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ON SUBVARIETIES OF ABELIAN VARIETIES

Olivier DEBARRE(*)

Let V be an irreducible subvariety of a complex abelian variety. When V is smooth, Sommese showed in his work that a large part of its geometry depends on “how ample” its normal bundle is (see § 1 for more details). Unfortunately, the only known way of measuring this ampleness uses rather strong properties of the ambient abelian variety X ; if X is for example *simple*, i.e. does not contain any proper abelian subvarieties, the normal bundle to any smooth V is ample and all is well, but in general, this measure can become absolutely meaningless.

We show that a notion of non-degeneracy due to Ran is a good substitute for ampleness of the normal bundle for arbitrary (singular) irreducible subvarieties. It can be defined as follows: *an irreducible subvariety V of an abelian variety X is geometrically non-degenerate if for any abelian variety Y quotient of X , the image of V in Y either is Y or has same dimension as V .* This property is rather easy to check and does not require V to be smooth; it was extensively used in [D1] to prove Fulton-Hansen-type connectedness results. For smooth subvarieties, it is (strictly) weaker than ampleness of the normal bundle.

Our main result is a Zak-type result for an arbitrary subvariety V of an abelian variety that relates the dimension of its “secant variety” (defined as being $V - V$) to that of its “tangential variety” (defined in the smooth case as the union of the projectivized tangent spaces to V , translated at the origin). Corollaries include a new proof of the finiteness of the Gauss map and an estimate on the ampleness of the normal bundle of a smooth geometrically non-degenerate subvariety.

We work over the field of complex numbers.

1. Geometrically non-degenerate subvarieties

Recall the following definition from [S1]: a line bundle L on an irreducible projective variety V is k -ample if, for some $m > 0$, the line bundle L^m is generated by its global sections and the fibers of the associated map $\phi_{L^m} : V \rightarrow \mathbf{P}^N$ are all of dimension $\leq k$. A vector bundle E on V is k -ample if the line bundle $\mathcal{O}_{\mathbf{P}E^*}(1)$ is k -ample. Ordinary ampleness coincide with 0-ample.

Let X be an abelian variety and let V be an irreducible variety with a morphism $f : V \rightarrow X$. Let V^0 be the open set of smooth points of V at which f is unramified. Define the normal bundle to f as the vector bundle on V^0 quotient of $f^*(TX)|_{V^0}$ by TV^0 .

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the map $\tau_{-f(v)}f$ at v is a linear map $T_vV \rightarrow T_0X$, which we will simply denote by f_* .

PROPOSITION 1.1.– *Under the above assumptions, let S be a complete irreducible subvariety of V^0 . The following properties are equivalent*

(i) *the restriction to S of the normal bundle to f is k -ample;*

(ii) *for any hyperplane H in T_0X , the set $\{s \in S \mid f_*(T_sV) \subset H\}$ has dimension $\leq k$.*

Proof. Let N be the restriction to S of the normal bundle to f and let $\iota : N^* \rightarrow f^*(T^*X)|_S$ be the canonical injection. The morphism

$$\phi : \mathbf{P}N^* \xrightarrow{\bar{\iota}} \mathbf{P}f^*(T^*X)|_S \simeq \mathbf{P}T_0^*X \times S \xrightarrow{pr_1} \mathbf{P}T_0^*X$$

satisfies

$$\phi^* \mathcal{O}_{\mathbf{P}T_0^*X}(1) = \bar{\iota}^* \mathcal{O}_{\mathbf{P}f^*(T^*X)|_S}(1) = \mathcal{O}_{\mathbf{P}N^*}(1).$$

It follows that N is k -ample if and only if the fibers of ϕ have dimension $\leq k$ ([S1], prop. 1.7.). The proposition follows, since the restriction of the projection $\mathbf{P}N^* \rightarrow S$ to any fiber of ϕ is injective. ■

When X is simple, the normal bundle to any smooth irreducible subvariety of X is ample ([H]). More generally, the normal bundle to any smooth irreducible subvariety of X is k -ample, where k is the maximum dimension of a proper abelian subvariety of X ([S1], prop. 1.20).

