HAL open science

The Collatz Problem and Its Generalizations: Experimental Data. Table 1. Primitive Cycles of $(3n+d)-$mappings.

Edward G. Belaga, Maurice Mignotte

► To cite this version:

Edward G. Belaga, Maurice Mignotte. The Collatz Problem and Its Generalizations: Experimental Data. Table 1. Primitive Cycles of $(3n+d)-$mappings.. 2006. hal-00129727

HAL Id: hal-00129727

https://hal.science/hal-00129727

Preprint submitted on 8 Feb 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

The Collatz Problem and Its Generalizations: Experimental Data.
Table 1. Primitive Cycles of $(3n+d)-$mappings.

Edward G. Belaga Maurice Mignotte
belaga@math.u-strasbg.fr *mignotte@math.u-strasbg.fr*

Université Louis Pasteur 7, rue René Descartes, F-67084 Strasbourg Cedex, FRANCE

Abstract. We study here experimentally, with the purpose to calculate and display the detailed tables of cyclic structures of dynamical systems $\mathcal{D}_d$ generated by iterations of the functions T_d acting, for all $d \geq 1$ relatively prime to 6, on positive integers :

$$T_d : \mathbf{N} \longrightarrow \mathbf{N}; \qquad T_d(n) = \begin{cases} \frac{n}{2} & \text{, if } n \text{ is even;} \\ \frac{3n+d}{2} & \text{, if } n \text{ is odd.} \end{cases}$$

In the case $d = 1$, the properties of the system $\mathcal{D} = \mathcal{D}_1$ are the subject of the well-known *Collatz*, or $3n + 1$ conjecture.

According to Jeff Lagarias, 1990, a cycle of the system $\mathcal{D}_d$ ($(3n+d)-$cycle, for short) is called *primitive* if its members have no common divisor > 1. For every one of 6667 systems $\mathcal{D}_d$, $1 \leq d \leq 19999$, we calculate its complete, as we argue, list of primitive cycles. Our calculations confirm, in particular, two long-standing conjectures of Lagarias, 1990, and suggest the plausibility of, and fully confirm several new deep conjectures of Belaga-Mignotte, 2000. Moreover, based on these calculations, the first author, Belaga 2003, advanced and proved a new deep conjecture concerning a sharp effective upper bound to the minimal member (*perigee*) of a primitive cycle,

§**1. Introduction.** From many points of view, the challenge posed by the *Collatz problem* – written down in his notebook by Lothar Collatz in 1937 and known today also as the *3n+1 conjecture*, *3n+1 mapping*, *3n+1 problem*, *Hasse's algorithm*, *Kakutani's problem*, *Syracuse algorithm*, *Syracuse problem*, *Thwaites conjecture*, and *Ulam's problem* — is unique in the history of modern mathematics. The Collatz conjecture affirms that the repeated iterations of the mapping

$$T : \mathbf{N} \longrightarrow \mathbf{N}; \qquad T(n) = \begin{cases} \frac{n}{2} & \text{, if } n \text{ is even;} \\ \frac{3n+1}{2} & \text{, if } n \text{ is odd.} \end{cases} \qquad (1:1)$$

produce ultimately the cycle $1 \to 2 \to 1$, whatever would be the initial number $n \in \mathbf{N}$.

The fact is — even if the Theorem of the Collatz problem can be easily understood and appreciated by children entering secondary school and in its classic simplicity and beauty it belongs more to the Euclidean than to modern era of mathematics – it remains still unresolved, after more than forty years of sustained theoretical and experimental efforts: cf. the *annotated bibliography*, by Jeff Lagarias, of 200 papers, books, and preprints (*arxiv.org/math.NT/0309224*, the last update: January 5, 2006).

Bibliographical Digression. To avoid bibliographical redundancy, all our references with numbers in square brackets direct the reader to corresponding items of the Lagarias bibliography, with his generally very helpful succinct comments. Four papers common to the Lagarias and our own bibliography, at the end of the paper, will be referred to by the double references, as in [Belaga, Mignotte 1999]–[22].

Not less surprisingly, the Collatz problem and its immediate elementary *generalizations* [21], [22], [23], [34], [37], [52], [53], [85], [105], [116], [121], were shown to be relevant in one or another substantial way to the deepest open questions in *algorithmic theory* [20], [52], [53], [98], [121], [131], [155] *and mathematical logic* [56], [116], [119], *Diophantine approximations and equations theory* [21], [22], [23], [36], [133], [154], [158], [159], [160], [165], *p-adic number theory* [1], [27], [28], [43], [137], [138-139], *multiplicative semigroup theory* [64], [13], [107], *discrete and continuous dynamical systems theory* [44], [59], [95], [115], [134], [141], and *discrete* (*random walks*) [32], [66, 67], [147], [182] *and continuous* *stochastic processes theories* [78], [96], [97], [106], [111], [112], [149], [161], [162], [163], [178], [181], [185].

