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ON FINITE RESOLUTIONS OF $K(n)$ - LOCAL SPHERES

HANS-WERNER HENN

Abstract. For odd primes $p$ we construct finite resolutions of the trivial module $\mathbb{Z}_p$ for the $n$-th Morava stabilizer group by (direct summands of) permutation modules with respect to finite $p$-subgroups. Furthermore we discuss the problem of realizing these resolutions by finite resolutions of the $K(n)$-local sphere by spectra which are (direct summands of) wedges of homotopy fixed point spectra for the action of these finite $p$-subgroups on the Lubin-Tate spectrum $E_n$.

0. Introduction

Let $p$ be a prime and let $K(n)$ be the $n$-th Morava $K$-theory at $p$. The category of $K(n)$-local spectra is a basic building block of the stable homotopy category of $p$-local spectra and, of course, the localization of the sphere, $L_{K(n)}S^0$, plays a central role in this category. The homotopy of $L_{K(n)}S^0$ can be studied by the Adams-Novikov spectral sequence: if $E_n$ denotes the periodic Landweber exact spectrum $E_n$ whose coefficients in degree 0 classify deformations (in the sense of Lubin and Tate) of the Honda formal group law over $\mathbb{F}_p$, then, up to a Galois extension, the $E_2$-page of the spectral sequence can be identified, by the Morava change of rings isomorphism, with the continuous cohomology of the Morava stabilizer group $\mathbb{S}_n$ with coefficients in $(E_n)_*$.

In this paper we discuss homological properties of the groups $\mathbb{S}_n$, in particular resolutions of the trivial module $\mathbb{Z}_p$, and show how they can be used to construct finite resolutions of $L_{K(n)}S^0$ in terms of spectra which are easier to understand.

For example, if $p - 1$ does not divide $n$, then the mod-$p$ cohomological dimension $cd_p(\mathbb{S}_n)$ of $\mathbb{S}_n$ is finite, equal to $n^2$, the trivial module $\mathbb{Z}_p$ admits a projective resolution of length $n^2$ and the $E_2$-term of the Adams-Novikov spectral sequence has a horizontal vanishing line. In homotopy theory this allows to construct a finite $E_n$-resolution of $L_{K(n)}S^0$ in the sense of Miller [Mi] (at least up to a Galois extension in case $n$ is still divisible by $p$).

If $p - 1$ divides $n$, then the mod-$p$ cohomological dimension of $\mathbb{S}_n$ is infinite, so $\mathbb{Z}_p$ does not admit a finite projective resolution, the $E_2$-term has no horizontal vanishing line and a finite $E_n$-resolution for $L_{K(n)}S^0$ cannot exist. However, in analogy with discrete groups of finite virtual cohomological dimension one can hope

This paper is a sequel to the joint paper [GHMR1]. It is inspired by that paper and the author is happy to acknowledge the influence of numerous discussions with Paul Goerss, Mark Mahowald and Charles Rezk on this subject. Thanks are also due to the referee for his suggestions.

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to construct resolutions of the trivial module $\mathbb{Z}_p$ by permutation modules on finite subgroups of $\mathbb{S}_n$ and then hope to realize those by finite resolutions (which will not be $E_n$-resolutions) of $L_{K(n)}S^n$ via homotopy fixed point spectra of the form $E_n^{bf}$ for suitable finite subgroups of $\mathbb{S}_n$. This is in fact the main subject of this paper.

There are various advantages of such resolutions. For instance, such a finite resolution gives rise to a spectral sequence with a horizontal vanishing line starting from the homotopy of the homotopy fixed point spectra and converging to $\pi_*(L_{K(n)}S^n)$. This spectral sequence should be more manageable than the Adams-Novikov spectral sequence. For example, some of the delicate differentials in the latter might already be accounted for by more transparent phenomena in the homotopy of the homotopy fixed point spectra $E_n^{bf}$. In fact, this is essentially what happened in the calculation of the homotopy of the Toda-Smith complex $V(1)$ at the prime $p = 3$ carried out in [GHM] (completing earlier work of Shimomura [Sh]). The resolutions can also be used to analyze the exotic part of Hopkins’ Picard groups (cf. [HMS]), i.e., the group of homotopy equivalence classes of invertible spectra in the category of $K(n)$-local spectra whose Morava module is isomorphic to that of the sphere $S^n$. This will be pursued in a separate paper. On a more philosophical level, one can say that these resolutions capture to what extent the presence of finite $p$-subgroups in the stabilizer group influences the homotopy of $\pi_*(L_{K(n)}S^n)$, very much in the same way as finite subgroups in a discrete group of finite virtual cohomological dimension influence the cohomology of the group.

Here is an outline of the paper. In section 1 we recall background material on $K(n)$-localization, the stabilizer groups and homotopy fixed point spectra. Section 2 discusses algebraic and homotopy theoretic resolutions in the case $n \neq 0 \mod p-1$; there is a general existence result (Theorem 4) and beyond that we discuss the few cases $n = 1$ and $p > 2$ as well as $n = 2$ and $p > 3$ in which finite resolutions are known in an explicit form. The results in this section are mostly reformulations or reinterpretations of results which have been known for more than 25 years. They are included for completeness and in order to help develop our ideas on the interplay between homological properties of the groups $\mathbb{S}_n$ and homotopical properties of $L_{K(n)}S^n$. In section 3 we discuss the much more difficult case $n \equiv 0 \mod p - 1$. We start by explaining how the well-known fibration

$$L_{K(1)}S^n \to KOZ_2 \xrightarrow{\Psi^2-\id} KOZ_2$$

(where $KOZ_2$ is 2-adic real $K$-theory) can be regarded as the realization of a particular permutation resolution of the trivial $\mathbb{S}_1$-module $\mathbb{Z}_2$. Then we use this example as a role model which suggests possible generalizations for larger $n$. In section 3.3 we survey recent joint work with Goerss, Mahowald and Rezk in which the case $n = 2, p = 3$ was studied. In section 3.4 we comment on joint work in progress with the same coauthors which concerns the case $n = 2, p = 2$. In the final two sections we present new general results on the existence of finite resolutions of the trivial module $\mathbb{Z}_p$ by permutation modules, at least for $p > 2$ (Proposition 17). If $n = p - 1$, we show how these algebraic resolutions can be realized by finite resolutions of $L_{K(n)}S^n$ (Theorem 25 and Theorem 26).
1. Background

1.1. Localization with respect to Morava $K$-theory.

1.1.1. Let $E$ be a spectrum and $E_*$ be the generalized homology theory determined by $E$. We recall that Bousfield localization with respect to $E_*$ is a functor $L_E$ from the homotopy category of spectra to itself together with a natural transformation $\lambda : X \to L_EX$ which is terminal among all $E_*$-equivalences. By [B] $L_E$ exists for each $E$. Classical examples are given by localization with respect to the Moore spectra $MZ_{(p)}$, for the $p$-local integers $Z_{(p)}$, resp. $M\mathbb{Q}$, for the rationals, in which case $L_E$ is the homotopy theoretic version of arithmetic localization with respect to $Z_{(p)}$ resp. with respect to $\mathbb{Q}$.

1.1.2. In this paper we will be concerned with the localization functors $L_{K(n)}$ with respect to Morava $K$-theory $K(n)$. We refer to [HS] for a good general reference. Here we recall that $K(0) = H\mathbb{Q}$ is the rational Eilenberg-Mac Lane spectrum and is independent of $p$. Otherwise, for any fixed prime $p$ we have $K(n)_* = F_p[v_n^{\pm1}]$ with generator $v_n$ in degree $2(p^n - 1)$. Furthermore, $K(n)^*$ is a multiplicative periodic cohomology theory which admits a theory of characteristic classes, and the associated formal group law $\Gamma_n$ is the Honda formal group law of height $n$.

Localization with respect to $K(n)$ plays a prominent role in stable homotopy theory because the functors $L_{K(n)}$ are elementary “building blocks” of the stable homotopy category of finite $p$-local complexes in the following sense.

• The localization functor $L_{K(n)}$ is “simple” in the sense that the category of $K(n)$-local spectra contains no nontrivial localizing subcategory [HS, Theorem 7.5].

• There is a tower of localization functors

\[ \ldots \to L_n \to L_{n-1} \to \ldots \to L_0 \]

(with $L_n = L_{K(0) \vee \ldots \vee K(n)}$) together with natural transformations $id \to L_n$ such that

\[ X \simeq \text{holim}_n L_n X \]

for every finite $p$-local spectrum $X$ [Ra2, Theorem 7.5.7].

Furthermore, for every $X$ there is a homotopy pullback diagram (a “chromatic square”)

\[
\begin{array}{ccc}
L_n X & \longrightarrow & L_{K(n)} X \\
\downarrow & & \downarrow \\
L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X
\end{array}
\]

i.e. $L_n$ is built from $L_{K(n)}$ and $L_{n-1}$. (The diagram is easily established by using that $L_{K(n)} L_{n-1} Z \simeq *$ for any $Z$. Its existence is implicit in [Ho].)
1.2 The stabilizer groups.