Following Ran, we will say that an irreducible subvariety V of X is *geometrically non-degenerate* if the restriction $H^0(X, \Omega_X^d) \rightarrow H^0(V_{\text{reg}}, \Omega_{V_{\text{reg}}}^d)$ contains no non-zero decomposable forms. Ran proved ([R1], lemma II.12) that V is geometrically non-degenerate if and only if for any abelian variety Y quotient of X , the image of V in Y either is Y or has same dimension as V .

Examples. 1) A divisor is geometrically non-degenerate if and only if it is ample; a curve is non-degenerate if and only if it generates X . Any non-degenerate subvariety of positive dimension generates X , but the converse is false in general.

2) Any irreducible subvariety of a *simple* abelian variety is geometrically non-degenerate.

3) If ℓ is a polarization on X and V is an irreducible subvariety of X with class a rational multiple of ℓ^c , it follows from [R1], cor. II.2 and II.3 that V is non-degenerate in the sense of [R1], II, hence geometrically non-degenerate. In particular, the subvarieties $W_d(C)$ of the Jacobian of a curve C are geometrically non-degenerate; it can be checked that their normal bundle is ample when they are smooth (use prop. 1.1).

We generalize this notion as follows. Let k be a non-negative integer.

non-degenerate if and only if for any abelian variety Y quotient of X , the image of V in Y either is Y or has dimension $\geq \dim(V) - k$.

PROPOSITION 1.3.– *In an abelian variety, any smooth irreducible subvariety with k -ample normal bundle is k -geometrically non-degenerate.*

Proof. Let $\pi : X \rightarrow Y$ be a quotient of X such that $\pi(V) \neq Y$. The tangent spaces to V along a smooth fiber of $\pi|_V$ are all contained in a fixed hyperplane, hence general fibers of $\pi|_V$ have dimension $\leq k$ by prop. 1.1. ■

The converse is not true, as the construction sketched below shows, but a partial converse will be obtained in 3.3. Roughly speaking, if Y is an abelian variety quotient of X , and if the image W of V in Y is not Y , then k -geometrical nondegeneracy requires that the general fibers of $V \rightarrow W$ be of dimension $\leq k$, whereas k -ampleness of the normal bundle requires that every fiber of $V \rightarrow W$ be of dimension $\leq k$.

Let L_E be an ample line bundle on an elliptic curve E , with linearly independent sections s_1, s_2 defining a morphism $E \rightarrow \mathbf{P}^1$ with ramification points $(e_1, 1), \dots, (e_4, 1) \in \mathbf{P}^1$. Let L_Y be an ample line bundle on a simple abelian variety Y of dimension ≥ 3 , with linearly independent sections t_1, t_2, t_3 such that $\text{div}(t_3)$, $F = \text{div}(t_1) \cap \text{div}(t_2) \cap \text{div}(t_3)$ and $\text{div}(e_i t_1 + t_2) \cap \text{div}(t_3)$ are smooth for $i = 1, \dots, 4$ (such a configuration can be constructed using results from [D2]). Set $X = E \times Y$ and define a subvariety V of X by the equations $s_1 t_1 + s_2 t_2 = t_3 = 0$; then V is smooth of codimension 2, geometrically non-degenerate, but its normal bundle is not ample (for all $e \in E$ and $f \in F$, one has $T_{(e,f)}V \subset T_f(\text{div}(t_3))$), only 1-ample (cor. 3.3).

PROPOSITION 1.4.– *Let X be an abelian variety and let V and W be irreducible subvarieties of X . Assume that V is k -geometrically non-degenerate and define a morphism $\phi : V^r \times W \rightarrow X^r$ by $\phi(v_1, \dots, v_r, w) = (v_1 - w, \dots, v_r - w)$. Then*

$$\dim \phi(V^r \times W) \geq \min(r \dim(X), r \dim(V) + \dim(W) - k) .$$

Proof. Assume first $r \dim(V) + \dim(W) - k \geq r \dim(X)$. Let $\pi : X \rightarrow X/K$ be a quotient of X . I claim that $r \dim \pi(V) + \dim \pi(W) \geq r \dim(X/K)$. If $\pi(V) = X/K$, this is obvious; otherwise, we have $\dim \pi(V) \geq \dim(V) - k_0$, where $k_0 = \min(k, \dim(K))$, hence

$$\begin{aligned} r \dim \pi(V) + \dim \pi(W) &\geq r(\dim(V) - k_0) + \dim(W) - \dim(K) \\ &\geq r \dim(X) + k - r k_0 - \dim(K) \geq r \dim(X/K) . \end{aligned}$$

It follows that (V, \dots, V, W) (where V is repeated r times) fills up X in the sense of [D1], (1.10); [D1], th. 2.1 then implies that ϕ is onto.