One of the most simple and natural generalizations of the Collatz problem, introduced by Jeff Lagarias [105] and, independently, by the authors [Belaga, Mignotte 1999]–[22], are so called $(3n+d)-$, or T_d-mappings acting, for all $d \geq 1$ relatively prime to 6, on the set $\mathbf{N}$ of natural numbers :

$$T_d : \mathbf{N} \longrightarrow \mathbf{N}; \qquad T_d(n) = \begin{cases} \frac{n}{2} & \text{, if } n \text{ is even;} \\ \frac{3n+d}{2} & \text{, if } n \text{ is odd.} \end{cases} \tag{1 : 2}$$

We study here experimentally, with the purpose to calculate and display the detailed tables of cyclic structures of dynamical systems $\mathcal{D}_d$ generated by iterations of the functions T_d. In the case $d = 1$, the cyclic properties of the system $\mathcal{D} = \mathcal{D}_1$ are the subject of the cyclic part of the Collatz conjecture.

According to Lagarias [105], a cycle of the system $\mathcal{D}_d$ ($(3n+d)-$cycle, for short) is called *primitive* if its members have no common divisor > 1. For every one of 6667 systems $\mathcal{D}_d$, $1 \leq d \leq 19999$, we calculate its complete, as we argue, list of primitive cycles. Our calculations confirm, in particular, two long-standing conjectures of Lagarias, 1990, and suggest the plausibility of, and fully confirm several new deep conjectures, Belaga, Mignotte, 2000. Moreover, based on these calculations, the first author, Belaga 2003, advanced and proved a new deep conjecture concerning a sharp effective upper bound to the minimal member (*perigee*) of a primitive cycle,

§**2. The $3n+d$ Generalization of the Collatz Problem and its Diophantine Interpretation.** We will need here a short list of Diophantine formulae related to the Collatz problem and its $3n+d$ generalization. We assume the acquaintance of the reader with the basic notions and reasonings leading to this interpretation. Any of the following papers of the authors will do : [Belaga, Mignotte 1999]–[22], [Belaga, Mignotte 2000]–[23], [Belaga, Mignotte, 2006^a], [Belaga, Mignotte, 2006^b].

In this section, we list the principal definitions necessary for an understanding of the Diophantine interpretation of the original and generalized Collatz problems, as well as the most important results, – without motivations, elaborations, and proofs, but, in a few cases, with supplementary references.

Let $\mathbf{D}$ be the set of odd natural numbers not divisible by 3. As we have already mentioned above, the iterations of the T_d-mapping (1:2) generates on the set of natural numbers $\mathbf{N}$ a dynamical system $\mathcal{D}_d$:

$$\mathcal{D}_d = \{\mathbf{N} \; ; \; T_d \; : \mathbf{N} \longrightarrow \mathbf{N}\} \; . \tag{2 : 1}$$

Definition 2.1 (1) For a given map T_d and a given $n \in \mathbf{N}$, the T_d-*trajectory* and, in particular, the T_d-*cycle of length* ℓ, *starting at* n are defined as follows:

$$\begin{cases} \textbf{(i)} \quad \tau_{T_d}(n) = \left\{n = T_d^0(n), \; n_1 = T_d(n), \; n_2 = T_d^2(n) = T_d(T_d(n)), \; \ldots \right\} \\ \textbf{(ii)} \quad \tau_{T_d}(n) \text{ is a } T_d\text{−cycle of the length } \ell \geq 1 \iff \begin{cases} n_\ell = T_d^\ell(n) = n, \\ \forall r \; (1 \leq r < \ell), \; n_r = T_d^r(n) \neq n \; . \end{cases} \end{cases} \tag{2 : 2}$$

(2) A non-cyclic T_d-trajectory starting at n is called *ultimately cyclic*, if there exist $j \geq 1$ such that the T_d-trajectory starting at $n_j = T_d^j(n)$ is cyclic.

(3) A T_d-trajectory $\tau_{T_d}(n)$ starting at n is *divergent* if it is neither cyclic, nor ultimately cyclic. Or, equivalently, if $|T_d^j(n)|_{j\to\infty}\to\infty$.

Definition 2.2. (1) For any integer r and any set of integers $\mathbf{K}$, let $\mathbf{K}_{(r)}$ denote the subset of all $q \in \mathbf{K}$ relatively prime to r. In this notations, $\mathbf{D} = \mathbf{N}_{(6)}$.

