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NONCOMMUTATIVE DESCENT AND NONABELIAN COHOMOLOGY

Philippe NUSS

In this paper we prove several results related to the descent theory for noncommutative rings. As an application, we get the following two theorems (precise definitions will be given in the core of the paper).

Theorem A. *Let \mathcal{H} be a bialgebra and $\psi : R \rightarrow S$ be an \mathcal{H} -Galois extension of noncommutative rings. The category $\mathfrak{BMod}_S^{\mathcal{H}}$ of right (\mathcal{H}, S) -bialgebra modules is then equivalent to the category \mathfrak{Mod}_R of right R -modules.*

Theorem B. *Let $K \rightarrow L$ be a Galois extension of skew-fields with Galois group G . Denote by $\mathbf{GL}_n(L)$ the group of right L -linear automorphisms of the vector space L^n . Then, for any positive integer n , the Galois nonabelian 1-cohomology set $H^1(G, \mathbf{GL}_n(L))$ is trivial.*

Theorem B provides for $n = 1$ a noncommutative version of Hilbert’s Theorem 90. Theorem A extends work of H.J. Schneider ([21]) and will be deduced from a general noncommutative descent theorem due to Cipolla ([4]). Recall that the descent problem may be phrased as follows:

Let $\psi : R \rightarrow S$ be any morphism of (noncommutative) rings. Given a right S -module M , under which assumptions on the extension ψ and the module M itself does there exist a right R -module N such that M is isomorphic to $N \otimes_R S$ as a right S -module?

Since Grothendieck ([8] [9]), it has been known, at least in the commutative case for faithfully flat extensions, that the existence of a solution for the descent problem is intimately related to the existence of a *descent datum* on M . In [4] Cipolla gave a noncommutative version of Grothendieck’s descent Theorem, using noncommutative descent data. Let \mathfrak{Desc}_ψ be the category whose objects are pairs (M, f) , where M is an S -module and f a descent datum on M . Cipolla’s result can be expressed by saying that, if the extension ψ is faithfully flat, the category \mathfrak{Desc}_ψ is equivalent to the category \mathfrak{Mod}_R . We deduce Theorem A from this by showing that the categories $\mathfrak{BMod}_S^{\mathcal{H}}$ and \mathfrak{Desc}_ψ are equivalent if the extension ψ is \mathcal{H} -Galois for some bialgebra \mathcal{H} .

The way to Theorem B passes through two reinterpretations of Cipolla’s descent data that are close to the geometrical intuition of the commutative case. The first one is in terms of what we call *twists* and *symmetry operators*. These objects play exactly the same rôle in the noncommutative case as Grothendieck’s gluing data and descent data do for commutative rings. The second interpretation relates the descent data to connections. We establish a dictionary whose main entries look as follows:

twist	\longleftrightarrow	gluing datum	\longleftrightarrow	connection
braid equation	\longleftrightarrow	cocycle condition	\longleftrightarrow	“flatness”
symmetry operator	\longleftrightarrow	descent datum	\longleftrightarrow	flat connection

The point of view involving connections allows us to show that the existence of gluing data on a given S -module M is measured by a cohomological obstruction, namely the *Atiyah class of M* . This extends work done by Laudal in the commutative case ([15]). Let us denote by $\mathfrak{Dp}\mathfrak{Sym}_\psi$ the category whose objects are pairs (M, ξ) , where ξ is a symmetry operator on the S -module M . Denote by \mathfrak{FlConn}_ψ the category whose objects are pairs (M, ∇) , with ∇ a flat connection on M . The keystone of the second part of the paper is the following result.

Theorem C. *Under the previous notations, there are equivalences of categories*

$$\mathfrak{Desc}_\psi \cong \mathfrak{Dp}\mathfrak{Sym}_\psi \cong \mathfrak{FlConn}_\psi$$

for any extension $\psi : R \rightarrow S$.

Moreover, when ψ is faithfully flat, one has the additional equivalence $\mathfrak{Desc}_\psi \cong \mathfrak{Mod}_R$ provided by Cipolla's descent Theorem.

As an application of Theorem C, we obtain the following extension of Knus's work ([12]) on hermitian modules.

Theorem D. *Let $\psi : (R, \sigma_R) \rightarrow (S, \sigma_S)$ be a faithfully flat extension of rings with involution. The category \mathfrak{DHer}_S of S -hermitian modules endowed with a symmetry operator and the category \mathfrak{Her}_R of R -hermitian modules are equivalent.*

The symmetry operators mentioned above are subject to a condition of cohomological nature, the *cocycle condition*, which, actually, is equivalent to the braid (or Yang-Baxter) equation. This leads us to the following two questions:

Question 1. *What kind of cohomology is involved in the cocycle condition?*

Question 2. *Given an R -module N_0 , classify all the R -modules N such that $N \otimes_R S \cong N_0 \otimes_R S$ and all the descent data on $N_0 \otimes_R S$.*

It turns out that the common answer is Amitsur cohomology. More precisely, we prove the following result from which we derive Theorem B.

Theorem E. *Let $\psi : R \rightarrow S$ be a faithfully flat extension and N_0 an R -module. Denote the set of descent data on N_0 by $H_{\text{desc}}^1(S/R, N_0)$ and the set of twisted forms of N_0 by $\text{Twist}(S/R, N_0)$. Then there are bijections*

$$H_{\text{desc}}^1(S/R, N_0) \cong \text{Twist}(S/R, N_0) \cong H^1(S/R, \mathcal{F}_{N_0}),$$

where the latter set is the Amitsur 1-cohomology set of ψ with values in a suitably defined functor \mathcal{F}_{N_0} . Moreover, if the extension ψ is G -Galois, then $H^1(S/R, \mathcal{F}_{N_0})$ is in bijection with the Galois 1-cohomology set $H^1(G, \mathcal{F}_{N_0}(S))$.

Let us briefly describe the content of the different sections.

The first section is devoted to the symmetry operators. We define an involutive automorphism τ of $S \otimes_R S$, canonically attached to the extension $\psi : R \rightarrow S$. The automorphism τ , which we call the *symmetry of the extension ψ* , plays a central rôle in this

paper; it verifies the braid equation and is a substitute for the flip $s \otimes s' \mapsto s' \otimes s$ in a noncommutative situation. We also define twists and symmetry operators.

In the second section we deal with noncommutative connections and the Atiyah class. In Section 3 we recall Cipolla's descent Theorem, a descent theorem for elements, and we state a descent theorem for homomorphisms. We then prove Theorem C.

In Section 4 we prove Theorem A after having defined the concept of a \mathcal{H} -Galois extension. This is a slight generalization of the concept of a Hopf-Galois extension studied in [19], [20], and [21]; it unifies the notions of both a classical Galois extension of fields and of a strongly graded ring. We also discuss an important subclass of bialgebra Galois extensions consisting of Galois extensions of noncommutative rings in the sense of [16]. We deduce a Galois descent theorem for modules. Section 5 is devoted to hermitian modules and to the proof of Theorem D.

The last section deals with nonabelian cohomology. We begin with recalling the definition of the Amitsur cohomology of an extension $\psi : R \rightarrow S$ with values in a functor \mathcal{F} from the category of algebras over R to the category of groups. We compute the nonabelian Amitsur cohomology sets $H^i(S/R, \mathcal{F})$ for $i = 0, 1$, where $\mathcal{F} = \text{Aut}(N_0)$ is the "automorphism" functor. It is defined on an algebra S over R by $\text{Aut}(N_0)(S) = \text{Aut}_S(N_0 \otimes_R S)$. We then prove Theorem E. In the case when N_0 is the R -module R itself, the Amitsur 1-cohomology set can be seen as a generalization of the relative Picard group $\text{Pic}(S/R)$ of commutative extensions ([13]). Finally, as an application, we deduce Theorem B.

I am grateful to O.A. Laudal who initiated me to his commutative algebraic setting of differential geometry during several visits in Norway. The link between descent theory and connections in the commutative case has been worked out by Arnfinn Laudal and explained by him in a series of unpublished lectures at the university of Oslo. I also would like to thank C. Kassel, M. Ojanguren, and M. Van den Bergh for helpful discussions.

0. Conventions and basic results

Let k be a commutative and unital ring, fixed throughout the paper. By *algebra* we always mean a unital, associative, but not necessarily commutative k -algebra. Unless otherwise stated, a module is always a *right* module and morphisms between modules are *right* linear. All modules are *unitary*, that is, $m \cdot 1 = m$ for all $m \in M$. A *bialgebra* \mathcal{H} is always defined over the ground ring k , and comodules are *right* \mathcal{H} -comodules. The unadorned symbol \otimes between a right k -module and a left k -module stands for \otimes_k .

Let A and B be two algebras. Denote by \mathfrak{Mod}_A the category of right A -modules, by ${}_B\mathfrak{Mod}$ the category of left B -modules and by ${}_B\mathfrak{Mod}_A$ the category of B - A -bimodules (A acting on the right and B on the left). Denote by \mathfrak{Gr} (respectively by \mathfrak{Ab}) the category of groups (respectively the category of abelian groups).

Let $\rho : A \rightarrow B$ be a morphism of algebras (also called an *extension*). We say that B is an *algebra over* A . Remark that B is in general not an A -algebra, unless the image $\rho(A)$ of A in B is central. Let \mathfrak{Alg}_A be the category of algebras over A . Thus an object of \mathfrak{Alg}_A is a morphism $\rho : A \rightarrow B$, and a morphism from $\rho : A \rightarrow B$ to $\rho' : A \rightarrow B'$ is a homomorphism of algebras θ from B to B' such that $\theta \circ \rho = \rho'$. A morphism of algebras $\rho : A \rightarrow B$ gives rise to two functors :

1) $\rho^* : \mathfrak{Mod}_B \longrightarrow \mathfrak{Mod}_A$ (the *restriction of the scalars*), where for any B -module M , $\rho^*(M)$ is the underlying abelian group M with the A -action given by $m \cdot a = m\rho(a)$ ($m \in M, a \in A$);

2) $\rho_! : \mathfrak{Mod}_A \longrightarrow \mathfrak{Mod}_B$ (the *extension of the scalars*), where for any A -module N , $\rho_!(N) = N \otimes_A B$, and B is a left A -module via the action $ab = \rho(a)b$ ($a \in A, b \in B$).

It is well known that the functor $\rho_!$ is left adjoint to the functor ρ^* , *i.e.*, there exists, for any $N \in \mathfrak{Mod}_A$ and $M \in \mathfrak{Mod}_B$, a natural bijection

$$\mathrm{Hom}_B(\rho_!(N), M) \cong \mathrm{Hom}_A(N, \rho^*(M)). \quad (1)$$

The same can be done for left modules. One may also consider the restriction of the scalars for bimodules. For instance, if $\psi : R \longrightarrow S$ is an extension, the S -bimodule S is an R -bimodule via

$$r \cdot s \cdot r' = \psi(r)s\psi(r'),$$

where $r, r' \in R$ and $s \in S$ (in the last term the multiplication is the product in S). The k -module $S \otimes_R S$ is obviously an S -bimodule by the formula

$$t \cdot (s \otimes s') \cdot t' = ts \otimes s't' \quad (2)$$

(here $s, s', t, t' \in S$), and hence an R -bimodule.

An extension $\psi : R \longrightarrow S$ is said to be *flat* (respectively *faithfully flat*) if S is flat (respectively faithfully flat) as a left R -module. If $\psi : R \longrightarrow S$ is faithfully flat, ψ is necessarily injective ([3]).

The symbol by which we denote an element should implicitly indicate to which set the element belongs. Finally, the definitions involving a tensor product will mostly be in terms of an indecomposable element, the extension to arbitrary elements being done by linearity.

1. The symmetry operators

1.1. The symmetry of an extension

Let $\psi : R \longrightarrow S$ be a morphism of algebras. The *multiplication of S* is the map μ from $S \otimes_R S$ to S defined by

$$\mu(s \otimes s') = ss'.$$

It has an R - S -linear section ε_1 as well as an S - R -linear section ε_2 given by $\varepsilon_1(s) = 1 \otimes s$ and $\varepsilon_2(s) = s \otimes 1$. Denote by $\Omega(S/R)$ the kernel of the map μ and call it the *module of (noncommutative) differential 1-forms of S over R* . It fits in the following fundamental exact sequence in ${}_S \mathfrak{Mod}_S$:

$$0 \longrightarrow \Omega(S/R) \xrightarrow{\iota} S \otimes_R S \xrightarrow{\mu} S \longrightarrow 0, \quad (3)$$

where ι is the canonical injection. For any element s of S , define the element ds belonging to $\Omega(S/R)$ by

$$ds = \varepsilon_1(s) - \varepsilon_2(s) = 1 \otimes s - s \otimes 1.$$

As for $R = k$ (see [11] for example), the module of differential 1-forms of S over R is generated by the elements ds (where $s \in S$) as a left S -module. The map d from S to $\Omega(S/R)$ is an R -derivation which is called the *universal derivation of S over R* . The retraction maps r_1 and r_2 from $S \otimes_R S$ to $\Omega(S/R)$ corresponding to ε_1 and ε_2 are given by $r_i = \text{id} - (\varepsilon_i \circ \mu)$ for $i = 1, 2$. Let u_1 be the R - S -linear isomorphism from $S \otimes_R S$ to $\Omega(S/R) \oplus S$ given by $u_1 = (-r_1, \mu)$. Similarly, let u_2 be the induced S - R -linear isomorphism from $S \otimes_R S$ to $\Omega(S/R) \oplus S$ given by $u_2 = (r_2, \mu)$. Explicitly, whenever s, s' belong to S , we have

$$u_1(s \otimes s') = (1 \otimes ss' - s \otimes s', ss') \quad \text{and} \quad u_2(s \otimes s') = (s \otimes s' - ss' \otimes 1, ss').$$

The inverse isomorphisms are $u_1^{-1} = (-\iota, \varepsilon_1)$ and $u_2^{-1} = (\iota, \varepsilon_2)$.

Lemma 1.1. *The following equality holds:*

$$u_1^{-1} \circ u_2 = u_2^{-1} \circ u_1.$$

Proof: An easy computation shows that

$$(u_1^{-1} \circ u_2)(s \otimes s') = (u_2^{-1} \circ u_1)(s \otimes s') = 1 \otimes ss' + ss' \otimes 1 - s \otimes s'.$$

□

Definition 1.2. The R - R -linear automorphism τ of $S \otimes_R S$ defined by

$$\tau = u_1^{-1} \circ u_2 = u_2^{-1} \circ u_1$$

is called the *symmetry of the extension ψ* . It is explicitly given by the formula

$$\tau(s \otimes s') = 1 \otimes ss' + ss' \otimes 1 - s \otimes s'. \quad (4)$$

One has $\tau(s \otimes 1) = 1 \otimes s$ and $\tau(1 \otimes s) = s \otimes 1$, whence the symmetry τ induces the map $-\text{id}$ on the submodule of the differential 1-forms $\Omega(S/R)$.

Proposition 1.3. *The symmetry τ is involutive and verifies $\mu \circ \tau = \mu$ and the braid (or Yang-Baxter) equation*

$$(\tau \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\tau \otimes \text{id}) = (\text{id} \otimes \tau) \circ (\tau \otimes \text{id}) \circ (\text{id} \otimes \tau) \quad (5)$$

in the group of R -linear automorphisms of $S \otimes_R S \otimes_R S$. Hence τ provides for each $n \geq 2$ a representation of the symmetric group \mathfrak{S}_n on $S^{\otimes_R n}$.

Proof: The symmetry is involutive by the very definition. It obviously verifies $\mu \circ \tau = \mu$. The braid equation follows from a tedious, but straightforward computation. Explicitly, one has for all $s, t, u \in S$

$$\begin{aligned} ((\tau \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\tau \otimes \text{id}))(s \otimes t \otimes u) &= ((\text{id} \otimes \tau) \circ (\tau \otimes \text{id}) \circ (\text{id} \otimes \tau))(s \otimes t \otimes u) \\ &= stu \otimes 1 \otimes 1 - st \otimes u \otimes 1 + s \otimes tu \otimes 1 - s \otimes t \otimes u - 1 \otimes s \otimes tu + 1 \otimes st \otimes u + 1 \otimes 1 \otimes stu. \end{aligned}$$

□

Remarks 1.4. 1) The involutive morphism τ will play the rôle of the flip in the noncommutative case. When $R \rightarrow S$ is a morphism of *commutative* algebras, then the flip and the map τ coincide infinitesimally; indeed, one checks that $\tau(s \otimes s') \equiv s' \otimes s \pmod{I^2}$. Here $I = \Omega(S/R)$ is the kernel of the multiplication map and I^2 is the square of I with respect to the product on $S \otimes_R S$ given by $(s_1 \otimes s_2) \cdot (t_1 \otimes t_2) = s_1 t_1 \otimes s_2 t_2$, where s_1, s_2, t_1, t_2 belong to S .

2) Notice that the isomorphism u_1 is induced by (ε_1, r_1) in a slightly different way as u_2 is from the pair (ε_2, r_2) . Let u'_1 be the R - S -linear isomorphism from $S \otimes_R S$ to $\Omega(S/R) \oplus S$ given by $u'_1 = (r_1, \mu)$. We derive an R - R -linear automorphism $v = u_2^{-1} \circ u'_1$ whose inverse is $v^{-1} = u_1'^{-1} \circ u_2$ and that is given by

$$v(s \otimes s') = s \otimes s' - 1 \otimes s s' + s s' \otimes 1 \quad \text{and} \quad v^{-1}(s \otimes s') = s \otimes s' + 1 \otimes s s' - s s' \otimes 1.$$

If the characteristic of S is different from 2, the automorphism v is no longer involutive. However v always verifies the braid equation and therefore provides for each $n \geq 2$ a representation of the braid group \mathfrak{B}_n on $S^{\otimes_R n}$.

1.2. The algebra structure on $S^{\otimes_R n}$

Let $S^{\otimes_R n}$ be the S -bimodule $\overbrace{S \otimes_R S \dots \otimes_R S}^{n \text{ times}}$. Using the braiding τ , the bimodule $S^{\otimes_R n}$ may be endowed with a ring structure. This is done in the following way. For $1 \leq i \leq n-1$, denote by τ_i the endomorphism of $S^{\otimes_R n}$ given by

$$\tau_i = \text{id}_S \otimes \dots \otimes \tau \otimes \dots \otimes \text{id}_S,$$

where τ acts on the $(i, i+1)$ -components. Let μ_i be the morphism from $S^{\otimes_R n}$ to $S^{\otimes_R (n-1)}$ defined by

$$\mu_i = \text{id}_S \otimes \dots \otimes \mu \otimes \dots \otimes \text{id}_S,$$

where the multiplication μ also acts on the $(i, i+1)$ -components. Define the product $\pi_n : S^{\otimes_R n} \otimes S^{\otimes_R n} \rightarrow S^{\otimes_R n}$ by

$$\begin{aligned} \pi_n = & (\mu_1 \circ \mu_2 \circ \dots \circ \mu_{2n-1}) \circ (\tau_{2n-2}) \circ (\tau_{2n-4} \circ \tau_{2n-3}) \circ \\ & (\tau_{2n-6} \circ \tau_{2n-5} \circ \tau_{2n-4}) \circ \dots \circ (\tau_4 \circ \dots \circ \tau_{n+1}) \circ (\tau_2 \circ \dots \circ \tau_n). \end{aligned}$$

This rather complicated formula becomes limpid in the commutative case, where τ is replaced by the flip $s \otimes s' \mapsto s' \otimes s$. It also has a nice pictorial representation in terms of braids. Indeed, the product is given by Figure 1.

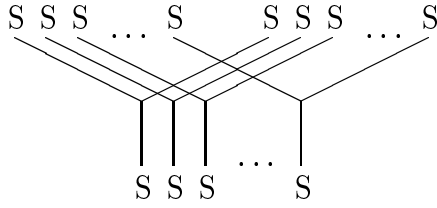


Figure 1

The multiplication map μ is represented by Figure 2, and the braid τ is symbolized by Figure 3. Remark that in Figures 2 and 3 we do not have to distinguish whether a strand passes behind another one or in front of it, since the braiding τ is involutive.