Assume now $s = r \text{codim}(V) - \dim(W) + k > 0$; let C be an irreducible curve in X that generates X . Let W' be the sum of W and s copies of C ; then $r \dim(V) + \dim(W') - k = r \dim(X)$ and the first case shows that the sum of the image of ϕ and s curves is X^r . The proposition follows. ■

Sommese has obtained very complete results on the homotopy groups of *smooth* subvarieties of an abelian variety. For example, he proved in [S2] that if V is a smooth subvariety of dimension d of an abelian variety X , with k -ample normal bundle, $\pi_q(X, V) = 0$ for $q \leq 2d - n - k + 1$. For arbitrary subvarieties, we have the following.

THEOREM 2.1.– *Let X be an abelian variety and let V be a k -geometrically non-degenerate normal subvariety of X of dimension $> \frac{1}{2}(\dim(X) + k)$. Then $\pi_1^{\text{alg}}(V) \simeq \pi_1^{\text{alg}}(X)$.*

Proof. The case $k = 0$ is cor. 4.2 of [D1]. The general case is similar, since the hypothesis implies that the pair (V, V) satisfies condition $(*)$ of [D1]. ■

Going back to smooth subvarieties, I will give an elementary proof of (a slight improvement of) the cohomological version of Sommese's theorem, based on the following vanishing theorem ([LP]) and the ideas of [SZ].

VANISHING THEOREM 2.2 (Le Potier, Sommese).– *Let E be a k -ample rank r vector bundle on a smooth irreducible projective variety V of dimension d . Then*

$$H^q(V, E^* \otimes \Omega_V^p) = 0 \quad \text{for} \quad p + q \leq d - r - k .$$

Recall also the following elementary lemma from [SZ]:

LEMMA 2.3.– *Let $0 \rightarrow F \rightarrow E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_k \rightarrow 0$ be an exact sequence of sheaves on a scheme V . Assume $H^s(V, E_i) = 0$ for $0 \leq i < k$ and $s \leq q$. Then $H^q(V, F) \simeq H^{q-k}(V, E_k)$.*

THEOREM 2.4.– *Let V be a smooth irreducible subvariety of dimension d of an abelian n -fold X and let \mathcal{L} be a nef line bundle on V . Assume that the normal bundle N of V in X is a direct sum $\oplus N_i$, where N_i is k_i -ample of rank r_i . Then, for $j > 0$*

$$H^q(V, S^j N^* \otimes \mathcal{L}^{-1}) = 0 \quad \text{for} \quad q \leq d - \max(r_i + k_i) .$$

Proof. Since $S^j N^*$ is a direct summand of $S^{j-1} N^* \otimes N^*$, it is enough to show, by induction on j , that $H^q(V, S^j N^* \otimes N_i^* \otimes \mathcal{L}^{-1})$ vanishes for $j \geq 0$ and $q \leq d - r_i - k_i$. Since $N_i \otimes \mathcal{L}$ is k_i -ample, the case $j = 0$ follows from Le Potier's theorem. For $j \geq 1$, tensor the exact sequence

$$0 \rightarrow S^j N^* \rightarrow S^{j-1} N^* \otimes \Omega_{X|V}^1 \rightarrow \dots \rightarrow \Omega_{X|V}^j \rightarrow \Omega_V^j \rightarrow 0$$

by $N_i^* \otimes \mathcal{L}^{-1}$. Since Ω_X^1 is trivial, the induction hypothesis and the lemma give $H^q(V, S^j N^* \otimes N_i^* \otimes \mathcal{L}^{-1}) \simeq H^{q-j}(V, \Omega_V^j \otimes N_i^* \otimes \mathcal{L}^{-1})$, and this group vanishes for $q \leq d - r_i - k_i$ by Le Potier's theorem. ■

We are now ready to prove our version of Sommese's result.

fold X . Assume that its normal bundle is a direct sum $\oplus N_i$, where N_i is k_i -ample of rank r_i . Then

a) $H^q(X, V; \mathbf{C}) = 0$ for $q \leq d - \max(r_i + k_i) + 1$;

b) for all nonzero elements P of $\text{Pic}^0(V)$, the cohomology groups $H^q(V, P)$ vanish for $q \leq d - \max(r_i + k_i)$.

