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Olivier Debarre

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# CODIMENSION 2 SUBVARIETIES OF ABELIAN VARIETIES

Olivier DEBARRE(\*)

Let  $(X, \theta)$  be a complex principally polarized abelian variety. We are interested in subvarieties of  $X$  whose class is  $d$  times the integral class  $\theta^2/2$ , where  $d$  is an integer. When  $(X, \theta)$  is *general*, any codimension 2 subvariety of  $X$  has this property ([Ma]).

The first question is the existence of such subvarieties with  $d$  *odd*. They can easily be constructed in Jacobians of curves, but to my knowledge, no other examples are known. On a general principally polarized abelian variety, this question can be reformulated in a more striking form: *is the integral class  $\theta^2/2$  the class of an algebraic cycle?* Although this question is not settled completely in this article, these subvarieties  $V$  are shown to be singular if  $\dim(X) \geq 8$ , and to satisfy  $\dim(\text{Sing}(V)) \geq \dim(X)/4$  for  $X$  general of dimension  $\geq 26$ . For  $V$  smooth, this is done by remarking that  $V$  is subcanonical; the Serre construction yields a rank 2 vector bundle on  $X$  with second Chern class  $[V]$ . We then prove that there are no topological complex rank 2 vector bundles  $E$  on a principally polarized abelian variety  $(X, \theta)$  of dimension  $\geq 8$  with second Chern class  $d\theta^2/2$ ,  $d$  odd, by showing that the expressions for  $\chi(X, E(-t\Theta))$ , as given by the Riemann-Roch theorem, cannot be all integers for  $t \in \mathbf{Z}$ . This is of course the way the Schwarzenberger conditions for vector bundles on the projective space are obtained ([H]).

The second question is whether there is an analog of Hartshorne's conjecture for codimension 2 subvarieties of abelian varieties. One has to make some adjustments here; first, in any abelian variety, there are codimension 2 subvarieties that are not complete intersections, because there are rank 2 vector bundles that do not split, such as the non-trivial extensions of a line bundle by itself. Call a codimension 2 subvariety an *almost complete intersection* if it is the zero set of a section of a rank 2 vector bundle which is the extension of a line bundle by another. Even then, one has to put additional restrictions on the ambient variety  $X$  or on the subvariety (if  $S$  is a simple abelian surface,  $Y$  is not an almost complete intersection in  $S \times Y$ ). A good place to start is to assume that the rank of the Néron-Severi group of  $X$  is 1. If  $(X, \theta)$  is a principally polarized abelian variety of dimension  $n \geq 7$  with this property, we prove that any smooth subvariety of  $X$  with class  $e\theta^2$ ,  $e \leq 3$ , is an almost complete intersection. The same holds for  $4 \leq e \leq 8$  and  $n \geq 11$ , if one assumes in addition that theta divisors are smooth.

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We gather in this section some results on abelian varieties. We refer to [S1] for the definition of  $k$ -ampleness, which will not be explicitly used here.

**THEOREM (Sommese).**— *Let  $X$  be an abelian variety of dimension  $n$  and let  $V$  be a smooth subvariety of  $X$  with  $k$ -ample normal bundle. If  $\dim(V) \geq (n + k + 2)/2$ , the restriction  $\text{Pic}(X) \rightarrow \text{Pic}(V)$  is an isomorphism.*

**Proof.** By [S2],  $\pi_q(X, V)$  vanishes for  $q \leq 3 \leq 2 \dim(V) - n - k + 1$ , hence also  $H^q(X, V; \mathbf{Z})$  by the relative Hurewicz theorem and the Universal Coefficient theorem. Since  $V$  is smooth, one also has  $H^q(X, \mathcal{O}_X) \simeq H^q(V, \mathcal{O}_V)$  for  $q \leq 2$  by Hodge theory, and the theorem follows from the exponential sequences for  $X$  and  $V$ . ■

When  $X$  is simple, any smooth subvariety of  $X$  of positive dimension has 0-ample (i.e. ample) normal bundle. The next theorem yields the integer  $k$  in Sommese's theorem for certain subvarieties.

**THEOREM 1.**— *Let  $(X, \ell)$  be a polarized abelian variety and let  $V$  be a subvariety of  $X$  with class a rational multiple of  $\ell^c$ . The restriction of the normal bundle to  $V$  in  $X$  to any complete subvariety of the set of smooth points of  $V$  is  $(c - 1)$ -ample.*

**Proof.** By [R], cor. II.2 and II.3,  $V$  is non-degenerate, hence geometrically non-degenerate. The theorem follows from [D], cor. 3.3. ■

Let  $W$  be a smooth codimension 2 subvariety of a smooth projective variety  $Y$ . Assume that  $W$  is *subcanonical*, i.e. that there exists a line bundle  $L$  on  $Y$  such that  $\omega_W \simeq L \otimes \mathcal{O}_W$ . The Serre construction yields a rank 2 vector bundle  $E$  on  $X$  with  $c_1(E) = c_1(L)$  and  $c_2(E) = [W]$ , a section  $\sigma$  of  $E$  with zero locus  $W$ , and an exact sequence

$$0 \longrightarrow \mathcal{O}_Y \xrightarrow{\sigma} E \longrightarrow \mathcal{I}_W \otimes L \longrightarrow 0 .$$