Figure 2

The fact that π_n induces a structure of algebra on $S^{\otimes n} = S \otimes \dots \otimes S$ is a general result on braids (see, *e.g.*, [2]), hence it induces a product on $S^{\otimes R^n}$. Denote this product by $*$, *i.e.*, $x * y = \pi_n(x \otimes y)$ for $x, y \in S^{\otimes R^n}$. For instance, $S \otimes_R S$ is an algebra over R by

$$(s_1 \otimes s_2) * (t_1 \otimes t_2) = s_1 s_2 t_1 \otimes t_2 + s_1 \otimes s_2 t_1 t_2 - s_1 s_2 \otimes t_1 t_2, \tag{6}$$

whereas $S \otimes_R S \otimes_R S$ is an algebra over R by

$$\begin{aligned} (s_1 \otimes s_2 \otimes s_3) * (t_1 \otimes t_2 \otimes t_3) = & (s_1 s_2 s_3 t_1 \otimes t_2 \otimes t_3) + (s_1 \otimes s_2 s_3 t_1 t_2 \otimes t_3) + (s_1 \otimes s_2 \otimes s_3 t_1 t_2 t_3) \\ & - (s_1 s_2 s_3 \otimes t_1 t_2 \otimes t_3) - (s_1 s_2 s_3 \otimes 1 \otimes t_1 t_2 t_3) + (s_1 s_2 s_3 \otimes t_1 \otimes t_2 t_3) \\ & - (s_1 \otimes s_2 s_3 \otimes t_1 t_2 t_3) + (s_1 s_2 \otimes s_3 \otimes t_1 t_2 t_3) - (s_1 s_2 \otimes s_3 t_1 \otimes t_2 t_3), \end{aligned}$$

where $s_1, s_2, s_3, t_1, t_2, t_3$ belong to S .

The product on $S \otimes_R S$ may also be constructed in the following way. On $S \oplus \Omega(S/R)$ there is an obvious algebra structure given by

$$(s, \omega) \cdot (s', \omega') = (ss', s\omega' + s'\omega). \tag{7}$$

An easy computation shows that the two algebra structures obtained on $S \otimes_R S$ by transporting the law (7) along the isomorphisms u_1 and u_2 coincide and gives (6). It readily follows that the multiplication map from $S \otimes_R S$ to S is a morphism of algebras whose kernel $\Omega(S/R)$ is a square zero ideal of $S \otimes_R S$. Notice that the two sections ε_1 and ε_2 of the multiplication map are morphisms of algebras.

Proposition 1.5. *The symmetry τ is an automorphism of the algebra $S \otimes_R S$. Moreover, τ is the image of ε_1 via the bijection given by the adjunction formula*

$$\mathrm{Hom}_S(S, \varepsilon_2^*(S \otimes_R S)) \cong \mathrm{Hom}_{S \otimes_R S}(\varepsilon_{2!}(S), S \otimes_R S).$$

Proof: A straightforward computation gives

$$\begin{aligned} \tau(s \otimes t) * \tau(u \otimes v) &= stuv \otimes 1 - stu \otimes v + st \otimes uv - s \otimes tuv + 1 \otimes stuv \\ &= \tau((s \otimes t) * (u \otimes v)). \end{aligned}$$

The adjunction formula is a particular case of (1). One easily checks that the map ε_1 from S to $\varepsilon_2^*(S \otimes_R S)$ is S -linear and $\varepsilon_1(s) * \varepsilon_2(t) = \tau(s \otimes t)$ for any $s, t \in S$. \square

1.3. The S -module structures on $M \otimes_R S$

Let $\psi : R \rightarrow S$ be an extension and M be an S -module. The right S -action μ from $M \otimes S$ to S given by $\mu(m \otimes s) = ms$ (with $m \in M$ and $s \in S$) is k -linear and factors through a morphism of S -bimodules from $M \otimes_R S$ to S still called S -action and still denoted by μ . Endow $S \otimes_R S$ with the product $*$ defined by Relation (6). Then, for $i = 1, 2$, the abelian group underlying $\varepsilon_{i!}(M)$ is isomorphic to $M \otimes_R S$.

Lemma 1.6. *Let $\psi : R \rightarrow S$ be an extension and M be an S -module. In the following assertions, m belongs to M and s, t, u to S .*

1) *There is an R -linear isomorphism $\varepsilon_{1!}(M) \cong M \otimes_R S$ given by*

$$m \otimes (s \otimes t) \longmapsto ms \otimes t + m \otimes st - mst \otimes 1.$$

Via this isomorphism $M \otimes_R S$ becomes an $S \otimes_R S$ -module denoted by $[M \otimes_R S]_1$. The $S \otimes_R S$ -action is defined by

$$(m \otimes s) *_1 (t \otimes u) = mst \otimes u + m \otimes stu - mst \otimes 1. \quad (8)$$

2) *There is an R -linear isomorphism $\varepsilon_{2!}(M) \cong M \otimes_R S$ given by the correspondence*

$$m \otimes (s \otimes t) \longleftarrow ms \otimes t.$$

Via this isomorphism $M \otimes_R S$ becomes an $S \otimes_R S$ -module denoted by $[M \otimes_R S]_2$. The $S \otimes_R S$ -action is defined by

$$(m \otimes s) *_2 (t \otimes u) = mst \otimes u + m \otimes stu - ms \otimes tu. \quad (9)$$

Proof: These easy computations are left to the reader. \square

One might expect *a priori* four right S -module structures on $M \otimes_R S$ induced by the sections ε_1 and ε_2 , namely $\varepsilon_i^*([M \otimes_R S]_j)$ for $i, j \in \{1, 2\}$. One easily checks that only three different structures appear, denoted by $(M \otimes_R S)_i$ for $i \in \{1, 2, 3\}$, and defined by

$$\begin{aligned} (M \otimes_R S)_1 &= \varepsilon_1^*([M \otimes_R S]_2) = \varepsilon_2^*([M \otimes_R S]_1), \\ (M \otimes_R S)_2 &= \varepsilon_2^*([M \otimes_R S]_2), \quad \text{and} \quad (M \otimes_R S)_3 = \varepsilon_1^*([M \otimes_R S]_1). \end{aligned}$$

The S -actions on $M \otimes_R S$ are denoted by \square_i ($i = 1, 2, 3$) and are given by the formulae

$$\begin{cases} (m \otimes s) \square_1 s' = (m \otimes s) *_2 \varepsilon_1(s') = (m \otimes s) *_1 \varepsilon_2(s') = m \otimes s s' \\ (m \otimes s) \square_2 s' = (m \otimes s) *_2 \varepsilon_2(s') = m \otimes s s' + m s s' \otimes 1 - m s \otimes s' \\ (m \otimes s) \square_3 s' = (m \otimes s) *_1 \varepsilon_1(s') = m \otimes s s' - m s s' \otimes 1 + m s \otimes s', \end{cases} \quad (10)$$

where $m \in M$ and $s, s' \in S$. The third structure is related to the automorphism ν defined in Remark 1.4.2. Since we want to generalize τ on $M \otimes_R S$, we just forget about \square_3 .

By Lemma 1.6, one gets

$$\begin{aligned} \varepsilon_k^* \varepsilon_{2!}(M) &\cong (M \otimes_R S)_k \quad \text{for } k = 1, 2, \\ \varepsilon_1^* \varepsilon_{1!}(M) &\cong (M \otimes_R S)_3, \quad \text{and} \quad \varepsilon_2^* \varepsilon_{1!}(M) \cong (M \otimes_R S)_1. \end{aligned} \quad (11)$$

Lemma 1.7. *Fix $i, j \in \{1, 2\}$. A map $\nu : [M \otimes_R S]_i \rightarrow [M \otimes_R S]_j$ is $S \otimes_R S$ -linear if and only if the map $\nu : (M \otimes_R S)_i \rightarrow (M \otimes_R S)_j$ is S -linear.*

Proof: Assume that $\nu : [M \otimes_R S]_i \rightarrow [M \otimes_R S]_j$ is $S \otimes_R S$ -linear and take $m \in M$ and $s, t \in S$. Then by (10)

$$\nu((m \otimes s) \square_i t) = \nu((m \otimes s) *_i \varepsilon_2(t)) = \nu(m \otimes s) *_j \varepsilon_2(t) = \nu(m \otimes s) \square_j t.$$

Conversely, assume that $\nu : (M \otimes_R S)_i \rightarrow (M \otimes_R S)_j$ is S -linear. Then, for $m \in M$ and $s, t, u \in S$, one has

$$\begin{aligned} \nu((m \otimes s) *_i (u \otimes v)) &= \nu((m \otimes s) *_i ((u \otimes 1) * (1 \otimes v))) = \nu(((m \otimes s) *_i (u \otimes 1)) *_i (1 \otimes v)) \\ &= \nu((m \otimes s) *_i (u \otimes 1)) *_j (1 \otimes v) = \nu(m \otimes s) *_j ((u \otimes 1) * (1 \otimes v)) = \nu(m \otimes s) *_j (u \otimes v). \end{aligned}$$

\square

Remarks 1.8. 1) If $R \rightarrow S$ is an extension, T the algebra $S \otimes_R S$ with the product $*$, and N an R -module, on $N \otimes_R T$ the canonical structure $(n \otimes t) \cdot t' = n \otimes t t'$ (with $n \in N$ and $t, t' \in T$) coincides with $[N \otimes_R S \otimes_R S]_2$.

2) Observe that $\varepsilon_{k!}(M)$ ($k = 1, 2$) are two liftings (in the sense of deformation theory, see for example [14]) of M to $S \otimes_R S$ with respect to the surjective morphism of algebras $\mu : S \otimes_R S \rightarrow S$.

1.4. The module $\Omega(S/R, M)$

Consider the usual S -bimodule structure (2) on $S \otimes_R S$. Then $M \otimes_S (S \otimes_R S)$ is naturally isomorphic to $(M \otimes_R S)_1$ via the S -linear map U , defined by $U(m \otimes s \otimes t) = ms \otimes t$ (the inverse map being $U^{-1}(m \otimes s) = m \otimes 1 \otimes s$). Via this isomorphism, the element $(m \otimes s) \square_1 s'$ corresponds to $m \otimes_S (1 \otimes s s')$ and the element $(m \otimes s) \square_2 s'$ to $m \otimes_S \tau(s \otimes s')$. Let

$$0 \longrightarrow M \otimes_S \Omega(S/R) \xrightarrow{\iota} M \otimes_S (S \otimes_R S) \xrightarrow{\text{id}_M \otimes \mu} M \otimes_S S \longrightarrow 0 \quad (12)$$

be the short exact sequence obtained by tensoring the fundamental exact sequence (3) with M over S . We still denote by ι the canonical map from $M \otimes_S \Omega(S/R)$ to $M \otimes_S (S \otimes_R S)$. It is injective, because $\text{Tor}_1^S(M, S) = 0$. Denote by $\Omega(S/R, M)$ the image of $M \otimes_S \Omega(S/R)$ in $M \otimes_R S$ via U . We then have the commutative diagram of S -morphisms

$$\begin{array}{ccccccc} 0 & \longrightarrow & M \otimes_S \Omega(S/R) & \xrightarrow{\iota} & M \otimes_S (S \otimes_R S) & \xrightarrow{\text{id}_M \otimes \mu} & M \otimes_S S & \longrightarrow & 0 \\ & & \downarrow \wr & & U \downarrow \wr & & \parallel \wr & & (13) \\ 0 & \longrightarrow & \Omega(S/R, M) & \longrightarrow & M \otimes_R S & \xrightarrow{\mu} & M & \longrightarrow & 0. \end{array}$$

Lemma 1.9. *Endow $\Omega(S/R, M)$ with the S -module structure induced by \square_1 . Then $\Omega(S/R, M)$ is generated as a right S -module by the elements of the form $m \otimes s - ms \otimes 1$. Moreover, the two S -module structures on $\Omega(S/R, M)$ induced by \square_1 and \square_2 coincide.*

Proof: The first assertion is a consequence of the fact that $\{ds ; s \in S\}$ generates $\Omega(S/R)$.

$$\begin{aligned} \text{Now } (m \otimes s - ms \otimes 1) \square_2 t &= m \otimes st + mst \otimes 1 - ms \otimes t - (ms \otimes t + mst \otimes 1 - ms \otimes t) \\ &= m \otimes st - ms \otimes t = (m \otimes s - ms \otimes 1) \square_1 t. \end{aligned}$$

□

1.5. The symmetry operators on $M \otimes_R S$

Let $\psi : R \rightarrow S$ be an extension and M be an S -module. We now propose a noncommutative analogue of the canonical map $\xi : M \otimes_R S \rightarrow S \otimes_R M$ given by $\xi(m \otimes s) = s \otimes m$ which is well-defined when R and S are commutative and M is symmetric as an R -module (*i.e.* $mr = rm$, whenever $r \in R$ and $m \in M$).

Let $\xi : M \otimes_R S \rightarrow M \otimes_R S$ be a morphism of R -modules.

Definition 1.10.

1) The map ξ is a *twist* on $M \otimes_R S$ if the map $\xi : (M \otimes_R S)_1 \rightarrow (M \otimes_R S)_2$ is an isomorphism of S -modules such that $\mu \circ \xi = \mu$;

2) The map ξ is said to verify the *Yang-Baxter equation* if the following equality of R -linear endomorphisms of $M \otimes_R S \otimes_R S$ holds:

$$(\xi \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\xi \otimes \text{id}) = (\text{id} \otimes \tau) \circ (\xi \otimes \text{id}) \circ (\text{id} \otimes \tau);$$

3) The map ξ is a *symmetry operator* on $M \otimes_R S$ (or shorter, a *symmetry* on $M \otimes_R S$) if it is a twist that verifies the Yang-Baxter equation.

Observe in view of Lemma 1.7 that a twist is the same as an $S \otimes_R S$ -linear isomorphism $\xi : [M \otimes_R S]_1 \rightarrow [M \otimes_R S]_2$ that verifies $\mu \circ \xi = \mu$. By analogy with the commutative descent theory (see [13] for example), for any R -linear map $\xi : M \otimes_R S \rightarrow M \otimes_R S$, define three R -linear endomorphisms $\xi^{(i)}$ of $M \otimes_R S \otimes_R S$ by

$$\xi^{(0)} = (\text{id} \otimes \tau) \circ (\xi \otimes \text{id}) \circ (\text{id} \otimes \tau), \quad \xi^{(1)} = (\xi \otimes \text{id}) \circ (\text{id} \otimes \tau), \quad \text{and} \quad \xi^{(2)} = \xi \otimes \text{id}.$$

Notice that ξ verifies the Yang-Baxter equation if and only if

$$\xi^{(0)} \circ \xi^{(2)} = \xi^{(1)}. \quad (14)$$

We call Relation (14) the *cocycle condition*.

Example 1.11. Let N be an R -module. On $M = N \otimes_R S$ there is a canonical involutive symmetry ξ_{can} given by

$$\xi_{\text{can}} = \text{id}_N \otimes \tau. \quad (15)$$

In particular, the symmetry τ of an extension $\psi : R \rightarrow S$ is an involutive symmetry on $S \otimes_R S$.

Lemma 1.12. *Any S -linear isomorphism $\xi : (M \otimes_R S)_1 \rightarrow (M \otimes_R S)_2$ that verifies the Yang-Baxter equation is necessarily a symmetry on $M \otimes_R S$.*

Proof: It remains to show that $\mu \circ \xi = \mu$. Let $m \otimes s$ and $\sum_i m_i \otimes s_i$ be elements in $M \otimes_R S$ such that $\xi(\sum_i m_i \otimes s_i) = m \otimes s$. Set $\xi(m_i \otimes 1) = \sum_j m_{ij} \otimes t_{ij}$. Using the equalities $m \otimes s = \xi(\sum_i m_i \otimes s_i) = \sum_i \xi(m_i \otimes 1) \square_2 s_i$, one deduces that

$$ms = \sum_i \sum_j m_{ij} t_{ij} s_i.$$

Write $\xi(m \otimes 1)$ as the element $\sum_h p_h \otimes u_h$ of $M \otimes_R S$. An easy computation gives

$$((\xi \otimes \text{id}_S) \circ (\text{id}_M \otimes \tau) \circ (\xi \otimes \text{id}_S))(\sum_i m_i \otimes s_i \otimes 1) = \sum_h p_h \otimes u_h \otimes s$$

and

$$\begin{aligned} & ((\text{id}_M \otimes \tau) \circ (\xi \otimes \text{id}_S) \circ (\text{id}_M \otimes \tau))(\sum_i m_i \otimes s_i \otimes 1) = \\ & \sum_i \sum_j (m_{ij} \otimes t_{ij} s_i \otimes 1 + m_{ij} \otimes 1 \otimes t_{ij} s_i - m_{ij} \otimes t_{ij} \otimes s_i), \end{aligned}$$

hence

$$\sum_h p_h u_h s = \sum_i \sum_j m_{ij} t_{ij} s_i,$$

which is equal to ms . On the other hand,

$$\mu \circ \xi(m \otimes s) = \mu(\xi(m \otimes 1) \square_2 s) = \sum_h p_h u_h s,$$

thus $\mu \circ \xi(m \otimes s) = ms = \mu(m \otimes s)$. □

Proposition 1.13. *Given any twist ξ on $M \otimes_R S$, there exists an involutive twist $\theta = \text{Invol}(\xi)$ defined for $m \otimes s \in M \otimes_R S$ by:*

$$\theta(m \otimes s) = \xi^{-1}(ms \otimes 1) - m \otimes s + ms \otimes 1.$$

The twists ξ and θ are also related by the equality

$$\theta(m \otimes s) = \xi^{-1}(m \otimes 1) \square_2 s. \quad (16)$$

Furthermore, if ξ is already involutive, then $\text{Invol}(\xi) = \xi$.

Proof: The commutation property $\mu \circ \xi = \mu$ implies that $\xi^{-1}(m \otimes 1) - m \otimes 1$ lies in $\Omega(S/R, M)$. By Lemma 1.9, one has

$$(\xi^{-1}(m \otimes 1) - m \otimes 1) \square_1 s = (\xi^{-1}(m \otimes 1) - m \otimes 1) \square_2 s$$

or, equivalently, $\xi^{-1}(ms \otimes 1) - m \otimes s = \xi^{-1}(m \otimes 1) \square_2 s - ms \otimes 1$. This implies (16).

Next, we prove that θ is an involutive twist. Evidently, $\mu \circ \theta = \mu$. The map θ from $(M \otimes_R S)_1$ to $(M \otimes_R S)_2$ is S -linear. Indeed,

$$\theta((m \otimes s) \square_1 t) = \theta(m \otimes st) = \xi^{-1}(m \otimes 1) \square_2 st = (\xi^{-1}(m \otimes 1) \square_2 s) \square_2 t = \theta(m \otimes s) \square_2 t.$$

In order to prove that θ is involutive, we set $\xi^{-1}(m \otimes 1) = \sum_i m_i \otimes v_i \in M \otimes_R S$. The relation $\mu \circ \xi^{-1} = \mu$ implies that $\sum_i m_i v_i = m$ and, hence, by S -linearity, that $\xi^{-1}(ms \otimes 1) = \sum_i m_i \otimes v_i s$. Let us now compute $\theta \circ \theta$.

$$\begin{aligned} \theta \circ \theta(m \otimes s) &= \theta(\xi^{-1}(ms \otimes 1) - m \otimes s + ms \otimes 1) \\ &= ms \otimes 1 + \xi^{-1}(ms \otimes 1) - ms \otimes 1 + \sum_i (m_i v_i s \otimes 1 + \xi^{-1}(m_i v_i s \otimes 1) - m_i \otimes v_i s) \\ &\quad - ms \otimes 1 - \xi^{-1}(ms \otimes 1) + m \otimes s = m \otimes s. \end{aligned}$$

Let us prove the last statement of Proposition 1.13. If ξ is already involutive, then

$$\xi^{-1}(m \otimes 1) \square_2 s = \xi^{-1}(m \otimes s) = \xi(m \otimes s)$$

and (16) implies that $\xi = \theta$. \square

Example 1.14. In the case $M = S$, then v is a twist and $\text{Invol}(v) = \tau$ (cf. Remark 1.4.2).