1.2.1. The functors $L_{K(n)}$ are controlled by cohomological properties of the Morava stabilizer group $\mathbb{S}_n$. We recall that $\mathbb{S}_n$ is the group of automorphisms of the $p$-typical formal group law $\Gamma_n$ over the field $\mathbb{F}_q$ (with $q = p^n$) whose $[p]$-series is given by $[p](x) = x^{p^n}$. The group $\mathbb{S}_n$ can be extended to a slightly larger group $\mathbb{G}_n$. In fact because $\Gamma_n$ is already defined over $\mathbb{F}_p$, the Galois group $\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ of the finite field extension $\mathbb{F}_p \subset \mathbb{F}_q$ acts in a natural way on $\text{Aut}(\Gamma_n) = \mathbb{S}_n$ and $\mathbb{G}_n$ can be identified with the semidirect product $\mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. In the sequel we will recall some of the basic properties of the group $\mathbb{S}_n$ resp. $\mathbb{G}_n$. The reader is referred to [Ha] or [Ral] for more details (see also [He] for a summary of what will be important in this paper).

1.2.2. The group $\mathbb{S}_n$ can be identified with the group of units in the maximal order $\mathcal{O}_n$ of the division algebra $\mathbb{D}_n$ over $\mathbb{Q}_p$ of dimension $n^2$ and Hasse invariant $\frac{1}{n}$. In more concrete terms, this maximal order can be described as follows: let $\mathcal{W}_{\mathbb{F}_q}$ denote the Witt vectors over $\mathbb{F}_q$. Then $\mathcal{O}_n$ is the non-commutative ring extension of $\mathcal{W}_{\mathbb{F}_q}$ generated by an element $S$ which satisfies $S^n = p$ and $Sw = w^\sigma S$, where $w \in \mathcal{W}_{\mathbb{F}_q}$ and $w^\sigma$ is the image of $w$ under the lift of the Frobenius automorphism of $\mathbb{F}_q$. The element $S$ generates a two sided maximal ideal $m$ in $\mathcal{O}_n$ with quotient $\mathcal{O}_n/m = \mathbb{F}_q$. Inverting $p$ in $\mathcal{O}_n$ yields the division algebra $\mathbb{D}_n$, and $\mathcal{O}_n$ is its maximal order.

The action of the Galois group $\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ on $\mathbb{S}_n$ is realized by conjugation by $S$ inside the group $\mathbb{D}_n^\times$ and the semidirect product $\mathbb{G}_n$ can therefore be described as the quotient of $\mathbb{D}_n^\times$ by the central subgroup generated by $S^n$, i.e. $\mathbb{G}_n \cong \mathbb{D}_n^\times / < S^n >$.

1.2.3. Reduction mod $m$ induces an epimorphism $\mathcal{O}_n^\times \twoheadrightarrow \mathbb{F}_q^\times$. Its kernel will be denoted by $S_n$ and is also called the strict Morava stabilizer group. The group $S_n$ is equipped with a canonical filtration by subgroups $F_i S_n$, $i = \frac{k}{n}$, $k = 1, 2, \ldots$, defined by

$$F_i S_n := \{g \in S_n | g \equiv 1 \mod S^n \} .$$

The intersection of all these subgroups contains only the element 1 and $S_n$ is complete with respect to this filtration, i.e. we have $S_n = \lim_i S_n / F_i S_n$. Furthermore, we have canonical isomorphisms

$$F_i S_n / F_{i+1} S_n \cong \mathbb{F}_q$$

induced by

$$x = 1 + a S^n \mapsto \bar{a} .$$

Here $a$ is an element in $\mathcal{O}_n$, i.e. $x \in F_i S_n$ and $\bar{a}$ is the residue class of $a$ in $\mathcal{O}_n/m = \mathbb{F}_q$.

The associated graded object $gr S_n$ with $gr_i S_n = F_i S_n / F_{i+1} S_n$, $i = \frac{1}{n}, \frac{2}{n}, \ldots$ becomes a graded Lie algebra with Lie bracket $[\bar{a}, \bar{b}]$ induced by the commutator $xyx^{-1}y^{-1}$ in $S_n$. Furthermore, if we define a function $\varphi$ from the positive real numbers to itself by $\varphi(i) := \min\{i + 1, pi\}$ then the $p$-th power map on $S_n$ induces maps $P : gr_i S_n \rightarrow gr_{\varphi(i)} S_n$ which define on $gr S_n$ the structure of a mixed Lie algebra in the sense of Lazard [La, Chap. II.1]. If we identify the filtration quotients with $\mathbb{F}_q$ as above then the Lie bracket and the map $P$ are explicitly given as follows (cf. Lemma 3.1.4 in [He]).
Lemma 1. Let $\bar{\alpha} \in gr_i S_n$, $\bar{\beta} \in gr_j S_n$. Then

\[ [\bar{\alpha}, \bar{\beta}] = \bar{\alpha} \bar{\beta}^{\alpha_i} \bar{\beta}^{-\alpha_j} \in gr_{i+j} S_n. \]

\[ P \bar{\alpha} = \begin{cases} \bar{\alpha}^1 + p^{\alpha_i} + \ldots + p^{(p-1)\alpha_i} & \text{if } i < (p-1)^{-1} \\ \bar{\alpha} + \bar{\alpha}^1 + p^{\alpha_i} + \ldots + p^{(p-1)\alpha_i} & \text{if } i = (p-1)^{-1} \\ \bar{\alpha} & \text{if } i > (p-1)^{-1}. \end{cases} \]

1.2.4. The groups $S_n$ contain non-trivial finite $p$-subgroups if and only if $p - 1$ divides $n$, and all abelian finite subgroups are cyclic. If furthermore $p > 2$, or $p = 2$ and $n$ is odd, then all finite $p$-subgroups are cyclic, and $S_n$ contains a cyclic subgroup of order $p^k$ if and only if $p^{k-1}(p-1)$ divides $n$. Such a cyclic subgroup $C_{p^k} \subset S_n$ is unique up to conjugacy.

The centralizer $C_{S_n}(C_{p^k})$ of $C_{p^k}$ in $S_n$ is the group of units in the maximal order of the centralizer $C_{D_n}(C_{p^k})$ of $C_{p^k}$ in $D_n$, and $C_{D_n}(C_{p^k})$ is again a division algebra. It is central over the cyclotomic extension of $\mathbb{Q}_p$ generated by $C_{p^k}$ and its dimension over its center is $m^2$, if $n = mp^{k-1}(p-1)$.

1.2.5. Recall that the cohomological $p$-dimension $cd_p(G)$ of a profinite group $G$ is defined as

\[ cd_p(G) = \sup \{ n \in \mathbb{N} \mid H^n(G, M) \neq 0 \text{ for some finite continuous } G - \text{module } M \} \]

where, here and elsewhere in this paper, $H^n(G, M)$ is always continuous cohomology with coefficients in a continuous profinite module over the completed group algebra $\mathbb{Z}_p[[G]] := \lim_{\to} \mathbb{Z}_p[G/U]$ (with $U$ running through all open subgroups of $G$). We refer to [SyW] for a discussion of the relevant homological algebra.

The cohomological dimension of the group $S_n$ resp. $G_n$ is infinite iff $S_n$ resp. $G_n$ contains non-trivial finite $p$-subgroups. By 1.2.4 this happens for $S_n$ iff $p-1$ divides $n$, and in the case of $G_n$ this happens iff $p$ or $p-1$ divides $n$. However, even in these cases $S_n$ and $G_n$ are still virtually of finite cohomological $p$-dimension and the virtual cohomological $p$-dimension $vcd_p(S_n)$ remains $n^2$. The reader is referred to [La] or [SyW] for more details on these notions.

1.3 Homotopy fixed point spectra.

1.3.1. By Hopkins-Miller (cf. [Re]) the group $G_n^+$ acts on the Lubin-Tate spectrum $E_n$; we recall that $E_n$ is the Landweber exact spectrum given by the 2-periodic theory with coefficients $\pi_i(E_n) = \pi_0(E_n)[u^\pm 1]$ (with $u \in \pi_{-2}(E)$) whose associated formal group law over $\pi_0(E_n)$ is a universal deformation of $\Gamma_n$ in the sense of Lubin and Tate [LT]. In particular there is a (non-canonical) isomorphism between $\pi_0(E_n)$ and $\mathbb{W}_F[[u_1, \ldots, u_{n-1}]]$, the ring of formal power series over $\mathbb{W}_F$ in the variables $u_1, \ldots, u_{n-1}$. We can and will choose the universal deformation $\Gamma_n$ to be $p$-typical with $p$-series

\[ [p]_{\Gamma_n}(x) = px + \frac{1}{\Gamma_n} u_1 x^p + \frac{1}{\Gamma_n} u_{n-1} x^{p^{n-1}} + \frac{1}{\Gamma_n} x^{p^n}, \]
in other words the classifying map $BP_* \to \pi_*(E_n)$ sends the Araki generator $v_i$ to $u_1 u_1^{1-p_i}$, if $i < n$, $v_n$ to $u_1^{1-p_n}$, and $v_i$ to 0 if $i > n$.

By Devinatz-Hopkins [DH2] there is a homotopy fixed point spectrum $E_n^{hK}$ if $K$ a closed subgroup of $G_n$ with an associated homotopy fixed point spectral sequence

$$E_2^{s,t} = H^s(K; \pi_t(E_n)) \Rightarrow \pi_{t-s}(E_n^{hK}) .$$

Furthermore, $E_n^{hG_n}$ can be identified with $L_{K(n)}S^0$ and the Adams-Novikov spectral sequence for $L_{K(n)}S^0$ can be identified with the associated homotopy fixed point spectral sequence

$$E_2^{s,t} \cong H^s(G_n; (E_n)_t) \Rightarrow \pi_{t-s}L_{K(n)}S^0 .$$

In fact, the action of the Galois group $Gal(F_q/F_p)$ on $H^s(G_n, (E_n)_*)$ is free and therefore the $E_2$-term can be identified with the invariants $H^s(G_n, (E_n)_*)^{Gal(F_q/F_p)}$.

