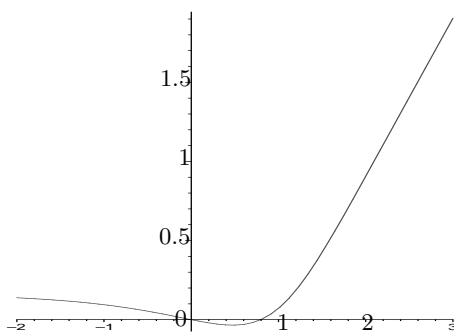


Figure 46.  $t \mapsto \log(r_{g,g^t}(\beta_1))$

It is interesting to note that the length function  $t \mapsto l_{g^t}(\alpha)$  of the horocyclic measured geodesic lamination is strictly decreasing and convex. We can see on the other figures that in the general case, the length function along a stretch line is neither monotonic nor convex. But the figures indicate that the function might be peakless in the sense of Busemann. We recall that a function  $f$  defined on a closed interval  $I \subset \mathbb{R}$  is said to be *peakless* if  $I$  can be decomposed into three subintervals  $I_l$ ,  $I_0$  and  $I_r$  whose interiors are pairwise disjoint and

Figure 47.  $t \mapsto l_{g^t}(\delta)$ Figure 48.  $t \mapsto \log(r_{g,g^t}(\delta))$ 

following each other in the order indicated, such that  $f$  is strictly decreasing on  $I_l$ , constant on  $I_0$  and strictly increasing on  $I_r$ . (Some of the intervals  $I_l$ ,  $I_0$  and  $I_r$  might be empty). In the case where  $I_0$  is empty, the function  $f$  is said to be *strictly peakless*. This notion was introduced by Busemann as a generalization of convexity (see [13] p. 109). Regarding the closed geodesics  $\gamma_1$  and  $\gamma_2$ , which are in the stump, we recall that their  $g^t$ -lengths (which are equal) are strictly convex and increasing.

In the sections that follow, we shall study the asymptotic behaviour of general stretch and anti-stretch lines in Teichmüller space.

### 3.11 The behaviour of the lengths of measured geodesic laminations along stretch and anti-stretch lines

In this section, we present some results that show that the estimates made in the examples studied in Sections 3.8, 3.9 and 3.10 about the lengths of measured geodesic laminations along stretch and anti-stretch lines, are valid in a general setting. We shall give estimates on the behaviour of lengths of geodesic laminations under stretch and anti-stretch rays.

The proofs are contained in [58] and [59].

In what follows, we shall use the term *topological measured geodesic lamination* to denote a geodesic lamination that admits a transverse measure in the usual sense, but which is not equipped with any particular such transverse measure. In other words, a topological measured geodesic lamination is the support of some measured geodesic lamination.

We first deal with the case of a stretch ray  $\{g^t \mid t \geq 0\}$ , and then we shall consider the case of an anti-stretch ray  $\{g^t \mid t \geq 0\}$ . Our aim is to determine, in each case, the limit, if it exists, of the length  $l_{g^t}(\alpha)$  of a measured geodesic lamination  $\alpha$ , as  $t \rightarrow \infty$ .



**Theorem 3.31** (Théret [58]). *Let  $g$  be a hyperbolic structure on  $S$ , let  $\mu$  be a complete geodesic lamination, let  $\{g^t \mid t \geq 0\}$  be a stretch ray directed by  $\mu$  and starting at  $g = g^0$  and let  $\lambda = \lambda_\mu(g)$  be the associated horocyclic measured geodesic lamination. Let  $\alpha$  be an arbitrary compactly supported measured geodesic lamination on  $S$ . We have the following:*

- (1) *If the support of  $\alpha$  is contained in the support of  $\lambda$ , then  $\lim_{t \rightarrow \infty} l_{g^t}(\alpha) = 0$ .*
- (2) *If  $i(\alpha, \lambda) \neq 0$ , then  $\lim_{t \rightarrow \infty} l_{g^t}(\alpha) = \infty$ .*
- (3) *If the support of  $\alpha$  is disjoint from the support of  $\lambda$ , then  $\{l_{g^t}(\alpha) \mid t \geq 0\}$  is a bounded subset of  $\mathbb{R}_+^*$ .*

The next result concerns the limits of the same quantities, but this time along anti-stretch rays.