2. Connections

2.1. Definitions

We start giving two equivalent definitions of a connection. Let $\psi : R \rightarrow S$ be a morphism of algebras, M an S -module, and $\psi^*(M)$ the underlying R -module. Classically (see [11] for example), a *connection on M* is an R -linear map ∇ from $\psi^*(M)$ to $M \otimes_S \Omega(S/R)$ such that

$$\nabla(ms) = \nabla(m)s + m \otimes ds. \quad (17)$$

Keeping in mind the identification derived from Diagram (13), we express all the notions attached to connections in terms of the module $\Omega(S/R, M)$. This will simplify further computations.

Lemma 2.1. *A connection on M is completely determined by an R -linear map $\tilde{\nabla}$ from $\psi^*(M)$ to $\Omega(S/R, M)$ such that*

$$\tilde{\nabla}(ms) = \tilde{\nabla}(m)s + m \otimes s - ms \otimes 1. \quad (18)$$

We shall call $\tilde{\nabla}$ a *connection* and shall write ∇ instead of $\tilde{\nabla}$; the context will clearly indicate which definition one has to consider.

Example 2.2. Let N be an R -module. On $M = N \otimes_R S$ there is a canonical connection ∇_{can} given by

$$\nabla_{\text{can}}(n \otimes s) = n \otimes 1 \otimes s - n \otimes s \otimes 1. \quad (19)$$

In particular, the universal derivation $d : S \rightarrow \Omega(S/R)$ is a connection on S .

The set $\text{Conn}_{S/R}(M)$ of the connections on M is easily seen to be a torsor under $\text{Hom}_S(M, \Omega(S/R, M))$ (or under $\text{Hom}_S(M, M \otimes_S \Omega(S/R))$ depending on which definition of a connection one takes).

Denote by $\Omega^n(S/R)$ the n -fold tensor product of $\Omega(S/R)$ over S , that is:

$$\Omega^n(S/R) = \overbrace{\Omega(S/R) \otimes_S \cdots \otimes_S \Omega(S/R)}^{n \text{ times}} \quad \text{for } n \geq 1 \quad \text{and} \quad \Omega^0(S/R) = S.$$

Clearly, for each n , the k -module $\Omega^n(S/R)$ is an S -bimodule.

Definition 2.3. An element in $\Omega^n(S/R)$ is called a *noncommutative differential n -form of S over R* . The tensor algebra

$$\Omega^*(S/R) = \bigoplus_{n \geq 0} \Omega^n(S/R),$$

which is a k -algebra by concatenation and an S -bimodule, is called the *algebra of noncommutative differential forms of S over R* .

The universal derivation d extends uniquely to an R -bimodule morphism of degree 1, still denoted by d , from $\Omega^*(S/R)$ to $\Omega^*(S/R)$ and of square zero verifying the generalized Leibniz rule:

$$d(\omega_p \otimes \omega_q) = d(\omega_p) \otimes \omega_q + (-1)^p \omega_p \otimes d(\omega_q)$$

(the indices indicate the degrees). Observe that the algebra of differential forms of S over R is, as in the case of k -algebras ([11]), solution of a universal property. For any $n \geq 1$, one has the following exact sequence:

$$0 \rightarrow \Omega^n(S/R) \rightarrow \Omega^{n-1}(S/R) \otimes_S (S \otimes_R S) \rightarrow \Omega^{n-1}(S/R) \rightarrow 0$$

obtained by setting $M = \Omega^{n-1}(S/R)$ in (13). The following two results are straightforward. The proofs are the same as for $R = k$ (compare [11]).

Lemma 2.4. *Let $\nabla : M \rightarrow M \otimes_S \Omega(S/R)$ be a connection on an S -module M . There exists a unique extension $\nabla_* : M \otimes_S \Omega^*(S/R) \rightarrow M \otimes_S \Omega^*(S/R)$ of degree 1 with $\nabla_0 = \nabla$ such that, for $\nu \in M \otimes_S \Omega^i(S/R)$ and $\omega \in \Omega^*(S/R)$, one has:*

$$\nabla_*(\nu \otimes \omega) = \nabla_*(\nu) \cdot \omega + (-1)^i \nu \cdot d\omega. \quad (20)$$

We shall write ∇ instead of ∇_* . By Lemma 2.4, $\nabla^2 = \nabla \circ \nabla$ makes sense, however $(M \otimes_S \Omega^*(S/R), \nabla)$ is not a complex in general.

Lemma 2.5. *The map ∇^2 from $M \otimes_S \Omega^*(S/R)$ to $M \otimes_S \Omega^{*+2}(S/R)$ is a morphism of $\Omega^*(S/R)$ -modules (in particular of S -modules).*

Thus ∇^2 is determined by the morphism of S -modules

$$\nabla_1 \circ \nabla_0 : M \rightarrow M \otimes_S \Omega^2(S/R)$$

which, by analogy with the classical case, is called the *curvature of the connection* ∇ .

Definition 2.6. A connection is said to be *flat* if its curvature is zero.

Example 2.7. Let N be an R -module. The canonical connection (19) on $M = N \otimes_R S$ is flat.

We define now the higher differential forms in terms of $\Omega(S/R, M)$. Let $\Omega^n(S/R, M)$ be the S -bimodule defined by induction by:

$$\Omega^n(S/R, M) = \Omega(S/R, \Omega^{n-1}(S/R, M)) \quad \text{and} \quad \Omega^0(S/R, M) = M.$$

The sequence (13) defining $\Omega(S/R, M)$ yields the short exact sequence

$$0 \rightarrow \Omega^n(S/R, M) \rightarrow \Omega^{n-1}(S/R, M) \otimes_R S \rightarrow \Omega^{n-1}(S/R, M) \rightarrow 0.$$

Lemma 2.8. *Let $\psi : R \rightarrow S$ be an extension of algebras. Then $\Omega^n(S/R, M)$ is a sub- S -module of $M \otimes_R S^{\otimes n}$.*

Proof: By induction on n . □

We now express the curvature viewed as the square of $\tilde{\nabla} : M \rightarrow \Omega(S/R, M)$ (cf. Formula (18)).

Lemma 2.9. *Under the previous isomorphisms the curvature $\tilde{\nabla}^2$ from M to $\Omega^2(S/R, M)$ is given by*

$$\tilde{\nabla}^2(m) = \sum_i (\tilde{\nabla}(m_i) \otimes u_i - \tilde{\nabla}(m_i) u_i \otimes 1),$$

where $m \in M$ and $\tilde{\nabla}(m) = \sum_i m_i \otimes u_i$.

Proof: Write $\nabla(m) = \sum_i m_i \otimes_S du_i$. Then by (20) one has $\nabla^2(m) = \sum_i \nabla(m_i) \otimes_S du_i$. The result follows from the identifications (13) and (18). □

2.2. The Atiyah class

Let $\psi : R \rightarrow S$ be a morphism of algebras and let M be an S -module.

Definition 2.10. The *Atiyah class* $\text{at}_{S/R}(M)$ of M is the class of the extension

$$0 \rightarrow \Omega(S/R, M) \rightarrow (M \otimes_R S)_1 \rightarrow M \rightarrow 0$$

deduced from (12) in the group $\text{Ext}_S^1(M, \Omega(S/R, M))$.

Theorem 2.11. *There exists a connection on M if and only if the Atiyah class $\text{at}_{S/R}(M)$ is zero in $\text{Ext}_S^1(M, \Omega(S/R, M))$.*

Proof: The R -linear section ε_2 enables us to write (12) viewed in the category \mathfrak{Mod}_R as

$$0 \rightarrow M \otimes_S \Omega(S/R) \rightarrow (M \otimes_S \Omega(S/R)) \oplus M \xrightarrow{\text{pr}} M \rightarrow 0, \quad (21)$$

where the direct sum is taken in \mathfrak{Mod}_R and pr is the canonical projection. We deduce an R -linear isomorphism β from $M \otimes_S (S \otimes_R S)$ to $(M \otimes_S \Omega(S/R)) \oplus M$ given by

$$\beta(m \otimes (s \otimes s')) = (m \otimes (s \otimes s') - m \otimes (s s' \otimes 1), m s s') \quad \text{and} \quad \beta^{-1}(\omega, m) = \omega + m \otimes (1 \otimes 1),$$

where $m \in M$, $s, s' \in S$ and $\omega \in M \otimes_S \Omega(S/R)$. Transporting the obvious S -module structure of $M \otimes_S (S \otimes_R S)$ on $(M \otimes_S \Omega(S/R)) \oplus M$ via β , we obtain the S -module structure given by

$$\begin{pmatrix} \omega \\ m \end{pmatrix} \cdot s = \begin{pmatrix} \omega \cdot s + m \otimes ds \\ ms \end{pmatrix}.$$

The class $\text{at}_{S/R}(M)$ is zero if and only if the sequence of S -modules (21) splits in \mathfrak{Mod}_S . The map σ from M to $(M \otimes_S \Omega(S/R)) \oplus M$ defined by

$$\sigma(m) = \begin{pmatrix} \nabla(m) \\ m \end{pmatrix}$$

is clearly an S -linear section of pr if and only if the map ∇ in the first component is a connection on M . □

Corollary 2.12. *Any projective S -module may be endowed with a connection.*

As in the commutative case, one can explicitly construct a connection (the Levi-Civita connection) on any finitely generated projective S -module (see [11]).

3. Noncommutative descent

3.1. The Amitsur complex

The essential tool for descent theory is the Amitsur complex. It plays a crucial rôle as well in the proof of Cipolla's Theorem as in Section 6 in order to classify descent data. Therefore and for the sake of completeness we now recall well-known facts (see, *e.g.*, [1] §6).

Let $\psi : R \rightarrow S$ be an extension of algebras. Define $(n+1)$ morphisms $\varepsilon_1, \dots, \varepsilon_{n+1}$ of R -bimodules from $S^{\otimes_R n}$ to $S^{\otimes_R(n+1)}$ by

$$\varepsilon_i(s_1 \otimes s_2 \dots \otimes s_n) = s_1 \otimes s_2 \dots \otimes s_{i-1} \otimes 1 \otimes s_i \otimes \dots \otimes s_n.$$

Then $[n] \mapsto S^{\otimes_R(n+1)}$ is a cosimplicial object in the category of R -bimodules, represented by the diagram

$$(0 \rightarrow R \rightarrow) S \rightrightarrows S \otimes_R S \rightrightarrows S \otimes_R S \otimes_R S \rightrightarrows \dots \quad (22)$$

The cofaces are the maps ε_i and the codegeneracies the maps μ_i already defined in the first section. Here R is in degree -1 and, consequently, S is in degree 0 .

Definition 3.1. The *Amitsur complex* $(\mathcal{A}^*(S/R), \Delta)$ of the morphism $\psi : R \rightarrow S$ is the complex of R -bimodules associated to the cosimplicial object $S^{\otimes_R *}$.

Thus $(\mathcal{A}^*(S/R), \Delta)$ is the complex

$$R \xrightarrow{\psi} S \xrightarrow{\Delta_0} S \otimes_R S \xrightarrow{\Delta_1} S \otimes_R S \otimes_R S \xrightarrow{\Delta_2} \dots,$$

where R is again in degree -1 and the differential in degree n is given by

$$\Delta_n = \sum_{i=1}^{n+2} (-1)^{i+1} \varepsilon_i.$$

Proposition 3.2. Assume that the extension $\psi : R \rightarrow S$ is faithfully flat. Then the Amitsur complex $(\mathcal{A}^*(S/R), \Delta)$ is acyclic.

Proof: By faithful flatness it is equivalent to showing that $(\mathcal{A}^*(S/R) \otimes_R S, \Delta \otimes \text{id})$ is acyclic. But $\Delta \otimes \text{id}$ is equal to $\Delta' = \Delta -$ (the last coface) given in degree n by

$$\Delta'_n = \sum_{i=1}^{n+1} (-1)^{i+1} \varepsilon_i.$$

The complex $(\mathcal{A}^*(S/R) \otimes_R S, \Delta')$ is acyclic, for the identity map is homotopic to the zero map since

$$\text{id}_{S^{\otimes_R(n+2)}} = (\Delta'_n \circ \mu_{n+1}) + (\mu_{n+2} \circ \Delta'_{n+1}).$$

□

Corollary 3.3. *Let $\psi : R \rightarrow S$ be a faithfully flat extension and N be an R -module. Then the complex $N \otimes_R \mathcal{A}^*(S/R)$ is acyclic.*

Proof: One uses the homotopy $\text{id}_N \otimes \mu_{n+1}$ to prove that $N \otimes_R \mathcal{A}^*(S/R) \otimes_R S$ is acyclic. The acyclicity of $N \otimes_R \mathcal{A}^*(S/R)$ follows again by faithful flatness. \square

3.2. Descent for elements and homomorphisms

We now state the noncommutative analogue of the descent theorem for elements and homomorphisms (see [13] II §2).

Theorem 3.4. *Let $\psi : R \rightarrow S$ be a faithfully flat extension. Let N and N' be two right R -modules.*

1) *An element $\sum_i (n_i \otimes s_i)$ of $N \otimes_R S$ is in the image of N under the natural injection $N \cong N \otimes_R R \hookrightarrow N \otimes_R S$ if and only if $\sum_i (n_i \otimes ds_i)$ is zero in $N \otimes_R S \otimes_R S$.*

2) *The following sequence of R -modules is exact:*

$$0 \rightarrow \text{Hom}_R(N, N') \xrightarrow{V} \text{Hom}_S(N \otimes_R S, N' \otimes_R S) \xrightarrow{W} \text{Hom}_S(N \otimes_R S \otimes_R S, N' \otimes_R S \otimes_R S),$$

where $V(h) = h \otimes \text{id}$ for $h \in \text{Hom}_R(N, N')$ and $W(g) = (\text{id} \otimes \tau) \circ (g \otimes \text{id}) - (g \otimes \text{id}) \circ (\text{id} \otimes \tau)$ for $g \in \text{Hom}_S(N \otimes_R S, N' \otimes_R S)$.

The first statement of Theorem 3.4 can be found in [4]. It implies a ‘‘pointwise’’ descent theorem for homomorphisms, namely $g : N \otimes_R S \rightarrow N' \otimes_R S$ comes from $h : N \rightarrow N'$ if and only if $(\text{id}_N \otimes d) \circ g = 0$.

Proof of Theorem 3.4:

1) It follows from the acyclicity of the Amitsur complex $N \otimes_R \mathcal{A}^*(S/R)$ (see Corollary 3.3).

2) The map V is injective. Indeed, if $h \otimes \text{id} = 0$ for $h \in \text{Hom}_R(N, N')$, then for any $n \in N$, one has $h(n) \otimes 1 = 0$, hence $h(n) = 0$ by faithful flatness. If g belongs to $\text{Hom}_S(N \otimes_R S, N' \otimes_R S)$ and n to N , one can write $g(n \otimes 1) = \sum_i n'_i \otimes u_i$ for some $n'_i \in N'$ and $u_i \in S$. Then

$$W(g)(n \otimes s \otimes t) = \sum_i n'_i \otimes 1 \otimes u_i s t - \sum_i n'_i \otimes u_i \otimes s t.$$

It is easy to see that $W \circ V = 0$ using the above expression of W . Let $g \in \text{Ker} W$. One has to show that there exists $h \in \text{Hom}_R(N, N')$ such that $h \otimes \text{id} = g$. By the descent theorem for elements, given $n \in N$, it suffices to prove that $(\text{id} \otimes d)(g(n \otimes 1)) = 0$. Again the above expression of W gives the result. \square

3.3. Cipolla’s noncommutative descent Theorem for modules

Let $\psi : R \rightarrow S$ be an extension and M be an S -module. For any R -linear morphism f from M to $M \otimes_R S$, using Cipolla’s notations, define three R -linear morphisms $d_f^{(i)}$ ($i = 1, 2, 3$) from $M \otimes_R S$ to $M \otimes_R S \otimes_R S$ by

$$d_f^{(1)}(m \otimes s) = m \otimes s \otimes 1, \quad d_f^{(2)}(m \otimes s) = m \otimes 1 \otimes s, \quad \text{and} \quad d_f^{(3)}(m \otimes s) = f(m) \otimes s.$$

Definition 3.5.

- 1) The map f is a *gluing datum* on M if f is an S -linear section of μ ;
- 2) The map f is said to verify the *cocycle condition* if the following equality holds

$$d_f^{(2)} \circ f = d_f^{(3)} \circ f;$$

3) (Cipolla, see [4]) The map f is a *descent datum* on M if it is a gluing datum which satisfies the cocycle condition.

Example 3.6. Let N be an R -module. On $M = N \otimes_R S$ there is a canonical descent datum f_{can} defined by

$$f_{\text{can}} = \text{id}_N \otimes \varepsilon_1. \quad (23)$$

In particular, if $N = R$, the S -linear morphism ε_1 is a descent datum on S .

Let \mathfrak{Desc}_ψ be the category whose objects are pairs (M, f) , where M is an S -module and f a descent datum on M . A morphism Φ from (M, f) to (M', f') is an S -linear map Φ from M to M' verifying the commutation rule:

$$(\Phi \otimes \text{id}_S) \circ f = f' \circ \Phi.$$

Since the map f (respectively f') is a section of $\mu : M \otimes_R S \rightarrow M$ (respectively of $\mu' : M' \otimes_R S \rightarrow M'$), one concludes that necessarily

$$\mu' \circ (\Phi \otimes \text{id}_S) = \Phi \circ \mu. \quad (24)$$

For any R -module N and for any object (M, f) in \mathfrak{Desc}_ψ , set

$$\mathcal{E}(N) = (N \otimes_R S, f_{\text{can}}) \in \mathfrak{Desc}_\psi \quad \text{and} \quad \mathcal{D}(M, f) = \text{Ker}(f - \varepsilon) \in \mathfrak{Mod}_R, \quad (25)$$

where $\varepsilon : M \rightarrow M \otimes_R S$ is the R -linear section of the S -action μ given on $m \in M$ by

$$\varepsilon(m) = m \otimes 1.$$

Lemma 3.7. *Under the previous notations, \mathcal{D} is a functor from \mathfrak{Desc}_ψ to \mathfrak{Mod}_R (the descent functor) and \mathcal{E} is a functor from \mathfrak{Mod}_R to \mathfrak{Desc}_ψ (the extension functor).*

Proof: A morphism $\varphi : N \rightarrow N'$ in \mathfrak{Mod}_R induces the morphism of S -modules $\varphi \otimes \text{id}_S$ from $N \otimes_R S$ to $N' \otimes_R S$ which trivially commutes with the canonical descent data. On the other hand, let Φ be an S -linear morphism from (M, f) to (M', f') in \mathfrak{Desc}_ψ . Then the equality $(\Phi \otimes \text{id}_S) \circ f = f' \circ \Phi$ and the fact that $\varepsilon \circ \Phi = (\Phi \otimes \text{id}_S) \circ \varepsilon$ imply that Φ induces an R -linear morphism from $\mathcal{D}(M, f)$ to $\mathcal{D}(M', f')$. \square

Cipolla's noncommutative descent Theorem, which is the noncommutative analogue of Grothendieck's Theorem 1 ([8]), is contained in the following result:

Theorem 3.8. *Let $\psi : R \rightarrow S$ be a faithfully flat extension of rings. The functors \mathcal{D} and \mathcal{E} induce an equivalence of categories between \mathfrak{Mod}_R and \mathfrak{Desc}_ψ .*

In fact, if N is a right R -module, there exists an isomorphism $N \cong \mathcal{D}[\mathcal{E}(N)]$. If f is a descent data on a right S -module M , there exists an isomorphism $M \cong \mathcal{D}(M, f) \otimes_R S$ and the descent is *effective* in the sense of Grothendieck. Theorem 3.8 illustrates a theorem of Barr-Beck for the comonad on \mathfrak{Mod}_S induced by the adjunction $\psi_! \dashv \psi^*$ (see, e.g., [5]).

Remark 3.9. If N is a projective (respectively finitely generated projective) R -module, then so is $\mathcal{E}(N)$ as an S -module. Hence the extension ψ gives rise to a map in K -theory $K_0(\psi) : K_0(R) \rightarrow K_0(S)$. Observe that the functor \mathcal{D} does not preserve in general the property of being projective or finitely generated projective (see [3] I.3.6), unless we make the additional assumption that R is central in S (*loc. cit.*, Proposition 12).