We can also consider the homotopy fixed point spectral sequence

$$E_2^{s,t} \cong H^s(G_n; (E_n)_t) \Rightarrow \pi_{t-s}E_n^{hG_n} .$$

There is an isomorphism $\pi_*E_n^{hG_n} \cong \pi_*L_{K(n)}S^0 \otimes \mathbb{Z}_p \mathbb{W}_F$, and we may therefore say that $E_n^{hG_n}$ is equal to $L_{K(n)}S^0$, up to a Galois extension.

1.3.2. Hopkins and Devinatz also showed that for any closed subgroup $K \subset G_n$ there is a natural isomorphism

$$\pi_*(L_{K(n)}(E_n \wedge E_n^{hK})) \cong map_{cts}(G_n/K, (E_n)_*)$$

(where $map_{cts}$ denotes continuous maps). The isomorphism is compatible with the obvious $(E_n)_*$-module structures on both sides. It is also compatible with the actions of $G_n$ on both sides, which is via the action on $E_n$ on the left hand side and via the diagonal action on the space of continuous maps on the right hand side. In other words, the isomorphism is one of Morava modules where a Morava module $M$ is a complete $(E_n)_*$-module with a continuous action of $G_n$ such that

$$g(ax) = g(a)g(x) \text{ for } g \in G_n, \ a \in (E_n)_*, \ x \in M .$$

By abuse of notation we will also say $M$ is an $(E_n)_*G_n$-module. Typical examples of such modules are given by $\pi_*(L_{K(n)}(E_n \wedge X))$, at least under suitable conditions on $X$, e.g. if $K(n)_*X$ is evenly graded (see [HS], or [GHMR1] for a summary of what is important for us). In order to keep our notation compact we will write in the sequel $(E_n)_*X$ instead of $\pi_*(L_{K(n)}(E_n \wedge X))$.

1.3.3. We will need information about maps between various homotopy fixed point spectra. In the following $F$ stands for function spectrum (which the reader may think of in the naive sense, i.e. as an object in the stable category). We recall the following results from [GHMR1].
1.3.3.1. Let $U$ be an open subgroup of $\mathbb{G}_n$. Functoriality of the homotopy fixed point spectra construction of [DH2] gives us a map $E_n^hU \land \mathbb{G}_n/U_+ \to E_n$ where as usual $\mathbb{G}_n/U_+$ denotes $\mathbb{G}_n/U$ with a disjoint base point added. Together with the product on $E_n$ we obtain a map

$$E_n \land E_n^hU \land \mathbb{G}_n/U_+ \to E_n$$

whose adjoint induces an equivalence

$$L_{K(n)}(E_n \land E_n^hU) \to \prod_{\mathbb{G}_n/U} E_n$$

realizing the isomorphism of 1.3.2 above. Similarly we get an equivalence

$$L_{K(n)}(E_n \land E_n \land E_n^{hU}) \to \prod_{\mathbb{G}_n/U} L_{K(n)}(E_n \land E_n) .$$

Now multiplication on $E_n$ defines obvious maps

$$F(E_n \land E_n^hU, E_n) \cong F(E_n \land E_n \land E_n^{hU}, E_n)$$

whose equalizer can be identified with $F(E_n^{hU}, E_n).$ \(^1\)

Because $E_n$ is $K(n)$-local we can replace the sources in our function spectra by their $K(n)$-localizations. In particular, we get equivalences

$$F(E_n \land E_n^hU, E_n) \cong F(L_{K(n)}(E_n \land E_n^hU), E_n) \cong F(E_n, E_n) \land \mathbb{G}_n/U_+$$

and then we can identify the equalizer with $E_n \land \mathbb{G}_n/U_+$, i.e. we end up with an equivalence

$$F(E_n^hU, E_n) \cong E_n \land \mathbb{G}_n/U_+ .$$

The same reasoning shows that we can replace $E_n$ in the target of the function spectrum by $L_{K(n)}(E_n \land I)$ where $I$ is any spectrum and we obtain an equivalence

$$F(E_n^{hU}, L_{K(n)}(E_n \land I)) \cong L_{K(n)}(E_n \land I) \land \mathbb{G}_n/U_+ .$$

1.3.3.2. More generally, let $K$ be any closed subgroup of $\mathbb{G}_n$. Then there exists a decreasing sequence $U_i$ of open subgroups $U_i$ with $K = \bigcap_i U_i$ and by [DH2] we have

$$E_n^{hK} \cong L_{K(n)} \hocolim_i E_n^{hU_i} .$$

By passing to the limit we obtain an equivalence

$$L_{K(n)}(E_n \land I)[[\mathbb{G}_n/K]] \cong F(E_n^{hK}, L_{K(n)}(E_n \land I))$$

where we have used the convention that if $E$ is a spectrum and $X = \lim_i X_i$ is an inverse limit of a sequence of finite sets with each $X_i$ finite then $E[[X]]$ is given as $\holim_i E \land (X_i)_+$, i.e. as the fibre of the usual self map of $\prod E \land (X_i)_+$. Note that if $X$ is such a profinite set with continuous $K$-action and if $E$ is a $K$-spectrum then $E[[X]]$ is a $K$-spectrum via the diagonal action.

If we concentrate (for simplicity) on the case $I = S^0$ and take homotopy fixed points in this equivalence with respect to another finite subgroup of $\mathbb{G}_n$ then we get the following result.

\(^1\)This can be either done in the stable category in which case we interpret the statement as saying that $F(E_n^{hU}, E_n)$ represents the equalizer of the two transformations between the functors $[-, F(E_n \land E_n^hU, E_n)]$ and $[-, F(E_n \land E_n \land E_n^{hU}, E_n)]$, or we can work in the category of $E_n$-modules in the sense of [EKMM] in which case the equalizer is to be interpreted on the point set level.
Proposition 2 ([GHMR1, Prop. 2.4]).

a) Let $K_1$ be a closed subgroup and $K_2$ a finite subgroup of $\mathbb{G}_n$. Then there is a natural equivalence (where the homotopy fixed points on the left hand side are formed with respect to the diagonal action of $K_2$)

$$E_n[[\mathbb{G}_n/K_1]]^{hK_2} \simeq F(E_n^{hK_1}, E_n^{hK_2}).$$

b) If $K_1$ is also an open subgroup then there is a natural decomposition

$$E_n[[\mathbb{G}_n/K_1]]^{hK_2} \simeq \prod_{K_2 \setminus \mathbb{G}_n/K_1} E_n^{hK_2},$$

where $K_x = K_2 \cap xK_1x^{-1}$ is the isotropy subgroup of the coset $xK_1$ and $K_2 \setminus \mathbb{G}_n/K_1$ is the finite (!) set of double cosets.

c) If $K_1$ is a closed subgroup and $K_1 = \bigcap_i U_i$ for a decreasing sequence of open subgroups $U_i$ then

$$F(E_n^{hK_1}, E_n^{hK_2}) \simeq \holim_i E_n[[\mathbb{G}_n/U_i]]^{hK_2} \simeq \holim_i \prod_{K_2 \setminus \mathbb{G}_n/U_i} E_n^{hK_{x,i}}.$$

where $K_{x,i} = K_2 \cap xU_i x^{-1}$ is, as before, the isotropy subgroup of the coset $xU_i$. □

Remark: If $U_i \subset U_j$ then the map

$$\prod_{K_2 \setminus \mathbb{G}_n/U_i} E_n^{hK_{x,i}} \to \prod_{K_2 \setminus \mathbb{G}_n/U_j} E_n^{hK_{x,j}}$$

in the inverse system of part (c) of the proposition can be described as follows: if $x \in \mathbb{G}_n/U_1$ has isotropy group $K_{x,i}$ and its image $x' \in \mathbb{G}_n/U_j$ has isotropy group $K_{x',j}$ then the restriction of the map to the factor determined by $x$ sends $E_n^{hK_{x,i}}$ via the transfer to the factor $E_n^{hK_{x',j}}$ determined by $x'$. In particular, this implies that homotopy groups the inverse system is Mittag-Leffler.

In the next result $\Hom(E_n)_{*[[\mathbb{G}_n]]}(-,-)$ denotes homomorphisms of Morava modules.

Proposition 3 ([GHMR1, Prop. 2.6]). Let $K_1$ and $K_2$ be closed subgroups of $\mathbb{G}_n$ and suppose that $K_2$ is finite. Then there is an isomorphism

$$((E_n)_{*[[\mathbb{G}_n/K_1]]})^{K_2} \cong \Hom(E_n)_{*[[\mathbb{G}_n]]}((E_n)_{*E_n^{hK_1}}, (E_n)_{*E_n^{hK_2}})$$

such that the following diagram commutes

$$\begin{array}{ccc}
\pi_* E_n[[\mathbb{G}_n/K_1]]^{hK_2} & \longrightarrow & ((E_n)_{*[[\mathbb{G}_n/K_1]]})^{K_2} \\
\downarrow \cong & & \downarrow \cong \\
\pi_* F(E_n^{hK_1}, E_n^{hK_2}) & \longrightarrow & \Hom(E_n)_{*[[\mathbb{G}_n]]}((E_n)_{*E_n^{hK_1}}, (E_n)_{*E_n^{hK_2}})
\end{array}$$
where the top horizontal map is the edge homomorphism in the homotopy fixed point spectral sequence, the left hand vertical map is the isomorphism given by Proposition 2 and the bottom horizontal map is the $E_n$-Hurewicz homomorphism. □

2. **The case $n \neq 0 \mod p - 1$**

In this section we begin our discussion of $L_{K(n)} S^0$. The case $n = 0$ is both exceptional and trivial: $K(0) = \mathbb{M} \mathbb{Q}$ and $L_{K(0)} S^0$ is the rationalized sphere spectrum. From now on we will therefore assume $n > 0$.