**Theorem 3.32** (Théret [59]). *Let  $\mu$  a complete geodesic lamination on  $S$  and let  $\{g^t \mid t \geq 0\}$  be an anti-stretch ray starting at a hyperbolic structure  $g = g^0$  and directed by  $\mu$ . Assume the stump of  $\mu$  is nonempty, and call it  $\gamma$ . Let  $\alpha$  be an arbitrary compactly supported measured geodesic lamination on  $S$ . We have the following:*

- (1) *If the support of  $\alpha$  is contained in the support of  $\gamma$ , then  $\lim_{t \rightarrow \infty} l_{g^t}(\alpha) = 0$ .*
- (2) *If  $i(\alpha, \gamma) \neq 0$ , then  $\lim_{t \rightarrow \infty} l_{g^t}(\alpha) = \infty$ . Note that in order to talk about  $i(\alpha, \gamma)$ , one has to put a transverse measure on  $\gamma$ . The condition  $i(\alpha, \gamma) \neq 0$  is then independent of the choice of the transverse measure on  $\gamma$ .*
- (3) *If the support of  $\alpha$  is disjoint from the support of  $\gamma$ , then  $\{l_{g^t}(\alpha) \mid t \geq 0\}$  is a bounded subset of  $\mathbb{R}_+^*$ .*

These two theorems suggest that the two measured geodesic laminations  $\gamma$  and  $\lambda$  play in some cases symmetric roles.

The two theorems show in particular that the length of a measured geodesic lamination that is disjoint from  $\gamma$  and from  $\lambda$  remains bounded along a stretch line.

A consequence of this fact is that subsurfaces of  $S$  that have empty intersection with the stump and the horocyclic measured geodesic lamination are distorted by a uniformly bounded amount as one follows a stretch line. Let us be more precise. Consider a stretch line directed by a complete geodesic lamination  $\mu$  whose stump  $\gamma$  is non-empty. Let  $\lambda$  be the support of any horocyclic measured geodesic lamination associated to this stretch line. Now assume that there exists a subsurface  $S'$  of  $S$  with nonempty boundary satisfying the following conditions:

- the Euler characteristic of  $S'$  is negative;
- $S'$  contains at least one simple closed curve which is essential in  $S$ ;
- $S' \cap \lambda = S' \cap \gamma = \emptyset$ .

Note that these requirements imply that any hyperbolic structure on  $S$  induces a hyperbolic structure with geodesic boundary on  $S'$  and that  $S'$ , equipped with such a structure, is isometrically embedded in  $S$ . The homotopy class of the identity map on  $S$  restricts to the homotopy class of the homotopy map on  $S'$ .

Let us say that two hyperbolic structures  $g$  and  $g'$  on  $S'$  are  $K$ -quasi-isometric if there exists a homeomorphism  $f$  of  $S'$  sending  $g$  to  $g'$ , which is isotopic to the identity of  $S'$ , and a number  $K > 0$  such that  $L(f) \leq K$  and  $L(f^{-1}) \leq K$ .

If  $g$  and  $g'$  on  $S'$  are  $K$ -quasi-isometric, then the ratios  $r_{g,g'}(\alpha)$  is bounded from above by  $K$  and from below by  $1/K$ . Conversely, one can show that if the ratios  $r_{g,g'}(\alpha)$ , as  $\alpha$  varies over the set of essential simple closed curves of  $S'$ , are bounded, then there exists some  $K$  for which the two structures  $g$  and  $g'$  are  $K$ -quasi-isometric. Therefore, we have the following:

**Corollary 3.33** (Théret [59]). *Let  $t \mapsto g^t$  be a stretch line directed by  $\mu$ , with stump  $\gamma$ . Assume that there exists a subsurface  $S'$  of  $S$  as above. Then there exists a positive number  $K$  such that for every  $t$  and  $t'$  in  $\mathbb{R}$ , the hyperbolic structures induced on  $S'$  by  $g^t$  and  $g^{t'}$  are  $K$ -quasi-isometric.*