Let $\psi : R \rightarrow S$ be an extension of rings and $\mathcal{O} : \mathfrak{Desc}_\psi \rightarrow \mathfrak{Mod}_S$ be the forgetful functor given by

$$\mathcal{O}(M, f) = M.$$

Denote by \mathcal{C} the functor from \mathfrak{Mod}_S to \mathfrak{Desc}_ψ defined on objects by

$$\mathcal{C}(M) = (M \otimes_R S, f_{\text{can}})$$

and in an obvious way on morphisms.

Proposition 3.10. *Under the previous notations, the functor \mathcal{O} is left-adjoint to \mathcal{C} .*

More precisely, the map α from $\text{Hom}_S(\mathcal{O}(M, f), M')$ to $\text{Mor}_{\mathfrak{Desc}_\psi}((M, f), \mathcal{C}(M'))$, defined on an element $\varphi \in \text{Hom}_S(\mathcal{O}(M, f), M')$ by

$$\alpha(\varphi) = (\varphi \otimes \text{id}_S) \circ f,$$

is a natural bijection. Its inverse is given by $\alpha^{-1}(\Phi) = \mu' \circ \Phi$.

Proof: One shows easily that α takes its values in $\text{Mor}_{\mathfrak{Desc}_\psi}((M, f), \mathcal{C}(M'))$: it is a consequence of the cocycle condition fulfilled by f . Let $\beta(\Phi) = \mu' \circ \Phi$. The S -linearity of $\beta(\Phi)$ is trivial and it is easy to see that $\beta \circ \alpha = \text{id}$. One observes that the equality $\alpha \circ \beta = \text{id}$ is a consequence of (24). \square

3.4. The equivalence theorem

Let $\psi : R \rightarrow S$ be a morphism of rings. Denote by \mathfrak{OpSym}_ψ the following category. Its objects are pairs (M, ξ) , where ξ is a symmetry operator on the S -module M . A morphism $\Phi : (M, \xi) \rightarrow (M', \xi')$ in \mathfrak{OpSym}_ψ is an S -linear map $\Phi : M \rightarrow M'$ such that $(\Phi \otimes \text{id}_S) \circ \xi = \xi' \circ (\Phi \otimes \text{id}_S)$. Let $\mathfrak{FlatConn}_\psi$ the category whose objects are pairs (M, ∇) , with ∇ a flat connection on M . A morphism $\Phi : (M, \nabla) \rightarrow (M', \nabla')$ in $\mathfrak{FlatConn}_\psi$ is an S -linear map $\Phi : M \rightarrow M'$ such that $(\Phi \otimes \text{id}_S) \circ \nabla = \nabla' \circ \Phi$.

We are now able to state the main theorem of this section.

Theorem 3.11. *Let $\psi : R \rightarrow S$ be a morphism of rings. Under the previous notations, there are equivalences of categories*

$$\mathbf{Desc}_\psi \cong \mathbf{OpSym}_\psi \cong \mathbf{FlatConn}_\psi.$$

This result follows readily from the following proposition:

Proposition 3.12. *Let $\psi : R \rightarrow S$ be a morphism of algebras and M a right S -module.*

1) *There are bijective correspondences between the set of gluing data on M , the set of involutive twists on $M \otimes_R S$, and the set of connections on M . Explicitly, $f : M \rightarrow M \otimes_R S$ is a gluing datum if and only if*

$$\nabla = f - \varepsilon$$

is a connection. Likewise, $f : M \rightarrow M \otimes_R S$ is a gluing datum if and only if the map ξ from $(M \otimes_R S)_1$ to $(M \otimes_R S)_2$ defined by

$$\xi(m \otimes s) = f(m) *_{2} (s \otimes 1) \tag{26}$$

is a twist. Moreover, this correspondence follows from the adjunction formula

$$\mathrm{Hom}_S(M, \varepsilon_1^*(\varepsilon_2!(M))) \cong \mathrm{Hom}_{S \otimes_R S}(\varepsilon_1!(M), \varepsilon_2!(M)).$$

2) *If f is a gluing datum, ξ (respectively ∇) the associated twist (respectively connection), then the following equality holds:*

$$\begin{aligned} & ((\xi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\xi \otimes \mathrm{id}) - (\mathrm{id} \otimes \tau) \circ (\xi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau))(m \otimes s \otimes t) \\ &= (d_f^{(3)} \circ f - d_f^{(2)} \circ f)(m)st = \nabla^2(m)st. \end{aligned} \tag{27}$$

3) *The correspondences stated in 1) induce bijective maps between the set of descent data on M , the set of involutive symmetries on $M \otimes_R S$ and the set of flat connections on M .*

Corollary 3.13. *If f is a gluing datum, it is related to the associated twist ξ and to the corresponding connexion ∇ by the equations:*

$$\begin{aligned} \xi(m \otimes s) &= \nabla(m)s + ms \otimes 1 = f(m)s - m \otimes s + ms \otimes 1, \\ \nabla(m) &= \xi(m \otimes 1) - m \otimes 1. \end{aligned}$$

The proof of Corollary 3.13 is straightforward in view of Relation (26) and the $S \otimes_R S$ -module structure of $M \otimes_R S$ given by (9).

Example 3.14. The canonical descent datum (23) on $N \otimes_R S$ corresponds, under the bijection of Proposition 3.12, to the canonical symmetry (15) and to the canonical connection (19). In particular on S , the descent datum ε_1 corresponds to the flat connection d and to the symmetry τ .

Remark 3.15. In Theorem 2.11, the section of μ annihilating at $S/R(M)$ is the gluing datum associated with the connection.

Proof of Proposition 3.12: Part3) readily follows from 1) coupled with formula (27). Let us show 1) and 2).

1) Let f be a gluing datum and ∇ be the map defined by $\nabla = f - \varepsilon$. The very definition implies that $\mu \circ \nabla = 0$, hence $\text{Im}(\nabla) \subset \Omega(S/R, M)$. Relation (18) is a straightforward computation. Conversely, if ∇ is a connection, the S -linearity of $f = \nabla + \varepsilon$ is clear, as well as the relation $\mu \circ f = \text{id}_M$.

We turn now to the question of the equivalence between involutive twists and connections. Let ξ be an involutive twist. Define $\nabla(m) = \xi(m \otimes 1) - m \otimes 1$. We claim that the corresponding ∇ is a connection. Indeed,

$$\nabla(ms) - \nabla(m)s = \xi(ms \otimes 1) - ms \otimes 1 - \xi(m \otimes 1) \square_1 s + m \otimes s = m \otimes s - ms \otimes 1.$$

Conversely, suppose that $\nabla : M \rightarrow \Omega(S/R, M)$ is a connection. Let $\xi(m \otimes s) = ms \otimes 1 + \nabla(m)s$. The map ξ is then an involutive twist. Indeed, the relation $\mu \circ \xi = \mu$ is obvious, since $\mu(\Omega(S/R, M)) = 0$. The S -linearity follows from the equalities

$$\xi((m \otimes s) \square_1 t) = \xi(m \otimes st) = mst \otimes 1 + \nabla(m)st = \xi(m \otimes s) \square_2 t.$$

We prove that ξ is involutive. Let $\nabla(m \otimes 1) = \sum (m_i \otimes u_i - m_i u_i \otimes 1) \in \Omega(S/R, M)$ and compute $\xi \circ \xi$.

$$\begin{aligned} \xi \circ \xi(m \otimes s) &= \xi(ms \otimes 1 + \nabla(m)s) = \xi(ms \otimes 1 + \sum_i (m_i \otimes u_i s - m_i u_i \otimes s)) \\ &= ms \otimes 1 + \nabla(ms) + \sum_i (m_i u_i s \otimes 1 + \nabla(m_i) u_i s - m_i u_i s \otimes 1 - \nabla(m_i u_i) s) \\ &= ms \otimes 1 + \nabla(m)s + m \otimes s - ms \otimes 1 + \\ &\quad \sum_i (\nabla(m_i) u_i s - \nabla(m_i) u_i s - m_i \otimes u_i s + m_i u_i \otimes s) = m \otimes s. \end{aligned}$$

Hence $\xi \circ \xi = \text{id}$.

By Lemma 1.7, a twist ξ belongs to $\text{Hom}_{S \otimes_R S}([M \otimes_R S]_1, [M \otimes_R S]_2)$, hence corresponds to an element $\tilde{\xi}$ in $\text{Hom}_{S \otimes_R S}(\varepsilon_{1!}(M), \varepsilon_{2!}(M))$. By the adjunction formula (1), we obtain the desired gluing datum $f \in \text{Hom}_S(M, \varepsilon_1^*(\varepsilon_{2!}(M)))$.

2) It remains to show (27). Let f be a gluing datum and ∇ the corresponding connection. For $m \in M$, set $\nabla(m) = \sum_i (m_i \otimes u_i - m_i u_i \otimes 1)$. An easy computation shows that

$$(d_f^{(3)} \circ f - d_f^{(2)} \circ f)(m) = \sum_i (\nabla(m_i) \otimes u_i - \nabla(m_i) u_i \otimes 1) = \nabla^2(m)$$

(in the last equality we use the result of Lemma 2.9).

Now let ξ be the twist associated with the descent datum f . For $m \in M$, write $f(m) = \sum_i (m_i \otimes u_i)$. Using the fact that $m = \sum_i (m_i u_i)$, a long and tedious computation gives

$$\begin{aligned} ((\xi \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\xi \otimes \text{id}))(m \otimes s \otimes t) &= m \otimes s t \otimes 1 - m s \otimes t \otimes 1 + m s t \otimes 1 \otimes 1 + \\ &\quad \sum_i (f(m_i) \otimes u_i s t + m_i \otimes u_i s \otimes t - m_i \otimes u_i \otimes s t) \end{aligned}$$

and

$$\begin{aligned} ((\text{id} \otimes \tau) \circ (\xi \otimes \text{id}) \circ (\text{id} \otimes \tau))(m \otimes s \otimes t) &= m s t \otimes 1 \otimes 1 + m \otimes s t \otimes 1 - m \otimes s \otimes t - m s \otimes t \otimes 1 + \\ &\quad \sum_i (m_i \otimes 1 \otimes u_i s t - m_i \otimes u_i \otimes s t + m_i \otimes u_i s \otimes t), \end{aligned}$$

hence

$$\begin{aligned} &((\xi \otimes \text{id}) \circ (\text{id} \otimes \tau) \circ (\xi \otimes \text{id}) - (\text{id} \otimes \tau) \circ (\xi \otimes \text{id}) \circ (\text{id} \otimes \tau))(m \otimes s \otimes t) \\ &= \sum_i (f(m_i) \otimes u_i s t - m_i \otimes 1 \otimes u_i s t) = (d_f^{(3)} \circ f - d_f^{(2)} \circ f)(m) s t, \end{aligned}$$

which proves Theorem 3.11. \square

Remarks 3.16.

1) The relationship between a twist ξ on $M \otimes_R S$ and a gluing datum f on M may be deduced from the adjunction formula

$$\text{Hom}_S(\psi_!(\psi^*(M)), (M \otimes_R S)_2) \cong \text{Hom}_R(\psi^*(M), \psi^*((M \otimes_R S)_2)),$$

since $\psi^*((M \otimes_R S)_2) = \psi^*((M \otimes_R S)_1) = \psi^*(M \otimes_R S)$. Observe that the natural transformation $\psi_! \psi^* \longrightarrow \varepsilon_1^* \varepsilon_{2!}$ is here an equivalence.

2) Let ξ be any twist, not necessarily involutive. Define ∇ by $\nabla(m) = \xi^{-1}(m \otimes 1) - m \otimes 1$, for $m \in M$. One easily checks that ∇ is a connection whose associated involutive twist is $\text{Invol}(\xi)$ (defined in Proposition 1.13).

3) Let \sim be the equivalence relation on the set of twists on $M \otimes_R S$, defined by

$$\xi^{-1} \sim \xi'^{-1} \text{ if and only if } \nabla = \nabla',$$

where ξ and ξ' are twists and ∇ (respectively ∇') is the connection corresponding to ξ (respectively to ξ') by Remark 3.16.2. Let D be the morphism of S -modules from $(M \otimes_R S)_1$ (or $(M \otimes_R S)_2$) to $\Omega(S/R, M)$ given by

$$D(m \otimes s) = m \otimes s - m s \otimes 1.$$

It is a retraction of the inclusion $\Omega(S/R, M) \hookrightarrow M \otimes_R S$. It is easy to see that the twists ξ and ξ' are equivalent if and only if the factorization

$$\xi^{-1} - \xi'^{-1} = (\xi^{-1} - \xi'^{-1}) \circ D$$

holds. To see this, use the equality

$$(\xi^{-1} - \xi'^{-1})(m \otimes s) = (\xi^{-1} - \xi'^{-1})(m s \otimes 1) + ((\xi^{-1} - \xi'^{-1}) \circ D)(m \otimes s).$$

4) Let f be a gluing datum on M and ξ the corresponding twist. Then $\xi \circ f = \varepsilon$. In fact, write $f(m) = \sum_i m_i \otimes u_i$ ($m, m_i \in M, u_i \in S$), then

$$\xi \circ f(m) = \sum_i (f(m_i) u_i - m_i \otimes u_i + m_i u_i \otimes 1) = m \otimes 1,$$

since $\sum_i m_i u_i = m$ and f is S -linear.

Corollary 3.17. *Let $\psi : R \rightarrow S$ be a faithfully flat extension. Let (M, f) be an object in \mathfrak{Desc}_ψ . Denote by ∇ the connection on M and by ξ the symmetry on $M \otimes_R S$ associated to f . Then the descended module $\mathcal{D}(M, f)$ is isomorphic to the kernel of ∇ and to the kernel of the double arrow $M \begin{smallmatrix} \xrightarrow{\varepsilon} \\ \xrightarrow{\varepsilon'} \end{smallmatrix} M \otimes_R S$, where $\varepsilon(m) = m \otimes 1$ and $\varepsilon'(m) = \xi(m \otimes 1)$.*

Proof: This is just the definition of \mathcal{D} (see (25)) and Theorem 3.11. □

4. Application to Galois descent

4.1. Noncommutative bialgebra Galois extensions

In this paragraph we apply the results of the previous sections to the case when $\psi : R \rightarrow S$ is a bialgebra Galois extension.

Let \mathcal{H} be a bialgebra over the ground ring k . Denote by $\Delta_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ its comultiplication and by $\epsilon : \mathcal{H} \rightarrow k$ its counit. We shall extend to this context some definitions given for Hopf algebras in [19] or [20]. Recall that a ring S is a \mathcal{H} -comodule algebra if S is as well a k -algebra as an \mathcal{H} -comodule, and the coaction $\Delta_S : S \rightarrow S \otimes \mathcal{H}$ is a morphism of algebras. Observe that $S \otimes \mathcal{H}$ is an algebra by the usual formula $(s \otimes h).(s' \otimes h') = ss' \otimes hh'$, and an S -module by $(s \otimes h).s' = ss' \otimes h$, with $s, s' \in S$ and $h, h' \in \mathcal{H}$. Let R be the subalgebra $S^{\text{co}\mathcal{H}}$ of \mathcal{H} -coinvariants of S consisting of all elements s of S such that $\Delta_S(s) = s \otimes 1$.

Definition 4.1. The inclusion map $\psi : R \rightarrow S$ is a *bialgebra Galois extension* or \mathcal{H} -Galois extension if ψ is faithfully flat and the map α_S from $S \otimes_R S$ to $S \otimes \mathcal{H}$ given by

$$\alpha_S(s \otimes t) = s \Delta_S(t)$$

is a k -linear isomorphism.

When \mathcal{H} is a Hopf algebra, we refer to \mathcal{H} -Galois extensions as *Hopf-Galois extensions*.

First examples 4.2 (see [21] for more examples). Fix a field F .

4.2.1. Let G be a group and G' be a normal subgroup. The induced morphism of group algebras $F[G'] \rightarrow F[G]$ is a Hopf-Galois extension with $\mathcal{H} = F[G/G']$. The objects in the category $\mathfrak{Mod}_{F[G]}$ are the G -modules and the functor $\psi_!$ from $\mathfrak{Mod}_{F[G']}$ to $\mathfrak{Mod}_{F[G]}$ is the induction functor $\text{Ind}_{G'}^G$.

4.2.2. Let \mathfrak{g} be a Lie algebra (respectively a p -Lie algebra, where p is a prime number) and \mathfrak{g}' a Lie (respectively a p -Lie) ideal of \mathfrak{g} . The morphism of universal enveloping algebras $U(\mathfrak{g}') \rightarrow U(\mathfrak{g})$ is a Hopf-Galois extension with $\mathcal{H} = U(\mathfrak{g}/\mathfrak{g}')$.

4.2.3. Let G be a group and S a strongly G -graded algebra. Let $R = S_e$, where e is the neutral element of G . Then the inclusion $R \rightarrow S$ is a Hopf-Galois extension with $\mathcal{H} = F[G]$.

Fundamental example 4.3: Noncommutative Galois extensions.

Let $\psi : R \rightarrow S$ be an extension of algebras and G be a finite group of order n acting by automorphisms of algebras on S . We say that $\psi : R \rightarrow S$ is a G -extension if G acts trivially on the image of ψ . Let e be the neutral element of G . For any S -bimodule Q , given two fixed elements g and h in G , there is an S -bimodule structure on Q denoted by \cdot and related to the initial structure by

$$s \cdot q \cdot s' = g(s)qh(s').$$

Denote by ${}_g Q_h$ the group Q endowed with this S -bimodule structure and by $\oplus S$ the S -bimodule $\bigoplus_{g \in G} {}_e S_g$. Let γ be the S - S -linear map from $S \otimes_R S$ to $\oplus S$ given by

$$\gamma(s \otimes t) = \bigoplus_{g \in G} (sg(t)).$$

A G -extension $\psi : R \rightarrow S$ is called *Galois* (see [16]) if S is a progenerator as a left R -module and the map γ is an S - S -linear isomorphism. In this case we denote G by $\text{Gal}(S/R)$. Recall ([6]) that a left module P over a ring R is called a *progenerator* if P is finitely generated, projective over R , and there exists $f_1, \dots, f_n \in \text{Hom}_R(P, R)$ and $p_1, \dots, p_n \in P$ such that $\sum_i f_i(p_i) = 1$.

Let $\mathcal{H} = k^G$ be the bialgebra with basis $\{\delta_g\}_{g \in G}$ and with multiplication \cdot and comultiplication $\Delta_{\mathcal{H}}$ defined by the formulae

$$\delta_g \cdot \delta_{g'} = \delta_g \text{ if } g = g', \quad \delta_g \cdot \delta_{g'} = 0 \text{ if } g \neq g', \quad \text{and} \quad \Delta_{\mathcal{H}}(\delta_g) = \sum_{ab=g} \delta_a \otimes \delta_b.$$

Notice that the unit in $\mathcal{H} = k^G$ is the element $1 = \sum_{g \in G} \delta_g$ and the counit is the map defined by $\epsilon(\delta_g) = 0$ if $g \neq e$ and $\epsilon(\delta_e) = 1$.

Let $\psi : R \rightarrow S$ be a G -Galois extension. Observe that S is an \mathcal{H} -comodule with the coaction given by $\Delta_S(s) = \sum_{g \in G} g(s) \otimes \delta_g$. This map is easily seen to be a morphism of algebras. Recall ([16]) that any G -Galois extension $\psi : R \rightarrow S$ is faithfully flat and that the algebra of invariants S^G is equal to R . Evidently $S^G = S^{\text{co}(k^G)} = R$. The map α_S is exactly the homomorphism γ via the obvious isomorphism $S \otimes \mathcal{H} \cong \oplus S$. Thus we have the well-known result:

Lemma 4.4. *A Galois extension $\psi : R \rightarrow S$ with Galois group G is the same as an \mathcal{H} -Galois extension, where $\mathcal{H} = k^G$ is the above defined bialgebra.*

Remark 4.5. The product on $\oplus S$, obtained by transporting the product (6) along the map γ , will still be denoted by $*$. Call again $\{\delta_g\}_{g \in G}$ the canonical basis of $\oplus S$. Then $*$ is given by

$$\begin{aligned} \delta_g * \delta_h &= 0 \quad \text{if } g, h \neq e, & \delta_e * \delta_h &= \delta_h * \delta_e = \delta_h \quad \text{if } h \neq e, \\ \text{and } \delta_e * \delta_e &= 2\delta_e - \sum_{g \in G} \delta_g. \end{aligned}$$

Since we do not deal with this product in the sequel, we leave the (easy) proof to the reader.