2.1 **Explicit examples I: the case $n = 1$ and $p > 2$.**

2.1.1. We briefly review the case $n = 1$ which is well understood. In this case we have $E_1 = K \mathbb{Z}_p$ ($p$-adic complex $K$-theory). The group $G_1 = S_1$ can be identified with $\mathbb{Z}_p^\times$, the group of units in the $p$-adic integers. If $p$ is odd then $\mathbb{Z}_p^\times \cong C_{p-1} \times \mathbb{Z}_p$, where $C_{p-1}$ denotes the cyclic group of order $p - 1$ given by the roots of unity in $\mathbb{Z}_p$. The homotopy fixed points $E_1^{hS^1}$ can be formed in two steps, first with respect to $C_{p-1}$ and then with respect to $\mathbb{Z}_p$. Thus we obtain the following fibration (cf. [HMS]) in which $\psi^{p+1}$ is the appropriate Adams operation and $K \mathbb{Z}_p^{hC_{p-1}}$ can be identified with the Adams summand of $p$-adic complex $K$-theory

$$L_{K(1)} S^0 \to K \mathbb{Z}_p^{hC_{p-1}} \xrightarrow{\psi^{p+1} - id} K \mathbb{Z}_p^{hC_{p-1}}.$$  

2.1.2. This fibration can also be considered as a suitable realization of a projective resolution of the trivial $\mathbb{Z}_p[[G_1]]$-module $\mathbb{Z}_p$, and it is this point of view which turns out to be useful for finding generalizations of the above fibration for larger $n$.

To get at this projective resolution we start with the following obvious short exact sequence of modules over the power series ring $\mathbb{Z}_p[[t]]$

$$0 \to \mathbb{Z}_p[[t]] \xrightarrow{\chi_1} \mathbb{Z}_p[[t]] \to \mathbb{Z}_p \to 0.$$

Now recall that there is an isomorphism of complete algebras $\mathbb{Z}_p[[t]] \cong \mathbb{Z}_p[[\mathbb{Z}_p]]$ induced by sending $t$ to $g - e \in \mathbb{Z}_p[[\mathbb{Z}_p]]$ if $g$ is a topological generator of $\mathbb{Z}_p$, e.g. if $g$ is the image of $p + 1 \in \mathbb{Z}_p^\times = G_1$ in the quotient group $G_1/C_{p-1} \cong \mathbb{Z}_p$. Therefore we can consider this sequence as an exact sequence of $\mathbb{Z}_p[[\mathbb{Z}_p]]$-modules, or even as an exact sequence of $\mathbb{Z}_p[[G_1]]$-modules, and as such we can write it as

$$0 \to \mathbb{Z}_p [G_1] \xrightarrow{e} \mathbb{Z}_p \to 0.$$

where $M [G_1]_{C_{p-1}}$ denotes the induced module of a $\mathbb{Z}_p[C_{p-1}]$-module $M$, i.e. the completed tensor product $\mathbb{Z}_p [G_1] \hat{\otimes}_{\mathbb{Z}_p[C_{p-1}]} M$. Because the trivial $\mathbb{Z}_p[C_{p-1}]$ module $\mathbb{Z}_p$ is projective we find that $\mathbb{Z}_p [G_1]_{C_{p-1}}$ is a projective $\mathbb{Z}_p[[G_1]]$-module and the
exact sequence is a projective resolution of the trivial \( \mathbb{Z}_p[[G]] \)-module \( \mathbb{Z}_p \) which is even split as a sequence of continuous \( \mathbb{Z}_p \)-modules. So if we apply the functor \( \text{Hom}_{\text{cts}}(\cdot, (K \mathbb{Z}_p)_*) \) of continuous homomorphisms into \((K \mathbb{Z}_p)_*\) to this sequence then we obtain another short exact sequence which, by 1.3.2, can be identified with the (a priori long) exact sequence which is associated to our fibration:

\[
0 \to (K \mathbb{Z}_p)_*(L_{K(1)}S^0) \cong (K \mathbb{Z}_p)_*, K\mathbb{Z}_p^{hC_p^{I-1}} \to (K \mathbb{Z}_p)_*, K\mathbb{Z}_p^{hC_p^{I-1}} \to 0.
\]

2.2 The general case.

2.2.1. Following Miller [Mi] we say that a \( K(n) \)-local spectrum \( I \) is \( E_n \)-injective if the map \( I = S \wedge I \to L_{K(n)}(E_n \wedge I) \) induced by the unit in \( E_n \) is split. Furthermore, a sequence \( X' \to X \to X'' \) of \( K(n) \)-local spectra is \( E_n \)-exact if the composition \( X' \to X'' \) is trivial and if

\[
[X', I] \leftarrow [X, I] \leftarrow [X'', I]
\]

is an exact sequence of abelian groups for each \( E_n \)-injective spectrum \( I \). Finally an \( E_n \)-resolution of a \( K(n) \)-local spectrum \( X \) is an \( E_n \)-exact sequence of \( K(n) \)-local spectra

\[
* \to X \to I^0 \to I^1 \to \ldots
\]

(i.e. every three term subsequence is \( E_n \)-exact) such that each \( I^s \), \( s \geq 0 \), is \( E_n \)-injective.

2.2.2. The following result is a folk theorem whose roots can be traced back to the work of Morava [Mo].

**Theorem 4.** If \( n \) is neither divisible by \( p - 1 \) nor by \( p \) then \( L_{K(n)}S^0 \) admits an \( E_n \)-resolution of length \( n^2 \). In fact, each of the \( E_n \)-injectives in the resolution can be chosen to be a direct summand of a finite wedge of \( E_n \)'s.

**Proof.** The idea of the proof is essentially to reverse the discussion in section 3.1, i.e. start with information in homological algebra and try to use this to construct an \( E_n \)-resolution. If neither \( p - 1 \) nor \( p \) divides \( n \) then \( \text{cd}_p(G_n) = n^2 \). Therefore the trivial \( \mathbb{Z}_p[[G]] \)-module \( \mathbb{Z}_p \) admits a projective resolution

\[
P_*: 0 \to P_{n+1} \to \ldots \to P_0 \to \mathbb{Z}_p \to 0
\]

of length \( n^2 \), and we may assume that each projective is finitely generated. We want to construct an \( E_n \)-resolution \( X_* \) of \( L_{K(n)}S^0 \)

\[
X_*: * \to X_{-1} \to X_0 \to \ldots \to X_{n+1} \to *
\]

with \( X_{-1} = L_{K(n)}S^0 \) such that the complex \( \text{Hom}_{\text{cts}}(P_*(E_n)_*) \) is isomorphic, as a complex of Morava modules, to the complex \((E_n)_*X_*\).
For this we note that if $F^r$ is a free $\mathbb{Z}_n[[G_n]]$-module of rank $r$ then we have an isomorphism of Morava modules

\[ \text{Hom}_{cts}(F^r, (E_n)_\ast) \cong (E_n)_\ast \left( \bigvee_{i=1}^{r} E_n \right) \]

and the $E_n$-Hurewicz homomorphism

\[ [E_n, E_n] \to \text{Hom}_{(E_n)_\ast}[[G_n]]((E_n)_\ast, (E_n)_\ast E_n) \]

is an isomorphism. In fact, (1) is a special case of 1.3.2 and (2) is a special case of Proposition 3.

Property (2) allows us now to construct the spectra $X_i$ (by lifting idempotents on finitely generated free $\mathbb{Z}_p[[G_n]]$-modules to homotopy idempotents on corresponding wedges of $E_n$'s) as well as the required maps between these spectra. Because $E_n$ is $K(n)$-local and $E_n$ is a ring spectrum, the spectra $X_i$ will all be $E_n$-injective. It remains to show that the sequence is $E_n$-exact.

For this it is enough to show that for any spectrum $Z$ of the form $Z := L_{K(n)}(E_n \wedge I)$ with some spectrum $I$ the complex $[X_{\bullet}, Z]$ is exact. By our discussion in 1.3.3.1 we know that

\[ [E_n, Z] \cong \lim_i \pi_0 Z \otimes \mathbb{Z}_p[[G_n/U_i]] \]
\[ \cong \lim_i \text{Hom}_{(E_n)_0}(\text{maps}(G_n/U_i, (E_n)_0), \pi_0 Z) \]
\[ \cong \text{Hom}_{(E_n)_0}(\text{colim}_i \text{maps}(G_n/U_i, (E_n)_0), \pi_0 Z) \]
\[ \cong \text{Hom}_{(E_n)_0}(\text{maps}_{cts}(G_n, (E_n)_0), \pi_0 Z) \]
\[ \cong \text{Hom}_{(E_n)_0}((E_n)_0 E_n, \pi_0 Z) \]

More generally, if $X_k$ is a direct summand in a finite wedge $\bigvee_{i=1}^{r} E_n$, then $[E_n, Z] \cong \text{Hom}_{(E_n)_0}(\bigvee_{i=1}^{r} (E_n)_0 X_k, \pi_0 Z)$ and we obtain an isomorphism of complexes

\[ [X_{\bullet}, Z] \cong \text{Hom}_{(E_n)_0}((E_n)_0 X_{\bullet}, \pi_0 Z) \]

Finally we observe that $(E_n)_0 X_{\bullet} \cong \text{Hom}_{cts}(P_{\bullet}, (E_n)_0)$ and because $P_{\bullet}$ is split as a complex of continuous $\mathbb{Z}_p$-modules, this is split as a complex of $(E_n)_0$-modules. Therefore $[X_{\bullet}, Z]$ is exact, i.e. the sequence $X_{\bullet}$ is $E_n$-exact. \hfill \Box

**Remark 1** This result is a pure existence result. It says nothing about an explicit form of such a resolution.

**Remark 2** If $n$ is divisible by $p$ (but not by $p-1$) we can offer the two following substitutes of Theorem 4.

Either we can use the existence of a finite projective resolution of the trivial $\mathbb{Z}_p[[G_n]]$-module $\mathbb{Z}_p$ to get one for the induced $\mathbb{Z}_p[[G_n]]$-module $\mathbb{Z}_p[\mathbb{G}_n^n]$. This resolution can then be realized as in the proof of Theorem 4 to give an $E_n$-resolution for $E_n^h\mathbb{G}_n^n$. For this 1.3.1 is, up to a Galois extension, equal to $L_{K(n)} S^0$.

