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HOCHSCHILD AND CYCLIC HOMOLOGY OF THE QUANTUM MULTIPARAMETRIC TORUS

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Nous calculons explicitement les groupes d'homologie de Hochschild du tore quantique multiparamétré en adaptant les *complexes de Koszul quantiques* construits antérieurement. Nous en déduisons les groupes d'homologie cyclique du tore quantique multiparamétré.

Abstract. — We explicitly compute the Hochschild homology groups of the quantum multiparametric torus, using previously constructed *quantum Koszul complexes*. We deduce the corresponding cyclic homology groups.

Keywords: Quantum algebras, quantum torus, Hochschild homology, cyclic homology, Koszul complexes.
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Introduction. — The *quantum multiparametric torus* is the algebra L_q generated over a field k by the $2N$ variables x_1, \dots, x_N and $x_1^{-1}, \dots, x_N^{-1}$ and the relations

$$x_i x_i^{-1} = 1 = x_i^{-1} x_i \quad \text{and} \quad x_i x_j = q_{ij} x_j x_i \quad (0.1)$$

for every $1 \leq i, j \leq N$ and where $\{q_{ij}\}_{1 \leq i, j \leq N}$ is a family of non-zero scalars of k satisfying the relations $q_{ii} = 1$ and $q_{ij} q_{ji} = 1$ for every $1 \leq i, j \leq N$. The name of quantum space comes from non-commutative geometry. The algebra L_q can actually be seen as a quantization of the algebra of the C^∞ functions over a N -dimensional torus cf. [R]. It also plays a universal role in the classification of certain Lie algebras cf. [BGK].

L. A. Takhtadjan [T] computed the Hochschild and cyclic homology of L_q in the case where all the scalars q_{ij} are equal for $i < j$. In [BGK], S. Berman, Y. Gao and Y. Krylyuk determined the cyclic homology group $HC_1(L_q)$. In this article, generalizing these results, we explicitly calculate the Hochschild and the cyclic homology of the algebra L_q .

For this purpose, we use results obtained in [W], where we calculated the homology of the *quantum multiparametric space* S_q (cf. Sec. 2). In the commutative case, in other words, when all the q_{ij} 's are equal to 1, the homology of the torus (*i.e.* of the Laurent

polynomials algebra) can immediately be deduced from the homology of the space (*i.e.* the polynomial algebra) by localization. In the non-commutatif case in which we are interested here, we can not localize *a priori*. Nevertheless, we shall show that the complexes which we used to compute the homology of S_q localize in a certain sense and may still serve to establish the homology groups $HH_*(L_q)$ and $HC_*(L_q)$.

1. Main results. — We state the two theorems which give explicitly the Hochschild and the cyclic homology of the quantum torus. Throughout the article, k will denote a zero characteristic field. We introduce the following notations.

Consider the set of all multi-integers $\alpha = (\alpha_i)_{i=1,\dots,N}$ of \mathbb{Z}^N endowed with its usual abelian group structure. For any $n \in \mathbb{Z}$, we denote by \bar{n} the multi-integer defined by $\bar{n}_i = n$ for every $i = 1, \dots, N$.

For any $\alpha \in \mathbb{Z}^N$, we set $x^\alpha = x_1^{\alpha_1} \dots x_N^{\alpha_N}$. It is clear that the set $\{x^\alpha\}_{\alpha \in \mathbb{Z}^N}$ is a basis of the algebra L_q . We denote by Z the subset of \mathbb{Z}^N formed by the multi-integers $\alpha \in \mathbb{Z}^N$ such that x^α belongs to the centre of L_q . In an equivalent way, by the commutation relations (0.1), we have

$$Z = \left\{ \alpha \in \mathbb{Z}^N \mid \prod_{k=1}^N q_{ik}^{\alpha_k} = 1 \text{ for all } i = 1, \dots, N \right\}.$$

The \mathbb{Z}^N -graduation of L_q induces a graduation on the homology groups of L_q . For any $\alpha \in \mathbb{Z}^N$, we denote by $HH_*(L_q)_\alpha$ (resp. by $HC_*(L_q)_\alpha$) the degree α component of the Hochschild homology (resp. the cyclic homology) groups of L_q . We have the following theorems.