4.2. The bialgebra Galois descent data

Let $\psi : R \rightarrow S$ be an \mathcal{H} -Galois extension and M be an S -module. We recall ([20]) the notion of right (\mathcal{H}, S) -bialgebra module. It is the natural generalization to our context of a Galois descent datum as it can be found in [13].

Definition 4.6. A k -module M is a *(right)- (\mathcal{H}, S) -bialgebra module* M if

- 1) M is an S -module,
- 2) M is an \mathcal{H} -comodule,
- 3) the coaction $\Delta_M : M \rightarrow S \otimes \mathcal{H}$ is S -linear, i.e., for any $m \in M, s \in S$, one has

$$\Delta_M(ms) = \Delta_M(m)\Delta_S(s). \quad (28)$$

Denote by $\mathfrak{BMod}_S^{\mathcal{H}}$ the category of (\mathcal{H}, S) -bialgebra modules, the morphisms being the S -linear and \mathcal{H} -colinear maps. Observe that if M is an (\mathcal{H}, S) -bialgebra module, then $M \otimes \mathcal{H}$ is endowed with an S -module structure given by

$$(m \otimes h) \cdot s = ms \otimes h,$$

for any $m \in M, h \in \mathcal{H}$ and $s \in S$; hence $M \otimes \mathcal{H}$ is also an R -module.

Example 4.7. Let $\psi : R \rightarrow S$ be an \mathcal{H} -Galois extension. Suppose that $M = N \otimes_R S$, where N is an R -module. Then M is endowed with a canonical (\mathcal{H}, S) -bialgebra module structure given by

$$\Delta_M^{\text{can}}(n \otimes s) = n \otimes \Delta_S(s).$$

Example 4.8. Let $\psi : R \rightarrow S$ be a Galois extension, $G = \text{Gal}(S/R)$ and $\mathcal{H} = k^G$ be as in Example 4.3. Then an (\mathcal{H}, S) -bialgebra module M is an S -module together with a left action of the group G on the abelian group underlying M such that, for any $g \in G, s \in S$ and $m \in M$, one has

$$g(ms) = g(m)g(s).$$

Call this a *Galois descent datum on the right S -module M* and say that M is a *Galois (right) S -module*. Let \mathfrak{Gal}_ψ be the category whose objects are the Galois S -modules, and whose morphisms are the S -linear maps commuting with the Galois actions. If M is a Galois S -module, we denote by M^G the R -submodule of M consisting of the elements invariant under the action of G .

If $M = N \otimes_R S$, where N is an R -module, then the canonical coaction Δ_M^{can} on M induces the canonical Galois descent datum given by

$$g_{\text{can}}(n \otimes s) = n \otimes g(s).$$

4.3. Noncommutative bialgebra Galois descent

We now state the main result of this section. It generalizes Theorem I in [20].

Theorem 4.9. *Let $\psi : R \rightarrow S$ be an \mathcal{H} -Galois extension. The category $\mathfrak{BMod}_S^{\mathcal{H}}$ of (\mathcal{H}, S) -bialgebra modules is equivalent to the category \mathfrak{Desc}_ψ and to the category \mathfrak{Mod}_R .*

Theorem 4.9 will follow from Propositions 4.10 and 4.12.

Proposition 4.10. *Let $\psi : R \rightarrow S$ be an \mathcal{H} -Galois extension. An (\mathcal{H}, S) -bialgebra module structure on a k -module M induces a descent datum f on M given by the composition*

$$M \xrightarrow{\Delta_M} M \otimes \mathcal{H} \xrightarrow{\beta} M \otimes_S (S \otimes \mathcal{H}) \xleftarrow{\alpha_M} M \otimes_S (S \otimes_R S) \xrightarrow{\beta'} M \otimes_R S,$$

where α_M is the k -isomorphism $\text{id}_M \otimes \alpha_S$ and β (respectively β') is the obvious k -linear (respectively S -linear) isomorphism.

Proof: In order to simplify the notations, from now on we neglect the canonical isomorphisms β and β' .

First we prove that f is a gluing datum. For any $m \in M$ and $s \in S$ compute $f(ms)$. One has

$$\alpha_M f(ms) = \Delta_M(ms) = \Delta_M(m) \Delta_S(s).$$

On the other hand observe that $\alpha_M(f(m)s)$ is also equal to $\Delta_M(m) \Delta_S(s)$, by definition of α_M . We deduce the S -linearity. The following diagram commutes.

$$\begin{array}{ccccc} M & \xrightarrow{\Delta_M} & M \otimes \mathcal{H} & \xleftarrow{\alpha_M} & M \otimes_R S \\ \text{id}_M \downarrow & & \downarrow \text{id}_M \otimes \epsilon & & \downarrow \mu \\ M & \xrightarrow{\text{id}_M} & M & \xleftarrow{\text{id}_M} & M \end{array}$$

Indeed, the left square commutes since M is an \mathcal{H} -comodule and the right square commutes since S is an \mathcal{H} -comodule. This ensures that $\mu \circ f = \text{id}_M$, thus f is a gluing datum.

We now prove that f satisfies the cocycle condition. We claim that the following diagram commutes:

$$\begin{array}{ccccc} M \otimes_R S & \xrightarrow{d_f^{(2)}} & M \otimes_R S \otimes_R S & \xrightarrow{\text{id}_M \otimes \alpha_S} & M \otimes_R (S \otimes \mathcal{H}) \cong (M \otimes_R S) \otimes \mathcal{H} \\ \alpha_M \downarrow & & & & \downarrow \alpha_M \otimes \text{id}_{\mathcal{H}} \\ M \otimes \mathcal{H} & \xrightarrow{\text{id}_M \otimes \Delta_{\mathcal{H}}} & & & M \otimes \mathcal{H} \otimes \mathcal{H}. \end{array}$$

This can easily be seen by the coassociativity of the coaction Δ_S . It implies the equality

$$d_f^{(2)} \circ f = (\text{id}_M \otimes \alpha_S^{-1}) \circ (\alpha_M^{-1} \otimes \text{id}_{\mathcal{H}}) \circ (\text{id}_M \otimes \Delta_{\mathcal{H}}) \circ \Delta_M. \quad (29)$$

By definition of $d_f^{(3)}$, we get

$$d_f^{(3)} \circ f = (\alpha_M^{-1} \otimes \text{id}_S) \circ (\Delta_M \otimes \text{id}_S) \circ f. \quad (30)$$

By (29) and (30), the cocycle condition is equivalent to the commutativity of the diagram

$$\begin{array}{ccccccc} M & \xrightarrow{\Delta_M} & M \otimes \mathcal{H} & \xrightarrow{\text{id}_M \otimes \Delta_{\mathcal{H}}} & M \otimes \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\alpha_M^{-1} \otimes \text{id}_{\mathcal{H}}} & M \otimes_R S \otimes \mathcal{H} \\ \Delta_M \downarrow & & & & & & \downarrow \text{id}_M \otimes \alpha_S^{-1} \\ M \otimes \mathcal{H} & \xrightarrow{\alpha_M^{-1}} & M \otimes_R S & \xrightarrow{\Delta_M \otimes \text{id}_S} & (M \otimes \mathcal{H}) \otimes_R S & \xrightarrow{\alpha_M^{-1} \otimes \text{id}_S} & M \otimes_R S \otimes_R S. \end{array} \quad (31)$$

Let us split this rectangle in three diagrams. The first one will be

$$\begin{array}{ccc} M & \xrightarrow{\Delta_M} & M \otimes \mathcal{H} \\ \Delta_M \downarrow & & \downarrow \text{id}_M \otimes \Delta_{\mathcal{H}} \\ M \otimes \mathcal{H} & \xrightarrow{\Delta_M \otimes \text{id}_{\mathcal{H}}} & M \otimes \mathcal{H} \otimes \mathcal{H}, \end{array}$$

which is commutative, by the coassociativity of Δ_M . The second diagram is

$$\begin{array}{ccc} M \otimes \mathcal{H} & \xrightarrow{\Delta_M \otimes \text{id}_{\mathcal{H}}} & M \otimes \mathcal{H} \otimes \mathcal{H} & \xleftarrow{\varphi_1} & M \otimes \mathcal{H} \otimes \mathcal{H} \\ \alpha_M \uparrow & & & & \uparrow \varphi_2 \\ M \otimes_R S & \xrightarrow{\Delta_M \otimes \text{id}_S} & (M \otimes \mathcal{H}) \otimes_R S, \end{array}$$

where φ_1 and φ_2 are defined by

$$\varphi_1(m \otimes h \otimes h') = m \otimes h \Delta_{\mathcal{H}}(h') \quad \text{and} \quad \varphi_2(m \otimes h \otimes s) = \sum_i m s_i \otimes h \otimes h_i,$$

with $m \in M$, $h, h' \in \mathcal{H}$, $s \in S$ and $\Delta_S(s) = \sum_i s_i \otimes h_i \in S \otimes \mathcal{H}$. Let us show that it commutes. Let $m \otimes s \in M \otimes_R S$ and set $\Delta_M(m) = \sum_j m_j \otimes h'_j$; one has

$$\begin{aligned} (\varphi_1 \circ \varphi_2 \circ (\Delta_M \otimes \text{id}_S))(m \otimes s) &= (\varphi_1 \circ \varphi_2) \left(\sum_j m_j \otimes h'_j \otimes s \right) = \varphi_1 \left(\sum_j \sum_i m_j s_i \otimes h'_j \otimes h_i \right) \\ &= \sum_j \sum_i m_j s_i \otimes h'_j \Delta_{\mathcal{H}}(h_i) = \sum_j \sum_i m_j \Delta_S(s_i) \otimes h'_j h_i, \end{aligned}$$

using the coassociativity of Δ_S in the last equality. Now, by the S -linearity of Δ_M (Relation (28)), it is clear that

$$((\Delta_M \otimes \text{id}_{\mathcal{H}}) \circ \alpha_M)(m \otimes s) = (\Delta_M \otimes \text{id}_{\mathcal{H}}) \left(\sum_i m s_i \otimes h_i \right) = \sum_j \sum_i m_j \Delta_S(s_i) \otimes h'_j h_i,$$

hence the second diagram commutes. Finally, the last diagram is

$$\begin{array}{ccccc} M \otimes \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\varphi_1} & M \otimes \mathcal{H} \otimes \mathcal{H} & \xrightarrow{\alpha_M^{-1} \otimes \text{id}_{\mathcal{H}}} & M \otimes_R S \otimes \mathcal{H} \\ \varphi_2 \uparrow & & & & \downarrow \text{id}_M \otimes \alpha_S^{-1} \\ M \otimes \mathcal{H} \otimes_R S & \xrightarrow{\alpha_M^{-1} \otimes \text{id}_S} & & & M \otimes_R S \otimes_R S. \end{array}$$

We show that it commutes. Let $m \otimes s \otimes t \in M \otimes_R S \otimes_R S$. Set

$$\Delta_S(s) = \sum_i s_i \otimes h_i \quad \text{and} \quad \Delta_S(t) = \sum_j t_j \otimes h'_j.$$

One then has

$$\begin{aligned} ((\alpha_M \otimes \text{id}_{\mathcal{H}}) \circ (\text{id}_M \otimes \alpha_S))(m \otimes s \otimes t) &= (\alpha_M \otimes \text{id}_{\mathcal{H}}) \left(\sum_j m \otimes s t_j \otimes h'_j \right) \\ &= \sum_j m \Delta_S(s) \Delta_S(t_j) \otimes h'_j = \sum_j m \Delta_S(s) t_j \otimes \Delta_{\mathcal{H}}(h'_j) \end{aligned}$$

using again the coassociativity of Δ_S in the last equality. We also have

$$\begin{aligned} ((\varphi_1 \circ \varphi_2 \circ (\alpha_M \otimes \text{id}_S))(m \otimes s \otimes t) &= (\varphi_1 \circ \varphi_2) (m \Delta_S(s) \otimes t) = \varphi_1 \left(\sum_i \sum_j m s_i t_j \otimes h_i \otimes h'_j \right) \\ &= \sum_i \sum_j m s_i t_j \otimes h_i \otimes \Delta_{\mathcal{H}}(h'_j), \end{aligned}$$

implying that the diagram commutes.

Patching now the three diagrams together, we get the commutativity of (31), hence Proposition 4.10. \square

Corollary 4.11. *Under the hypothesis of Proposition 4.10, the descended module $\mathcal{D}(M, f)$ is isomorphic to the R -module*

$$\{m \in M ; \sum_i m_i \Delta_S(u_i) = m \otimes 1\},$$

where the elements m_i and u_i are defined by $f(m) = \sum_i m_i \otimes u_i$ for $m \in M$.

Proposition 4.12. *Let $\psi : R \rightarrow S$ be an \mathcal{H} -Galois extension. A descent datum f on M endows M with an (\mathcal{H}, S) -bialgebra module structure. The coaction Δ_M is given by the composition*

$$M \xrightarrow{f} M \otimes_R S \xrightarrow{\beta'^{-1}} M \otimes_S (S \otimes_R S) \xrightarrow{\alpha_M} M \otimes_S (S \otimes \mathcal{H}) \xrightarrow{\beta^{-1}} M \otimes \mathcal{H},$$

where α_M , β and β' have been defined above.

Proof: As before, we omit the obvious isomorphisms β and β' . We begin by checking the S -linearity of Δ_M . Let $m \in M$, $s \in S$ and set $f(m) = \sum_i m_i \otimes u_i$. Then

$$\Delta_M(ms) = \alpha_M(f(m)s) = \sum_i m_i \otimes \Delta_S(u_i s) = \sum_i m_i \otimes \Delta_S(u_i) \Delta_S(s) = \Delta_M(m) \Delta_S(s).$$

Let us show that Δ_M is a coaction. To this end we prove that the following diagram commutes:

$$\begin{array}{ccccccc} M & \xrightarrow{f} & M \otimes_R S & \xrightarrow{\alpha_M} & M \otimes \mathcal{H} & \xrightarrow{f \otimes \text{id}_{\mathcal{H}}} & M \otimes_R S \otimes \mathcal{H} \\ f \downarrow & & & & & & \downarrow \alpha_M \otimes \text{id}_{\mathcal{H}} \\ M \otimes_R S & \xrightarrow{\alpha_M} & M \otimes \mathcal{H} & \xrightarrow{\text{id}_M \otimes \Delta_{\mathcal{H}}} & M \otimes \mathcal{H} \otimes \mathcal{H} & & \end{array}$$

A direct computation, which uses the S -linearity of f , shows that

$$(\alpha_M \otimes \text{id}_{\mathcal{H}} \circ f \otimes \text{id}_{\mathcal{H}} \circ \alpha_M \circ f)(m) = \sum_i \sum_{j=1}^{n_i} \sum_{l=1}^{n'_i} m_{il} \Delta_S(w_{il} v_{ij}) \otimes h_{ij},$$

where $f(m) = \sum_i m_i \otimes u_i$, $\Delta_S(u_i) = \sum_{j=1}^{n_i} v_{ij} \otimes h_{ij}$ and $f(m_i) = \sum_{l=1}^{n'_i} m_{il} \otimes w_{il}$. Since Δ_S is a coaction, we have

$$\sum_{l=1}^{n'_i} m_{il} \Delta_S(w_{il} v_{ij}) \otimes h_{ij} = \sum_{l=1}^{n'_i} m_{il} w_{il} v_{ij} \otimes \Delta_{\mathcal{H}}(h_{ij}),$$

and since $f \circ \mu = \text{id}_M$, this is equal to $\sum_i \sum_{j=1}^{n_i} m_i v_{ij} \otimes \Delta_{\mathcal{H}}(h_{ij})$, which is clearly the same as $(\text{id}_M \otimes \Delta_{\mathcal{H}} \circ \alpha_M \circ f)(m)$. We deduce Proposition 4.12. \square

Proof of Theorem 4.9: Denote by $\mathcal{E}_{\mathcal{H}}$ the functor from $\mathfrak{BMod}_S^{\mathcal{H}}$ to \mathfrak{Desc}_{ψ} and \mathcal{B} the functor from \mathfrak{Desc}_{ψ} to $\mathfrak{BMod}_S^{\mathcal{H}}$ respectively given by $\mathcal{E}_{\mathcal{H}}(M) = (M, f)$ (with f given by Proposition 4.10) and $\mathcal{B}(M, f) = M_{\mathcal{H}}$, the S -module M endowed with the (\mathcal{H}, S) -bialgebra module structure described in Proposition 4.12. By the very definition of f and Δ_M respectively, we deduce the natural isomorphisms $\mathcal{B} \circ \mathcal{E}_{\mathcal{H}} \cong \text{id}$ and $\mathcal{E}_{\mathcal{H}} \circ \mathcal{B} \cong \text{id}$. Thus $\mathfrak{BMod}_S^{\mathcal{H}}$ is equivalent to \mathfrak{Desc}_{ψ} , which is equivalent to \mathfrak{Mod}_R by Theorem 3.8. \square

Example 4.13. Suppose that M is equal to $N \otimes_R S$, where N is an R -module. Then M is endowed with the canonical descent datum f_{can} (see Example 3.6). The corresponding (\mathcal{H}, S) -bialgebra module structure on M is Δ_M^{can} of Example 4.7.

4.4. Galois descent

Corollary 4.14. (*Bialgebra Galois descent for elements*). *Let $\psi : R \rightarrow S$ be an \mathcal{H} -Galois extension. An element $\sum_i n_i \otimes s_i$ of $N \otimes_R S$ is in the image of N under the natural injection $N \cong N \otimes_R R \hookrightarrow N \otimes_R S$ if and only if,*

$$\Delta_{N \otimes_R S}^{\text{can}}(n \otimes s) = n \otimes s \otimes 1.$$

Proof: It suffices to remark that in Theorem 3.4 the universal derivation corresponds to the canonical descent datum (Example 3.14), that is the same (*cf.* Example 4.13) as the canonical coaction $\Delta_{N \otimes_R S}^{\text{can}}$ of Example 4.7. \square

Corollary 4.15. (*Galois descent for elements and homomorphisms*). *Let $\psi : R \rightarrow S$ be a G -Galois extension. Let N and N' be R -modules.*

1) *An element $\sum_i n_i \otimes s_i$ of $N \otimes_R S$ is in the image of N under the natural injection $N \cong N \otimes_R R \hookrightarrow N \otimes_R S$ if and only if, for any $g \in G$, the following equality holds:*

$$\sum_i n_i \otimes g(s_i) = \sum_i n_i \otimes s_i.$$

2) *For $g \in G$, denote by \tilde{g} the automorphism of $\text{Hom}_S(N \otimes_R S, N' \otimes_R S)$ defined by*

$$\tilde{g}(h) = g \circ h \circ g^{-1},$$

where $h \in \text{Hom}_S(N \otimes_R S, N' \otimes_R S)$. Then $h = j \otimes \text{id}_S$ with $j \in \text{Hom}_R(N, N')$ if and only if, for any $g \in G$,

$$\tilde{g}(h) = h.$$

Proof: Observe that a Galois extension is faithfully flat (since S is a left R -progenerator); so we can apply Theorem 3.4. The first part is trivial. To prove the second part, remark that if $h = j \otimes \text{id}_S$, then clearly $g(h) = h$ for any $g \in G$. Conversely, assume that for any $g \in G$, $\tilde{g}(h) = h$. Write $g \circ h \circ g^{-1}(n \otimes 1) = h(n \otimes 1)$. Since $g^{-1}(n \otimes 1) = n \otimes 1$, we get $g(h(n \otimes 1)) = h(n \otimes 1)$. The first statement of the theorem implies that $h(n \otimes 1) = j(n) \otimes 1$. By S -linearity of f , we deduce that j is R -linear and that $h = j \otimes \text{id}_S$. \square

As an immediate consequence of Theorem 4.9, of Lemma 4.4, and of Corollary 4.11, we get

Corollary 4.16. (*Galois descent for modules*). Let $\psi : R \rightarrow S$ be a Galois extension with group $\text{Gal}(S/R) = G$. The category \mathfrak{Gal}_ψ of Galois S -modules is equivalent to the category of R -modules \mathfrak{Mod}_R .