### 3.12 Thurston's boundary

Teichmüller space  $\mathcal{T}(S)$  is embedded in the function space  $\mathbb{R}_+^{\mathcal{S}}$  by the length functional  $l : \mathcal{T} \rightarrow \mathbb{R}_+^{\mathcal{S}}$  defined in (3.2) above. This embedding, composed with the natural map from  $\mathbb{R}_+^{\mathcal{S}}$  onto its projectivization  $P\mathbb{R}_+^{\mathcal{S}}$ , gives an embedding (see [19])

$$\mathcal{T}(S) \rightarrow P\mathbb{R}_+^{\mathcal{S}}. \quad (3.10)$$

We also recall that the intersection number functional associates to each  $\lambda \in \mathcal{ML}(S)$  the element  $i(\lambda, \cdot) \in \mathbb{R}_+^{\mathcal{S}}$ , and that this defines a map from  $\mathcal{ML}(S)$  into  $\mathbb{R}_+^{\mathcal{S}}$ . At the level of projectivizations, we obtain a map

$$\mathcal{PL}(S) \rightarrow P\mathbb{R}_+^{\mathcal{S}}$$

which restricts to an embedding

$$\mathcal{PL}_0(S) \rightarrow P\mathbb{R}_+^{\mathcal{S}}, \quad (3.11)$$

which induces a topology on the space  $\mathcal{PL}_0(S)$ , by restriction of the projectivization of the weak topology on  $P\mathbb{R}_+^{\mathcal{S}}$ .

We have the following result of Thurston (cf. [19], where the result is described using measured foliations instead of measured laminations).

**Theorem 3.34** (Thurston). *The images of the spaces  $\mathcal{T}(S)$  and  $\mathcal{PL}_0(S)$  in  $P\mathbb{R}_+^{\mathfrak{S}}$  by the embeddings (3.10) and (3.11) are disjoint. With the space  $P\mathbb{R}_+^{\mathfrak{S}}$  being equipped with the quotient of the weak topology, the closure of the image of  $\mathcal{T}(S)$  in  $P\mathbb{R}_+^{\mathfrak{S}}$  is compact, and the complement of this image in the closure coincides with the image of  $\mathcal{PL}_0(S)$ . Equipped with the induced topology, the union  $\overline{\mathcal{T}(S)} = \mathcal{T}(S) \cup \mathcal{PL}_0(S)$  is homeomorphic to a closed ball of dimension  $6g - 6 + 2n$ , whose boundary is the image of  $\mathcal{PL}_0(S)$ .*

For this reason, the space  $\mathcal{PL}_0(S)$  is also called Thurston's boundary of Teichmüller space.

The following convergence criterion is useful:

Let  $\lambda$  be an element of  $\mathcal{PL}_0(S)$ . A sequence  $(g_n)$  in  $\mathcal{T}(S)$  converges to the projective class  $[\lambda] \in \mathcal{PL}_0(S)$  if and only if there exists a sequence  $x_n \in \mathbb{R}_+^*$  such that for all  $\alpha \in \mathfrak{S}$ , we have

$$\lim_{n \rightarrow \infty} x_n l_{g_n}(\alpha) = i(\lambda, \alpha).$$

### 3.13 Converging to Thurston's boundary

Questions about the convergence of stretch and anti-stretch rays were already considered in [47] where the following is proved:

**Theorem 3.35** (Papadopoulos [47], Theorem 5.1 and the remark following it). *Let  $g$  be an element of Teichmüller space  $\mathcal{T}(S)$  and let  $\mu$  be a complete geodesic lamination in  $S$ . Then, the stretch ray directed by  $\mu$  and starting at  $g$  converges to the projective class of the associated horocyclic measured geodesic lamination  $\lambda_\mu(g)$ , considered as an element of Thurston's boundary  $\mathcal{PL}_0(S)$  of  $\mathcal{T}(S)$ .*

We shall say that a measured geodesic lamination  $\mu$  is *uniquely ergodic* if as a topological lamination,  $\mu$  admits a unique transverse measure up to a scalar multiple. In particular, if  $\mu$  consists of a weighted simple closed geodesic, then it is uniquely ergodic. A uniquely ergodic measured geodesic lamination is *minimal* in the sense that every leaf is dense in its support.

Note that our definitions of unique ergodicity for measured geodesic laminations and for measured foliations (Definition 2.8 above) do not coincide under the natural correspondence between the spaces  $\mathcal{ML}_0$  and  $\mathcal{MF}_0$ . (This is so because of the case of foliations which contain cylindrical components.)

As in the case of measured foliations, one can talk about uniquely ergodic projective classes of measured laminations.