THEOREM 1.1. — *The Hochschild homology of the quantum torus L_q is given by*

$$HH_*(L_q) = \bigoplus_{\alpha \in \mathbb{Z}^N} HH_*(L_q)_\alpha$$

where

$$\begin{cases} HH_n(L_q)_\alpha = 0 & \text{for } n \in \mathbb{N} \text{ when } \alpha \notin Z, \\ HH_n(L_q)_\alpha \cong k^{\binom{N}{n}} & \text{for } 0 \leq n \leq N \text{ when } \alpha \in Z, \\ HH_n(L_q)_\alpha = 0 & \text{for } n > N \text{ when } \alpha \in Z. \end{cases}$$

THEOREM 1.2. — *The cyclic homology of the quantum torus L_q is given by*

$$HC_*(L_q) = \bigoplus_{\alpha \in \mathbb{Z}^N} HC_*(L_q)_\alpha$$

where

$$\begin{aligned}
(a) \quad & HC_n(L_q)_\alpha = 0 \quad \text{for } n \in \mathbb{N} \text{ when } \alpha \notin Z, \\
(b) \quad & \begin{cases} HC_n(L_q)_{\bar{0}} \cong k^{\binom{N}{n}} \oplus k^{\binom{N}{n-2}} \oplus \dots & \text{for } 0 \leq n \leq N, \\ HC_n(L_q)_{\bar{0}} \cong k^{2^{N-1}} & \text{for } n > N \text{ and } n \text{ even,} \\ HC_n(L_q)_{\bar{0}} \cong k^{2^{N-1}-1} & \text{for } n > N \text{ and } n \text{ odd,} \end{cases} \\
(c) \quad & \begin{cases} HC_n(L_q)_\alpha \cong k^{\binom{N-1}{n}} & \text{for } 0 \leq n < N, \\ HC_n(L_q)_\alpha = 0 & \text{for } n \geq N, \end{cases} \quad \text{when } \alpha \in Z \setminus \{\bar{0}\}.
\end{aligned}$$

The proofs of Theorems 1.1 and 1.2 are detailed in the two following sections. For the definitions, the main properties of Hochschild and cyclic homology, and the notations, we refer to [L].

2. Hochschild homology of the quantum torus. — In this section, we prove Theorem 1.1. In order to do that, we use the results of [W] to construct a free resolution of L_q by L_q -bimodules in the form of a quantum Koszul complex. This resolution furnishes a “small” complex computing the Hochschild homology of L_q and from which we deduce the results of Theorem 1.1.

We need two further algebras. Let S_q (resp. Λ_q) be the algebra generated over k by the variables x_1, \dots, x_N and the relations $x_i x_j = q_{ij} x_j x_i$ (resp. the relations $x_i x_j = -q_{ij} x_j x_i$) for every $1 \leq i, j \leq N$. The algebra S_q is called the *multiparametric quantum space*. It is a deformation of the symmetric algebra $S(V)$ of a N -dimensional vector space V over k , whereas Λ_q is a deformation of the exterior algebra $\Lambda(V)$ of V . The family $\{x^\alpha\}_{\alpha \in \mathbb{N}^N}$ (resp. the family $\{x^\beta\}_{\beta \in \{0,1\}^N}$) forms a basis of S_q (resp. of Λ_q). These two algebras are naturally \mathbb{N} -graded, S_q is \mathbb{N}^N -graded and Λ_q is $\{0,1\}^N$ -graded. We denote by Λ_q^n the subspace spanned by the n -degree (for the \mathbb{N} -graduation) homogeneous elements Λ_q .

In [W], we constructed complexes, *the quantum Koszul complexes*, yielding resolutions for certain deformations of $S(V)$. The case of S_q was treated as an example. We recall the form of the quantum Koszul complex for the algebra S_q . It is the complex $K_*(S_q) = (S_q \otimes \Lambda_q^n \otimes S_q, d)$ where the differential d is defined on the homogeneous elements by:

$$\begin{aligned}
d(x^\alpha \otimes x_{i_1} \dots x_{i_n} \otimes x^{\alpha'}) &= \sum_{i=k}^n (-1)^{k+1} \prod_{s=1}^{k-1} q_{i_s i_k} x^\alpha x_{i_k} \otimes x_{i_1} \dots \widehat{x_{i_k}} \dots x_{i_n} \otimes x^{\alpha'} \\
&+ \sum_{i=k}^n (-1)^{n-k} \prod_{s=k+1}^n q_{i_k i_s} x^\alpha \otimes x_{i_1} \dots \widehat{x_{i_k}} \dots x_{i_n} \otimes x_{i_k} x^{\alpha'}
\end{aligned} \tag{2.1}$$

where $\alpha, \alpha' \in \mathbb{N}^N$, where $1 \leq i_1 < \dots < i_n \leq N$, and where $\widehat{x_{i_k}}$ means that the element x_{i_k} is omitted from the product of Λ_q . Proposition 4.1 of [W] states that the complex $K_*(S_q)$ is a free resolution of S_q by S_q -bimodules.