Let  $E$  be a rank 2 vector bundle on an abelian variety  $X$ . Writing formally  $c_1(E) = \lambda + \mu$  and  $c_2(E) = \lambda\mu$ , the Riemann-Roch theorem gives ([F], ex. 15.2.6, page 291):

$$\begin{aligned} \chi(X, E) &= p_n(c_1(E), c_2(E)) = (\lambda^n + \mu^n)/n! \\ &= \frac{1}{2^{n-1}n!} \sum_{0 \leq 2k \leq n} c_1(E)^{n-2k} (c_1(E)^2 - 4c_2(E))^k \binom{n}{2k} . \end{aligned}$$

## 2. Calculations

For any real number  $x$ , let  $[x]$  be the largest integer  $\leq x$  and let  $\lceil x \rceil$  be the smallest integer  $\geq x$

and  $\ell \in H^2(X, \mathbf{Z})$ , set  $\ell_j = \ell^j/j!$  and

$$\begin{aligned} p_n(\ell, d, t) &= p_n(\ell - 2t\theta, d\theta^2/2 - t\ell\theta + t^2\theta^2) \\ &= \frac{1}{2^{n-1}n!} \sum_{0 \leq 2k \leq n} (\ell - 2t\theta)^{n-2k} (\ell^2 - 2d\theta^2)^k \binom{n}{2k}. \end{aligned}$$

Note that if  $E$  is a rank 2 vector bundle on  $X$  with  $c_1(E) = \ell$  and  $c_2(E) = d\theta^2/2$ , then  $p_n(\ell, d, t) = \chi(X, E(-t\Theta))$ .

LEMMA 2.– Let  $s$  and  $t$  be non-negative integers such that  $2s \leq t$ ; set  $u = [t/2]$ . Then

$$\sum_{s \leq i \leq t/2} \binom{t}{2i} \binom{i}{s} = \binom{t}{2s} 2^{u-s} \rho_{s,t},$$

where

$$\rho_{s,t} = \frac{(u+1)(u+2)\cdots(2u-s)}{(2u-1)(2u-3)\cdots(2s+1)}$$

when  $t$  is odd, and

$$\rho_{s,t} = \frac{u(u+1)\cdots(2u-s-1)}{(2u-1)(2u-3)\cdots(2s+1)}$$

when  $t$  is even.

**Proof.** Maple gives

$$\sum_{s \leq i \leq t/2} \binom{t}{2i} \binom{i}{s} = 2^{t-1} \binom{t}{2s} \frac{\Gamma(t-s)\Gamma(s+\frac{1}{2})}{\Gamma(t)\Gamma(\frac{1}{2})},$$

which is equal to the expressions in the lemma. ■

PROPOSITION 3.– The following equality holds

$$p_n(\ell, d, t) = \sum_{\substack{0 \leq h \leq n \\ 0 \leq j \leq h/2}} (-1)^{h-j} t^{h-2j} 2^{-[(n-h)/2]-j+1} d^j \ell_{n-h} \theta_h \binom{h}{2j} \rho_{j,2j+n-h}.$$

**Proof.** One has

$$\begin{aligned} p_n(\ell, d, t) &= \frac{1}{2^{n-1}n!} \sum_{0 \leq 2k \leq n} (\ell - 2t\theta)^{n-2k} (\ell^2 - 2d\theta^2)^k \binom{n}{2k} \\ &= \frac{1}{2^{n-1}n!} \sum_{\substack{0 \leq 2k \leq n \\ 0 \leq i \leq n-2k}} \ell^{n-2k-i} (-2t\theta)^i \binom{n-2k}{i} \sum_{j=0}^k \ell^{2k-2j} (-2d\theta^2)^j \binom{k}{j} \binom{n}{2k} \\ &= \frac{1}{2^{n-1}n!} \sum_{\substack{0 \leq i \leq n-2k \\ 0 \leq j \leq k \leq n/2}} \ell^{n-i-2j} \theta^{i+2j} (-2t)^i (-2d)^j \binom{n-2k}{i} \binom{k}{j} \binom{n}{2k} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n!} \sum_{\substack{0 \leq h \leq n \\ 0 \leq j \leq h/2}} (-1)^{h-j} t^{h-2j} 2^{-j+1-\lceil (n-h)/2 \rceil} d^j \ell_{n-h} \theta_h \binom{h-2j}{j \leq k \leq j+(n-h)/2} \binom{2k}{2k} \binom{j}{j} \\
&= \sum_{\substack{0 \leq h \leq n \\ 0 \leq j \leq h/2}} (-1)^{h-j} t^{h-2j} 2^{-j+1-\lceil (n-h)/2 \rceil} d^j \ell_{n-h} \theta_h \binom{h}{2j} \rho_{j,2j+n-h} ,
\end{aligned}$$

which proves the proposition. ■

### 3. Subvarieties with class $d\theta^2/2$ , $d$ odd

Let  $(X, \theta)$  be a principally polarized abelian variety and let  $V$  be an irreducible subvariety of  $X$  with class  $d\theta^2/2$ , with  $d$  odd. Pick a very ample line bundle  $M$  on  $X$ . When  $s > \dim(\text{Sing}(V))$ , the intersection  $W$  of  $V$  with a general smooth complete intersection  $Y$  of  $s$  elements of  $|M|$  is smooth.