In the paper [47], it is shown (Proposition 5.2) that if the complete geodesic lamination  $\mu$  is measured and uniquely ergodic, then the anti-stretch ray directed by  $\mu$  converges to the projective class of  $\mu$ . A more general result has been obtained by Th eret in [59], where the following is proved:

**Theorem 3.36** (Th eret [59]). *Let  $\mu$  be a complete geodesic lamination on  $S$  whose stump  $\gamma$  is nonempty and is uniquely ergodic. Then, any anti-stretch ray directed by  $\mu$  converges to the projective class of  $\gamma$ .*

Note that there are instances where the anti-stretch line converges to a point in Teichm uller space (and not on its boundary). This occurs for any anti-stretch line directed by a complete geodesic lamination  $\mu$  which has finitely many leaves, and where all of these leaves converge to punctures at both ends, and it occurs only for such complete geodesic laminations. (In other words,  $\mu$  is an ideal triangulation.) Note that  $\mu$  is an ideal triangulation if and only if the stump of  $\mu$  is empty. The limiting hyperbolic structure  $g$  is the one for which all the distances between the distinguished points on that lamination are zero, that is,  $\lambda_\mu(g) = \emptyset$ .

To state the next result, we say that a measured geodesic lamination  $\mu'$  is *totally transverse* to the complete geodesic lamination  $\mu$  if the measure-equivalence class of a measured foliation representing  $\mu'$  is totally transverse to  $\mu$  in the sense of Definition 3.9.

One application of Theorems 3.35 and 3.36 is the following

**Corollary 3.37.** *Let  $\alpha$  and  $\beta$  be two points in the boundary  $\mathcal{P}\mathcal{L}_0(S)$  of  $\mathcal{T}(S)$  that can be represented by totally transverse measured geodesic laminations, and suppose that  $\alpha$  is uniquely ergodic. Then, there exists a line in Teichm uller space which is geodesic for Thurston's asymmetric metric and which converges to  $\alpha$  in the negative direction and to  $\beta$  in the positive direction.*

In general, this geodesic is not unique: it suffices to consider a uniquely ergodic stump which possesses several completions (see the discussion in Section 3.3). The stretch lines that are directed by these various complete geodesic laminations, passing through hyperbolic structures which have the same associated horocyclic measured geodesic laminations, converge in the negative direction and in the positive direction to the same points,  $\alpha$  and  $\beta$  respectively, in the boundary  $\mathcal{P}\mathcal{L}_0(S)$  of  $\mathcal{T}(S)$ . (To see that it is possible to find such hyperbolic structures, one can use Thurston's parametrization  $\phi_\mu$  of Teichm uller space described in Section 3.5 above.)

It is possible to permute the points  $\alpha$  and  $\beta$ , obtaining geodesics that converge to  $\alpha$  in the positive direction and to  $\beta$  in the negative direction (at least if  $\beta$  is uniquely ergodic). Note that *a priori* these geodesics will be distinct from the preceding ones.

### 3.14 Stretch lines and earthquakes

There is no attempt here to present any of the basic results on earthquakes. We consider earthquakes only in their relation to stretch maps. We start by recalling the definition.

**Definition 3.38** (Fenchel-Nielsen flow). This is a flow on Teichmüller space, which is defined as follows. Let  $S$  be a hyperbolic surface and let  $\alpha$  be a homotopy class of an essential simple closed curve on  $S$ . The *normalized left Fenchel-Nielsen flowline* through  $S$  is the one-parameter family  $(\mathcal{E}_\alpha^t(S))_{t \in \mathbb{R}}$  of hyperbolic surfaces such that for each  $t \geq 0$  (respectively  $t \leq 0$ ),  $\mathcal{E}_\alpha^t(S)$  is the hyperbolic surface obtained by cutting the hyperbolic surface  $S$  along the closed geodesic in the class  $\alpha$  and gluing back the two boundary geodesics of the resulting surface after a left (respectively right) twist of magnitude  $|t|l_g(\alpha)$ . We say that the hyperbolic surface  $\mathcal{E}_\alpha^t(S)$  is obtained from  $S$  by a time- $t$  normalized Fenchel-Nielsen twist along  $\alpha$ .

We note that this “normalized” parametrization of the Fenchel-Nielsen flow is different from the parametrization used by Kerckhoff in [28]; it is the one used in Papadopoulos [45].