We will now construct a free resolution $K_*(L_q)$ of L_q by L_q -bimodules. We use the natural S_q -bimodule structure of S_q -bimodule of L_q to state

PROPOSITION 2.1. — *The complex $K_*(L_q) = L_q \otimes_{S_q} K_*(S_q) \otimes_{S_q} L_q$ is a free L_q -bimodule resolution of L_q .*

Proof. — The complex $K_*(L_q)$ may be identified with the complex $(L_q \otimes \Lambda_q^n \otimes L_q, d)$ where the differential d is still defined by Formula (2.1) on an homogeneous element $x^\alpha \otimes x_{i_1} \dots x_{i_n} \otimes x^{\alpha'}$ where $\alpha, \alpha' \in \mathbb{Z}^N$. This complex is clearly a complex of free L_q -bimodules. We shall prove that the homology group $H_n(K_*(L_q))$ is zero when $n > 0$ and is isomorphic to L_q when $n = 0$.

Let $\gamma \in \mathbb{Z}^N$. Consider the degree γ sub-complex $K_*(L_q)_\gamma$ of $K_*(L_q)$. It is spanned by all elements of the form $x^\alpha \otimes x^\beta \otimes x^{\alpha'}$ where $\alpha, \alpha' \in \mathbb{Z}^N$ and $\beta \in \{0, 1\}^N$, and where $\alpha + \beta + \alpha' = \gamma$. Furthermore, it is the union of the increasing sequence of the subcomplexes $K_*(L_q)_\gamma^r$ spanned by the elements $x^\alpha \otimes x^\beta \otimes x^{\alpha'}$ where, we have $\alpha_i, \alpha'_i \geq -r$ for any $i = 1, \dots, N$, and where r runs over \mathbb{N} .

Let us show that the group $H_n(K_*(L_q)_\gamma^r)$ is zero when $n > 0$ and is isomorphic to kx^γ when $n = 0$. To this end, we shall construct an isomorphism between $K_*(L_q)_\gamma^r$ and $K_*(S_q)_{\gamma+2\bar{r}}$. For any homogeneous element $x = x^\alpha \otimes x^\beta \otimes x^{\alpha'}$ where $\alpha_i, \alpha'_i \geq -r$ and $\alpha + \beta + \alpha' = \gamma$, set $\psi(x) = x^{\bar{r}} x^\alpha \otimes x^\beta \otimes x^{\alpha'} x^{\bar{r}}$. One checks that ψ induces an isomorphism $\psi : K_*(L_q)_\gamma^r \rightarrow K_*(S_q)_{\gamma+2\bar{r}}$. Since $K_*(S_q)$ is a resolution of S_q , we have

$$H_n(K_*(L_q)_\gamma^r) \cong H_n(K_*(S_q)_{\gamma+2\bar{r}}) = 0$$

for $n > 0$ and

$$H_0(K_*(L_q)_\gamma^r) \cong H_0(K_*(S_q)_{\gamma+2\bar{r}}) \cong kx^{\gamma+2\bar{r}}.$$

The first isomorphism of the last line is given by $H(\psi)(x^\gamma) = x^{\bar{r}} x^\gamma x^{\bar{r}}$. Hence $H_0(K_*(L_q)_\gamma^r) \cong kx^\gamma$.