(2) For a given $d \in \mathbf{D}$, let $T = T_d$ be the corresponding map (1:2), and let, for a given $n \in \mathbf{N}$, $\tau_d(n) = \tau_{T_d}(n)$ be, according to (2:2), the T_d-trajectory/cycle starting at n. Then,

$$q = \gcd(d,n) > 1 \quad \Longrightarrow \quad \forall k \geq 1 \ . \ \ q = \gcd(d, T_d^k(n)) = \gcd(\tau_d(n)), \ \ \tau_d(n) = q \cdot \tau_{d/q}(n), \qquad (2:3)$$

with $\tau_d(n)$ being called *$q-$multiple* of the $T_{d/q}-$trajectory/cycle $\tau_{d/q}(n)$. This reduces the study of non-primitive T_d-trajectories, and in particular, T_d-cycles, to the study of their underlying primitive T_q-cycles, for all proper divisors q of d.

(3) Otherwise, $n \in \mathbf{N}_{(d)}$, and the trajectory $\tau_d(n)$ is called *primitive*,

$$\gcd(d,n) = 1 \quad \Longrightarrow \quad \forall k \geq 1 \ . \quad \gcd(d, T_d^k(n)) = \gcd(\tau_d(n) = 1 \ . \qquad (2:4)$$

Primitive trajectories are numerous: according to (2:4), the T_d-trajectory starting at a natural number n relatively prime to d is primitve. But how many of such trajectories are *primitive cycles*? More specifically, do primitive cycles exist for all systems $d \in \mathbf{D}$? And what are the chances of an integer $n \in \mathbf{D}_{(d)}$ to belong to such a cycle?

These are the subjects of the original conjectures of Lagarias [105] :

Conjecture 2.3. (1) Existence of a Primitive T_d-cycle. For any $d \in \mathbf{D}$, there exists at least one primitive T_d-cycle. In the case of the Collatz problem, it is the cycle $\mathbf{C}^o = \{1 \to 2 \to 1\}$.

(2) Finiteness of the Set of Primitive T_d-cycles. For any $d \in \mathbf{D}$, the number of primitive T_d-cycles is finite. In the case of the Collatz problem, one conjectures $\mathbf{C}^o$ being the only primitive cycle.

In contrast to the first two conjectures of Lagarias, the following conjecture has no immediately apparent $(3n+1)-$precursor or analogue :

Conjecture 2.4. For any $n \in \mathbf{D}$, the set of primitive T_d-cycles, $d \in \mathbf{D}_{(n)}$, meeting n is infinite.

We have interpreted this conjecture as dual to the first conjectures of Lagarias, with the emerging *disparate duality* between $d-$ and $n-$related phenomena being an important new and all-pervading intuition behind the present study [Belaga, Mignotte 2000]–[23], [Belaga, Mignotte 2006^a]. This duality finds its precise formal description in the related Diophantine framework, Theorem 2.10(4) below.

Definition 2.5. (1) For any positive integer $n \in \mathbf{N}$, define the number $\mathrm{Odd}(n)$ obtained by factoring out of n the highest possible power of 2; thus $\mathrm{Odd}(n)$ is odd and $m = \mathrm{Odd}(n) \cdot 2^j$, for some j. Hence the notation:

$$\forall n \in \mathbf{N} \ , \quad \nu_2(n) = \max\big\{j \geq 0 \ \big| \ n \cdot 2^{-j} \in \mathbf{N}\big\}, \quad n = \mathrm{Odd}(n) \cdot 2^{\nu_2(n)}. \qquad (2:5)$$

(2) Let $\tau_d(n) = \{n, T_d(n), T_d(T_d(n)), \ldots\}$ be a primitive T_d-trajectory starting at an odd n not divisible by 3 and relatively prime to d, $n \in \mathbf{D}_{(\mathbf{6})}$. The full sequence of odd members, in the order of their appearance in $\tau_d(n)$ is called *the odd frame* of $\tau_d(n)$ and denoted by $\mathrm{Oddframe}(n,d)$, with the full sequence of the corresponding exponents of 2, denoted by $\mathrm{Evenframe}(n,d)$ (2:5):

$$\forall n, \ d \in \mathbf{D}, \quad \begin{cases} \textbf{(i)} \quad \tau_d(n) = \big\{n, n_1, n_2, \ldots\big\}; \\ \textbf{(ii)} \quad \mathrm{Evenframe}(n,d) = \big\{p_1, p_2, \ldots\big\} \subset \mathbf{N}, \ \forall j \geq 1, \ p_j = \nu_2(3n_{j-1} + d) \geq 1; \\ \textbf{(iii)} \quad \mathrm{Oddframe}(n,d) = \big\{m_0 = n, m_1, m_2, \ldots\big\} \subset \mathbf{D}_{(d)} \ , \\ \qquad\qquad \forall j \geq 1, \ m_j = \mathrm{Odd}(3m_{j-1} + d) = n_{r_j}, \ r_j = \displaystyle\sum_{1 \leq i \leq j} p_i \ . \end{cases} \qquad (2:6)$$