Remarks 2.6. 1) It is likely that a similar result should hold for cohomology with integral coefficients, but it does not seem to follow from Sommese's work.

2) If the normal bundle is k -ample, we get $H^q(X, V; \mathbf{C}) = 0$ for $q \leq 2d - n - k + 1$. If the normal bundle is a sum of ample line bundles, $H^q(X, V; \mathbf{C}) = 0$ for $q \leq d$; in particular, the restriction $H^0(X, \Omega_X^d) \rightarrow H^0(V, \Omega_V^d)$ is injective and V is non-degenerate in the sense of [R1], II, hence also geometrically non-degenerate.

3) By [GL], $H^q(V, P) = 0$ for P outside of a subset of codimension $\geq d - q$ of $\text{Pic}^0(V)$. By ([S]), this subset is a union of translates of abelian subvarieties of X by torsion points.

Proof of the theorem. For a), it is enough by Hodge theory to study the maps

$$H^i(X, \Omega_X^j) \longrightarrow H^i(V, \Omega_{X|V}^j) \xrightarrow{\psi} H^i(V, \Omega_V^j).$$

Since Ω_X^j is trivial, we only need look at $\phi : H^i(X, \mathcal{O}_X) \rightarrow H^i(V, \mathcal{O}_V)$ and ψ . We begin with ψ . We may assume $j > 0$. Let M_j be the kernel of the surjection $\Omega_{X|V}^j \rightarrow \Omega_V^j \rightarrow 0$. The long exact sequence above gives

$$0 \rightarrow S^j N^* \rightarrow S^{j-1} N^* \otimes \Omega_{X|V}^1 \rightarrow \dots \rightarrow N^* \otimes \Omega_{X|V}^{j-1} \rightarrow M_j \rightarrow 0.$$

The lemma and the theorem then yield

$$H^i(V, M_j) \simeq H^{i+j-1}(V, S^j N^*) = 0$$

for $i + j - 1 \leq d - \max(k_i + r_i)$, since $j > 0$. This implies that ψ has the required properties.

For $i = 0$, the map ψ is $H^0(X, \Omega_X^j) \rightarrow H^0(V, \Omega_V^j)$. By Hodge symmetry, this proves that ϕ also has the required properties, hence the first point.

For b), we may assume $d - \max(k_i + r_i) \geq 1$, in which case the first point implies $\text{Pic}^0(X) \simeq \text{Pic}^0(V)$. Let $P \in \text{Pic}^0(X)$ be nonzero. The same proof as above yields $H^0(V, \Omega_V^q \otimes P|_V) = 0$ for $q \leq d - \max(k_i + r_i)$. The lemma then follows from the existence of an anti-linear isomorphism $H^0(V, \Omega_V^q \otimes P|_V) \simeq H^q(V, P|_V^*)$ ([GL]). ■

I will end this section with an amusing consequence of th. 2.5. If C is a curve in an abelian variety X , write C_d for the subvariety $C + \dots + C$ (d times) of X . Recall that

JC has dimension $2d - n - 2$.