Or, if we insist on a resolution of $L_{K(n)} S^0$, we can consider the formal group $\Gamma_n$ over the algebraic extension $K := \bigcup_{r \geq 0} F_{q^r}$ of $F_q$ with Galois group $\text{Gal}(K, F_q) \cong \mathbb{Z}_p$. 

This will have the effect of replacing $G_n = S_n \rtimes \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ by the group $G_n(K) := S_n \rtimes \text{Gal}(K/\mathbb{F}_p)$. The advantage of doing this is that while $G_n$ has elements of order $p$ and therefore infinite mod-$p$ cohomological dimension, the group $G_n(K)$ has no elements of order $p$ and its mod-$p$ cohomological dimension is finite, equal to $n^2 + 1$. As a consequence one gets a projective resolution of the trivial $G_n(K)$-module $\mathbb{Z}_p$ of length $n^2 + 1$. If we also replace $E_n$ by the corresponding Lubin-Tate spectrum $E_n(K)$ whose homotopy groups in degree 0 classify deformations of $\Gamma_n$ over $K$ then our proof carries over verbatim: we only need to remark that 1.3.2 and the two properties (1) and (2) in the proof of Theorem 4 hold with $E_n$ and $G_n$ replaced by $E_n(K)$ and $G_n(K)$.

2.3. Explicit examples II: the case $n = 2$ and $p > 3$.

As before we let $q = p^n$. We start with an observation valid for all $n > 1$ (cf. section 1.3 of [GHMR1]). The reduced norm $S_n \to \mathbb{Z}_p^\times$ admits a canonical extension $G_n \to \mathbb{Z}_p^\times \rtimes \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and by composing with the evident projection we obtain a homomorphism $G_n \to \mathbb{Z}_p^\times / C_{p-1}$ where $C_{p-1}$ is the cyclic subgroup of $\mathbb{Z}_p$ of order $p - 1$. The kernel of this homomorphism will be denoted $G_n^1$ and the kernel of its restriction to $S_n$ will be denoted $S_n^1$. $G_n^1$ contains a cyclic subgroup of order $q - 1$ (the roots of unity in $\mathbb{W}_{F_q} \subset \mathcal{O}_n$) and this subgroup is invariant with respect of the action of $\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. Therefore $G_n^1$ contains the semidirect product $C_q \rtimes \text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$. We will denote this finite subgroup by $F_{\lambda(q-1)}$ in the sequel.

For more information on $\pi_*(E_n^{hF_{\lambda(q-1)}})$ we refer to the appendix.

**Theorem 5.** Assume $n = 2$ and $p > 3$.

a) There exists a fibration

$$L_{K(2)}S^0 \to E_2^{hG_2^1} \to E_2^{hG_2^1}$$

and an $E_n$-resolution

$$\ast \to E_2^{hG_2^1} \to E_2^{hF_{\lambda(q-1)}} \to X \to E_2^{hF_{\lambda(q-1)}} \to \ast$$

where $X \simeq \Sigma^2(q-1)E_2^{hF_{\lambda(q-1)}} \vee \Sigma^2(p^2-1)E_2^{hF_{\lambda(q-1)}}$.

b) There exists an $E_n$-resolution of the form

$$\ast \to L_{K(2)}S^0 \simeq E_2^{hG_2} \to E_2^{hF_{\lambda(q-1)}} \to E_2^{hF_{\lambda(q-1)}} \vee X \to \vee X \to E_2^{hF_{\lambda(q-1)}} \to E_2^{hF_{\lambda(q-1)}} \to \ast$$

Given Theorem 4, this is a fairly straightforward consequence of the following purely algebraic result in which $\lambda_{p-1}$ denotes the $\mathbb{Z}_p[F_{2(q-1)}]$-module whose underlying $\mathbb{Z}_p$-module is $\mathbb{W}_{\mathbb{F}_q}$, on which $C_{q-1}$ acts via

$$C_{q-1} \times \mathbb{W}_{\mathbb{F}_q} \to \mathbb{W}_{\mathbb{F}_q}, \ (u, w) \mapsto u^{p-1}w$$

and on which the group $\text{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ acts via the lift of Frobenius (cf. appendix). The algebraic result is in turn a consequence of the calculation of the cohomology of the relevant Morava stabilizer algebra in [Ra1, Theorem 6.3.22]. Proposition 7 below is the group theoretic version of Ravenel’s result.

---

1 I would like to thank Ethan Devinatz for a reassuring discussion of this point.
Theorem 6. Assume $n = 2$ and $p > 3$.
a) There exists a short exact sequence of $\mathbb{Z}_p[[G_2]]$-modules

$$0 \to \mathbb{Z}_p \xrightarrow{G_2^1} \mathbb{Z}_p \xrightarrow{G_2^1} \mathbb{Z}_p \to 0$$

and a projective resolution of the trivial $\mathbb{Z}_p[[G_2]]$-module $\mathbb{Z}_p$

$$0 \to \mathbb{Z}_p \xrightarrow{G_2^1} \mathbb{Z}_p \xrightarrow{G_2^1} \mathbb{Z}_p \to 0.$$  

b) There exists a projective resolution of the trivial $\mathbb{Z}_p[[G_2]]$-module of the form

$$0 \to \mathbb{Z}_p \xrightarrow{G_2^1} \mathbb{Z}_p \xrightarrow{G_2^1} \mathbb{Z}_p \to 0.$$  

Proposition 7. Assume $n = 2$ and $p > 3$.
a) $H^*(S^1_2; F_p)$ is a 3-dimensional Poincaré duality algebra and

$$H^3(S^1_2; F_p) \cong F_p$$

is trivial as module over $F_{2(q-1)}$.

b) There is a canonical isomorphism of $\mathbb{Z}_p[F_{2(q-1)}]$-modules

$$H^1(S^1_2; F_p) \cong \lambda_{p-1} \otimes \mathbb{Z}_p F_p.$$  

c) The Bockstein homomorphism induces an isomorphism of $\mathbb{Z}_p[F_{2(q-1)}]$-modules

$$H^1(S^1_2; F_p) \cong H^2(S^1_2; F_p).$$  

Proof of the Proposition. $S^1_2$ is a torsionfree $p$-adic Lie group of dimension 3; so by [La, V.2.5.8] $H^*(S^1_2; F_p)$ is a 3-dimensional Poincaré duality algebra which is therefore additively determined by $H^1(S^1_2; F_p)$ resp. its dual $H^1(S^1_2; F_p)$.

The filtration of $S^1_2$ described in section 1.2.3 induces one of $S^1_2$. From the splitting $S^1_2 \cong S^2_2 \times \mathbb{Z}_p$ (with $\mathbb{Z}_p$ being central) we deduce that $gr_i S^1_2 = gr_i S^2_2$ if $i = \frac{k}{2}$ with $k$ odd and $gr_i S^1_2 = F_q / F_p$ if $i = \frac{k}{2}$ with $k$ even. In particular, if $i = \frac{k}{2}$ with $k$ odd there is an element in $gr_1 S^1_2$ with $(\overline{b}^{2i} - \overline{b}) \neq 0$ and then Lemma 1a shows that the commutator map $gr_i S^1_2 \times gr_1 S^1_2 \to gr_{i+1} S^1_2$ given by $[\overline{a}, \overline{b}] = \overline{a(b^{2i} - b)}$ is onto. Furthermore, if $i = \frac{k}{2}$ with $k$ odd and $j = \frac{l}{2}$ with $l$ odd then the commutator map $gr_i S^1_2 \times gr_j S^1_2 \to gr_{i+j} S^1_2$ given by $[\overline{a}, \overline{b}] = \overline{a(b^i - b)}$ is again onto. This implies that there is a canonical isomorphism

$$H_1(S^1_2; \mathbb{Z}_p) \cong F_{1/2} S^1_2 / F_1 S^1_2 \cong F q \cong H_1(S^1_2; F_p).$$  

Furthermore, the conjugation action of $u \in C_{q-1}$ is induced by $(u, 1 + aS) \mapsto u(1 + aS)u^{-1} = 1 + u^{-1}aS$ while the action of Frobenius is induced by $(S, 1 + aS) \mapsto S(1 + aS)S^{-1} = 1 + a^2 S$ and this implies that our isomorphism is an isomorphism
of \( \mathbb{Z}_p[F_{2(q-1)}] \)-modules \( H_1(S^1_2; \mathbb{Z}_p) \cong H_1(S^1_2; \mathbb{F}_p) \cong \lambda_{(-1)} \otimes_{\mathbb{Z}_p} \mathbb{F}_p \). By dualizing we get an isomorphism of \( \mathbb{Z}_p[F_{2(q-1)}] \)-modules \( H^1(S^1_2; \mathbb{F}_p) \cong (\lambda_{(-1)})^* \otimes_{\mathbb{Z}_p} \mathbb{F}_p \).