(3) If $\tau_d(n)$ is a cycle $\mathbf{C}$ of the length ℓ, then the periodic sequence (2:5(iii)) is called the *Oddcycle* associated with $\mathbf{C}$ and denoted by Oddframe($\mathbf{C}$), with the period $k,\ 1 < k < \ell$, called the *Oddlength* of the cycle $\mathbf{C}$, with the respective list Evenframe($\mathbf{C}$) of exponents defined by the formula (2:5(ii)), and with the parameters *length*, *Oddlength*, and Evenframe($\mathbf{C}$) satisfying the following obvious relationship implied by the cyclic condition (2:2(ii)):

$$\ell = p_1 + \cdots + p_k \geq \lceil k \cdot \log_2 3 \rceil \iff B_{k,\ell} = 2^\ell - 3^k > 0\ , \quad \Lambda = \{(k, \ell) \in \mathbf{N} \mid \ell \geq \lceil k \cdot \log_2 3 \rceil\}. \qquad (2:7)$$

(4) Let Δ be the subset of the set of pairs $(n, d) \in \mathbf{D} \times \mathbf{D}$, with relatively prime integers $n,\ d$. If a primitive T_d-cycle $\mathbf{C}$ meets a number $n, (n, d) \in \Delta$, then the pair $\langle n, d\rangle$ is called a *primitive membership pair*, or simply *membership*.

Definition 2.6. Let $\langle n, d\rangle \in \mathcal{M}$ be a primitive membership. Using the notations (7:2,3), we define:

$$\begin{cases} a,\ b,\ f,\ g,\ h\ :\quad \mathcal{M} \longrightarrow \mathbf{D}\,; \\ \forall \langle n, d\rangle \in \mathcal{M},\ A = a(\langle n, d\rangle),\ B = B_{k,\ell} = b(\langle n, d\rangle),\ F = f(\langle n, d\rangle),\ G = g(\langle n, d\rangle),\ H = h(\langle n, d\rangle)\,; \\ A = \begin{cases} 1, & \text{if } k = 1\,; \\ 3^{k-1} + 2^{p_1} \cdot 3^{k-2} + \cdots + 2^{p_1+\cdots+p_{k-2}} \cdot 3 + 2^{p_1+\cdots+p_{k-1}}, & \text{otherwise;} \end{cases} \\ B = B_{k,\ell} = 2^\ell - 3^k = 2^{p_1+p_2+\ldots+p_k} - 3^k > 0\ \ (cf.(4:3)); \quad B \text{ is called } \textit{Collatz}\ \ (\mathrm{k}, \ell) - \textit{number}; \\ H = \gcd(A, B)\ ; \quad F = \dfrac{A}{H}\ ; \quad G = \dfrac{B}{H}\ ; \quad \gcd(F, G) = 1. \end{cases} \qquad (2:8)$$

The following theorem translates the iterative "cyclic walk" language for $(3n + d)-$maps into its Diophantine equivalent:

Theorem 2.7. [Belaga, Mignotte 1999]–[22] **(1) Diophantine Cyclic Walk.** Let $\langle n, d\rangle \in \mathcal{M}$ be a primitive membership, $\mathbf{C} = \mathbf{C}(d, n)$ be the corresponding primitive T_d-cycle starting at $m_0 = n$, of the *Oddlength* $k > 1$, let $m_1 = \mathrm{Odd}\,(3m_0 + d)$ (4:3), and let $\langle m_1, d\rangle$ be the corresponding membership. Let σ_k be the circular counterclockwise permutation on $k-$tuples of positive integers $\mathbf{P} = \mathrm{Even}\,(n, d)$. Then the set $\mathbf{P_1} = \mathrm{Even}\,(m_1, d)$ corresponding to the membership $\langle m_1, d\rangle$ is defined by the formula

$$\mathbf{P}_1 = \mathrm{Even}\,(m_1, d) = \sigma_k\big(\mathbf{P}\big) = \sigma_k\big(p_1, p_2, \ldots, p_{k-1}, p_k\big) = \big(p_2, p_3, \ldots, p_k, p_1\big).$$

(2) From Primitive Membership to its Diophantine Representation. Let $\langle n, d\rangle$ be a membership, and let the functions a, b, f, g, h be defined as above (2:8). Then $F = f(\langle n, d\rangle) = n$, $G = g(\langle n, d\rangle) = d$.