Since the complex $K_*(L_q)_\gamma$ is the union of the complexes $K_*(L_q)_\gamma^r$, we obtain

$$H_*(K_*(L_q)_\gamma) = H_*\left(\varinjlim_r K_*(L_q)_\gamma^r\right) = \varinjlim_r H_*(K_*(L_q)_\gamma^r).$$

The homology groups are equal to 0 when $n \geq 1$ and isomorphic to kx^γ when $n = 0$. Observing that the homology of $K_*(L_q)$ is given by the sum $\bigoplus_{\gamma \in \mathbb{Z}^N} H_*(K_*(L_q)_\gamma)$, we deduce that $H_n(K_*(L_q)) = 0$ for $n > 0$ and that $H_0(K_*(L_q)) = L_q$. This proves that $K_*(L_q)$ is a resolution of L_q . \square

Proof of Theorem 1.1. — Proposition 2.1 implies that the Hochschild homology of L_q is isomorphic to the homology of the complex

$$\tilde{K}_*(L_q) = L_q \otimes_{L_q \otimes L_q^{\text{op}}} K_*(L_q) = (L_q \otimes \Lambda_q^*, \tilde{d})$$

where the differential \tilde{d} is given by

$$\tilde{d}(x^\alpha \otimes x^\beta) = \sum_{i=1}^N \Omega(\alpha, \beta, i) x^{\alpha+[i]} \otimes x^{\beta-[i]}$$

on an homogeneous element and where $[i]$ is the multi-integer of \mathbb{Z}^N defined by $x^{[i]} = x_i$ and where, for any $\alpha \in \mathbb{Z}^N$ and $\beta \in \{0, 1\}^N$, we set $\Omega(\alpha, \beta, i) = 0$ when $\beta_i = 0$ and

$$\Omega(\alpha, \beta, i) = (-1)^{\sum_{k=1}^{i-1} \beta_k} \left(\prod_{u < i} q_{ui}^{\beta_u} \prod_{v > i} q_{vi}^{\alpha_v} - \prod_{u > i} q_{iu}^{\beta_u} \prod_{v < i} q_{iv}^{\alpha_v} \right)$$

otherwise. Remark that $\Omega(\alpha, \beta, i) = 0$ for every $i = 1, \dots, N$ if and only if $\alpha + \beta \in Z$. This situation is similar to what occurs in the proof of Theorem 6.1 of [W].

Set $K = \tilde{K}_*(L_q)$. The complex K is still \mathbb{Z}^N -graded in a natural way. Let us first deal with the case when $\gamma \notin Z$. We shall prove that K_γ is acyclic in this case by producing an homotopy.

For any $\alpha \in \mathbb{Z}^N$, $\beta \in \{0, 1\}^N$ such that $\alpha + \beta \notin Z$ and $i = 1, \dots, N$, we set

$$\omega(\alpha, \beta, i) = \begin{cases} 0 & \text{when } \beta_i = 1 \\ 0 & \text{when } \Omega(\alpha - [i], \beta + [i], i) = 0 \\ \Omega(\alpha - [i], \beta + [i], i)^{-1} & \text{otherwise} \end{cases}$$

and

$$\tilde{h}(x^\alpha \otimes x^\beta) = \frac{1}{N} \sum_{i=1}^N \omega(\alpha, \beta, i) x^{\alpha-[i]} \otimes x^{\beta+[i]}.$$

The reader may check, with the help of the calculations of the proof of Theorem 6.1 of [W], that $\tilde{d}\tilde{h} + \tilde{h}\tilde{d} = \text{id}$. This proves that $HH_*(L_q)_\gamma = 0$ when $\gamma \notin Z$.

In the case where $\gamma \in Z$, we check that the differential \tilde{d} is zero on the subcomplexes K_γ . Hence, for any $\gamma \in Z$, we have $HH_*(L_q)_\gamma \cong \tilde{K}_*(L_q)_\gamma$. It is clear that, as a vector space, the complex $\tilde{K}_*(L_q)_\gamma$ admits $\{x^\gamma (x^\beta)^{-1} \otimes x^\beta\}_{\beta \in \{0, 1\}^N}$ as a basis. Therefore, $HH_n(L_q)_\gamma \cong k^{\binom{N}{n}}$ when $0 \leq n \leq N$ and $HH_n(L_q) = 0$ otherwise. This ends the proof of Theorem 1.1. \square

3. Cyclic homology of the quantum torus. — This section is devoted to the proof of Theorem 1.2. We will deduce the cyclic homology of L_q from its Hochschild homology. Let us recall that the cyclic homology of an algebra A is computed by a mixed complex $CC_*(A) = (A^{\otimes(*+1)}, b, B)$ cf. [L]. This complex is graded in a natural way when A is graded.