This implies that if M is an S -module endowed with a Galois descent datum, then there exists an R -module N and an isomorphism $\eta : N \otimes_R S \rightarrow M$ in \mathfrak{Gal}_ψ , where $N \otimes_R S$ is endowed with the Galois action of Example 4.8. Moreover, one has

$$N \cong M^G \quad \text{and} \quad \eta(n \otimes s) = ns.$$

Remark 4.17 (due to C. Kassel). Corollary 4.16 can be proved directly in some special cases using Morita equivalence. Indeed, introduce the crossed product ring $S \rtimes G$ defined as the free left S -module with basis G and product given by

$$(sg) \rtimes (th) = h(s)t hg,$$

where $s, t \in S$ and $g, h \in G$. It is easy to see that a Galois right S -module is the same as a right $S \rtimes G$ -module, the correspondence being given by

$$m \cdot (sg) = g(m)s.$$

When the ring $S \rtimes G$ is simple and when the order of G is a unit in S , it is well-known (cf. [17] 7.8.2) that the rings $S \rtimes G$ and R are Morita equivalent. This establishes an equivalence between their categories of modules, namely between \mathfrak{Gal}_ψ and \mathfrak{Mod}_R .

5. Descent for hermitian modules

5.1. Rings with involution

Let A be a ring. Recall ([12]) that an *involution on A* is an involutive antiautomorphism σ of A . This means that $\sigma(a + a') = \sigma(a) + \sigma(a')$, $\sigma(1) = 1$, $\sigma^2(a) = a$, and $\sigma(aa') = \sigma(a')\sigma(a)$ for $a, a' \in A$. A *morphism $\rho : (A, \sigma_A) \rightarrow (B, \sigma_B)$ of rings with involution* is a ring homomorphism $\rho : A \rightarrow B$ such that $\sigma_B \circ \rho = \rho \circ \sigma_A$.

Let $\psi : (R, \sigma_R) \rightarrow (S, \sigma_S)$ be a morphism of rings with involution. From the canonical R -bimodule structure of S we obtain the equalities

$$\sigma_S(sr) = \sigma_R(r)\sigma_S(s) \quad \text{and} \quad \sigma_S(rs) = \sigma_S(s)\sigma_R(r)$$

which hold for $r \in R$ and $s \in S$.

Proposition 5.1. *The map σ given on the algebra $(S \otimes_R S, *)$ by*

$$\sigma(s \otimes t) = \sigma_S(t) \otimes \sigma_S(s)$$

defines an involution. The multiplication map $\mu : (S \otimes_R S, \sigma) \rightarrow (S, \sigma_S)$ as well as the two sections $\varepsilon_i : (S, \sigma_S) \rightarrow (S \otimes_R S, \sigma)$ ($i = 1, 2$) are morphisms of rings with involution.

Proof: One checks that σ is well-defined and a straightforward computation shows that

$$\sigma((s \otimes t) * (u \otimes v)) = (\sigma_S(v) \otimes \sigma_S(u)) * (\sigma_S(t) \otimes \sigma_S(s)) = \sigma(u \otimes v) * \sigma(s \otimes t).$$

The last sentence of Proposition 5.1 is a trivial assertion. □

5.2. Hermitian modules and descent

We recall a few definitions and facts from [12]. Let R be a ring with involution σ_R . Any right R -module N becomes a left R -module setting $r \cdot n = n\sigma_R(r)$ for $r \in R$ and $n \in N$. A *sesquilinear form on N* is a biadditive map $b : N \times N \rightarrow R$ such that

$$b(nr, n'r') = \sigma_R(r)b(n, n')r',$$

where $r, r' \in R$ and $n, n' \in N$. Let Z_R be the centre of R . Denote by $G(Z_R)$ the multiplicative subgroup of the group Z_R^\times of the units of Z_R consisting of elements ζ which verify $\zeta\sigma_R(\zeta) = 1$.

A sesquilinear form b on N is ζ -hermitian if for any $n, n' \in N$, one has

$$b(n, n') = \zeta\sigma_R(b(n', n)).$$

The set of ζ -hermitian sesquilinear forms on N is a group denoted by $\text{Sesq}^\zeta(N)$. Finally, a ζ -hermitian R -module is a pair (N, b) , where N is a R -module and $b \in \text{Sesq}^\zeta(N)$. Observe that in [12], N is always finitely generated and projective. We do not assume this because of difficulties arising in the descent of finitely generated projective modules (see Remark 3.9). Denote by \mathfrak{Herm}_R^ζ the category of ζ -hermitian R -modules. A morphism $(N, b) \rightarrow (N', b')$ in this category is an R -linear isomorphism $\varphi : N \rightarrow N'$ such that $b' \circ (\varphi, \varphi) = b$ (i.e., φ is an *isometry*).

Let $\zeta \in G(Z_R)$. Assume that $\zeta' = \psi(\zeta) \in G(Z_S)$ (which holds in the usual case $\zeta = \pm 1$). Therefore $\varepsilon_1(\zeta') = \varepsilon_2(\zeta') \in G(Z_{S \otimes_R S})$. Denote by ζ'' the element $\varepsilon_1(\zeta')$. It is well-known (see [12] I. § 7) that an extension $\psi : (R, \sigma_R) \rightarrow (S, \sigma_S)$ of rings with involution induces a functor

$$\psi_! : \mathfrak{Herm}_R^\zeta \rightarrow \mathfrak{Herm}_S^{\zeta'},$$

with $\psi_!(N) = N \otimes_R S$. The extended sesquilinear form \tilde{b} on $N \otimes_R S$ is given by

$$\tilde{b}(n \otimes s, n' \otimes s') = \sigma(s)b(n, n')s',$$

where b is the sesquilinear form on N , $n, n' \in N$, and $s, s' \in S$. For example, when (M, b) is an S -module, one checks that $\varepsilon_1(M) \cong [M \otimes_R S]_1$ (see Lemma 1.6) is a ζ -hermitian $S \otimes_R S$ -module by the formula

$$\tilde{b}(m \otimes s, m' \otimes s') = \tau(\sigma(s) \otimes b(m, m'))s',$$

where $m, m' \in M$ and $s, s' \in S$. Hence $(M \otimes_R S)_1$ is a ζ -hermitian S -module by

$$\tilde{b}(m \otimes s, m' \otimes s') = \sigma(s)b(m, m')s' = b(ms, m's').$$

Denote by $\mathfrak{DHerm}_S^{\zeta'}$ the category whose objects are triples (M, b, ξ) , where (M, b) is an object of $\mathfrak{Herm}_S^{\zeta'}$ and $\xi : \varepsilon_{1!}(M) \rightarrow \varepsilon_{2!}(M)$ is simultaneously a symmetry operator and a morphism in the category $\mathfrak{Herm}_{S \otimes_R S}^{\zeta''}$.

We now state a noncommutative analogue of the theorem mentioned in [12] (III (2.2)) for hermitian modules over commutative rings.

Theorem 5.2. (*Descent theorem of hermitian modules*). Let $\psi : (R, \sigma_R) \rightarrow (S, \sigma_S)$ be a faithfully flat extension of rings with involution. With the above notations, the descent functors \mathcal{D} and the extension functor \mathcal{E} induce an equivalence of categories between $\mathfrak{D}\mathfrak{Herm}_S^{\zeta'}$ and $\mathfrak{Herm}_R^{\zeta}$. Furthermore, if R is central in S , the equivalence restricts to the subcategories of $\mathfrak{D}\mathfrak{Herm}_S^{\zeta'}$ and $\mathfrak{Herm}_R^{\zeta}$ of finitely generated projective modules.

Recall that the functors \mathcal{D} and \mathcal{E} are defined by Relation (25).

Proof: By Proposition 5.1, $S \otimes_R S$ is a ring with involution. We make use of Proposition 3.12 in order to introduce symmetry operators rather than descent data, which is here much more convenient. Let $(M, b, \xi) \in \mathfrak{D}\mathfrak{Herm}_S^{\zeta'}$. One checks that $\mathcal{E}(M)$, which is by Corollary 3.17 the R -module $\{m \in M ; \xi(m \otimes 1) = m \otimes 1\}$, is a ζ -hermitian module with the sesquilinear form induced by b . If R is central in S , one applies Remark 3.9. \square

6. Nonabelian cohomology

In this section we shall classify descent data and related objects by nonabelian cohomology sets.

6.1. The Amitsur cohomology

Let $\psi : R \rightarrow S$ be an extension. Let \mathcal{F} be a functor from the category \mathfrak{Alg}_R to the category of groups \mathfrak{Gr} . Applying \mathcal{F} to (22), one gets a cosimplicial group

$$(0 \rightarrow \mathcal{F}(R) \rightarrow \mathcal{F}(S) \rightrightarrows \mathcal{F}(S \otimes_R S) \rightrightarrows \mathcal{F}(S \otimes_R S \otimes_R S) \rightrightarrows \cdots)$$

with cofaces $\mathcal{F}(\varepsilon_i)$ and codegeneracies $\mathcal{F}(\mu_i)$. One now has to distinguish whether \mathcal{F} takes its values in the category of abelian groups or not.

- **First case:** \mathcal{F} is a functor from \mathfrak{Alg}_R to the category of abelian groups \mathfrak{Ab} .

Definition 6.1. The *Amitsur complex* $(\mathcal{A}^*(S/R, \mathcal{F}), \Delta)$ of the morphism $\psi : R \rightarrow S$ with values in \mathcal{F} is the complex of abelian groups associated to the cosimplicial object $\mathcal{F}(S^{\otimes_R *})$. Its cohomology groups are called the *Amitsur cohomology groups of $\psi : R \rightarrow S$ with values in \mathcal{F}* and are denoted by $H^i(S/R, \mathcal{F})$.

Example 6.2. Let $\mathcal{F} = \mathbf{G}_a$ be the functor “additive group”, which associates to each algebra S over R the underlying abelian group $(S, +)$. Then by Proposition 3.2, when the extension $R \rightarrow S$ is faithfully flat, we have

$$H^0(S/R, \mathbf{G}_a) = R \quad \text{and} \quad H^i(S/R, \mathbf{G}_a) = 0 \quad \text{for } i \geq 1.$$

- **Second case:** \mathcal{F} is a functor from \mathfrak{Alg}_R to the category of groups \mathfrak{Gr} .

We first give two examples of such functors.

Example 6.3. Let $\mathcal{F} = \mathbf{G}_m$ be the functor “multiplicative group”, which associates to each algebra S over R the group S^\times of invertible elements of S .

Example 6.4. For any S -module M , let $\text{Aut}_S(M)$ be the group of S -linear automorphisms of M . Fix an R -module N_0 . We form, as in [12], the “automorphism” functor $\mathcal{A}ut(N_0)$

from \mathfrak{Alg}_R to the category of groups \mathfrak{Gr} . It is defined on the objects by $\mathcal{A}ut(N_0)(S) = \mathcal{A}ut_S(N_0 \otimes_R S)$ for $S \in \mathfrak{Alg}_R$ and on the morphisms by $\mathcal{A}ut(N_0)(h) = \tilde{h}$, with $h : S \rightarrow S'$ and $\tilde{h}(\varphi) = \varphi \otimes_{S \text{ id}_{S'}}$ for any $\varphi \in \mathcal{A}ut_S(N_0 \otimes_R S)$. Example 6.3 is a particular case of this, since $\mathcal{A}ut(R) = \mathbf{G}_m$.

For any $n \geq 1$, define also the functor “ n -linear group” \mathbf{GL}_n by $\mathbf{GL}_n = \mathcal{A}ut(R^n)$. Then $\mathbf{G}_m = \mathbf{GL}_1$ is a subfunctor of \mathbf{GL}_n : for any algebra S over R , there is a monomorphism of groups

$$1 \longrightarrow \mathbf{G}_m(S) \xrightarrow{l} \mathbf{GL}_n(S),$$

where $l(s)$ is the left multiplication by $s \in S$ on S^n . Denote by \mathbf{PGL}_n the left quotient functor $\mathbf{PGL}_n = \mathbf{G}_m \backslash \mathbf{GL}_n$ with values in the category of pointed sets. Observe that, unlike the commutative case, since there is no Skolem-Noether Theorem, \mathbf{PGL}_n is not necessarily isomorphic to $\mathcal{A}ut(M_n(R))$. Here $M_n(R)$ denotes the algebra over R of $n \times n$ matrices with coefficients in R (see, *e.g.*, [18]).

When \mathcal{F} takes its values in the category of groups, there are definitions of cohomology sets only in degree 0 and 1. They are given in the following way. To simplify, denote by x_i the image in $\mathcal{F}(S^{\otimes_R(n+1)})$ of $x \in \mathcal{F}(S^{\otimes_R n})$ under $\mathcal{F}(\varepsilon_i)$. Set

$$H^0(S/R, \mathcal{F}) = \{x \in \mathcal{F}(S); x_1 = x_2\}.$$

This set $H^0(S/R, \mathcal{F})$ has an obvious group structure. As in the commutative case, call it the *Amitsur 0-cohomology group of $\psi : R \rightarrow S$ with values in \mathcal{F}* . An *Amitsur 1-cocycle* is an element $y \in \mathcal{F}(S \otimes_R S)$ subject to the relation $y_2 = y_1 y_3$. Denote by $Z^1(S/R, \mathcal{F})$ the set of all Amitsur 1-cocycles. Two Amitsur 1-cocycles y and y' are said to be *cohomologous* if there exists $x \in \mathcal{F}(S)$ such that $y' = x_2 y x_1^{-1}$. The quotient of $Z^1(S/R, \mathcal{F})$ by this equivalence relation is denoted by $H^1(S/R, \mathcal{F})$ and is called the *Amitsur 1-cohomology set of $\psi : R \rightarrow S$ with values in \mathcal{F}* . It is a pointed set, the class of the neutral element of $\mathcal{F}(S \otimes_R S)$ being the distinguished point.

Remark 6.5. If the functor \mathcal{F} takes its values only in the category of sets (respectively of pointed sets), then $H^0(S/R, \mathcal{F})$ is still defined, but only as a set (respectively as a pointed set). For example, let $n \geq 2$ and let μ_n be the functor “ n -roots of the unity” defined by

$$\mu_n(S) = \{x \in \mathbf{G}_m(S); x^n = 1\}$$

for any algebra S over R . In fact $\mu_n(S)$ is a pointed $\mathbf{G}_m(S)$ -set, the group acting by conjugation. Let us briefly compute $H^0(S/R, \mu_n)$.

Proposition 6.6. *Let $\psi : R \rightarrow S$ be a faithfully flat extension and $n \geq 2$ an integer. Then there exists an isomorphism of pointed sets*

$$H^0(S/R, \mu_n) \cong \mu_n(R).$$

Proof: By definition, we have

$$H^0(S/R, \mu_n) = \{x \in S^\times; x^n = 1 \quad \text{and} \quad 1 \otimes x = x \otimes 1\}.$$

From the descent theorem for elements (Theorem 3.4) and since ψ is injective, it follows that

$$H^0(S/R, \boldsymbol{\mu}_n) \cong \psi(R) \cap \boldsymbol{\mu}_n(S) = \{\psi(r) \in \psi(R); \psi(r^n) = 1\} \cong \boldsymbol{\mu}_n(R).$$

□

The goal of the following subsection is to compute $H^i(S/R, \mathcal{A}ut(N_0))$ ($i = 0, 1$) for a given R -module N_0 .

6.2. The Amitsur cohomology sets $H^i(S/R, \mathcal{A}ut(N_0))$ for $i = 0, 1$

Computation of $H^0(S/R, \mathcal{A}ut(N_0))$

Proposition 6.7. *Let $\psi : R \rightarrow S$ be a faithfully flat extension of algebras and N_0 an R -module. Then there exists an isomorphism of groups*

$$H^0(S/R, \mathcal{A}ut(N_0)) \cong \text{Aut}_R(N_0).$$

Proof: The R -bimodule $S \otimes_R S$ has two left S -module structures respectively denoted by ${}_1(S \otimes_R S)$ and ${}_2(S \otimes_R S)$, induced by the morphisms of algebras ε_1 and ε_2 . There exist two isomorphisms of R -modules $\phi_i : (N_0 \otimes_R S) \otimes_{S_i} (S \otimes_R S) \rightarrow N_0 \otimes_R S \otimes_R S$ ($i = 1, 2$) explicitly given by

$$\begin{aligned} \phi_1(n \otimes s) \otimes (u \otimes v) &= n \otimes \tau(s \otimes u)v, & \phi_1^{-1}(n \otimes s \otimes t) &= (n \otimes 1) \otimes (s \otimes t), \\ \phi_2(n \otimes s) \otimes (u \otimes v) &= n \otimes s u \otimes v, & \phi_2^{-1}(n \otimes s \otimes t) &= (n \otimes 1) \otimes (s \otimes t). \end{aligned}$$

An easy computation shows that the isomorphisms ϕ_1 and ϕ_2 are related by

$$\phi_1 = (\text{id}_{N_0} \otimes \tau) \circ \phi_2 \circ (\text{id}_{N_0 \otimes_R S} \otimes \tau).$$

Let h be in $\text{Aut}_S(N_0 \otimes_R S)$ such that $h_1 = h_2$, and let $i = 1$ or 2 . The map h_i is by definition the $S \otimes_R S$ -linear automorphism of $N_0 \otimes_R S \otimes_R S$ given by $h_i = \phi_i \circ g_i \circ \phi_i^{-1}$, where g_i is the automorphism of $(N_0 \otimes_R S) \otimes_{S_i} (S \otimes_R S)$ equal to $h \otimes \text{id}_{i(S \otimes_R S)}$. Then,

$$\begin{aligned} h_1 &= (\text{id}_{N_0} \otimes \tau) \circ \phi_2 \circ (\text{id}_{N_0 \otimes_R S} \otimes \tau) \circ g_1 \circ (\text{id}_{N_0 \otimes_R S} \otimes \tau) \circ \phi_2^{-1} \circ (\text{id}_{N_0} \otimes \tau) \\ &= (\text{id}_{N_0} \otimes \tau) \circ \phi_2 \circ g_2 \circ \phi_2^{-1} \circ (\text{id}_{N_0} \otimes \tau) = (\text{id}_{N_0} \otimes \tau) \circ h_2 \circ (\text{id}_{N_0} \otimes \tau). \end{aligned}$$

The second equality follows from the fact that $(\text{id}_{N_0 \otimes_R S} \otimes \tau) \circ g_1 = g_2 \circ (\text{id}_{N_0 \otimes_R S} \otimes \tau)$, for τ is an S -linear map from ${}_1(S \otimes_R S)$ to ${}_2(S \otimes_R S)$.