(3) From the Diophantine Formulae to their Membership Interpretation. Let $k \geq 2$ and $\mathbf{P} = \big(p_1, \ldots, p_k\big) \subset \mathbf{N}^k$ be a $k-$tuple of positive integers satisfying the inequality

$$\ell = |\mathbf{P}| = p_1 + \cdots + p_k \geq \lceil k \cdot \log_2 3 \rceil\ , \qquad (2:9)$$

and let the numbers A, B, F, G, H be defined as in (2:8). Then $\langle F, G\rangle$ is a primitive membership.

The following deep theorem demonstrates the strength of the above Diophantine formalism (2:5-9) ; we are not aware of an alternative, non-Diopnantine proof of the below inequality (2:10):

Theorem 2.8. [Belaga 2003]–[21] **(1)** Let $\langle n, d\rangle \in \mathcal{M}$ be a primitive membership of the length ℓ and *Oddlength* k, and let $\tau_d(n), (n, d) \in \Delta$ be the corresponding T_d-cycle. If n is the minimal member of the Oddframe(n, d), then the inequality holds :

$$1 \leq n < \frac{d}{2^{\frac{\ell}{k}} - 3}\ . \qquad (2:10)$$

(2) The inequality (2:10) is sharp : for any $K > 1$ and any $\epsilon > 0$, there exists the primitive membership $\langle n, d\rangle \in \mathcal{M}$, such that n is the minimal member of the corresponding primitive cycle of the *Oddlength* $k > K$ and length ℓ satisfying the inequality :

$$0 < \frac{n \cdot (2^{\frac{\ell}{k}} - 3)}{d} - 1 < \epsilon \ . \tag{2 : 11}$$

The following statement demonstrates the particular importance of Collatz numbers for our problems :

Definition 2.9. **(1)** The pair $(k, \ell) \in \Lambda$ (2:7) and the corresponding *Collatz number* $B = B_{k,\ell}$ (2:8) are called *narrow* if $\gcd(k, \ell) = 1$ and $\ell = \lceil k \cdot \log_2 3\rceil$. We denote by Λ_0 the set of narrow pairs (k, ℓ) :

$$\Lambda_0 = \{(k, \ell) \in \Lambda \mid \ \gcd(k, \ell) = 1 \ \&\ \ell = \lceil k \cdot \log_2 3\rceil\} \ . \tag{2 : 12}$$

(2) Let $\langle n, d\rangle \in \mathcal{M}$ be a primitive $(k, \ell)-$membership with the corresponding Collatz number B. We associate with $\langle n, d\rangle \in \mathcal{M}$ its *Collatz corona* $\mathcal{A} = \mathcal{A}_{k,\ell}$, a finite set of natural numbers depending only on the *Oddlength* and *length* of $\langle n, d\rangle \in \mathcal{M}$, as follows :

(a) If $k = 1$, then

$$\mathcal{A}_{1,\ell} = \{1\} \ . \tag{2 : 13)}$$

(b) If $k > 1$, then for any *aperiodic* $(k - 1)-$tuple $\mathbf{P} = \{p_1, \ldots, p_{k-1}\}$ of positive integers satisfying the inequality (cf. (2:9))

$$p_1 + \ldots + p_{k-1} < \ell \ , \tag{2 : 14}$$

the following number belongs to $\mathcal{A}$:

$$A = A(\mathbf{P}) = 3^{k-1} + 3^{k-2} \cdot 2^{p_1} + \ldots + 3 \cdot 2^{p_1+\ldots+p_{k-2}} + 2^{p_1+\ldots+p_{k-1}} \ . \tag{2 : 15}$$

By definition, members of Collatz corona are odd positive integers not divisible by 3. The Collatz corona corresponding to a narrow pair (k, ℓ) is called *narrow*, too.

Theorem 2.10. **(1)** Collatz corona $\mathcal{A}_{k,\ell}$ is a one-element set *iff* $k = 1$ (2:13))