By th. 1, the restriction of the normal bundle  $N_{V/X}$  to  $W$  is 1-ample (0-ample if  $X$  is simple). The normal bundle  $N_{Y/X}$  is a direct sum of  $s$  copies of  $M|_Y$ , hence  $N_{W/X} \simeq (N_{V/X})|_W \oplus (M|_W)^{\oplus s}$  is 1-ample ([S1], cor. 1.10). By Sommese's theorem, the restriction  $\text{Pic}(X) \rightarrow \text{Pic}(W)$  is bijective for  $n - 2 - s = \dim(W) \geq (n + 3)/2$  ( $\geq (n + 2)/2$  if  $X$  is simple).

The restriction  $\text{Pic}(X) \rightarrow \text{Pic}(Y)$  is bijective for  $n - s = \dim(Y) \geq 3$  by Lefschetz's theorem; for  $s \leq (n - 7)/2$  ( $s \leq (n - 6)/2$  if  $X$  is simple), so is the restriction  $\text{Pic}(Y) \rightarrow \text{Pic}(W)$ , and  $W$  is *subcanonical in  $Y$* .

The exact sequence  $0 \rightarrow \text{TY} \rightarrow \text{TX}|_Y \rightarrow N_{Y/X} \rightarrow 0$  yields

$$td(\text{TY}) = td(N_{Y/X})^{-1} = \left( \frac{1 - e^{-m|_Y}}{m|_Y} \right)^s ,$$

where  $m = c_1(M)$ . Performing the Serre construction on  $Y$ , we get a rank 2 vector bundle  $E$  on  $Y$ , such that  $c_1(E) = \ell|_Y$ , for some  $\ell \in H^2(X, \mathbf{Z})$  (because of the isomorphism  $H^2(X, \mathbf{Z}) \simeq H^2(Y, \mathbf{Z})$ ), and  $c_2(E) = [W] = [V]|_Y$ . In particular,  $ch(E)$  is the restriction to  $Y$  of a class  $ch_X(E) \in H^\bullet(X, \mathbf{Z})$ . The Riemann-Roch theorem gives:

$$\chi(Y, E) = \int_Y td(\text{TY}) ch(E) = \left( \frac{1 - e^{-m}}{m} \right)^s ch_X(E) [Y] = \int_X (1 - e^{-m})^s ch_X(E) .$$

Note that  $e^{-m} = ch(M^{-1})$ , so this formula means that we can do as if  $E$  were the restriction to  $Y$  of a vector bundle  $E_X$  on  $X$  with Chern classes  $\ell$  and  $[V]$ :

$$\chi(Y, E) = \int_X \sum_{i=0}^s (-1)^i \binom{s}{i} ch(E_X \otimes M^{-i}) = \sum_{i=0}^s (-1)^i \binom{s}{i} \chi(X, E_X \otimes M^{-i}) .$$

In particular, if  $m = b\theta$ , the rational  $\Delta^s p_n(\ell, d, b) = \sum_{i=0}^s (-1)^i \binom{s}{i} p_n(\ell, d, bi)$  must be an integer.

Any subvariety of  $X$  with class  $d\theta^2/2$  with  $d$  odd is singular, and its singular locus has dimension  $\geq n - 4\lfloor \frac{n}{4} \rfloor$  for  $n \neq 10, 11$ , dimension  $\geq 1$  for  $n = 10$  and dimension  $\geq 2$  for  $n = 11$ .

*Remarks.* 1) In a generic  $n$ -dimensional Jacobian  $(JC, \theta)$ , the subvariety  $V = W_{n-2}(C)$  has class  $\theta^2/2$  and its singular locus has dimension  $n - 6$  for  $n \geq 6$ . For  $3 \leq n \leq 5$ ,  $V$  is smooth, but *not* subcanonical in  $X$ : one has  $c_1(\omega_V) = x + \theta$ , where  $x$  is the class of the divisor  $W_{n-3}(C)$  in  $V$  ([M]).

2) It is likely that Sommese's theorem holds for local complete intersections. In that case, the conclusion of the theorem would be that  $V$  is not a local complete intersection on a set of dimension  $\geq n - 4\lfloor \frac{n}{4} \rfloor$ .

3) Because of the Atiyah-Singer theorem ([H], th. 24.5.4), the proof actually shows that there are no topological complex rank 2 vector bundles  $E$  on a principally polarized abelian variety  $(X, \theta)$  of dimension 4, 5, or  $\geq 8$ , with second Chern class  $d\theta^2/2$ ,  $d$  odd. One can show that this method gives no restrictions for  $d$  even and  $c_1(E) = a\theta$ .

**Proof of the theorem.** Let  $R$  be the localization of  $\mathbf{Z}$  at the prime ideal  $(2)$ , viewed as a subring of  $\mathbf{Q}$ , and let  $2R$  be the maximal ideal of  $R$ .

Let  $V$  be a subvariety of  $X$  with class  $d\theta^2/2$ ; assume that  $\dim(\text{Sing}(V)) < s$ , where  $s = n - 4\lfloor \frac{n}{4} \rfloor$  (except for  $n = 10$ , where we take  $s = 1$ , and  $n = 11$ , where we take  $s = 2$ ). Since  $s \leq (n - 7)/2$ , the discussion above applies; we choose  $m = 3\theta$ .