**Definition 3.39** (Earthquake flow, see [28] and [62]). Let  $S$  be a hyperbolic surface, let  $\gamma$  be a compactly supported measured geodesic lamination on  $S$  and let  $\alpha_n$  be a sequence of elements in  $\mathbb{R}_+^* \times \mathcal{S}$  converging to  $\gamma$  in the topology of  $\mathcal{ML}_0(S)$ . Then, for each  $t \in \mathbb{R}$ , the sequence of hyperbolic structures  $\mathcal{E}_{\alpha_n}^t(S)$  converges to a hyperbolic structure  $\mathcal{E}_\gamma^t(S)$  that does not depend on the choice of the sequence  $\alpha_n$  converging to  $\gamma$ . We say that the hyperbolic structure  $\mathcal{E}_\gamma^t(S)$  is obtained from  $S$  by a time- $t$  normalized earthquake along  $\gamma$ . The earthquake is said to be a *left* (respectively *right*) earthquake if  $t \geq 0$  (respectively  $t \leq 0$ ).

**Theorem 3.40** (Earthquake and stretch commute, Th  ret [59]). *Let  $\mu$  be a complete geodesic lamination on  $S$  and let  $\gamma$  be a sublamination of  $\mu$  equipped with a transverse measure. Let  $\mathcal{E}_\gamma^t$  denote as above the normalized earthquake flow along  $\gamma$  and let  $\mathcal{S}_\mu^t$  denote the stretch flow directed by  $\mu$ . Then, for every  $t$  and  $s$  in  $\mathbb{R}$ , we have  $\mathcal{E}_\gamma^t \circ \mathcal{S}_\mu^s = \mathcal{S}_\mu^s \circ \mathcal{E}_\gamma^t$ .*

In the paper [47], there is a definition and a study of the extension of the normalized earthquake flow to Thurston’s boundary  $\mathcal{PL}_0(S)$  which is a quotient flow of a flow defined on the unprojectivized space  $\mathcal{ML}_0(S)$ . In fact, in the paper [47], in the case where  $S$  is closed, these flows are defined on the space  $\mathcal{MF}_0(S)$  of equivalence classes of measured foliation, and on the space  $\mathcal{PMF}_0(S)$  of projective classes of measured foliations on  $S$ . We can define the flows on  $\mathcal{ML}_0(S)$  and  $\mathcal{PL}_0(S)$  by using the natural correspondence between

foliations and laminations. As an application of Theorem 3.40, we have the following:

**Corollary 3.41** (Th  ret [59]). *Assume that the surface  $S$  is closed. Let  $\gamma$  be a measured geodesic lamination on  $S$  and let  $\mu$  be a completion of  $\gamma$  with stump  $\gamma$ . Then, the time- $t$  normalized earthquake (defined on  $\mathcal{ML}_0(S)$ ) along  $\gamma$  of the horocyclic geodesic lamination  $\lambda_\mu(S)$  is the horocyclic geodesic lamination  $\lambda_\mu(\mathcal{E}_t^\alpha(S))$ . In other words, we have, for all  $t \in \mathbb{R}$ ,  $\lambda_\mu(\mathcal{E}_t^\alpha(S)) = \mathcal{E}_t^\alpha(\lambda_\mu(S))$ .*

Theorem 3.40 on the commutativity of the earthquake flow along components of the stump with the stretch flow (along completions of the stump) can provide a class of examples of converging anti-stretch lines. Indeed, as soon as we know that some anti-stretch line directed by a complete geodesic lamination  $\mu$  converges to a point on Thurston's boundary, all the other anti-stretch lines obtained by the action of an earthquake along a component of the stump also converge to the same point.

## 4 Problems

In this last section, we have collected a few problems, which concern mainly Thurston's asymmetric metric. Some of them may be easy.

**Problem I: On the non-symmetry of Thurston's metric.** There are several natural questions that arise directly from the fact that Thurston's metric  $K$  is non-symmetric. For instance: characterize the pairs of hyperbolic structures  $g$  and  $h$  that satisfy  $K(g, h) = K(h, g)$ . In other words, study the locus in  $\mathcal{T}(S) \times \mathcal{T}(S)$  defined by the equation  $K(g, h) = K(h, g)$ . Since  $K$  is a geodesic metric, another natural question is: give necessary and/or sufficient conditions on the hyperbolic surfaces  $g$  and  $h$  under which there exists a geodesic segment from  $g$  to  $h$  which is also (up to reparametrization) a geodesic segment from  $h$  to  $g$ .