Now \( \lambda_{(-1)} \) and \( \lambda_{(1)} \) are self-dual (the isomorphism is induced by the pairing \( (x, y) \mapsto xy^{-1} + (xy^{-1})^* \)) and they are both isomorphic (the isomorphism is given by \( x \mapsto x^* \)) and therefore we obtain (b).

Because of \( H_1(S^1_2; \mathbb{Z}_p) \cong H_1(S^1_2; \mathbb{F}_p) \cong \mathbb{F}_p \) we see that the Bockstein homomorphism \( \beta : H_2(S^1_2; \mathbb{F}_p) \to H_1(S^1_2; \mathbb{F}_p) \) is onto, and hence \( \beta : H^1(S^1_2; \mathbb{F}_p) \to H^2(S^1_2; \mathbb{F}_p) \) is mono. On the other hand Poincaré duality gives an additive isomorphism \( H^1(S^1_2; \mathbb{F}_p) \cong H^2(S^1_2; \mathbb{F}_p) \) and hence the Bockstein gives an isomorphism \( H^1(S^1_2; \mathbb{F}_p) \cong H^2(S^1_2; \mathbb{F}_p) \) which is clearly \( \mathbb{Z}_p[F_{2(q-1)}] \)-linear and thus (c) is proved.

To prove (a) we consider the subgroup \( F_1S^1_2 \) of \( S^1_2 \) and note that this subgroup is invariant by the conjugation action of \( F_{2(q-1)} \). Using Lemma 1a as above shows that the closure of the commutator subgroup of \( F_1S^1_2 \) is \( F_2S^1_2 \) and then Lemma 1b gives that the closure of the subgroup generated by \( p \)-th powers and commutators is \( F_2S^1_2 \). It follows that \( H_1(F_1S^1_2; \mathbb{Z}_p) \) is torsion (\( \cong \mathbb{Z}/p^2 \oplus \mathbb{Z}/p \oplus \mathbb{Z}/p \)) and that we have an isomorphism of \( \mathbb{Z}_p[F_{2(q-1)}] \)-modules

\[
H_1(F_1S^1_2; \mathbb{F}_p) \cong gr_1S^1_2 \oplus gr_2S^1_2.
\]

Identifying the \( \mathbb{Z}_p[F_{2(q-1)}] \)-module structure on \( gr_1S^1_2 \oplus gr_2S^1_2 \) as before shows that we have an isomorphism of \( \mathbb{Z}_p[F_{2(q-1)}] \)-modules

\[
H_1(F_1S^1_2; \mathbb{F}_p) \cong \mathbb{F}_p/F_p \oplus \lambda_{p-1}
\]

where \( C_{p-1} \) acts trivially on \( F_p/F_p \) and Frobenius acts by multiplication by \(-1\).

Again this module is self-dual so that we have an isomorphism of \( \mathbb{Z}_p[F_{2(q-1)}] \)-modules

\[
H^1(F_1S^1_2; \mathbb{F}_p) \cong \mathbb{F}_p/F_p \oplus \lambda_{p-1}
\]

Now we use that \( F_1S^1_2 \) is équi-p-valé in the sense of Lazard, hence its cohomology is the exterior algebra on \( H^1(F_1S^1_2; \mathbb{F}_p) \) [La, Proposition V.2.5.7.1]. So if \( \alpha \) is any endomorphism of \( H^1(F_1S^1_2; \mathbb{F}_p) \) then the induced homomorphism on \( H^3(F_1S^1_2; \mathbb{F}_p) \) is given by multiplication with the determinant of \( \alpha \). In particular, for the action of \( u \in C_{p-1} \) the determinant is \( 1 \) (it is obviously \( 1 \) on \( F_p/F_p \) and it is \( 1 \) on \( \lambda_{p-1} \) because \( u \) acts via multiplication by \( u^{p-1} \)) and for the action of Frobenius the determinant is again \( 1 \) (it is \( -1 \) on both \( F_p/F_p \) and on \( \lambda_{p-1} \)). Therefore \( H^3(F_1S^1_2; \mathbb{F}_p) \) is trivial as \( \mathbb{Z}_p[F_{2(q-1)}] \)-module.

Because \( H_1(F_1S^1_2; \mathbb{Z}_p) \) is a torsion group we deduce from the mod-\( p \) calculation and the universal coefficient theorem an isomorphism \( H^3(F_1S^1_2; \mathbb{Z}_p) \cong \mathbb{Z}_p \). Likewise we find \( H^3(S^1_2; \mathbb{Z}_p) \cong \mathbb{Z}_p \). Now a restriction-transfer argument shows that the \( \mathbb{Z}_p[F_{2(q-1)}] \)-module structure on \( H^3(S^1_2; \mathbb{Z}_p) \) is trivial if and only if it is trivial on \( H^3(F_1S^1_2; \mathbb{Z}_p) \). We have already seen that the latter is trivial after mod-\( p \) reduction and this implies that it was trivial before. \( \square \)

**Proof of Theorem 6.** a) The exact sequence of \( \mathbb{Z}_p[G_2] \)-modules

\[
(3) \quad 0 \to \mathbb{Z}_p[\hat{G}_2] \to \mathbb{Z}_p[\hat{G}_2] \to \mathbb{Z}_p \to 0.
\]
is an immediate consequence of the isomorphism $G_2 / G^1_2 \cong \mathbb{Z}^X / C_{p-1} \cong \mathbb{Z}_p$. (Note that this sequence is essentially the same exact sequence as that in section 2.1!)

The projective resolution of $\mathbb{Z}_p$ as $\mathbb{Z}_p[[G^1_2]]$-module is now constructed by using Proposition 7 as follows. The map $\mathbb{Z}_p \uparrow_{F_2(q-1)}^G \mathbb{Z}_p$ is just the $G^1_2$-linear extension of the identity of $\mathbb{Z}_p$ (considered as an $F_2(q-1)$-linear map). If $N_0$ is its kernel then we can compute $H_0(S^1_2; N_0/(p))$ from the long exact homology sequence associated to the short exact sequence

$$0 \rightarrow N_0 \rightarrow \mathbb{Z}_p \uparrow_{F_2(q-1)}^{G^1_2} \rightarrow Z_p \rightarrow 0$$

of $\mathbb{Z}_p[F_{2(q-1)}]$-modules and identify it with $H_1(S^1_2; F_p) \cong \lambda_{p-1} \otimes \mathbb{Z}_p F_p$. Because $\lambda_{p-1}$ is projective (the order of $F_2(q-1)$ is prime to $p$) we can lift the resulting map from $\lambda_{p-1} \rightarrow H_0(S^1_2; N_0/(p))$ to an $F_2(q-1)$-linear map $\lambda_{p-1} \rightarrow N_0$.

Let $N_1$ be the kernel of the $G^1_2$-linear extension $\lambda_{p-1} \uparrow_{F_2(q-1)}^{G^1_2} \rightarrow N_0$. By a Nakayama Lemma type argument with $H_0$ we see that this extension is onto (cf. Lemma 4.3 of [GHRM1]) and then we can compute $H_0(S^1_2; N_1/(p)) \cong H_2(S^1_2; F_p)$ as $\mathbb{Z}_p[F_2(q-1)]$-module.

By iterating the procedure we can construct a $\mathbb{Z}_p[[G^1_2]]$-surjection $\lambda_{p-1} \uparrow_{F_2(q-1)}^{G^1_2} \rightarrow N_1$ whose kernel $N_2$ satisfies $H_0(S^1_2; N_2/(p)) \cong H_3(S^1_2; F_p) \cong F_p$ as $\mathbb{Z}_p[F_2(q-1)]$-module and $H_i(S^1_2; N_2/(p)) = 0$ if $i > 0$.

Finally by using the Nakayama Lemma once more we see that the $G^1_2$-linear extension $\mathbb{Z}_p \uparrow_{F_2(q-1)}^{G^1_2} \rightarrow N_2$ of the $F_2(q-1)$-linear projection $\mathbb{Z}_p \rightarrow H_0(S^1_2; N_2/(p))$ is an isomorphism. By splicing together the short exact sequences we obtain the projective resolution of Theorem 6a.

b) We take the projective resolution obtained in (a) and induce it up to get one of the $\mathbb{Z}_p[[G^1_2]]$-module $\mathbb{Z}_p \uparrow_{G^1_2}^{G^1_2}$. Then we use the exact sequence (3) and construct the obvious double complex whose columns are these induced projective resolutions. The resulting double complex gives the projective resolution of (b). □

**Proof of Theorem 5.** a) For the fibration we can refer to Proposition 7.1 in [DH2]. In fact, the fibration “realizes” (in the same sense as before) the short exact sequence of $\mathbb{Z}_p[[G^1_2]]$-modules in Theorem 6a. The $E_n$-resolution of $E_n^{G^1_2}$ is now obtained as in the proof of Theorem 4 as the realization of the projective resolution of $\mathbb{Z}_p \uparrow_{G^1_2}^{G^1_2}$ which is induced from the one given in Theorem 6a. To finish the proof of (a) it remains to identify the spectrum which corresponds to the module $\lambda_{p-1} \uparrow_{F_2(q-1)}^{G^1_2}$.