The numbers  $\Delta^{s'} p_n(\ell, d) = \Delta^{s'} p_n(\ell, d, 3)$  must be integers for all  $s' \geq s$ . We will use the expression for  $p_n(\ell, d, t)$  given in prop. 3. For all  $h$ , the classes  $\ell_h$  and  $\theta_{n-h}$  are integral, hence their product is an integer. We consider four cases.

**Case  $n = 4m$ .**

The rational  $p_n(\ell, d, 0)$  must be an integer. In the sum given by prop. 3, one need only consider  $2j = h$ , for which the exponent of 2 is  $-2m + 1$ . Note that  $\rho_{j, 4m}$  is in  $2R$  when  $j < 2m$ . It follows that  $2^{2m-1} p_n(\ell, d, 0)$  is congruent to  $d^{2m} \rho_{2m, n} \theta_n = d^m$  modulo  $2R$ , hence that  $d$  must be even for  $m \geq 1$ .

**Case  $n = 4m + 1$ .**

The rational  $\Delta^1 p_n(\ell, d) = p_n(\ell, d, 0) - p_n(\ell, d, 3)$  must be an integer. In the sum given by prop. 3, one need only consider  $2j \leq h - 1$ . The exponent of 2 is then  $\geq -2m + 1$ , with equality if and only if either  $h = 2j + 2$  or  $h = 2j + 1$ . In the former case, the term in the sum is

$$(-1)^{j+1} 3^2 d^j 2^{-2m+1} \ell_{n-h} \theta_h \binom{2j+2}{2j} \rho_{j, 4m-1}.$$

For  $j < 2m - 1$ , it follows from lemma 2 that  $\rho_{j, 4m-1}$  is in  $2R$ . For  $j = 2m - 1$ ,  $\binom{2j+2}{2j} = 2m(4m - 1)$  is even.

$$(-1)^j 3d^j 2^{-2m+1} \ell_{n-h} \theta_h \binom{2j+1}{2j} \rho_{j,4m} .$$

For  $j < 2m$ , it follows from lemma 2 that  $\rho_{j,4m}$  is in  $2\mathbb{R}$ . For  $j = 2m$ , the term is  $3d^{2m} 2^{-2m+1} (4m+1) \theta_n$ .

It follows that  $2^{2m-1} p_n(\ell, d, 0)$  is congruent to  $3d^{2m} (4m+1)$  modulo  $2\mathbb{R}$ , hence that  $d$  must be even for  $m \geq 1$ .

**Case**  $n = 4m + 2$ .

The rational  $\Delta^2 p_n(\ell, d) = p_n(\ell, d, 0) - 2p_n(\ell, d, 3) + p_n(\ell, d, 6)$  must be an integer. In the sum given by prop. 3, one need only consider  $2j \leq h - 2$ . The exponent of 2 is then  $\geq -2m + 1$ , with equality if and only if either  $h = 2j + 3$  or  $h = 2j + 2$ . In the former case, the term in the sum is

$$(-1)^{j+1} (6^3 - 2 \cdot 3^3) d^j 2^{-2m+1} \ell_{n-h} \theta_h \binom{2j+3}{2j} \rho_{j,4m-1} .$$

For  $j < 2m - 1$ , it follows from lemma 2 that  $\rho_{j,4m-1}$  is in  $2\mathbb{R}$ . For  $j = 2m - 1$ ,  $\binom{2j+3}{2j} = 2m(4m-1)(4m+1)/3$  is even. The corresponding terms are therefore all in  $2^{-2m+3}\mathbb{R}$ .

When  $h = 2j + 2$ , the term in the sum is

$$(-1)^j (6^2 - 2 \cdot 3^2) d^j 2^{-2m+1} \ell_{n-h} \theta_h \binom{2j+2}{2j} \rho_{j,4m} .$$

For  $j < 2m$ , it follows from lemma 2 that  $\rho_{j,4m}$  is in  $2\mathbb{R}$ . For  $j = 2m$ , the term is  $18d^{2m} 2^{-2m+1} (2m+1)(4m+1) \theta_n$ .

It follows that  $2^{2m-2} \Delta^2 p_n(\ell, d)$  is congruent to  $9d^{2m} (2m+1)(4m+1)$  modulo  $2\mathbb{R}$ , hence that  $d$  must be even for  $m \geq 2$ .

**Case**  $n = 4m + 3$ .

The rational  $\Delta^3 p_n(\ell, d) = p_n(\ell, d, 0) - 3p_n(\ell, d, 3) + 3p_n(\ell, d, 6) - p_n(\ell, d, 9)$  must be an integer. In the sum given by prop. 3, one need only consider  $2j < h - 2$ . The exponent of 2 is then  $\geq -2m + 1$ , with equality if and only if either  $h = 2j + 4$  or  $h = 2j + 3$ . In the former case, the term in the sum is

$$(-1)^{j+1} (9^4 - 3 \cdot 6^4 + 3 \cdot 3^4) d^j 2^{-2m+1} \ell_{n-h} \theta_h \binom{2j+4}{2j} \rho_{j,4m-1} .$$

Since  $4 \mid 9^4 - 3 \cdot 6^4 + 3 \cdot 3^4$ , the corresponding terms are all in  $2^{-2m+3}\mathbb{R}$ .