**Problem II: On the symmetrization of Thurston's asymmetric metric.** There are several definitions for the symmetrization of an asymmetric metric  $K$ , none of them being more natural than the others. Two such options are

$$\sigma K(g, h) = \max\{K(g, h), K(h, g)\}$$

and

$$SK(g, h) = \frac{1}{2}(K(g, h) + K(h, g)).$$

Each of these symmetrizations is a genuine metric. In the case where  $K$  is Thurston's asymmetric metric, it is natural to try to relate these two associated symmetrizations to other known metrics on that space.

In this respect, we mention that one can adapt Thurston's definition of his asymmetric metric to the context of the Teichmüller space  $\mathcal{T}_{1,0}$  of  $S_{1,0}$ , that is, of the torus,  $\mathcal{T}_{1,0}$  is the space of flat metrics up to isotopy and homothety on the torus. *A priori*, we have two distinct asymmetric metrics,  $L$  and  $K$ , defined as in (3.4) and (3.5) above, on  $\mathcal{T}_{1,0}$ , hyperbolic length being replaced here by Euclidean length, with a suitable normalization that takes care of homothety. Such a study has been carried out in the paper [7], where it is shown that  $K = L$ , as in the case of surfaces of negative Euler characteristic, but with a different kind of proof. Recall that there is a natural identification between  $\mathcal{T}_{1,0}$  and  $\mathbb{H}^2$ . With this identification, we obtain a nonseparating and nonsymmetric metric  $\delta = K = L$  on  $\mathbb{H}^2$ . An explicit formula for  $\delta$  is given in [7], and it is shown there that its symmetrization  $S\delta$  is the Poincaré metric of  $\mathbb{H}^2$ , which, as is well-known, is also the Teichmüller metric on  $\mathcal{T}_{1,0}$ .

It is unlikely that for surfaces of negative Euler characteristic, some particular symmetrization of Thurston's asymmetric metric is Teichmüller's metric, but one can ask the reverse question, that is, to find an interesting asymmetric metric on Teichmüller space whose symmetrization is Teichmüller's metric. In fact, one can ask the same question for the other known metrics on Teichmüller space.

**Problem III: Comparing Thurston's asymmetric metric with other metrics on Teichmüller space.** As a first approach, one can think that Thurston's asymmetric metric is very different from Teichmüller's metric, because the first one is defined using hyperbolic geometry, whereas the definition of the latter is based on complex analysis. But, as is well-known, there are many tools that make the relation between conformal and hyperbolic geometry. One of the basic tools is a result of Wolpert ([67] p. 326) stating that given any two hyperbolic structures  $g$  and  $h$  on  $S$ , then, for any quasiconformal homeomorphism  $f : g \rightarrow h$  with quasiconformal dilatation  $K(f)$  and for any homotopy class  $\gamma$  of essential simple closed curves on  $S$ , we have

$$l_h(f(\gamma)) \leq K(f)l_g(\gamma).$$

This gives the following inequality between Thurston's asymmetric metric  $L$  and the Teichmüller metric:

$$L(g, h) \leq 2d_T(g, h).$$

Indeed, let  $f : (S, g) \rightarrow (S, h)$  be the Teichmüller map, where  $g$  and  $h$  are considered as conformal structures. Wolpert's result implies

$$\frac{l_h(\gamma)}{l_g(\gamma)} \leq K(f) = e^{2d_T(g, h)}$$

which gives

$$\log \frac{l_h(\gamma)}{l_g(\gamma)} \leq \log K(f) = 2d_T(g, h),$$

hence  $L(g, h) \leq 2d_T(g, h)$ .

Thus, the remaining question is to try to find a reverse inequality (may be with an additive constant), or to prove that there is no such inequality. In other words, the question is to see whether there is some large-scale quasi-isometric relation between Thurston's asymmetric metric and Teichmüller's metric.

In this respect, we note that in [7], it is shown that in the case of the Teichmüller space of the torus, Thurston's asymmetric metric and Teichmüller's metric are not quasi-isometric. (In that paper, there is a discussion of the notion of quasi-isometry in the context of non-symmetric metrics.)