PROPOSITION 2.7.— *Let X be an abelian variety of dimension n and let C be a smooth irreducible curve in X . Assume that C generates X and that its Gauss map is birational onto its image. Then, for $d < n$, the singular locus of C_d has dimension $\geq 2d - n - 1$ unless X is isomorphic to the Jacobian of C and C is canonically embedded in X .*

Proof. Let $\gamma : C_{\text{reg}} \rightarrow \mathbf{PT}_0X$ be the Gauss map and let $\pi : C^{(d)} \rightarrow C_d$ be the sum map. The image of the differential of π at the point $(c_1 \bullet \dots \bullet c_d)$ is the linear subspace of T_0X generated by $\gamma(c_1), \dots, \gamma(c_d)$. Since C generates X , the curve $\gamma(C)$ is non-degenerate; it follows that for c_1, \dots, c_d general, the points $\gamma(c_1), \dots, \gamma(c_d)$ span a $(d-1)$ -plane whose intersection with the curve $\gamma(C)$ consists only of these points. Thus π is birational. Moreover, if $x = c_1 + \dots + c_d$ is smooth on C_d , then $\gamma(c_i) \subset \tau_x^*(T_x C_d) \cap \gamma(C)$ hence $\pi^{-1}(x)$ is finite. Zariski's Main Theorem implies that π induces an isomorphism between $\pi^{-1}((C_d)_{\text{reg}})$ and $(C_d)_{\text{reg}}$.

Let s be the dimension of the singular locus of C_d and assume $-1 \leq s \leq 2d - n - 2$; let L be a very ample line bundle on X and let W be the intersection of C_d with $(s+1)$ general elements of $|L|$. Then W is smooth of dimension ≥ 2 and contained in $(C_d)_{\text{reg}}$. If H is a hyperplane in T_0X and $x = c_1 + \dots + c_d \in W$ then $T_x C_d \subset H$ implies $\gamma(c_i) \in \mathbf{PH} \cap \gamma(C)$; prop. 1.1 implies that the restriction of $N_{C_d/X}$ to W is ample. Since $N_{W/X}$ is the direct sum of this restriction and of $(s+1)$ copies of L , th. 2.5 implies that the restriction $H^1(X, \mathcal{O}_X) \rightarrow H^1(W, \mathcal{O}_W)$ is bijective.

On the other hand, the line bundle π^*L is nef and big on $C^{(d)}$, hence the Kawamata-Viehweg vanishing theorem ([K], [V]) implies

$$H^1(C^{(d)}, \mathcal{O}_{C^{(d)}}) \subset H^1(\pi^{-1}(W), \mathcal{O}_{\pi^{-1}(W)}) \simeq H^1(W, \mathcal{O}_W).$$

Since $H^1(C, \mathcal{O}_C) \simeq H^1(C^{(d)}, \mathcal{O}_{C^{(d)}})$ ([M]), we get $h^1(C, \mathcal{O}_C) \leq h^1(X, \mathcal{O}_X)$ and there must be equality because C generates X . Thus, the inclusion $C \subset X$ factors through an isogeny $\phi : JC \rightarrow X$. Since π is birational, the inverse image $\phi^{-1}(C_d)$ is the union of $\deg(\phi)$ translates of $W_d(C)$. But any two translates of $W_d(C)$ meet along a locus of dimension $\geq 2d - n > s$, hence ϕ is an isomorphism. ■

3. A Zak-type result

A lot of information has been obtained in the last fifteen years about the secant and tangential varieties of subvarieties of the projective space ([FL1], [Z1], [Z3], [L], ...).

We will prove a result analogous to th. 2 of [Z1] (see also [FL1], cor. 5.5) for a subvariety of an abelian variety.

Let X be an abelian variety and let V be a variety with a morphism $f : V \rightarrow X$. Recall that f is unramified along a subvariety S of V if $\Delta_S = \{(v, s) \in V \times S \mid v = s\}$

weakly unramified along S if Δ_S is a connected component of $V \times_X S$, ignoring scheme structures. In that case, if $p : V \times S \rightarrow X$ is the morphism defined by $p(v, s) = f(v) - f(s)$ and $\epsilon : \tilde{X} \rightarrow X$ is the blow-up of the origin, there exists a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{p}} & \tilde{X} \\ \alpha \downarrow & & \downarrow \epsilon \\ V \times S & \xrightarrow{p} & X \end{array}$$

where α is the blow-up of $V \times_X S$. Let E be the exceptional divisor above Δ_S and set $T(V, S) = \tilde{p}(E)$. It is a subscheme of \mathbf{PT}_0X contained in $\bigcup_{s \in S} \mathbf{PT}_sV$ and equal to the latter when V is smooth along S and f is unramified along S . Loosely speaking, $T(V, S)$ is the set of limits in \tilde{X} of $(v - s)$, as $v \in V$ and $s \in S$ converge to the same point. Obviously, $\dim T(V, S) < \dim(f(V) - f(S))$.