When  $h = 2j + 3$ , the term in the sum is

$$(-1)^j (9^3 - 3 \cdot 6^3 + 3 \cdot 3^3) d^j 2^{-2m+1} \ell_{n-h} \theta_h \binom{2j+3}{2j} \rho_{j,4m} .$$

For  $j < 2m$ , it follows from lemma 2 that  $\rho_{j,4m}$  is in  $2\mathbb{R}$ . For  $j = 2m$ , the term is  $54d^{2m} 2^{-2m+1} (2m+1)(4m+1)(4m+3) \theta_n$ .

It follows that  $2^{2m-2} \Delta^3 p_n(\ell, d)$  is congruent to  $27d^{2m} (2m+1)(4m+1)(4m+3)$  modulo  $2\mathbb{R}$ , hence that  $d$  must be even for  $m \geq 2$ . ■

general with this method. However, if one makes the additional assumption  $\text{NS}(X) \simeq \mathbf{Z}\theta$ , one can write  $\ell = a\theta$ , where  $a$  is an integer. The intersection numbers  $\ell_{n-h}\theta_h = a^{n-h}\binom{n}{h}$  then have further divisibility properties.

PROPOSITION 5.— *Let  $(X, \theta)$  be a principally polarized abelian variety of dimension  $n$  with  $7 \leq n \leq 31$  and  $\text{NS}(X) \simeq \mathbf{Z}\theta$ . Any subvariety of  $X$  with class  $d\theta^2/2$  with  $d$  odd has a singular locus of dimension  $\geq s(n)$ , where the numbers  $s(n)$  are given in the following table*

$n$	$s(n)$	$n$	$s(n)$	$n$	$s(n)$	$n$	$s(n)$	$n$	$s(n)$
7	0	12	2	17	1 (*)	22	6	27	9
8	0 (*)	13	1 (*)	18	2	23	7	28	8
9	1 (*)	14	2 (*)	19	5	24	6	29	9
10	1 (*)	15	4	20	4	25	5	30	10
11	2 (*)	16	2	21	5	26	6	31	11

The numbers marked with (\*) are no better than the ones given in th. 4.

**Proof.** We proceed exactly as in th. 4: let  $V$  be a subvariety of  $X$  with class  $d\theta^2/2$ . If  $s > \dim(\text{Sing}(V))$  and  $s \leq (n-6)/2$  (note that  $X$  is simple), the numbers

$$\Delta^{s'} p_n(a\theta, d) = \sum_{i=0}^{s'} (-1)^i \binom{s'}{i} p_n(a\theta, d, 3i)$$

must be integers for all  $s' \geq s$ . The corresponding calculations were performed using Maple; I will only give three representative examples.

For  $n = 7$ ,  $4\Delta^1 p_7(a\theta, d)$  is congruent to  $d^3 + d^2(a^2 + a)$  modulo  $2\mathbf{R}$ . Since  $d$  is odd,  $\Delta^1 p_7(a\theta, d)$  cannot be an integer, hence  $\dim(\text{Sing}(V)) \geq 0$ .

For  $n = 12$ ,  $4\Delta^2 p_{12}(a\theta, d)$  is congruent to  $d^4 + d^5 + a^2 d^4$  modulo  $2\mathbf{R}$ , and  $2\Delta^3 p_{12}(a\theta, d)$  is congruent to  $d^4 + ad^4$  modulo  $2\mathbf{R}$ . Since  $d$  is odd,  $\Delta^2 p_{12}(a\theta, d)$  and  $\Delta^3 p_{12}(a\theta, d)$  cannot be both integers, hence  $\dim(\text{Sing}(V)) \geq 2$ .

For  $n = 21$ ,  $8\Delta^5 p_{21}(a\theta, d)$  is congruent to  $d^8$  modulo  $2\mathbf{R}$ . Since  $d$  is odd,  $\Delta^5 p_{21}(a\theta, d)$  cannot be an integer, hence  $\dim(\text{Sing}(V)) \geq 5$ .

The other cases are similar. ■

The three cases in the proof of the proposition correspond to three different patterns which emerge from the calculations for low values of  $n$ . The next lemma generalizes one of these patterns.



such that  $2^{a_{m,s}} \Delta^s p_{4m+s}(a\theta, d)$  is congruent to  $d^{2m}$  modulo  $2\mathbb{R}$ .

**Sketch of proof.** The proof consists in a careful (and lengthy) analysis of the terms of the sum

$$\Delta^s p_n(a\theta, d) = \sum_{\substack{0 \leq h \leq n \\ 0 \leq j \leq h/2}} \Delta_{h-2j}^s (-1)^{h-j} 3^{h-2j} 2^{-\lceil (n-h)/2 \rceil - j + 1} d^j a^{n-h} \binom{n}{h} \binom{h}{2j} \rho_{j, 2j+n-h},$$

where  $n = 4m + s$  and  $\Delta_k^s = \sum_{i=0}^s \binom{s}{i} (-1)^i i^k$ . One shows that  $\Delta_k^s = 0$  for  $k < s$ , that  $\Delta_s^s = (-1)^s s!$  and that

$$\Delta_{s+u}^s = (-1)^s (s+u)! sR_u(s),$$

for  $u > 0$ , where  $R_u$  is a polynomial of degree  $u - 1$ . Set  $T_u(X) = (u+1)!XR_u(X)$ ; then  $T_u(1) = 1$  and the following relation holds

$$T_u(X+1) - T_u(X) = 1 + \frac{1}{u+2} \sum_{0 < j < u} \binom{u+2}{j+1} T_j(X)$$

and implies that for all integers  $x$ , the number  $2^{a(u)}T_u(x)$  is in  $\mathbb{R}$ , where  $a(1) = 0$ ,  $a(2) = 1$  and  $a(u) = \lfloor u/2 \rfloor - 1$  for  $u \geq 3$ .