We will use Connes's long exact sequence. For any $\gamma \in \mathbb{Z}^N$, it gives rise to the exact sequence

$$\cdots \longrightarrow HH_n(L_q)_\gamma \xrightarrow{I} HC_n(L_q)_\gamma \xrightarrow{S} HC_{n-2}(L_q)_\gamma \xrightarrow{B} HH_{n-1}(L_q)_\gamma \longrightarrow \cdots \quad (3.1)$$

The following lemma is a generalization to the \mathbb{Z}^N -graded case of Proposition 2.4. of [V].

LEMMA 3.1. — *For any $\gamma \in \mathbb{Z}^N$, $\gamma \neq \bar{0}$, we have the exact sequence*

$$0 \longrightarrow HC_n(L_q)_\gamma \xrightarrow{B} HH_{n+1}(L_q)_\gamma \xrightarrow{I} HC_{n+1}(L_q)_\gamma \longrightarrow 0 \quad (3.2)$$

for $n \geq 0$.

Proof. — For any integer $k = 1, \dots, N$, let us define a derivation D_k on L_q by setting $D_k(x^\alpha) = \alpha_k x^\alpha$ for any homogeneous element $x^\alpha \in L_q$. As in [G, II.4], we consider the endomorphism L_{D_k} on the mixed complex $CC_*(L_q)$ defined by

$$L_{D_k}(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^n a_0 \otimes \cdots \otimes D_k a_i \otimes \cdots \otimes a_n$$

where $a_0, \dots, a_n \in L_q$ and $n \geq 1$. This endomorphism induces an operator, still denoted by L_{D_k} , on the cyclic homology groups $HC_*(L_q)$.

Let $\gamma \in \mathbb{Z}^N$. The restriction of L_{D_k} to the subcomplex $CC_\gamma(L_q)_\gamma$ generated by the degree γ homogeneous elements of $L_q^{\otimes(*+1)}$ is nothing else than multiplication by γ_k . When $\gamma \neq \bar{0}$, we may choose an integer ℓ such that $\gamma_\ell \neq 0$. The endomorphism L_{D_ℓ} is then an isomorphism and induces an isomorphism on $HH_*(L_q)_\gamma$. Corollary II.4.6 of [G] yields $L_{D_\ell} \circ S = 0$, hence $S = 0$. The exact sequence (3.2) is then an immediate consequence of Connes's exact sequence (3.1). \square

We prove another lemma. Call L the algebra of Laurent polynomials in N variables, in other words, the commutative algebra generated over k by the variables x_1, \dots, x_N and $x_1^{-1}, \dots, x_N^{-1}$ and the relations $x_i x_i^{-1} = 1$. The algebra L is \mathbb{Z}^N -graded in a natural way.

LEMMA 3.2. — *When $\gamma \in \mathbb{Z}^N$, the mixed complexes $CC_*(L)_\gamma$ and $CC_*(L_q)_\gamma$ are isomorphic.*

Proof. — The modules of the complex $CC_*(L_q)_\gamma$ are generated by the degree γ homogeneous elements of the tensor powers of L_q . We say that an homogeneous element x of degree γ of $L_q^{\otimes(n+1)}$ is normalized when it is of the form

$$x = x^\gamma (x^{\alpha^n})^{-1} \cdots (x^{\alpha^1})^{-1} \otimes x^{\alpha^1} \otimes \cdots \otimes x^{\alpha^n}$$

where $\alpha^1, \dots, \alpha^n \in \mathbb{Z}^N$. Every degree γ homogeneous element of $L_q^{\otimes(n+1)}$ is equal to a normalized homogeneous element up to multiplication by a scalar. We define the map $\psi : L_q^{\otimes(n+1)} \rightarrow L^{\otimes(n+1)}$ as the identity map on the normalized homogeneous elements. We shall prove that ψ is an isomorphism of mixed complexes, *i.e.*, ψ commutes with b and B . The boundary b splits in the sum $b = \sum_{i=0}^n (-1)^i b_i$. For any $i = 1, \dots, n$, we have $b_i \psi = \psi b_i$. Actually, remark that, for all $\alpha, \beta \in \mathbb{Z}^N$, Relations (0.1) imply the relation $(x^\beta)^{-1} (x^\alpha)^{-1} \otimes x^\alpha x^\beta = (x^{\alpha+\beta})^{-1} \otimes x^{\alpha+\beta}$ in $L_q^{\otimes 2}$. Hence, for $i < n$, on the normalized homogeneous element x , we have