THEOREM 3.1.— *Let X be an abelian variety and let V be an irreducible projective variety with a morphism $f : V \rightarrow X$. Let S be a complete irreducible subvariety of V along which f is weakly unramified. Then $\dim T(V, S) = \dim(f(V) - f(S)) - 1$.*

We begin with a lemma.

LEMMA 3.2.— *Let C be an irreducible projective curve with a morphism $g : C \rightarrow X$ such that $g(C)$ is a smooth curve through the origin. Assume that g is unramified at some point $c_0 \in C$ with $g(c_0) = 0$ and that $\mathbf{PT}_0g(C) \notin T(V, S)$. The morphism $h : V \times C \rightarrow X$ defined by $h(v, c) = f(v) - g(c)$ is weakly unramified along $S \times \{c_0\}$ and $T(V \times C, S \times \{c_0\})$ is contained in the cone over $T(V, S)$ with vertex $\mathbf{PT}_0g(C)$.*

One can prove that $T(V \times C, S \times \{c_0\})$ is actually equal to the cone.

Proof. Let Γ be a smooth irreducible curve, let γ_0 be a point on Γ and let $q = (q_1, q_2, q_3) : \Gamma \rightarrow (V \times C) \times_X S$ be a morphism with $q(\gamma_0) = (s_0, c_0, s_0)$. We need to prove that $q(\Gamma) \subset \Delta'_S$, where $\Delta'_S = \{(s, c_0, s) \mid s \in S\}$; since Δ_S is a connected component of $V \times_X S$, it suffices to show that q_2 is constant. Suppose the contrary; then (q_1, q_3) lifts to a morphism $\tilde{q}_{13} : \Gamma \rightarrow \tilde{Y}$ and g to a morphism $\tilde{g} : C \rightarrow \tilde{X}$. Since $p(q_1, q_3) = gq_2$, one has $\tilde{p}\tilde{q}_{13} = \tilde{g}q_2$ hence $\tilde{g}(c_0) = \tilde{p}(\tilde{q}_{13}(\gamma_0)) \in T(V, S)$. This contradicts the hypothesis since $\tilde{g}(c_0)$ is the point $\mathbf{PT}_0g(C)$ of \mathbf{PT}_0X . This proves the first part of the lemma.

The second part is similar: let $\tilde{Z} \rightarrow (V \times C) \times S$ be the blow-up of $(V \times C) \times_X S$, let Γ be a smooth irreducible curve with a point $\gamma_0 \in \Gamma$ and let $\tilde{q} : \Gamma \rightarrow \tilde{Z}$ be a morphism such that $\tilde{q}(\gamma_0)$ is in the exceptional divisor above Δ'_S . Write $q = \alpha\tilde{q} = (q_1, q_2, q_3)$ and keep the same notation as above. Then $pq(\Gamma)$ is contained in the surface $pq_{13}(\Gamma) - g(C)$

This proves the lemma. ■

Proof of the theorem. We proceed by induction on the codimension of $f(V) - f(S)$. Assume $f(V) - f(S) = X$; if $T(V, S) \neq \mathbf{PT}_0X$, pick a point $u \notin T(V, S)$ and a smooth projective curve C' in X tangent to u at 0 , and such that the restriction induces an injection $\text{Pic}^0(X) \rightarrow \text{Pic}^0(C')$. Let C be a smooth curve with a connected ramified double cover $g: C \rightarrow C'$ unramified at a point c_0 above 0 ; the map $\text{Pic}^0(C') \rightarrow \text{Pic}^0(C)$ induced by g is injective.

Since p is surjective, C' generates X and C is smooth, th. 3.6 of [D1] implies that $(V \times S) \times_X C$ is connected. If $h: V \times C \rightarrow X$ is defined by $h(v, c) = f(v) - g(c)$, it follows that $(V \times C) \times_X S$ is also connected. On the other hand, the lemma implies that the set $\{(s, c_0, s) \mid s \in S\}$ is a connected component of, hence is equal to, $(V \times C) \times_X S$. It follows that $h^{-1}(f(S)) = S \times \{c_0\}$. Since $g^{-1}(0)$ consists of 2 distinct points, this is absurd, hence $T(V, S) = \mathbf{PT}_0X$.