Lemma 6.8. *The map $\phi_2 \circ g_2 \circ \phi_2^{-1}$ is equal to $h \otimes \text{id}_S$.*

Proof: Let $n \in N_0$ and $s, t \in S$. Set $h(n \otimes s \otimes t) = \sum_i n_i \otimes s_i \otimes t_i \in N_0 \otimes_R S \otimes_R S$. Then,

$$\phi_2 \circ g_2 \circ \phi_2^{-1}(n \otimes s \otimes t) = \phi_2 \circ g_2((n \otimes 1) \otimes (s \otimes t)) = \phi_2 \circ g_2((n \otimes s) \otimes (1 \otimes t)) = h(n \otimes s) \otimes t.$$

Here we use the equality

$$(n \otimes 1) \otimes (s \otimes t) = (n \otimes s) \otimes (1 \otimes t)$$

which holds in the module $(N_0 \otimes_R S) \otimes_{S^2} (S \otimes_R S)$. \square

We then have the following equivalences:

$$\begin{aligned} h_1 = h_2 &\iff (\text{id}_{N_0} \otimes \tau) \circ \phi_2 \circ (\text{id}_{N_0} \otimes \tau) \circ g_1 \circ (\text{id}_{N_0} \otimes \tau) \circ \phi_2^{-1} \circ (\text{id}_{N_0} \otimes \tau) = h \otimes \text{id}_S \\ &\iff (\text{id}_{N_0} \otimes \tau) \circ \phi_2 \circ g_2 \circ \phi_2^{-1} \circ (\text{id}_{N_0} \otimes \tau) = h \otimes \text{id}_S \\ &\iff (\text{id}_{N_0} \otimes \tau) \circ (h \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \tau) = h \otimes \text{id}_S \\ &\iff (\text{id}_{N_0} \otimes \tau) \circ (h \otimes \text{id}_S) = (h \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \tau). \end{aligned}$$

One concludes by the descent theorem for homomorphisms (Theorem 3.4). \square

Corollary 6.9. *Let $\psi : R \rightarrow S$ be a faithfully flat extension. For any $n \geq 1$, there exists an isomorphism of groups*

$$H^0(S/R, \mathbf{GL}_n) \cong \mathbf{GL}_n(R).$$

In particular, for $n = 1$ we have

$$H^0(S/R, \mathbf{G}_m) \cong R^\times.$$

Computation of $H^1(S/R, \mathcal{A}ut(N_0))$

Denote from now on $[N_0 \otimes_R S \otimes_R S]_2$ by $N_0 \otimes_R S \otimes_R S$ (see Remark 1.8.1 for a justification). Let ν be an $S \otimes_R S$ -linear automorphism of $N_0 \otimes_R S \otimes_R S$. Then ν is an S -linear isomorphism from $(N_0 \otimes_R S \otimes_R S)_2$ to $(N_0 \otimes_R S \otimes_R S)_2$ (see Lemma 1.7). We call an element $\nu \in \mathcal{A}ut(N_0)(S \otimes_R S)$ a *descent 1-cocycle* if $\nu \circ (\text{id}_{N_0} \otimes \tau)$ is a symmetry on $N_0 \otimes_R S \otimes_R S$. Denote by $Z_{\text{desc}}^1(S/R, N_0)$ the set of descent 1-cocycles. Two descent 1-cocycles ν and ν' are *cohomologous* if the two associated symmetries $\xi = \nu \circ (\text{id}_{N_0} \otimes \tau)$ and $\xi' = \nu' \circ (\text{id}_{N_0} \otimes \tau)$ are related by

$$\xi' = (\gamma \otimes \text{id}_S) \circ \xi \circ (\gamma \otimes \text{id}_S)^{-1} \quad (32)$$

for some $\gamma \in \mathcal{A}ut(N_0)(S)$. Denote by ∇ (respectively ∇') the connection associated to ξ (respectively to ξ') by Corollary 3.13. It is easy to see that Relation (32) means $\nabla' = (\gamma \otimes \text{id}_S) \circ \nabla \circ \gamma^{-1}$. The relation defined above on descent 1-cocycles clearly is an equivalence relation. Denote by $H_{\text{desc}}^1(S/R, N_0)$ the pointed set of the descent 1-cocycle classes modulo this equivalence relation and by $\langle \nu \rangle$ the class of a 1-cocycle ν . The distinguished point of $H_{\text{desc}}^1(S/R, N_0)$ is $\langle \text{id}_{N_0 \otimes_R S \otimes_R S} \rangle$.

Theorem 6.10. *Let $\psi : R \rightarrow S$ be a faithfully flat extension and N_0 an R -module. Then there exists an isomorphism of pointed sets*

$$H^1(S/R, \mathcal{A}ut(N_0)) \cong H_{\text{desc}}^1(S/R, N_0)$$

given by $[h] \mapsto \langle h \rangle$.

Proof: This result will follow from Propositions 6.11 and 6.13 below.

Proposition 6.11. *Under the same hypothesis as in Theorem 6.10, there exists an isomorphism of pointed sets $Z^1(S/R, \mathcal{A}ut(N_0)) \cong Z_{\text{desc}}^1(S/R, N_0)$.*

Proof: The R -bimodule $S \otimes_R S \otimes_R S$ has three left $S \otimes_R S$ -module structures respectively denoted by ${}_1(S \otimes_R S \otimes_R S)$, ${}_2(S \otimes_R S \otimes_R S)$ and ${}_3(S \otimes_R S \otimes_R S)$, induced by the morphisms of algebras ε_1 , ε_2 and ε_3 . There exist three isomorphisms of R -modules

$$\phi_i : (N_0 \otimes_R S \otimes_R S) \otimes_{S \otimes_R S} {}_i(S \otimes_R S \otimes_R S) \longrightarrow N_0 \otimes_R S \otimes_R S \quad (i = 1, 2, 3)$$

explicitly given by

$$\begin{aligned} \phi_1(n \otimes s \otimes t) \otimes (u \otimes v \otimes w) &= n \otimes (\varepsilon_1(s \otimes t) * (u \otimes v \otimes w)) \\ &= n \otimes (stu \otimes v \otimes w + 1 \otimes stuv \otimes w + 1 \otimes s \otimes tuv \\ &\quad - st \otimes uv \otimes w - st \otimes 1 \otimes uvw + st \otimes u \otimes vw \\ &\quad - 1 \otimes st \otimes uvw + s \otimes t \otimes uvw - s \otimes tu \otimes vw), \\ \phi_1^{-1}(n \otimes s \otimes t \otimes u) &= (n \otimes 1 \otimes 1) \otimes (s \otimes t \otimes u), \\ \phi_2(n \otimes s \otimes t) \otimes (u \otimes v \otimes w) &= n \otimes (\varepsilon_2(s \otimes t) * (u \otimes v \otimes w)) \\ &= n \otimes (stu \otimes v \otimes w + s \otimes tuv \otimes w + s \otimes 1 \otimes uvw - st \otimes uv \otimes w \\ &\quad - st \otimes 1 \otimes uvw + st \otimes u \otimes vw - s \otimes tu \otimes vw), \\ \phi_2^{-1}(n \otimes s \otimes t \otimes u) &= (n \otimes 1 \otimes 1) \otimes (s \otimes t \otimes u), \\ \phi_3(n \otimes s \otimes t) \otimes (u \otimes v \otimes w) &= n \otimes (\varepsilon_3(s \otimes t) * (u \otimes v \otimes w)) \\ &= n \otimes (stu \otimes v \otimes w + s \otimes tuv \otimes w - st \otimes uv \otimes w), \\ \phi_3^{-1}(n \otimes s \otimes t \otimes u) &= (n \otimes 1 \otimes 1) \otimes (s \otimes t \otimes u), \end{aligned}$$

where $n \in N$ and $s, t, u, v, w \in S$. An other long and tedious computation shows that the following equalities hold:

$$\begin{aligned} \phi_2 &= (\text{id}_{N_0} \otimes (\text{id}_S \otimes \tau)) \circ \phi_3 \circ (\text{id}_{N_0 \otimes_R S \otimes_R S} \otimes (\text{id}_S \otimes \tau)), \\ \phi_1 &= (\text{id}_{N_0} \otimes (\tau \otimes \text{id}_S)) \circ \phi_2 \circ (\text{id}_{N_0 \otimes_R S \otimes_R S} \otimes (\tau \otimes \text{id}_S)), \end{aligned}$$

whence

$$\phi_1 = (\text{id}_{N_0} \otimes ((\tau \otimes \text{id}_S) \circ (\text{id}_S \otimes \tau))) \circ \phi_3 \circ (\text{id}_{N_0 \otimes_R S \otimes_R S} \otimes ((\text{id}_S \otimes \tau) \circ (\tau \otimes \text{id}_S))).$$

For $h \in \text{Aut}_{S \otimes_R S}(N_0 \otimes_R S \otimes_R S)$, the map h_i ($i = 1, 2, 3$) is by definition the $S \otimes_R S \otimes_R S$ -linear automorphism of $N_0 \otimes_R S \otimes_R S \otimes_R S$ given by

$$h_i = \phi_i \circ g_i \circ \phi_i^{-1},$$

where g_i is the automorphism of $(N_0 \otimes_R S \otimes_R S) \otimes_{S \otimes_R S} {}_i(S \otimes_R S \otimes_R S)$ defined by

$$g_i = h \otimes_{S \otimes_R S} \text{id}_i(S \otimes_R S \otimes_R S).$$

One has

$$g_3 = (\text{id}_{N_0 \otimes_R S \otimes_R S} \otimes ((\text{id}_S \otimes \tau) \circ (\tau \otimes \text{id}_S))) \circ g_1 \circ (\text{id}_{N_0 \otimes_R S \otimes_R S} \otimes ((\text{id}_S \otimes \tau) \circ (\tau \otimes \text{id}_S)))$$

and

$$g_3 = (\text{id}_{N_0 \otimes_R S \otimes_R S} \otimes (\text{id}_S \otimes \tau)) \circ g_2 \circ (\text{id}_{N_0 \otimes_R S \otimes_R S} \otimes (\text{id}_S \otimes \tau)).$$

We shall express all the h_i in terms of g_3 , the third structure being the simplest one. We have

$$\begin{aligned} h_1 &= \phi_1 \circ g_1 \circ \phi_1^{-1} \\ &= (\text{id}_{N_0} \otimes ((\tau \otimes \text{id}_S) \circ (\text{id}_S \otimes \tau))) \circ \phi_3 \circ (\text{id}_{N_0 \otimes_R S \otimes_R S} \otimes ((\text{id}_S \otimes \tau) \circ (\tau \otimes \text{id}_S))) \circ g_1 \circ \\ &\quad (\text{id}_{N_0 \otimes_R S \otimes_R S} \otimes ((\tau \otimes \text{id}_S) \circ (\text{id}_S \otimes \tau))) \circ \phi_3^{-1} \circ (\text{id}_{N_0} \otimes ((\text{id}_S \otimes \tau) \circ (\tau \otimes \text{id}_S))) \\ &= (\text{id}_{N_0} \otimes ((\tau \otimes \text{id}_S) \circ (\text{id}_S \otimes \tau))) \circ \phi_3 \circ g_3 \circ \phi_3^{-1} \circ (\text{id}_{N_0} \otimes ((\text{id}_S \otimes \tau) \circ (\tau \otimes \text{id}_S))), \text{ and} \\ h_2 &= \phi_2 \circ g_2 \circ \phi_2^{-1} \\ &= (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ \phi_3 \circ (\text{id}_{N_0 \otimes_R S \otimes_R S} \otimes \text{id}_S \otimes \tau) \circ g_2 \circ \\ &\quad (\text{id}_{N_0 \otimes_R S \otimes_R S} \otimes \text{id}_S \otimes \tau) \circ \phi_3^{-1} \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \\ &= (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ \phi_3 \circ g_3 \circ \phi_3^{-1} \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau). \end{aligned}$$

Lemma 6.12. *The map $\phi_3 \circ g_3 \circ \phi_3^{-1}$ coincides with $h \otimes \text{id}_S$.*

Proof: Let $n \in N_0$ and $s, t, u \in S$. Set $h(n \otimes s \otimes t) = \sum_i n_i \otimes s_i \otimes t_i \in N_0 \otimes_R S \otimes_R S$. Then,

$$\begin{aligned} \phi_3 \circ g_3 \circ \phi_3^{-1}(n \otimes s \otimes t \otimes u) &= \phi_3 \circ g_3((n \otimes 1 \otimes 1) \otimes (s \otimes t \otimes u)) \\ &= \phi_3 \circ g_3((n \otimes s \otimes t) \otimes (1 \otimes 1 \otimes u)) = \phi_3(\sum_i n_i \otimes (s_i t_i \otimes 1 \otimes u + s_i \otimes t_i \otimes u - s_i t_i \otimes 1 \otimes u)) \\ &= h(n \otimes s \otimes t) \otimes u. \end{aligned}$$

Here we use the equality

$$(n \otimes 1 \otimes 1) \otimes (s \otimes t \otimes u) = (n \otimes s \otimes t) \otimes (1 \otimes 1 \otimes u)$$

which holds in $(N_0 \otimes_R S \otimes_R S) \otimes_{S \otimes_R S} {}_3(S \otimes_R S \otimes_R S)$. \square

Thus we can write

$$\begin{aligned} h_1 &= (\text{id}_{N_0} \otimes \tau \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (h \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (\text{id}_{N_0} \otimes \tau \otimes \text{id}_S), \\ h_2 &= (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (h \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau), \\ h_3 &= h \otimes \text{id}_S. \end{aligned}$$

We then have the following equivalences:

$$\begin{aligned} h_2 &= h_3 \circ h_1 \\ \iff (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (h \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \\ &= (h \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \tau \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (h \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (\text{id}_{N_0} \otimes \tau \otimes \text{id}_S) \\ \iff (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (h \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (\text{id}_{N_0} \otimes \tau \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \\ &= (h \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \tau \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (h \otimes \text{id}_S) \\ \iff (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (h \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \tau \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (\text{id}_{N_0} \otimes \tau \otimes \text{id}_S) \\ &= (h \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \tau \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (h \otimes \text{id}_S). \end{aligned}$$

The last equivalence comes from the braid equation (5) of τ . Setting $\xi = h \circ (\text{id}_N \otimes \tau)$, the last equality can be rewritten as

$$(\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (\xi \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) = (\xi \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \text{id}_S \otimes \tau) \circ (\xi \otimes \text{id}_S),$$

which is the cocycle condition for ξ . It is clear that ξ is an isomorphism from $(N \otimes_R S \otimes_R S)_1$ to $(N \otimes_R S \otimes_R S)_2$, hence by Lemma 1.12, it is a symmetry. This proves Proposition 6.11. \square

Proposition 6.13. *Under the same hypothesis as in Theorem 6.10, let h and h' be two Amitsur (or descent) 1-cocycles. Then h and h' are Amitsur-cohomologous if and only if they are descent-cohomologous.*

Proof: Let $\xi = h \circ (\text{id}_{N_0} \otimes \tau)$ and $\xi' = h' \circ (\text{id}_{N_0} \otimes \tau)$ be the corresponding symmetries. Assume that h and h' are Amitsur-cohomologous. Thus there exists a map γ in $\text{Aut}_S(N_0 \otimes_R S)$ such that $h' = \gamma_2 \circ h \circ \gamma_1^{-1}$. One has

$$\gamma_2 = \phi_2 \circ (\gamma \otimes \text{id}_S) \circ \phi_2^{-1} = \gamma \otimes \text{id}_S$$

and

$$\gamma_1 = (\text{id}_{N_0} \otimes \tau) \circ (\gamma \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \tau).$$

Then the following equivalences hold:

$$\begin{aligned} h' &= \gamma_2 \circ h \circ \gamma_1^{-1} \\ \iff \xi' \circ (\text{id}_{N_0} \otimes \tau) &= (\gamma \otimes \text{id}_S) \circ \xi \circ (\text{id}_{N_0} \otimes \tau) \circ (\text{id}_{N_0} \otimes \tau) \circ (\gamma^{-1} \otimes \text{id}_S) \circ (\text{id}_{N_0} \otimes \tau) \\ \iff \xi' &= (\gamma \otimes \text{id}_S) \circ \xi \circ (\gamma^{-1} \otimes \text{id}_S), \end{aligned}$$

hence h and h' are descent-cohomologous. \square

This proves Theorem 6.10. \square

6.3. Twisted forms

Suppose now given a morphism of algebras $\psi : R \rightarrow S$ and an R -module N_0 . A *twisted form* of N_0 is a pair (N, φ) such that N is an R -module and φ is an S -linear isomorphism from $N \otimes_R S$ to $N_0 \otimes_R S$. Two twisted forms (N, φ) and (N', φ') of N_0 are *isomorphic* if there exists a homomorphism of R -modules $\theta : N \rightarrow N'$ such that $\varphi' \circ (\theta \otimes \text{id}_S) = \varphi$. When the extension ψ is faithfully flat, then necessarily θ is an isomorphism. Denote by $\text{Twist}(S/R, N_0)$ the pointed set of isomorphism classes of twisted forms. The distinguished point is the class of $(N_0, \text{id}_{N_0} \otimes \text{id}_S)$.

Lemma 6.14. *Let (N, φ) be a twisted form of N_0 . The R -linear endomorphism ξ of $N_0 \otimes_R S \otimes_R S$ defined by*

$$\xi = (\varphi \otimes \text{id}_S) \circ (\text{id}_N \otimes \tau) \circ (\varphi \otimes \text{id}_S)^{-1}$$

is an involutive symmetry on $N_0 \otimes_R S \otimes_R S$. We refer to ξ as the symmetry corresponding to the twisted form (N, φ) .

Proof: It is obvious that ξ is involutive. The map $\xi : (N_0 \otimes_R S \otimes_R S)_1 \longrightarrow (N_0 \otimes_R S \otimes_R S)_2$ is S -linear. Indeed, the morphisms $\varphi \otimes \text{id}_S : (N_0 \otimes_R S \otimes_R S)_i \longrightarrow (N_0 \otimes_R S \otimes_R S)_i$ ($i = 1, 2$) and $\text{id}_{N_0} \otimes \tau : (N_0 \otimes_R S \otimes_R S)_1 \longrightarrow (N_0 \otimes_R S \otimes_R S)_2$ are S -linear. By Lemma 1.12, it remains to show the cocycle condition. One has

$$\begin{aligned}
& (\xi \otimes \text{id}_S) \circ (\text{id}_N \otimes \text{id}_S \otimes \tau) \circ (\xi \otimes \text{id}_S) \\
&= (\varphi \otimes \text{id}_S \otimes \text{id}_S) \circ (\text{id}_N \otimes \tau \otimes \text{id}_S) \circ (\varphi^{-1} \otimes \text{id}_S \otimes \text{id}_S) \circ (\text{id}_N \otimes \text{id}_S \otimes \tau) \circ (\varphi \otimes \text{id}_S \otimes \text{id}_S) \circ \\
&\quad (\text{id}_N \otimes \tau \otimes \text{id}_S) \circ (\varphi^{-1} \otimes \text{id}_S \otimes \text{id}_S) \\
&= (\varphi \otimes \text{id}_S \otimes \text{id}_S) \circ (\text{id}_N \otimes \tau \otimes \text{id}_S) \circ (\varphi^{-1} \otimes \text{id}_S \otimes \text{id}_S) \circ (\varphi \otimes \text{id}_S \otimes \text{id}_S) \circ (\text{id}_N \otimes \text{id}_S \otimes \tau) \circ \\
&\quad (\text{id}_N \otimes \tau \otimes \text{id}_S) \circ (\varphi^{-1} \otimes \text{id}_S \otimes \text{id}_S) \quad (\varphi \otimes \text{id}_{S \otimes_R S} \text{ and } \varphi^{-1} \otimes \text{id}_{S \otimes S} \text{ commute with } \text{id}_{N \otimes_R S \otimes \tau}) \\
&= (\varphi \otimes \text{id}_S \otimes \text{id}_S) \circ (\text{id}_N \otimes \tau \otimes \text{id}_S) \circ (\text{id}_N \otimes \text{id}_S \otimes \tau) \circ (\text{id}_N \otimes \tau \otimes \text{id}_S) \circ (\varphi^{-1} \otimes \text{id}_S \otimes \text{id}_S) \\
&\hspace{15em} \text{(braid equation for } \tau) \\
&= (\varphi \otimes \text{id}_S \otimes \text{id}_S) \circ (\text{id}_N \otimes \text{id}_S \otimes \tau) \circ (\text{id}_N \otimes \tau \otimes \text{id}_S) \circ (\text{id}_N \otimes \text{id}_S \otimes \tau) \circ (\varphi^{-1} \otimes \text{id}_S \otimes \text{id}_S) \\
&= (\text{id}_N \otimes \text{id}_S \otimes \tau) \circ (\varphi \otimes \text{id}_S \otimes \text{id}_S) \circ (\text{id}_N \otimes \tau \otimes \text{id}_S) \circ (\varphi^{-1} \otimes \text{id}_S \otimes \text{id}_S) \circ (\text{id}_N \otimes \text{id}_S \otimes \tau) \\
&= (\text{id}_N \otimes \text{id}_S \otimes \tau) \circ (\xi \otimes \text{id}_S) \circ (\text{id}_N \otimes \text{id}_S \otimes \tau), \text{ hence } \xi \text{ is a symmetry.} \quad \square
\end{aligned}$$

Lemma 6.15. *Let (N, φ) be a twisted form of N_0 and ξ the corresponding symmetry and f the corresponding descent datum. The associated flat connection ∇ on $N_0 \otimes_R S$ is given by the equation*

$$\nabla = (\varphi \otimes \text{id}_S) \circ \nabla_{\text{can}} \circ \varphi^{-1}.$$

Hence, when the extension $\psi : R \longrightarrow S$ is faithfully flat, the descended module $\mathcal{D}(N_0 \otimes_R S, f)$ is isomorphic to N .