In the above sum, set  $u = h - 2j - s$ . Since  $\Delta_k^s = 0$  for  $k < s$ , the sum is over the set of  $u, j$  such that  $u \geq 0$ ,  $j \geq 0$  and  $2j + u \leq 4m$ . Using lemma 1, it is easy to see that among the non-zero terms corresponding to fixed  $u$ , the lowest valuation at 2 occurs for  $j$  maximal, i.e.  $j = 2m - \lfloor u/2 \rfloor$ , and that it occurs only there when  $4 \mid u$ . For  $u = 0$ , the term is the product of

$$A_0 = 2^{-2m+1} \frac{n!}{(4m)!}$$

and a unit of  $\mathbb{R}$ . For  $u > 0$ , it is the product of

$$sR_u(s) 2^{-2m + \lfloor u/2 \rfloor + 1} \frac{n!}{4m - u - \epsilon}$$

and a unit of  $\mathbb{R}$ , where  $\epsilon = 1$  if  $u$  is odd and 0 otherwise. The latter can be rewritten as the product of

$$A_0 T_u(s) 2^{\lfloor u/2 \rfloor} \binom{4m}{u + \epsilon}$$

and a unit of  $\mathbb{R}$ . Since  $2^{\lfloor u/2 \rfloor - 1} T_u(x)$  is in  $\mathbb{R}$  for  $u \geq 3$ , the valuation at 2 of the corresponding term is less than the one of  $A_0$ ; for  $u = 1$  and 2,  $2^{\lfloor u/2 \rfloor} T_u(x)$  is in  $\mathbb{R}$ , and  $\binom{4m}{u+\epsilon} = 2m(4m-1)$  is even, so that the valuation at 2 is still less than the one of  $A_0$ .

In sum, there is only one term with the lowest valuation at 2; it corresponds to  $u = 0$  and  $j = 2m$  and is equal to (up to a unit of  $\mathbb{R}$ )

$$2^{-2m+1} \frac{(4m+s)!}{(4m)!} d^{2m}.$$

The only thing to check is that this valuation  $-a_{m,s}$  is negative. This is where the inequality  $s \leq 2m - \log_2(6m-5)$  comes into play. The reader will check that it implies that  $2^{2m-1}$  does not divide  $(4m+s)!/(4m)!$ . This bound is not optimal. ■

divide  $(6m-3)!/(4m)!$ . This implies that in this range, the conclusion of the lemma holds for  $s \leq 2m-3$ .

**THEOREM 7.**— *Let  $(X, \theta)$  be a principally polarized abelian variety of dimension  $n \geq 26$  with  $\text{NS}(X) \simeq \mathbf{Z}\theta$ . Any subvariety of  $X$  with class  $d\theta^2/2$  with  $d$  odd has a singular locus of dimension  $\geq n/4$ .*

**Proof.** We proceed as before: let  $V$  be a subvariety of  $X$  with class  $d\theta^2/2$  whose singular locus has dimension  $< n/4$ . Set  $m = \lceil 3n/16 \rceil$  and  $s = n - 4m$ ; then  $s > \dim(\text{Sing}(V))$  and  $s \leq (n-6)/2$ , hence  $\Delta^s p_n(a\theta, d)$  must be an integer. By the lemma, this implies:

$$2m - \log_2(6m-4) < s = n - 4m \leq 4m/3.$$

It follows that  $m < 9$ . In that range, the remark gives  $n - 4m = s > 2m - 3$ , hence  $m \leq 4$  and  $n \leq 26$ . The case  $n = 26$  is covered by prop. 5. ■

### 3. Subvarieties with class $d\theta^2/2$ , $d$ even

Let  $r$  be a non-negative integer and let  $q_r(c_1, c_2)$  be the determinant of the  $r \times r$  matrix

$$\begin{pmatrix} c_1 & 1 & 0 & & \dots & 0 \\ c_2 & c_1 & 1 & 0 & & \dots & 0 \\ 0 & c_2 & c_1 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & c_2 & c_1 & 1 & 0 \\ 0 & \dots & & 0 & c_2 & c_1 & 1 \\ 0 & \dots & & & 0 & c_2 & c_1 \end{pmatrix}.$$

Writing formally  $c_1 = \lambda + \mu$  and  $c_2 = \lambda\mu$ , one checks easily that

$$q_r(c_1, c_2) = \frac{\lambda^{r+1} - \mu^{r+1}}{\lambda - \mu} = \frac{1}{2^r} \sum_{0 \leq 2k \leq r} c_1^{r-2k} (c_1^2 - 4c_2)^k \binom{r+1}{2k+1}.$$

After I had proved it, I found that the following lemma had already appeared in [Sc].