It may also be possible to relate Thurston's asymmetric metric to the Weil-Petersson metric on Teichmüller space  $\mathcal{T}(S)$ . This question is motivated by a characterization due to Thurston and Wolpert of the Weil-Petersson metric that is based, like Thurston's asymmetric metric, on the length-spectrum of closed curves. Indeed, Thurston introduced a Riemannian metric on  $\mathcal{T}(S)$  where the scalar product of two tangent vectors at some hyperbolic surface (considered as an element of  $\mathcal{T}(S)$ ) is defined as the second derivative with respect to the earthquake flows along these vectors of the length of a uniformly distributed sequence of closed geodesics on the given hyperbolic surface, and Wolpert showed later on that this metric coincides with the Weil-Petersson metric; see [68].

We note that if  $d_{qi}$  is the metric introduced in [65] by Thurston on Teichmüller space, whose definition we recalled in (3.1) above, and if  $d_T$  denotes as before Teichmüller's metric, then there exists a constant  $C > 0$  such that

$$d_T \leq d_{qi} \leq Cd_T$$

(see [65] p. 268 where Thurston attributes this result to Douady and Earle).

We note finally that it is easy to make definitions of (symmetric or asymmetric) metrics on Teichmüller space that are based on the comparison of lengths of closed geodesics between hyperbolic surfaces, but what is interesting is to be able to define metrics that have nice geometrical properties. For instance, one can take any finite collection  $\{\gamma_1, \dots, \gamma_n\}$  of simple closed curves on the surface whose lengths for an arbitrary hyperbolic metric completely determine that metric, and using these curves, one can define a distance between two hyperbolic metrics  $g$  and  $h$  by taking

$$\log \sup_{i=1, \dots, n} \left\{ \frac{l_g(\gamma_i)}{l_h(\gamma_i)}, \frac{l_h(\gamma_i)}{l_g(\gamma_i)} \right\}.$$



Clearly, this defines a metric on Teichmüller space, but does it have interesting geometric properties?

**Problem IV: Isometries.** Isometries between spaces equipped with asymmetric metrics can be defined in the same way as between usual metric spaces, except that one has to be careful about the order of the variables in the case of an asymmetric metric. A well-known result of H. L. Royden [50] states that the group of isometries of the Teichmüller metric is the mapping class group. Masur and Wolf proved an analogous result for the Weil-Petersson metric, see [41]. Is the same statement true for Thurston’s asymmetric metric? It is easy to see that the elements of the mapping class group are isometries for Thurston’s metric. Therefore, the question is about the reverse inclusion.

We also recall that Bers obtained in [11] a classification of the isometries of the Teichmüller metric in terms of the properties of the displacement function and of the intersection pattern of the minimal displacement sets of the isometries. (Bers’ classification is based on Thurston’s classification of mapping classes into pseudo-Anosov, reducible and parabolic, but it also constitutes an independent approach to the classification.) Likewise, Daskalopoulos and Wentworth obtained in [16] a classification of isometries of the Weil-Petersson metric, again in terms of the displacement function and the displacement sets. This result is also described in [17]. It is natural to ask for an analogous classification for the isometries of Thurston’s asymmetric metric. Note that this is not equivalent to the problem of showing that the isometry group of Thurston’s asymmetric metric is the mapping class group.

**Problem V: Geodesics.** Thurston showed that stretch lines are geodesics for Thurston’s asymmetric metric and that any two points in Teichmüller space can be joined by a geodesic path that is a concatenation of stretch segments (see Theorem 8.5 of [61] or Theorem 3.21 above for a precise statement). *A priori*, there may exist other types of geodesics. Thus, an interesting problem is to describe an arbitrary geodesic. In particular, is any geodesic a limit of a concatenations of stretch segments? Thurston also proved that a geodesic path joining two points in Teichmüller space is in general not unique. Therefore, another natural question is to characterize the set of ordered pairs of points such that the geodesic joining them is unique. (This question would be easy if one knew that every geodesic path is a concatenation of stretch paths).

**Problem VI: The dual Thurston asymmetric metric.** Work out an asymptotic formula linking  $K(g, h)$  and  $K(h, g)$ . To find a precise formula is probably not a reasonable problem. Along a stretch line, we suspect a formula reminding the collar formula  $\sinh(aK(g, h)) \sinh(bK(h, g)) \simeq c$  with some constants  $a, b, c$  depending on the genus and on the number and punctures of the surface  $S$ , and on the “complexity” of the complete lamination directing the stretch line. We already know that Thurston’s asymmetric metric and its dual are not Lipschitz equivalent. In fact, there are no constants  $C_1$  and  $C_2$

such that for all  $g$  and  $h$  in  $\mathcal{T}$ , we have  $K(g, h) \leq C_1 K(h, g) + C_2$  (Proposition 3.30 above).