Assume now $f(V) - f(S) \neq X$. Take a curve C' as above; by the lemma, the morphism $f': V \times C' \rightarrow X$ defined by $f'(v, c') = f(v) + c'$ is weakly unramified along $S \times \{0\}$, and $\dim T(V \times C', S \times \{0\}) \leq \dim T(V, S) + 1$. It follows from the induction hypothesis that

$$\dim T(V, S) \geq \dim(f(V) + C' - f(S)) - 2 = \dim(f(V) - f(S)) - 1,$$

which proves the theorem. ■

The following corollary provides a partial converse to prop. 1.3.

COROLLARY 3.3.— *Let X be an abelian variety of dimension n and let V be an irreducible projective variety of dimension d with a morphism $f: V \rightarrow X$ such that $f(V)$ is k -geometrically non-degenerate. Let V^0 be the open set of smooth points of V at which f is unramified. The restriction of the normal bundle to f to any complete irreducible subvariety S of V^0 is $(n - d - 1 + k)$ -ample.*

Proof. By prop. 1.1, we must show that for any hyperplane H in T_0X , any irreducible component S_H of $\{s \in S \mid f_*(T_v V) \subset H\}$ has dimension $\leq n - d - 1 - k$. But $T(V, S_H)$ is contained in H and the theorem gives $f(V) - f(S_H) \neq X$. Since f is unramified along S_H and $f(V)$ is k -geometrically non-degenerate, prop. 1.4 implies that $f(V) - f(S_H)$ has dimension $\geq d + \dim(S_H) - k$; this proves the corollary. ■

It should be noted that the corollary also follows from the main result of [Z2] (cor. 1), whose proof is unfortunately so sketchy (to say the least) that I could not understand it.

COROLLARY 3.4.— *Let X be an abelian variety and let V be an irreducible projective variety with a morphism $f: V \rightarrow X$. Let L be a linear subspace of T_0X and let S*

$\dim(f_*(T_s V) \cap L) < m$ for all $s \in S$, and let $\Delta_{f(S)}$ be the small diagonal in $f(S)^m$. Then

$$\dim(f(V)^m - \Delta_{f(S)}) < m \dim(X) - \dim(L) + m .$$

In particular, if $m \leq \dim(L)$ and $f(V)$ is k -geometrically non-degenerate,

$$m \dim(V) + \dim(S) < m \dim(X) - \dim(L) + m + k .$$

Proof. Let $r = \dim(L)$; the variety $N = \{ [t_1, \dots, t_m] \in \mathbf{P}(L^m) \mid t_1 \wedge \dots \wedge t_m = 0 \}$ has codimension $r - m + 1$ in $\mathbf{P}(L^m)$. Consider the morphism $f^m : V^m \rightarrow X^m$ and the subvariety Δ_S of V^m . The hypothesis imply that in $\mathbf{P}(T_0 X^m)$, the intersection of $T(V^m, \Delta_S)$ and $\mathbf{P}(L^m)$ is contained in N . It follows that

$$\begin{aligned} \dim T(V^m, \Delta_S) &\leq \dim(N) + \dim \mathbf{P}(T_0 X^m) - \dim \mathbf{P}(L^m) \\ &= \dim \mathbf{P}(T_0 X^m) - (r - m + 1) . \end{aligned}$$

The first inequality of the corollary follows from th. 3.1., and the second from prop.

1.4. ■

4. Applications to the Gauss map

We keep the same setting: X is an abelian variety and V an irreducible projective variety of dimension d with a morphism $f : V \rightarrow X$. Let V^0 be the open set of smooth points of V at which f is unramified; define the *Gauss map* $\gamma : V^0 \rightarrow G(d, T_0 X)$ by $\gamma(v) = f_*(T_v V)$. The following result was first proved by Ran ([R2]), and by Abramovich ([A]) in all characteristics.