$$\begin{aligned} b_i(x) &= x^\gamma (x^{\alpha^n})^{-1} \dots (x^{\alpha^1})^{-1} \otimes x^{\alpha^1} \otimes \dots \otimes x^{\alpha^i} x^{\alpha^{i+1}} \otimes \dots \otimes x^{\alpha^n} \\ &= x^\gamma (x^{\alpha^n})^{-1} \dots (x^{\alpha^i + \alpha^{i+1}})^{-1} \dots (x^{\alpha^1})^{-1} \otimes x^{\alpha^1} \otimes \dots \otimes x^{\alpha^i + \alpha^{i+1}} \otimes \dots \otimes x^{\alpha^n}. \end{aligned}$$

By definition, when $\gamma \in Z$, the element x^γ belongs to the centre of L_q . So, for b_n , we obtain

$$\begin{aligned} b_n(x) &= x^{\alpha^n} x^\gamma (x^{\alpha^n})^{-1} \dots (x^{\alpha^1})^{-1} \otimes x^{\alpha^1} \otimes \dots \otimes x^{\alpha^{n-1}} \\ &= x^\gamma (x^{\alpha^{n-1}})^{-1} \dots (x^{\alpha^1})^{-1} \otimes x^{\alpha^1} \otimes \dots \otimes x^{\alpha^{n-1}}. \end{aligned}$$

We have proved that $b\psi = \psi b$. A similar calculation holds for Connes's boundary B . Hence the map ψ is a morphism of mixed complexes. It is clearly an isomorphism. The inverse map is the map sending an homogeneous element of $L^{\otimes(n+1)}$ to the same element under its normalized form in $L_q^{\otimes(n+1)}$. \square

Proof of Theorem 1.2. — We distinguish two cases: the case where $\gamma \neq \bar{0}$ and the case where $\gamma = \bar{0}$. Consider the first one. Set $hh_n^\gamma = \dim(HH_n(L_q)_\gamma)$ and $hc_n^\gamma = \dim(HC_n(L_q)_\gamma)$. Since the Hochschild and cyclic homology groups are isomorphic at the zero level, one has $hc_0^\gamma = hh_0^\gamma$. Sequence (3.2) of Lemma 3.1 gives the recurrence relation $hc_{n+1}^\gamma = hh_{n+1}^\gamma - hc_n^\gamma$ for every $n \in \mathbb{N}$. When $\gamma \neq Z$, Theorem 1.1 yields $hh_n^\gamma = 0$ for every $n \in \mathbb{N}$. This proves Point (a) of Theorem 1.2, namely that $hc_n^\gamma = 0$ for every $n \in \mathbb{N}$. When $\gamma \in Z$, Theorem 1.1 implies $hh_n^\gamma = \binom{N}{n}$ when $0 \leq n \leq N$ and $hh_n^\gamma = 0$ otherwise. One then deduces, from the recurrence relation, that $hc_n^\gamma = \binom{N-1}{n}$ when $0 \leq n \leq N-1$ and $hc_n^\gamma = 0$ when $n \geq N$. This proves Point (c) of Theorem 1.2.

Consider now the case where $\gamma = \bar{0}$. According to Lemma 3.2, the problem reduces to a question in commutative algebra, namely to the computation of the cyclic homology of L in degree $\bar{0}$. Since the algebra L is smooth, its cyclic homology is the homology of the mixed complex $(\Omega_{L|k}^*, 0, \partial)$ where $\Omega_{L|k}^*$ is the algebra of the differential forms on L and ∂ the de Rham differential (for more details see [L]). The \mathbb{Z}^N -graduation of $CC_*(L)$ induces a \mathbb{Z}^N -graduation on $(\Omega_{L|k}^*, 0, \partial)$. As a vector space, $\Omega_{L|k}^*$ is isomorphic to $L \otimes \Lambda$ where Λ is the exterior algebra generated by the variables x_1, \dots, x_N . Denote by Λ^n the subspace spanned by its degree n homogeneous elements (for the \mathbb{N} -graduation). In degree $\bar{0}$, the differential ∂ is zero and we have

$$HC_n(L_q)_{\bar{0}} \cong HC_n(L)_{\bar{0}} \cong \Lambda^n \oplus \Lambda^{n-2} \oplus \dots$$

for every $n \in \mathbb{N}$. This proves Point (b) of Theorem 1.2. \square

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