**Problem VII: Anti-stretch lines.** Find conditions for a hyperbolic metric equipped with a complete geodesic lamination so that the anti-stretch line starting at that point and directed by that lamination is also a stretch line (up to reparametrization).

**Problem VIII: Curvature.** Study the various existing notions of curvature of Teichmüller space equipped with Thurston's asymmetric metric (Finsler curvature, Gromov hyperbolicity, boundedness of curvature in the sense of Alexandroff and so on). Note that this metric is not nonpositively curved in the sense of Busemann since there may be several geodesic segments joining two points.

**Problem IX: Convergence of anti-stretch rays.** For a given complete geodesic lamination  $\mu$  with nonempty stump  $\gamma$ , find weaker hypotheses than those of Theorem 3.36 that imply the convergence of an anti-stretch ray directed by  $\mu$  to the projective class of  $\gamma$ . The convergence result in Theorem 3.36 may suggest that if the measured geodesic lamination  $\gamma$  is not uniquely ergodic, then either an anti-stretch ray directed by  $\mu$  does not converge to a definite point on Thurston's boundary, or it converges to the projective class of the geodesic lamination  $\gamma$  equipped with some transverse measure which would be a "barycenter" of all transverse measures carried by  $\gamma$ . Thus, a natural question would be to find conditions under which one of these two cases occurs. One can reason by analogy with Masur's result regarding Teichmüller's metric stated as Theorem 2.42 above, where a whole family of rays corresponding to different transverse measures on a given foliation converge to a single point, corresponding to a special transverse measure on that foliation.

**Problem X: Convexity.** Kerckhoff showed in [28] that the geodesic-length function on Teichmüller space associated to a closed geodesic is convex along earthquake paths and that the sum of geodesic-length functions associated to a finite family of geodesics that fills up the surface is strictly convex. Wolpert showed in [69] that geodesic-length functions are strictly convex along Weil-Petersson geodesics. Such convexity results have been used by several authors to solve various problems. For instance, Kerckhoff used the convexity of geodesic-length functions along earthquake paths to obtain a solution of the Nielsen realization problem [28]. Wolpert, in his paper [69], used the strict convexity of geodesic-length functions along Weil-Petersson geodesics to obtain a new proof of the fact that Teichmüller space is Stein, and a new solution of the Nielsen realization problem. Kerckhoff, in his paper [30] used the convexity properties that he obtained in [28] to develop his theory of *lines of minima* in Teichmüller space, which opened a new geometrical point of view on that space. Thurston, in his preprint [63], constructed a mapping class group-equivariant

spine for the Teichmüller space of a closed surface based on the convexity of the geodesic-length function. P. Schmutz further developed such a theory in his paper [51], where he also used the convexity of the geodesic-length function to study what he called a *systole function* on Teichmüller space. In view of all that, it is natural to study convexity properties of length functions along stretch lines. The graphs that we plotted in 3.10 show that this function is not convex in general, but one can conjecture that it is peakless in the sense of Busemann. At the infinitesimal level, Thurston's asymmetric metric is convex in the sense that it is a Finsler metric, that is, it is defined by a length structure induced by a norm on each tangent space whose closed ball is a convex body (which is not strictly convex).

**Problem XI: The visual boundary.** Describe the visual boundary at any point (or at some class of points) of Teichmüller space equipped with Thurston's asymmetric metric. For that, one needs first to understand all the geodesic rays starting at a point (cf. Problem V above). Does the visual boundary depend on the choice of the basepoint? Does the action of the mapping class group extend to the space union its visual boundary? One can ask similar questions about the visual boundary of the dual asymmetric metric  $K^*(x, y) = K(y, x)$ .

**Problem XII: Stretch maps between general metric spaces.** Work out a theory of stretch maps between general (i.e. not necessarily hyperbolic) metrics on a surface. This problem is mentioned by Thurston in his paper [61]. A particularly interesting class of metrics on surfaces is the class of Euclidean metrics with cone singularities. It is also an interesting problem to study stretch maps between higher-dimensional manifolds equipped with metrics of constant curvature, or between singular spaces (graphs, two-complexes and so on).

**Problem XIII: Moduli space.** Study the behaviour of stretch and anti-stretch lines in moduli space.

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