Proof: Recall that the canonical connection ∇_{can} on $N \otimes_R S$ is given by Formula (19). Let n_0 be an element in N_0 and write $\varphi^{-1}(n_0 \otimes 1)$ as $\sum n_i \otimes s_i \in N \otimes_R S$. Setting $\nabla(n_0 \otimes s) = \xi(n_0 \otimes s \otimes 1) - n_0 \otimes s \otimes 1$, an easy computation shows that

$$\nabla(n_0 \otimes s) = ((\varphi \otimes \text{id}_S) \circ \nabla_{\text{can}} \circ \varphi^{-1})(n_0 \otimes s) = \varphi(n_i \otimes 1) \otimes u_i s - n_0 \otimes s \otimes 1.$$

The descended module is $\text{Ker} \nabla$ which is isomorphic as an R -module to $\varphi(N \otimes 1) \cong N$, for $\text{Ker} \nabla_{\text{can}} = N_0$. \square

Lemma 6.16. *Let (N, φ) and (N', φ') be two twisted forms of N_0 and let ξ and ξ' be the corresponding symmetries. Suppose that N' is isomorphic to N via the R -linear isomorphism $h : N' \longrightarrow N$. Then, setting $\gamma = \varphi' \circ (h \otimes \text{id}_S)^{-1} \circ \varphi^{-1}$, we have*

$$\xi' = (\gamma \otimes \text{id}_S) \circ \xi \circ (\gamma \otimes \text{id}_S)^{-1},$$

Proof: Let $\gamma = \varphi' \circ (h \otimes \text{id}_S)^{-1} \circ \varphi^{-1}$, then

$$\begin{aligned}
& (\gamma \otimes \text{id}_S) \circ \xi \circ (\gamma \otimes \text{id}_S)^{-1} \\
&= (\varphi' \otimes \text{id}_S) \circ (h \otimes \text{id}_S \otimes \text{id}_S)^{-1} \circ (\varphi \otimes \text{id}_S)^{-1} \circ (\varphi \otimes \text{id}_S) \circ (\text{id}_N \otimes \tau) \circ (\varphi \otimes \text{id}_S)^{-1} \circ (\varphi \otimes \text{id}_S) \circ \\
&\quad (h \otimes \text{id}_S \otimes \text{id}_S) \circ (\varphi' \otimes \text{id}_S)^{-1} \\
&= (\varphi' \otimes \text{id}_S) \circ (h \otimes \text{id}_S \otimes \text{id}_S)^{-1} \circ (\text{id}_N \otimes \tau) \circ (h \otimes \text{id}_S \otimes \text{id}_S) \circ (\varphi' \otimes \text{id}_S)^{-1} \\
&= (\varphi' \otimes \text{id}_S) \circ (\text{id}_N \otimes \tau) \circ (\varphi' \otimes \text{id}_S)^{-1} = \xi'. \quad \square
\end{aligned}$$

Theorem 6.17. *Let $\psi : R \rightarrow S$ be a faithfully flat extension and N_0 an R -module. There exists an isomorphism of pointed sets*

$$\Delta : \text{Twist}(S/R, N_0) \rightarrow H^1(S/R, \text{Aut}(N_0))$$

given by $\Delta(N, \varphi) = [\xi \circ (\text{id}_{N_0} \otimes \tau)]$, where ξ is the symmetry corresponding to the twisted form (N, φ) .

Proof: The map Δ is well-defined by Lemma 6.16 and Proposition 6.13.

Surjectivity. Let $[\nu] \in H^1(S/R, \text{Aut}(N_0))$, let ξ be the symmetry $\nu \circ (\text{id}_{N_0} \otimes \tau)$ and f be the corresponding descent datum. By Theorem 3.8, $N = \mathcal{D}(N_0 \otimes S, f)$ is such that $N \otimes_R S$ is isomorphic to $N_0 \otimes_R S$ by an isomorphism φ , hence (N, φ) is a twisted form of N_0 . Now $\Delta(N, \varphi) = [\nu]$, for $\mathcal{ED}(N_0 \otimes_R S, f)$ is isomorphic to $(N \otimes_R S, \text{id}_N \otimes \tau)$ in \mathcal{Desc}_ψ .

Injectivity. Suppose that $\Delta(N, \varphi) = \Delta(N', \varphi')$. Denote by ξ and ξ' the symmetries corresponding to (N, φ) and (N', φ') respectively. By definition, $\xi' = (\gamma \otimes \text{id}_S) \circ \xi \circ (\gamma \otimes \text{id}_S)^{-1}$. Let $\sigma \in \text{Hom}_S(N \otimes_R S, N' \otimes_R S)$ defined by $\sigma = \varphi^{-1} \circ \gamma \circ \varphi$. We claim

$$(\sigma \otimes \text{id}_S) \circ (\text{id}_N \circ \tau) = (\text{id}_{N'} \circ \tau) \circ (\sigma \otimes \text{id}_S).$$

$$\begin{aligned} \text{Indeed, } & (\sigma \otimes \text{id}_S) \circ (\text{id}_N \circ \tau) = (\varphi \otimes \text{id}_S)^{-1} \circ (\gamma \otimes \text{id}_S) \circ (\varphi \otimes \text{id}_S) \circ (\text{id}_N \circ \tau) \\ & = (\varphi \otimes \text{id}_S)^{-1} \circ (\gamma \otimes \text{id}_S) \circ \xi \circ (\varphi \otimes \text{id}_S) = (\varphi \otimes \text{id}_S)^{-1} \circ \xi' \circ (\gamma \otimes \text{id}_S) \circ (\varphi \otimes \text{id}_S) \\ & = (\text{id}_{N'} \circ \tau) \circ (\varphi \otimes \text{id}_S)^{-1} \circ (\gamma \otimes \text{id}_S) \circ (\varphi \otimes \text{id}_S) = (\text{id}_{N'} \circ \tau) \circ (\sigma \otimes \text{id}_S). \end{aligned}$$

Hence, by the descent theorem for homomorphisms (Theorem 3.4), σ descends to an isomorphism $N \cong N'$, and (N, φ) is equal to (N', φ') in $\text{Twist}(S/R, N_0)$. \square

6.4 Galois case

Suppose that G and A are two finite groups and that G acts on A on the left such that

$$g(ab) = g(a)g(b)$$

for $g \in G$ and $a, b \in A$. Classically, one defines (see, e.g., [24]) two cohomology sets, denoted by $H^i(G, A)$ ($i = 0, 1$), as follows.

Let $H^0(G, A) = A^G$ be the set of invariant elements of A under the action of G . Then $H^0(G, A)$ is evidently a group. A (*Galois*) *1-cocycle* is a map $\alpha : G \rightarrow A$ such that

$$\alpha(gg') = \alpha(g)g(\alpha(g'))$$

for any $g, g' \in G$. Denote by $Z^1(G, A)$ the set of all the 1-cocycles. Two 1-cocycles α and α' are *cohomologous* if there exists $a \in A$ such that

$$\alpha'(g) = a^{-1}\alpha(g)g(a)$$

for any $g \in G$. Denote by $H^1(G, A)$ the pointed set of 1-cocycle classes modulo this equivalence relation. The distinguished point of $H^1(G, A)$ is the class of the constant map from G to A which associates to any element of G the neutral element of A . It is well-known ([24]) that $H^1(G, A)$ classifies the torsors under A .

Suppose that $\psi : R \rightarrow S$ is a G -Galois extension (see Example 4.3) and let N_0 be a fixed R -module. The group G acts by automorphisms on $\text{Aut}(N_0)(S) = \text{Aut}_S(N_0 \otimes_R S)$ by

$$g(h) = (\text{id}_{N_0} \otimes g) \circ h \circ (\text{id}_{N_0} \otimes g^{-1}),$$

where $g \in G$ and $h \in \text{Aut}_S(N_0 \otimes_R S)$. In this section we shall compute the cohomology sets $H^i(G, \text{Aut}(N_0)(S))$ ($i = 0, 1$), which are called the *nonabelian Galois cohomology sets*.

Proposition 6.18. *Let $\psi : R \rightarrow S$ be a G -Galois extension of algebras and N_0 an R -module. The groups $H^0(S/R, \text{Aut}(N_0))$ and $H^0(G, \text{Aut}(N_0)(S))$ are isomorphic.*

Proof: The Galois descent theorem for homomorphisms (see Corollary 4.15) implies that $H^0(G, \text{Aut}(N_0)(S))$ is isomorphic to $\text{Aut}_R(N_0)$. We conclude by Proposition 6.7. \square

Denote here by $\{\bar{g}\}$ a Galois descent datum on a module M and by $\bar{g}(m)$ the action of $g \in G$ on the element $m \in M$ (see Example 4.8).

Lemma 6.19. *Let α be a Galois 1-cocycle. For any $g \in G$, the R -automorphism \bar{g} of $N_0 \otimes_R S$ defined by*

$$\bar{g} = \alpha(g) \circ (\text{id}_{N_0} \otimes g)$$

is a Galois descent datum. Conversely, if $\{\bar{g}\}$ is a Galois descent datum on $N_0 \otimes_R S$, then the map $\alpha : G \rightarrow \text{Aut}_S(N_0 \otimes_R S)$ given by $\alpha(g) = \bar{g} \circ (\text{id}_{N_0} \otimes g^{-1})$ is a Galois 1-cocycle.

Proof: Let α be a 1-cocycle. Let us show that \bar{g} is a Galois descent datum. One has

$$\overline{gh} = \alpha(g) \circ g(\alpha(h)) \circ (\text{id}_{N_0} \otimes gh) = \alpha(g) \circ (\text{id}_{N_0} \otimes g) \circ \alpha(h) \circ (\text{id}_{N_0} \otimes g^{-1}) \circ (\text{id}_{N_0} \otimes gh) = \bar{g} \circ \bar{h}.$$

It is trivial that $\bar{e} = \text{id}$, where e is the neutral element of G . For any $n \in N_0$, $s, t \in S$, one has, using the S -linearity of $\alpha(g)$, the equalities

$$\bar{g}(n \otimes st) = \alpha(g)(n \otimes g(s)g(t)) = \alpha(g)(n \otimes g(s))g(t) = \bar{g}(n \otimes s)g(t).$$

Conversely, if $\{\bar{g}\}$ is a Galois descent datum on $N_0 \otimes_R S$ and $\alpha(g) = \bar{g} \circ (\text{id}_{N_0} \otimes g^{-1})$, one has for any $g, g' \in G$,

$$\begin{aligned} \alpha(gg') &= \bar{g} \circ \bar{g}' \circ (\text{id}_{N_0} \otimes g'^{-1}) \circ (\text{id}_{N_0} \otimes g^{-1}) = \bar{g} \circ \alpha(g') \circ (\text{id}_{N_0} \otimes g^{-1}) \\ &= \bar{g} \circ (\text{id}_{N_0} \otimes g^{-1}) \circ (\text{id}_{N_0} \otimes g) \circ \alpha(g') \circ (\text{id}_{N_0} \otimes g^{-1}) = \alpha(g) \circ g(\alpha(g')), \end{aligned}$$

which proves that $\alpha(g)$ is a Galois 1-cocycle. \square

Theorem 6.20. *Let $\psi : R \rightarrow S$ be a G -Galois extension and N_0 an R -module. Then there exists a bijection of pointed sets*

$$H^1(S/R, \text{Aut}(N_0)) \cong H^1(G, \text{Aut}(N_0)(S)).$$

Proof: By Theorem 6.17, it is enough to prove that there exists a bijection of pointed sets between $H^1(G, \text{Aut}(N_0)(S))$ and $\text{Twist}(S/R, N_0)$. Let $[\alpha] \in H^1(G, \text{Aut}(N_0)(S))$ and $\{\bar{g}\}$ the corresponding Galois descent datum given by Lemma 6.19. Let N be the descended module $N = \{x \in N_0 \otimes_R S; \bar{g}(x) = x, \forall g \in G\}$ and φ the isomorphism given by the Galois descent theorem. Set $\Gamma([\alpha]) = (N, \varphi)$. The map Γ from $H^1(G, \text{Aut}(N_0)(S))$ to $\text{Twist}(S/R, N_0)$ is then well defined. Indeed, assume α cohomologous to α' via $h \in \text{Aut}(N_0)(S)$. Let $\{\bar{g}\}$ and $\{\tilde{g}\}$ be the corresponding Galois descent data. It is easy to show that $\tilde{g} = h^{-1} \circ \bar{g} \circ h$, hence the descended modules are isomorphic.

Surjectivity. Let (N, φ) be a twisted form of N_0 . Let $\{\bar{\gamma}\}$ be the canonical Galois descent datum on $N \otimes_R S$ (see Example 4.8). It is associated to the canonical symmetry

$\text{id}_N \otimes \tau$. Then $\bar{g} = \varphi \circ \bar{\gamma} \circ \varphi^{-1}$ is a Galois descent datum on $N_0 \otimes_R S$, and the corresponding 1-cocycle α is such that $\Gamma([\alpha]) = (N, \varphi)$.

Injectivity. Suppose that $\Gamma([\alpha]) = (N, \varphi)$ is isomorphic to $\Gamma([\alpha']) = (N', \varphi')$. Let $\{\bar{g}\}$ and $\{\tilde{g}\}$ be the Galois descent data corresponding respectively to α and α' . Let $h = \varphi \circ \varphi'^{-1} \in \mathcal{A}ut(N_0)(S)$. Then α' is cohomologous to α via h . Indeed,

$$\begin{aligned} \alpha'(g) &= \tilde{g} \circ (\text{id}_{N_0} \otimes g^{-1}) = \varphi' \circ \bar{\gamma} \circ \varphi'^{-1} \circ (\text{id}_{N_0} \otimes g^{-1}) = \varphi' \circ \varphi^{-1} \circ \bar{g} \circ \varphi \circ \varphi'^{-1} \circ (\text{id}_{N_0} \otimes g^{-1}) \\ &= h^{-1} \circ \bar{g} \circ (\text{id}_{N_0} \otimes g^{-1}) \circ (\text{id}_{N_0} \otimes g) \circ h \circ (\text{id}_{N_0} \otimes g^{-1}) = h^{-1} \circ \alpha(g) \circ g(h). \quad \square \end{aligned}$$

Corollary 6.21. *Let $\psi : K \rightarrow L$ be a G -Galois extension of skew-fields. Then, for any integer $n \geq 1$, we have*

$$H^1(G, \mathbf{GL}_n(L)) = \{1\}.$$

We denote here by 1 the distinguished point of $H^1(G, \mathbf{GL}_n(L))$. Observe that the case $n = 1$ is a noncommutative version of Hilbert's Theorem 90.

Proof: By Theorem 6.20 and Theorem 6.17, one has

$$H^1(G, \mathbf{GL}_n(L)) \cong H^1(L/K, \mathcal{A}ut(K^n)) \cong \text{Twist}(L/K, K^n).$$

An element of the latter set is a pair (P, φ) , where P is a K -vector space and φ is an isomorphism from $P \otimes_K L$ to L^n . The vector space P is equal to $K^{[I]}$ for some set I . Thus φ gives an isomorphism $L^{[I]} \cong L^n$. Since L is a field, the cardinality of I is n . Hence $\text{Twist}(L/K, K^n) = \{(K^n, K^n \otimes_K L \xrightarrow{\varphi} L^n)\}$, where φ is the canonical map. \square

REFERENCES

- [1] M. ARTIN, *On Azumaya algebras and finite dimensional representations of rings*, Journal of Algebra 11 (1969), 532 - 563.
- [2] J.C. BAEZ, *Hochschild homology in a braided tensor category*, Trans. A.M.S. 344 (1994), 885 - 906.
- [3] N. BOURBAKI, *Algèbre commutative*. Chap. 1. Modules plats. Hermann, Paris, (1961).
- [4] M. CIPOLLA, *Discesa fedelmente piatta dei moduli*, Rendiconti del Circolo Matematico di Palermo, Serie II - tomo XXV (1976).
- [5] P. DELIGNE, *Catégories tannakiennes*, in *The Grothendieck Festschrift Volume II*, Progress in Mathematics 187, Birkhäuser, Boston - Basel - Berlin (1990).
- [6] F. DEMEYER - E. INGRAHAM, *Separable algebras over commutative rings*, Lecture Notes in Mathematics 181. Springer-Verlag, Berlin - Heidelberg - New York (1971).
- [7] J. GIRAUD, *Méthode de la descente*, Mémoires Soc. Math. France 2 (1964).
- [8] A. GROTHENDIECK, *Technique de descente et théorèmes d'existence en géométrie algébrique. I. Généralités. Descente par morphismes fidèlement plats*, Séminaire Bourbaki (1959-1960) (190).
- [9] A. GROTHENDIECK, *Technique de descente et théorèmes d'existence en géométrie algébrique. II. Le théorème d'existence en théorie formelle des modules*, Séminaire Bourbaki (1959-1960) (195).
- [10] A. JOYAL - M. TIERNEY, *An extension of the Galois theory of Grothendieck*, Memoirs of the American Mathematical Society, Number 309 (1984).
- [11] M. KAROUBI, *Homologie cyclique et K-théorie*, Astérisque 149, Société Mathématique de France (1987).
- [12] M.A. KNUS, *Quadratic and hermitian forms over rings*, Grundlehren der mathematischen Wissenschaften 294, Springer-Verlag, Berlin - Heidelberg - New York (1991).
- [13] M.A. KNUS - M. OJANGUREN, *Théorie de la descente et algèbres d'Azumaya*, Lecture Notes in Mathematics 389, Springer-Verlag, Berlin - Heidelberg - New York (1974).
- [14] O.A. LAUDAL, *Formal moduli of algebraic structures*, Lecture Notes in Mathematics 754, Springer-Verlag, Berlin - Heidelberg - New York (1979).
- [15] O.A. LAUDAL, *A framework for change* (to appear).
- [16] L. LE BRUYN - M. VAN DEN BERGH - F. VAN OYSTAEYEN, *Graded orders*, Birkhäuser, Boston - Basel (1988).
- [17] J.C. MCCONNELL - J.C. ROBSON, *Noncommutative noetherian rings*, Wiley - Interscience Publication, Chichester - New York - Brisbane - Toronto - Singapore (1987).

- [18] J.S. MILNE, *Étale cohomology*, Princeton Mathematical Series, Princeton University Press, Princeton (1980).
- [19] S. MONTGOMERY, *Hopf algebras and their actions on rings*, CMBS Regional conference series in mathematics, Number 82, American Mathematical Society (1993).
- [20] H.-J. SCHNEIDER, *Principal homogeneous spaces for arbitrary Hopf algebras*, Israel J. Math., Vol. 72, Nos 1-2, (1990), 167 - 195.
- [21] H.-J. SCHNEIDER, *Representation theory of Hopf Galois extensions*, Israel J. Math., Vol. 72, Nos 1-2, (1990), 196 - 231.
- [22] Séminaire de géométrie algébrique du Bois-Marie, *Revêtements étales et groupe fondamental*, Lecture Notes in Mathematics 224, Springer-Verlag, Berlin - Heidelberg - New York (1971).
- [23] Séminaire de géométrie algébrique du Bois-Marie, *Théorie des topos et cohomologie étale des schémas*, Volume 2, Lecture Notes in Mathematics 270, Springer-Verlag, Berlin - Heidelberg - New York (1972).
- [24] J.P. SERRE, *Cohomologie galoisienne*, Lecture Notes in Mathematics 5, Springer-Verlag, Berlin - Heidelberg - New York (1973).

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