(2) Otherwise, Collatz corona $\mathcal{A}_{k,\ell}$ of a Collatz number $B = B_{k,\ell}$ is a finite set of mutually distinct positive integers not equal to B, and its cardinality $\alpha_{k,\ell} = \#\mathcal{A}_{k,\ell}$ satisfies the following formula (cf. (2:7) for the definition of the set Λ) :

$$\forall (k, \ell) \in \Lambda \quad \begin{cases} \alpha_{k,\ell} = \#\mathcal{A}_{k,\ell} = \begin{cases} \binom{\ell-1}{k-1} \ , & \text{if } \gcd(k, \ell) = 1 \ , \\ \sum_{r \mid \gcd(k,\ell)} \mu(r) \cdot \binom{\frac{\ell}{r}-1}{\frac{k}{r}-1}, & \text{otherwise} \ , \end{cases} \\ \alpha_{k,\ell} \equiv 0 \pmod{k} \ , \end{cases} \tag{2 : 16}$$

where μ is the Möbius function :

$$\mu(m) = \begin{cases} 1 \ , & \text{if } m = 1 \ , \\ (-1)^q \ , & \text{if } m \text{ is the product of } q \text{ distinct primes} \ , \\ 0 \ , & \text{if } m \text{ is divisible by a square of a prime} \ . \end{cases}$$

The low part of the formula (2:16), defined for pairs (k, ℓ) with the property $\gcd(k, \ell) > 1$, is universal and covers – but also obscures – the special upper case $\gcd(k, \ell) = 1$.

(3) The below lower and upper bounds to members A of Collatz corona are sharp :

$$\left.\begin{matrix} \forall (k, \ell) \in \Lambda \\ \forall A \in \mathcal{A}_{k,\ell} \end{matrix}\right\} \begin{cases} \underline{A}_{k,\ell} = 3^k - 2^k \ , \ \overline{A}_{k,\ell} = 2^{\ell-k+1} \cdot \big(3^{k-1} - 2^{k-1}\big) + 3^{k-1} \ ; \\ 3^{k-1} < \underline{A}_{k,\ell} \le A \le \overline{A}_{k,\ell} < 2^{\ell-k+1} \cdot 3^{k-1} \ . \end{cases} \tag{2 : 17}$$

(3) Since the function a in the definition (2:8) and the corresponding construction ((2:15) do not actually depend on the $k-$th component p_k of the correspnding Collatz configuration, their values for two $k-$configurations of different lengths can be equal. More precisely,

$$\forall (k,\ell) \in \Lambda \quad \forall j \geq 1 \ , \qquad \mathcal{A}_{k,\ell} \subset \mathcal{A}_{k,\ell+j} \ . \tag{2 : 18}$$

(4) With the exception of the trivial case, corresponding to the primitive T_1-cycle $1 \to 2 \to 1$,

$$k = 1, \ \ell = 2, \ \mathcal{A}_{1,2} = \{1\}, \ B = 1 \ ,$$

no Collatz number ever belongs to the respective Collatz corona and, if $k \geq 4$, it is located below the upper bound (2:17) of the corona,

$$\forall (k,\ell) \in \Lambda \ , \ B = B_{k,\ell} < 2^{\ell} \ \big(< \overline{A}_{k,\ell} \ , \ \text{if } k \geq 4 \big) \ . \tag{2 : 19}$$

(5) Moreover, if $(k,\ell) \in \Lambda$ is not narrow, then $\mathcal{A}_{k,\ell}$ extends both below and above B.

Consequently, the original Collatz, or $3n+1$, conjecture is equivalent to the following claim :

The (Diophantine version of the) Collatz conjecture 2.11. *For any pair* $(k,\ell) \in \Lambda \setminus \{(1,2)\}$, *no member of the Collatz corona* $\mathcal{A}_{k,\ell}$ *is devisible by the Collatz number* $B_{k,\ell}$.

§**3. Experimental Study of the Cyclic Structure of Dynamical Systems $\mathcal{D}_d$ generated by the corresponding $(3n+d)-$mappings.** For all 6667 dynamical systems $\mathcal{D}_d$ within the range $1 \leq d \leq 19999$, $d \in \mathbf{D}$, our computations have confirmed Conjectures 2.3, 2.4, – see our main Table below – albeit inevitably with different degrees of certainty. Moreover, as it has been already mentioned above, the numerical data obtained in our computations suggest the plausibility of previously unknown general laws of the iterative behaviour of $(3m+d)-$maps, with one of such laws formulated and proved in [Belaga 2003]–[21].

(1) Withing the chosen range, Conjecture 2.3(1) has been fully confirmed: all such systems $\mathcal{D}_d$ have at least one primitive cycle.

(2) The confirmation of Conjecture 2.3(2) is more problematic: *a priori*, no quantity of computations could confirm that a certain (sub)set is finite. Thus, the plausibility of such a claim hinges on the quality of availble evidence that the search was exhaustive. Such an evidence is discussed below, §4.

(i) The total number of experimentally discovered T_d-cycles, $1 \leq d \leq 19999, \ d \in \mathbf{D}$, is equal to 42765.