For this we refer to the appendix.

b) The $E_n$-resolution of part (b) is nothing but the realization of the resolution of Theorem 6b obtained via (the proof of) Theorem 4. □

**Remarks** a) Ravenel [Rai, Theorem 6.3.3.1] resp. Yanaguchi [Y] have also studied $H^*(S^3, F_p)$ for $p \geq 5$ resp. $p \geq 3$. In principle this can be used to obtain an explicit resolution for $L_{K(3)} S^0$ if $p \geq 3$.

b) No other explicit resolutions seem to be known if $n \neq 0 \text{ mod } p - 1$. 


3. The case $n \equiv 0 \mod p - 1$


This case is again well understood. The isomorphism $G_1 = S_1 \cong \mathbb{Z}_2^2 \cong C_2 \times \mathbb{Z}_2$ allows, as before, to form the homotopy fixed points in two stages and we obtain the following fibration (cf. [HMS]) in which $\psi^3$ is again given by the appropriate Adams operation:

\[ L_{K(1)}S^0 \to K\mathbb{Z}_2^{hC_2} \xrightarrow{\psi^3} K\mathbb{Z}_2^{hC_2} . \]

The homotopy fixed points $K\mathbb{Z}_2^{hC_2}$ can be identified with 2-adic real $K$-theory $KO\mathbb{Z}_2$. Note, however, that this is not an example of Theorem 4. In fact, an $E_n$-resolution of finite length cannot exist in this case because the cohomological dimension $cd_2(G_1)$ is infinite. Nevertheless this is a very good substitute of such a resolution.

3.2. The general problem.

The natural question arises whether there are generalizations of the fibre sequence (4) for higher $n$. What should they look like? In other words, can we explain the appearance of $KO\mathbb{Z}_2$ in (4) so that it fits into a more general framework?

A good point of view is again provided by homological algebra as follows: the fibre sequence (4) is a homotopy theoretic analogue of the exact sequence of $\mathbb{Z}_2[[G_1]]$-modules

\[ 0 \to \mathbb{Z}_2 \left[ \frac{G_1}{C_2} \right] \to \mathbb{Z}_2 \left[ \frac{G_1}{C_2} \right] \to \mathbb{Z}_2 \to 0 . \]

This is not a free (neither a projective) resolution but rather a resolution by the permutation modules $\mathbb{Z}_2[[G_1/C_2]]$.

This suggests that we should look for a resolution of the trivial $G_n$-module $\mathbb{Z}_p$ in terms of something like permutation modules on finite subgroups and try to realize those by appropriate homotopy fixed point spectra where realization is again in the sense of the isomorphism of Morava modules of 1.3.2 which gives us for each finite subgroup $F$ of $G_n$ a canonical isomorphism $E_nL_{hF} \cong \text{Hom}_{cts}(\mathbb{Z}_p \left[ \frac{G_n}{F} \right], E_n)$.

This leads to the following questions.

Questions.

(Q1) Are there resolutions of finite length of the trivial $G_n$-module $\mathbb{Z}_p$ by (finite) direct sums of permutation modules on finite $(p)$-subgroups of $G_n$?

(Q2) Can these algebraic resolutions be realized by “resolutions” of spectra? What do we mean by a “resolution” of spectra?

(Q3) If the answers to (Q1) and (Q2) are yes, how unique are these resolutions?

Remark The group $S_n$ resp. $G_n$ are of finite $vcd_p$. If $G$ is a discrete group which is virtually of finite cohomological dimension, then a permutation resolution of finite length can be obtained from the cellular chains of a contractible finite dimensional
$G$-CW-complex on which $G$ acts with finite stabilizers. Such spaces always exist and hence such resolutions always exist [Se]. In case $G$ is profinite and $\text{vcd}_p(G) < \infty$ then some sort of positive answer to (a) may be given by algebraically mimicking Serre’s construction: one considers a finite index open subgroup $H$ with $\text{cd}_p(H) < \infty$, then one takes a projective resolution of finite length of the trivial $\mathbb{Z}_p[[H]]$-module $\mathbb{Z}_p$ and finally one obtains the desired resolution by tensor induction from $H$ to $G$. This ensures existence, but the drawback of this construction is that it tends to be not very efficient. In particular, the length of the resolution would be much larger than necessary (the vcd?) and the modules in the resolution would not be finitely generated.

In the case of the stabilizer groups we will describe a construction which gives better qualitative (and in favorable cases quantitative) information on the form of such resolutions. Before we turn to the general theory we will survey recent joint work with Goerss and Mahowald [GHM] resp. with Goerss, Mahowald and Rezk [GHMRI] in which we construct explicit and efficient resolutions in the case $n = 2$ and $p = 3$.

### 3.3. Explicit examples IV: the case $n = 2$ and $p = 3$.

Throughout this section we assume $n = 2$ and $p = 3$. In this case there are two different explicit resolutions which we will call duality resolution resp. centralizer resolution. We will see in sections 3.5 and 3.6 below that the centralizer resolution can be generalized both algebraically and homotopy theoretically. The justification for its name will become clear in section 3.5. The duality resolution has the advantage of being more efficient and having an intriguing symmetry.

In the sequel we describe both resolutions. For more details in the case of the duality resolution we refer to [GHMRI]. The centralizer resolution is implicit in [GHM] and is a special case of Theorem 26 below.

#### 3.3.1. In the following results we use the phrase “resolution of spectra” in the following weak sense: we call a sequence of spectra

$$* \to X_{-1} \to X_0 \to X_1 \to \ldots$$

a resolution of $X_{-1}$ if the composite of any two consecutive maps is null-homotopic and if each of the maps $X_i \to X_{i+1}$, $i \geq 0$ can be factored as $X_i \to C_i \to X_{i+1}$ such that $C_{i-1} \to X_i \to C_i$ is a calibration for every $i \geq 0$ (with $C_{-1} := X_{-1}$). We say that the resolution is of length $n$ if $C_n \simeq X_n$ and $X_i \simeq *$ if $i > n$.

#### 3.3.2. Before we can describe our resolutions we need to introduce certain finite subgroups of $\mathbb{G}_2^1$ and some of their representations (cf. [GHMRI] for more details).

The group $\mathbb{G}_2^1$ contains a group $G_{24}$ of order 24 which is isomorphic to the semidirect product $C_3 \rtimes Q_8$ such that $Q_8$ acts non-trivially on $C_3$. If $\omega$ is a primitive 8-th root of unity then $C_3$ is generated by the element $s = \frac{1}{2}(1 + \omega S)$ while $Q_8$ is generated by $\omega^2$ and $\omega S$.

$\mathbb{G}_2^1$ contains also the semidirect product $C_8 \rtimes \text{Gal}(F_8/F_2)$ generated by $\omega$ and $S$. This group which was denoted $F_2(3^r-1)$ in section 2.3 can be identified with the
semidihedral group $SD_{16}$ of order 16. It has a unique non-trivial one dimensional representation $\chi$ over $\mathbb{Z}_3$ which is trivial on the subgroup $< \omega^2, \omega S >$. ($\chi$ agrees with the representation $\lambda_{4-}$ that is discussed in the appendix.)

3.3.3 The duality resolution. The following results are proved in [GHMR1].

**Theorem 8 (Algebraic duality resolution).** There exists a short exact sequence of $\mathbb{Z}_3[[G_2]]$-modules

$$0 \to \mathbb{Z}_3 \downarrow G_{2i} \to \mathbb{Z}_3 \downarrow G_{2} \to \mathbb{Z}_3 \to 0,$$

an exact complex of $\mathbb{Z}_3[[G_2]]$ - modules

$$0 \to \mathbb{Z}_3 \downarrow G_{24} \to \mathbb{Z}_3 \downarrow G_{2} \to \mathbb{Z}_3 \to 0,$$

and an exact complex of $\mathbb{Z}_3[[G_2]]$ - modules

$$0 \to \mathbb{Z}_3 \downarrow G_{24} \to (\mathbb{Z}_3 \downarrow G_{2} \to \mathbb{Z}_3 \downarrow G_{24} \to \mathbb{Z}_3 \to 0.$$

**Theorem 9 (Homotopy theoretic duality resolution).** There exists a fibration

$$L_{K(2)}S^0 \to E_2^{hG_2} \to E_2^{hG_2}$$

and resolutions of spectra of length 3 resp. 4

$$* \to E_2^{hG_2} \to E_2^{hG_2} \to \Sigma^8 E_2^{hSD_{16}} \to \Sigma^{40} E_2^{hSD_{16}} \to \Sigma^{48} E_2^{hG_{24}} \to *$$

resp.

$$* \to L_{K(2)}S^0 \to E_2^{hG_2} \to E_2^{hG_{24}} \to \Sigma^8 E_2^{hSD_{16}} \to \Sigma^{40} E_2^{hSD_{16}} \to \Sigma^{48} E_2^{hG_{24}} \to *.$$

**Remarks**

a) The spectrum $E_2^{hG_{24}}$ is a version of the Hopkins-Miller higher real $K$-theory spectrum $EO_2$ at $p = 3$. Its coefficients are described in detail in [GHMR1]. The coefficients of $E_2^{hSD_{16}}$ are given by the completion of $\mathbb{Z}_3[v_1^\pm v_2^\pm]$ with respect to the ideal generated by $v_1^\pm v_2^{-1}$ (cf. the discussion in the appendix).

b) The appearance of the 8-fold suspension is forced by the character $\chi$, while the 40-fold suspension is there for purely aesthetic reasons (note that $\Sigma^{40} E_2^{hSD_{16}} \simeq \Sigma^8 E_2^{hSD_{16}}$ by periodicity), so that the homotopy theoretic resolution displays a similar kind of duality as the algebraic resolution. The appearance of the 48-fold suspension, however, is a genuinely homotopy theoretic phenomenon and cannot be avoided.

c) The resolution appears to be self dual but there is no satisfactory explanation of this duality yet. And it is not at all clear whether there are generalizations, say to the case $n = p - 1$, $p > 3$, and what they could look like.