**LEMMA 8.**— *Let  $N$  be a vector bundle of rank 2 on an irreducible projective variety  $V$  of dimension  $m$ , generated by its global sections, and such that  $c_1(N) = a\ell$  and  $c_2(N) = e\ell^2$ , for some polarization  $\ell$  on  $V$ . Then  $a^2 \geq 4e \cos^2 \frac{\pi}{m+1}$ , and the inequality is strict if  $N$  is ample.*

**Proof.** Since  $N$  is generated by its global sections, the cycle classes  $q_r(a\ell, e\ell^2)$  on  $V$  are non-negative for  $0 \leq r \leq m$ , and positive if  $N$  is ample ([FL]). Write  $q_r(a\ell, e\ell^2) = q_r(a, e) \ell^r$ . We may assume  $a^2 < 4e$ ; setting  $u = \arccos\left(\frac{a}{2\sqrt{e}}\right)$ , the formula above yields

$q_r(a, e) = e^{r/2} \frac{\sin(r+1)u}{\sin u}$ . Since  $0 < u < \pi/2$ , the inequalities  $q_r(a, e) \geq 0$  (resp.  $> 0$ ) for  $0 \leq r \leq m$  are equivalent to  $(m+1)u \leq \pi$  (resp.  $< \pi$ ), hence the lemma. ■

a smooth subvariety of  $X$  with class  $e\theta^2$  and normal bundle  $N$ . Assume that there exists a line bundle  $L \sim a\theta$  on  $X$  such that  $\omega_V \simeq L|_V$ ; the Serre construction yields an exact sequence

$$0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{I}_V(L) \rightarrow 0 ,$$

where  $E$  is a vector bundle on  $X$  whose Chern classes are  $a\theta$  and  $e\theta^2$ . Since  $N$  is generated by its global sections, the lemma implies  $a^2 \geq 4e$  for  $n \geq 1 + 2\pi\sqrt{e/3}$ . Actually, the weaker inequality  $\cos \frac{\pi}{n-1} > \frac{[\sqrt{4e-1}]}{2\sqrt{e}}$  is enough and yields the following lower bounds for  $n$ :

$e$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n \geq$	5*	6*	8*	6	8	7	11	8	7	11	9	8*	14	10	9

When  $X$  is *simple*,  $N$  is ample, and the bounds marked with a \* in the table can be lowered by 1.

The next ingredient we want to use is Bogomolov's inequality for stable bundles. Recall that a rank 2 vector bundle  $E$  on  $X$  is  $\theta$ -stable (resp.  $\theta$ -semi-stable) if for all line bundles  $M \subset E$ , one has  $2c_1(M) \cdot \theta^{n-1} < c_1(E) \cdot \theta^{n-1}$  (resp.  $\leq$ ).

**PROPOSITION 9.**— *Let  $(X, \theta)$  be a principally polarized abelian variety with  $\text{NS}(X) \simeq \mathbf{Z}\theta$ , and let  $E$  be a vector bundle of rank 2 on  $X$  with  $c_1(E) = a\theta$  and  $c_2(E) = e\theta^2$ . If  $a^2 \geq 4e$ , there exist an integer  $b$  with  $b \geq a/2$  and  $b(a-b) \leq e$ , and a theta divisor  $\Theta$  such that  $E(-b\Theta)$  has a section whose zero set has codimension  $\geq 2$ .*

**Proof.** If  $a^2 > 4e$ , the vector bundle  $E$  is not  $\theta$ -semi-stable ([B], [MR], th. 4.3), hence there exist an integer  $b > a/2$  and a theta divisor  $\Theta$  such that  $H^0(X, E(-b\Theta)) \neq 0$ .

If  $a^2 = 4e$ , then  $a' = a/2$  is an integer and the Chern classes of the vector bundle  $E(-a'\Theta)$  all vanish. Since the fundamental group of  $X$  is abelian, it has no irreducible representations of dimension 2 and the vector bundle  $E(-a'\Theta)$  is not  $\theta$ -stable ([MR], th. 5.1). Hence there exist  $b \geq a'$  and a theta divisor  $\Theta$  such that  $H^0(X, E(-b\Theta)) \neq 0$ .

In either case, if  $b$  is maximal with this property, the zero sets of non-zero sections of  $E(-b\Theta)$  are empty or of codimension 2. It follows that

$$0 \leq c_2(E(-b\Theta)) \cdot \theta^{n-2} = (e - b(a-b))n! ,$$

hence the proposition. ■

Recall that *almost complete intersections* in  $X$  were defined as codimension 2 subvarieties that are zero sets of sections of a rank 2 vector bundles which are extensions

$$0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0 ,$$

get complete intersections. Assume  $L = M$ , and let  $e_E \in H^1(X, \mathcal{O}_X) \simeq \text{Ext}^1(L, L)$  be the class defining the extension. Let  $\sigma \in H^0(E)$  and let  $s$  be its image in  $H^0(L)$ . One has  $e_E \smile s = 0$  in  $H^1(X, L)$ ; the zero locus  $Z(\sigma)$  of  $\sigma$  is contained in the divisor  $Z(s)$  of  $s$  and is the zero locus of a section of  $L|_{Z(s)}$ . For any  $D \in H^0(X, \text{TX})$ , one can define a section  $Ds$  of  $L|_{Z(s)}$  ([BD]); there is an exact sequence