(ii) The numbers $\omega(d)$ of such cycles for individual systems $\mathcal{D} - d$ are ranging from $\omega(d) = 1$, for 1481 systems $\mathcal{D}_d$ out of 6667, to 2, for 1507 of them, to 3, for 1005, etc., to 944 for the system $\mathcal{D}_{14303}$:

Table 3.1. Eleven systems $\mathcal{D}_d$ with the numbers of primitive cycles bigger than 160.

d	7463	18359	7727	15655	10289	9823	17021	14197	13085	6487	14303
$\omega(d)$	162	164	198	207	214	241	258	329	335	534	944

(3) Compared to the case of the two Conjectures 2.3, our claim of the confirmation of Conjecture 2.4 is even more problematic: no quantity of (always *finite*) computations could confirm that a certain (sub)set is *infinite* !

Still, taking in account relatively small total numbers of memberships in primitive cycles for corresponding numbers n compared to the total number 42765 of primitive cycles found in the chosen range, – the total numbers ranging from 18 to 452 (for $n = 1$), – we have a strong evidence of the plausibility of Conjecture 2.4.

§4. Algorithm Searching for Primitive Cycles.

According to the generalized Collatz conjecture, all trajectories of the system $\mathbf{D}_d, d \in \mathbf{D}$ are either cyclic, or ultimately cyclic, This claim became the basic assumption of our algorithm searching for primitive T_d-cycles :

Working Assumption 4.1. Let $\Delta = \{(u,v) \in \mathbf{D} |\, \gcd(u,v) = 1\}$. The following algorithm always halts :

$$\mathbf{A}_{3x+d}^{\text{prim.cycl}} \quad : \quad \begin{cases} \forall (u,v) \in \Delta \\ d := u;\ m := n := v \\ \textbf{while } m \neq n \textbf{ do} \\ \quad m := T_d(m);\ n := T_d(n);\ n := T_d(n); \\ \textbf{endwhile} \ . \end{cases} \tag{4 : 1}$$

The algorithm (4:1) computes at the j-th step the iterations $T_d^j(n)$ and $T_d^{2j}(n)$, and then compares them. See Proposition 4.2 below for a proof that even the full (i. e., not necessarily restricted to primitive cycles) version $\mathbf{A}_{3x+d}^{\text{cyclic}}$ of this algorithm detects all T_d-cycles.

This classical cycle detection device, remarkable for its simplicity, has been invented by Robert W. Floyd. (Never published by the author. The standard reference is [Knuth 1969], pp. 4-7, Exercise 7. See also [Cohen 1993], §8.5.2, for an update on cycles detection methods.) :

Proposition 4.2. Let $f : \mathbf{N} \longrightarrow \mathbf{N}$ be a function. A f-trajectory starting at a positive integer m,

$$\tau(m) = \left\{ f^0(m) = m, f(m), f^2(m) = f(f(m)), f^3(m), \dots \right\} \tag{4 : 2}$$

is ultimately cyclic *iff*, for some $j \geq 1$,

$$f^j(m) = f^{2j}(m) \ , \tag{4 : 3}$$

Proof : **(1)** The *if* (or sufficiency) condition (4:3) is obvious.

(2) The *only if* (or necessary) condition. Suppose $\tau(m)$ runs at the point $s = f^z(m)$, $z \geq 1$, into a cycle of the length w, so that, for any $x \geq 0$, $f^{z+x}(n) = f^{z+x+w}(m)$.

Assuming $t = z + x$, the condition (4:2) will be satisfied if, for some positive integers $x, y \geq 1$, the equality holds :

$$2t = 2(z+x) = z + x + y \cdot w = t + y \cdot w.$$

The choice $y = \left\lceil \frac{z}{w} \right\rceil$ and $x = y \cdot w - z$ would do. *End of Proof.*

The above Working Assumption has been verified in more than $75,000,000$ cases within the intervals

$$d \in \mathbf{I}_{1,19999} = [1, 19999] \cap \mathbf{D} \ , \quad n \in \mathbf{I}_{1,600 \cdot d - 1} \ . \tag{4 : 4}$$

and separately, in a huge controlling check, in more than 24,000,000,000 cases within the intervals

$$d \in \mathbf{I}_{1,4999} \ , \quad n \in \mathbf{I}_{1,3000 \cdot d - 1} \ . \tag{4 : 5}$$

To speed the search, trajectories have been traced in the increasing order of the starting number $m \in \mathbf{D}$. Then, if the trajectory $\tau_d(m)$ fell at some point behind the initial point $m,\ n = T_d^j(m) < m$, the search along $\tau_d(m)$ was ceased, since its continuation, the trajectory $\tau_d(n)$ starting at n, has been already treated before. Every one of the thus traced trajectories either run ultimately into a primitive cycle, or descended under m, and thus, has been already traced to a cycle earlier.