PROPOSITION 4.1.— *Let X be an abelian variety and let V be an irreducible projective variety with a morphism $f : V \rightarrow X$. Assume that S is a complete irreducible variety contained in a fiber of the Gauss map; then $f(V) + \langle f(S) \rangle = f(V)$. In particular, the Gauss map of a smooth projective subvariety of X invariant by translation by no non-zero abelian subvariety of X is finite.*

Proof. Under the hypothesis of the proposition, $T(V, S)$ has dimension $\dim(V) - 1$; th. 3.1 implies $f(V) - f(S) = f(V)$, hence the proposition. ■

For any linear subspace L of $T_0 X$ and any integer $m \leq \dim(L)$, let $\Sigma_{L,m}$ be the Schubert variety $\{M \in G(d, T_0 X) \mid \dim(L \cap M) \geq m\}$; its codimension in $G(d, T_0 X)$ is $m(\text{codim}(L) - d + m)$.

PROPOSITION 4.2.— *Let X be an abelian variety and let V be an irreducible projective variety of dimension d with a morphism $f : V \rightarrow X$ such that $f(V)$ is k -geometrically non-degenerate. Let $\gamma : V^0 \rightarrow G(d, T_0 X)$ be the Gauss map, let L be a linear subspace of $T_0 X$ and let m be an integer $\leq \dim(L)$. Then any complete subvariety S of V^0 of dimension $\geq \text{codim } \Sigma_{L,m} + (m - 1)(\dim(L) - m) + k$ meets $\gamma^{-1}(\Sigma_{L,m})$.*

Proof. Apply cor. 3.4. ■

proposition); the proposition gives that for $m = 1$ or $\dim(L)$. The corresponding Schubert varieties are $\Sigma_{L,1} = \{M \in G(d, T_0X) \mid L \cap M \neq 0\}$ and $\Sigma_{L,\dim(L)} = \{M \in G(d, T_0X) \mid L \subset M\}$.

More generally, a result of Fulton and Lazarsfeld imposes strong restrictions on the image of the Gauss map of smooth subvarieties with ample normal bundle which I believe should also hold for geometrically non-degenerate subvarieties.

PROPOSITION 4.3.— *Let X be an abelian variety and let V be a smooth irreducible projective variety of dimension d with an unramified morphism $f : V \rightarrow X$ and Gauss map $\gamma : V \rightarrow G(d, T_0X)$. Assume that the normal bundle to f is ample; any subvariety S of $\gamma(V)$ meets any subvariety of $G(d, T_0X)$ of codimension $\leq \dim(S)$.*

Proof. If Q is the universal quotient bundle on $G(d, T_0X)$, the pull-back $\gamma^*(Q)$ is isomorphic to the normal bundle f^*TX/TV , hence is ample. It follows from [FL2] that for each Schubert variety Σ_λ of codimension m in $G(d, T_0X)$ and each irreducible subvariety S of V of dimension m , one has $\int_S \gamma^*[\Sigma_\lambda] > 0$. Now the class of any irreducible subvariety Z of $G(d, T_0X)$ of codimension m is a linear combination with non-negative coefficients (not all zero) of the Schubert classes; this implies $\int_S \gamma^*[Z] > 0$, hence $S \cap \gamma^{-1}(Z) \neq \emptyset$. ■

Regarding the Gauss map of a smooth subvariety of an abelian variety, Sommese and Van de Ven also proved in [SV] a strong result for higher homotopy groups of pull-backs of smooth subvarieties of the Grassmannian.

THEOREM (Sommese, Van de Ven).— *Let X be an abelian variety and let V be a smooth irreducible subvariety of X of dimension d with k -ample normal bundle and Gauss map $\gamma : V \rightarrow G(d, T_0X)$. For any smooth subvariety Z of $G(d, T_0X)$, the pull-back $\gamma^{-1}(Z)$ is non-empty for $\dim(V) \geq \text{codim}(Z) + k$, connected for $\dim(V) > \text{codim}(Z) + k$ and*

$$\gamma_* : \pi_j(V, \gamma^{-1}(Z)) \longrightarrow \pi_j(G(d, T_0X), Z)$$

is bijective for $j \leq d - \text{codim}(Z) - k$ and surjective for $j = d - \text{codim}(Z) - k + 1$.

My feeling is that all these results should still be valid for k -geometrically non-degenerate local complete intersections.

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