$$0 \longrightarrow H^0(X, L)/\mathbf{C}s \longrightarrow H^0(Z(s), L|_{Z(s)}) \xrightarrow{\partial} H^1(X, \mathcal{O}_X) \xrightarrow{\smile s} H^1(X, L)$$

and a local calculation shows that  $\partial(Ds) = 2i\pi(D \smile c_1(L))$  (*loc.cit.*). Any section of  $L|_{Z(s)}$  can be written as  $s' + Ds$ , for some  $s' \in H^0(X, L)$  and  $D \in H^0(X, \text{TX})$ , and  $Z(\sigma)$  can be described by the “equations”  $s = s' + Ds = 0$ , where  $D \smile c_1(L) \in \mathbf{C}e_E$ .

**THEOREM 10.**— *Let  $(X, \theta)$  be a principally polarized abelian variety of dimension  $n \geq 6$  with  $\text{NS}(X) \simeq \mathbf{Z}\theta$ , and let  $V$  be a smooth subvariety of  $X$  with class  $e\theta^2$ . If  $n \geq 1 + 2\pi\sqrt{e/3}$ , there exist integers  $a$  and  $c$ , a line bundle  $L \sim a\theta$  and a divisor  $D \sim c\theta$  on  $X$ , such that:*

- a)  $a \geq 2\sqrt{e}$ ,  $0 < c \leq a/2$  and  $c(a - c) \leq e$ ;
- b)  $\omega_V \simeq L|_V$ ;
- c)  $V \subset D$ .

*If moreover  $c(a - c) = e$ ,  $V$  is an almost complete intersection.*

For  $e \leq 15$ , the table above gives better bounds for  $n$ .

**Proof.** Since  $X$  is simple, the normal bundle of  $V$  is ample and since  $n \geq 6$ , Sommese’s theorem implies that  $V$  is subcanonical; since  $\text{NS}(X) \simeq \mathbf{Z}\theta$ , there exists a line bundle  $L \sim a\theta$  such that  $\omega_V \simeq L|_V$ . If  $n \geq 1 + 2\pi\sqrt{e/3}$ , lemma 8 yields  $a^2 \geq 4e$  and prop. 9 the existence of an integer  $b \geq a/2$  such that  $b(a - b) \leq e$ , and of a theta divisor  $\Theta$  such that  $E(-b\Theta)$  has a section  $\sigma$  whose zero set has codimension  $\geq 2$ . Set  $c = a - b$ ; the exact sequence from the Serre construction yields  $H^0(X, \mathcal{I}_V(L(-b\Theta))) \neq 0$ . In particular,  $c > 0$  and there exists a divisor  $D \in |L(-b\Theta)|$  that contains  $V$ . Finally, if  $e = c(a - c)$ , then  $c_2(E(-b\Theta)) = 0$  and  $\sigma$  vanishes nowhere. This implies that  $E$  is an extension of a line bundle by another. ■

**COROLLARY 11.**— *Let  $(X, \theta)$  be a principally polarized abelian variety of dimension  $n \geq 7$  with  $\text{NS}(X) \simeq \mathbf{Z}\theta$ , and let  $V$  be a smooth subvariety of  $X$  with class  $e\theta^2$ , where  $e \leq 8$ . If  $e \geq 4$ , assume that  $n \geq 11$  (or see better numbers in table above) and that theta divisors are non-singular in codimension 3; then  $V$  is an almost complete intersection.*

**Proof.** Keep the notation of the proposition and assume  $c(a - c) < e$ ; if  $c \geq 2$ , we get  $2a - 4 < e \leq a^2/4$ , which implies  $a \geq 6$  and contradicts the hypothesis  $e \leq 8$ . If  $c = 1$ , we get  $a - 1 < e \leq a^2/4$ , which implies  $e \geq a \geq 4$ . The theta divisor  $D$  is non-singular in codimension 3, hence locally factorial ([G]), and  $V$  is Cartier in  $D$ . Since the class of  $V$  is  $e\theta^2$ , it follows that  $\omega_V \sim (e + 1)\theta|_V$ , hence  $a = e + 1$ , contradiction. We have proved  $e = c(a - c)$ , hence the corollary. ■

$a = e = 4$ ,  $c = 1$ . There is a theta divisor  $\Theta$  such that  $H^0(X, E(-3\Theta)) \neq 0$  and a point  $x \in X$  such that  $L \simeq \mathcal{O}_X(3\Theta + \Theta_x)$ , so that  $D = \Theta_x$ . Serre's exact sequence yields  $h^0(X, E(-3\Theta)) = 1$ ; let  $\tau$  be a generator. Its zero set  $Z(\tau)$  has class  $\theta^2$ ,  $\sigma \wedge \tau = \theta_x$  and  $H^0(X, E) = \mathbf{C}\sigma \oplus H^0(X, 3\Theta)\tau$ , and  $\Theta_x$  contains both  $V$  and  $Z(\tau)$ . It might be possible to exclude this case even when theta divisors are singular.

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OLIVIER DEBARRE  
MATHÉMATIQUE – UNIVERSITÉ LOUIS PASTEUR  
7, rue René Descartes  
67084 STRASBOURG CEDEX – FRANCE  
e-mail: debarre@math.u-strasbg.fr