We have carried out some additional controlling checks, too :

(1) In 82 cases, for all $d \in \mathbf{I}_{2755,2999}$, an independent search has been carried out, with the bigger initial interval $1 \leq n \leq 5000 \cdot d$. No new primitive cycles have been discovered.

(2) In one particularly interesting case, $d = 343$, mentioned below, the chosen initial controlling interval was $1 \leq n \leq 60,000 \cdot 343 > 2 \cdot 10^7$. The calculations have confirmed the existence of only three primitive cycles discovered earlier.

Note also that the minimal members of all primitive cycles, discovered thus far, fall under the upper limit $600 \cdot d$ in (4:4), with only two cases coming relatively close to this limit :

(1) $d = 343$; one of primitive T_{343}-cycles (out of three discovered) has the minimal member (cf. (10:3)) $n_o = 177,337$; $517 \cdot 343 < n_o < 518 \cdot 343 < 600 \cdot 343$ (and the maximal member $m^o = 159,053,606$).

(2) $d = 551$; one of primitive T_{551}-cycles (out of ten) has the minimal member $n_o = 212,665$; $385 \cdot 551 < n_o < 386 \cdot 551 < 600 \cdot 551$ (with $m^o = 8,332,648$).

§5. Commentaries to the Table of Primitive T_d-cycles, $1 \leq d \leq 19999$ (d odd and not divisible by 3).

Notations :

(1) The data corresponding to a given shift number d is framed by the double backslash \\\\.

(2) The first line of the block of data corresponding to a given shift number d, for example,

$$d = 5; a = 5; lg = [3, 5, 5, 27, 27]; LG = [1, 3, 3, 17, 17];$$

includes the value of d, in this case, $d = 5$, the number a of primitive cycles found, $a = 5$, and the lists of *lengths*, lg, and *Oddlengths*, LG, of corresponding cycles, respectively : lg= [. . .], LG= [. . .].

(3) Then, for a given d, follows the detaled descriptions of the corresponding primitive T_d-cycles, in the order of the values of their *perigees* (the minimal members).

(4) The list S[j] of the length *Oddlength* represents the list of odd members of the cycle listed in the order of the index j, $1 \leq j \leq a$, according to the above definintion of *Oddframe* of the cycle, (2:6(iii)).

(5) The list P[j], j, $1 \leq j \leq a - 1$, , represents the first $Oddlength - 1$ members of the list of exponents of 2, corresponding to the passage from one odd member of the $Oddframe$ of the cycle to the next one (cf. (2:6)), with the exponent for the passage from the last odd member of the cycle to the first one omitted. The full list has been defined above (2:6(ii)) as $Evenframe$ of the cycle.

(6) The last member of the $Evenframe$ of the cycle, missing in P[j], is presented immediately after P[j], as p= . . .. The reason of such an independent presentation of this member (p_k in the notations (2:7)) is its irrelevance to the construction of numbers A of the formulae (2:8).

References

[Belaga 1995] **Edward G. Belaga [1995]** : Probing into the $3x + d$ World, Preprint 95/03, Univ. Louis Pasteur, Strasbourg.

[Belaga 1998]**Edward G. Belaga [1998]:** Reflecting on the $3x + 1$ Mystery. Outline of a Scenario - Improbable or Realistic ?

[Belaga 2003]–[21] **Edward G. Belaga [2003]:** 21. Effective polynomial upper bounds to perigees and numbers of $(3x + d)-$cycles of a given *Oddlength*, *Acta Arithmetica* **106**, 197206.

[Belaga, Mignotte 1999]–[22] **Edward G. Belaga, Maurice Mignotte [1999]:** Embedding the $3m + 1$ Conjecture in a $3x + d$ Context, *Experimental Mathematica* **2:5**, 145-151.

[Belaga, Mignotte 2000]–[23] **Edward G. Belaga, Maurice Mignotte [2000]:** Cyclic Structure of Dynamical Systems Associated with $3x + d$ Extensions of Collatz Problem.
http://www-irma.u-strasbg.fr/irma/publications/2000/00017.shtml

[Belaga, Mignotte 2006^a] **Edward G. Belaga, Maurice Mignotte [2006]:** Walking Cautiously Into the Collatz Wilderness: Algorithmically, Number Theoretically, and Randomly.
http://www-irma.u-strasbg.fr/irma/publications/2006/06009.shtml

[Belaga, Mignotte 2006^b] **Edward G. Belaga, Maurice Mignotte [2006]:** The Collatz Problem and Its Generalizations: Experimental Data. Table 2. Factorization of Collatz Numbers $2^\ell - 3^k$.
http://www-irma.u-strasbg.fr/irma/publications/2006/06016.shtml