3.3.4 The centralizer resolution. For describing the centralizer resolution we will make use of the $\mathbb{Z}_3[SD_{16}]$-module $\lambda_{2}$ (cf. 2.3 and appendix) and the unique non-trivial one dimensional representation $\check{\chi}$ over $\mathbb{Z}_3$ of $G_{24}$ which is trivial on $s$ and on $\omega S$.

The following results are implicit in [GHM]. We will give a proof in section 3.6.6.
Theorem 10 (Algebraic centralizer resolution). There exists a short exact sequence of $\mathbb{Z}_3[[G_2]]$-modules

$$0 \rightarrow Z_3 \mid G_2 \rightarrow Z_3 \mid G_2 \rightarrow Z_3 \rightarrow 0,$$

an exact complex of $\mathbb{Z}_3[[G_2]]$ - modules

$$0 \rightarrow Z_3 \mid G_2 \rightarrow \lambda_2 \mid G_2 \rightarrow \chi \mid G_2 \nabla \nabla \chi \mid G_2 \rightarrow Z_3 \mid G_2 \rightarrow Z_3 \rightarrow 0,$$

and an exact complex $\mathbb{Z}_3[[G_2]]$ - modules

$$0 \rightarrow Z_3 \mid G_2 \rightarrow (Z_3 \oplus \lambda_2) \mid G_2 \rightarrow (\lambda_2 \oplus \chi) \mid G_2 \nabla \nabla \chi \mid G_2 \rightarrow$$

$$\rightarrow \chi \mid G_2 \nabla \nabla (\chi \oplus Z_3) \mid G_2 \rightarrow Z_3 \mid G_2 \rightarrow Z_3 \rightarrow 0.$$

Theorem 11 (Homotopy theoretic centralizer resolution). There exists a fibration

$$L_{K(2)}S^0 \rightarrow E_2^{|G_2|} \rightarrow E_2^{|G_2|}$$

and resolutions of spectra of length 3 resp. 4

$$\ast \rightarrow E_2^{|G_2|} \rightarrow E_2^{|G_2|} \rightarrow \Sigma^8 E_2^{hS_{D_{16}}} \vee \Sigma^{36} E_2^{hG_{24}} \rightarrow$$

$$\rightarrow \Sigma^4 E_2^{hS_{D_{16}}} \vee \Sigma^2 E_2^{hS_{D_{16}}} \rightarrow E_2^{hSD_{16}} \rightarrow \ast,$$

resp.

$$\ast \rightarrow L_{K(2)}S^0 \rightarrow E_2^{hG_{24}} \rightarrow \Sigma^8 E_2^{hS_{D_{16}}} \vee \Sigma^{36} E_2^{hG_{24}} \rightarrow$$

$$\rightarrow \Sigma^4 E_2^{hS_{D_{16}}} \vee \Sigma^2 E_2^{hS_{D_{16}}} \rightarrow E_2^{hSD_{16}} \rightarrow E_2^{hS_{D_{16}}} \rightarrow \ast.$$  

3.4. Work in progress (the case $n = p = 2$).

In this case we do have an algebraic duality resolution but we have not yet completely succeeded in realizing it. However, there is an algebraic centralizer resolution which can be realized.

To describe the homotopy theoretic resolutions we note that, for $p = 2$, $S_2$ contains a subgroup of order $24$, also denoted $G_{24}$ (but not isomorphic to the group with the same label that we used in the last section). For $p = 2$ the group $G_{24}$ is isomorphic to the semidirect product $Q_8 \rtimes C_3$ of the quaternion group $Q_8$ with the cyclic group $C_3$ or order 3 which cyclically permutes $i$, $j$ and $k$. $G_{24}$ contains cyclic subgroups of order 2, 4 and of order 6, We fix such subgroups and denote them by $C_2$ resp. $C_4$ resp. $C_6$.

Then we have the following results which, up to Galois extension, give resolutions of $L_{K(2)}S^0$ at $p = 2$. Details will appear in [GHMR2].
Theorem 12 (Centralizer resolution). There exists a fibration

\[ E_2^{hS_2} \longrightarrow E_2^{hS_2^1} \longrightarrow E_2^{hS_2^3} \]

and a resolution of spectra of length 3

\[ \ast \longrightarrow E_2^{hS_2^1} \longrightarrow E_2^{hG_{24}} \lor E_2^{hG_{24}} \longrightarrow E_2^{hC_6} \lor E_2^{hC_4} \longrightarrow E_2^{hC_2} \longrightarrow E_2^{hC_0} \longrightarrow \ast \]

Theorem 13 (Duality resolution). There exists a fibration

\[ E_2^{hS_2} \longrightarrow E_2^{hS_2^1} \longrightarrow E_2^{hS_2^3} \]

and a resolution of spectra of length 3

\[ \ast \longrightarrow E_2^{hS_2^1} \longrightarrow E_2^{hG_{24}} \longrightarrow E_2^{hC_6} \longrightarrow E_2^{hC_6} \longrightarrow X_3 \longrightarrow \ast \]

together with an isomorphism of Morava modules \( E_\ast(X_3) \cong E_\ast(E_2^{hG_{24}}) \). \( \square \)

Remarks a) The spectrum \( E_2^{hG_{24}} \) is a version of the higher real \( K \)-theory spectrum \( EO_2 \) at \( p = 2 \). We have not been able yet to further identify the spectrum \( X_3 \).
b) We expect that the resolutions described in this and the previous section will help to better understand the Shimomura-Wang calculation [ShW] of \( \pi_\ast L_{K(2)} S^0 \) at \( p = 3 \) and that they will be crucial for calculating \( \pi_\ast L_{K(2)} S^0 \) at \( p = 2 \).

3.5. Permutation resolutions in the case \( n = k(p - 1) \) for \( p \) odd.

In this section we will give a positive answer to question (Q1) and the algebraic part of question (Q3) of section 3.2 above, at least if \( p \) is odd.

3.5.1. We start by introducing some relative homological algebra (cf. [EM]) in a form which parallels Miller’s discussion of \( E_\ast \)-injective spectra and \( E_\ast \)-injective resolutions of spectra (cf. section 2.2.1).

Let \( p \) be a fixed prime. If \( G \) is a profinite group we denote the collection of finite \( p \)-subgroups of \( G \) by \( \mathcal{F}_p(G) \), or simply by \( \mathcal{F}(G) \) or even \( \mathcal{F} \) if \( G \) and \( p \) are clear from the context. Throughout this section we will make the following

Assumption: \( G \) contains only finitely many conjugacy classes of finite \( p \)-subgroups.

We recall that all our modules will be profinite continuous modules for the completed group algebras and that induced modules are formed by using the completed tensor product.

A \( \mathbb{Z}_p[[G]] \)-module \( P \) will be called \( \mathcal{F} \)-projective if the canonical \( \mathbb{Z}_p[[G]] \)-linear map

\[ \bigoplus_{(F) \in \mathcal{F}} P \xrightarrow{G \cdot F} P \]

is a split epimorphism (where the sum is taken over conjugacy classes of finite \( p \)-subgroups). It is clear that each \( \mathbb{Z}_p[[G]] \)-module which is induced from a \( \mathbb{Z}_p[F] \)-module for some \( F \in \mathcal{F} \) is \( \mathcal{F} \)-projective, and that a \( \mathbb{Z}_p[[G]] \)-module \( P \) is \( \mathcal{F} \)-projective
if and only if \( P \) is a retract of some module of the form \( \bigoplus_{(F) \in \mathcal{F}} M_F \uparrow^G_F \) where each
\( M_F \) is a \( \mathbb{Z}_p[F]- \)module.

The class of \( \mathcal{F} \)-projectives determines in the usual way a class of \( \mathcal{F} \)-exact sequences: a sequence of \( \mathbb{Z}_p[[G]] \)-modules \( M' \to M \to M'' \) is called \( \mathcal{F} \)-exact if the composition \( M' \to M'' \) is trivial and

\[
\text{Hom}_{\mathbb{Z}_p[[G]]}(P, M') \to \text{Hom}_{\mathbb{Z}_p[[G]]}(P, M) \to \text{Hom}_{\mathbb{Z}_p[[G]]}(P, M'')
\]

is an exact sequence of abelian groups for each \( \mathcal{F} \)-projective \( \mathbb{Z}_p[[G]] \)-module \( P \). It is obvious that the category of \( \mathbb{Z}_p[[G]] \)-modules has enough \( \mathcal{F} \)-projectives (and it is easy to see that we get a projective class in the sense of relative homological algebra).

Finally an \( \mathcal{F} \)-resolution of a \( \mathbb{Z}_p[[G]] \)-module \( M \) is a sequence of \( \mathbb{Z}_p[[G]] \)-modules

\[
\ldots \to P_i \to P_{i-1} \to M \to 0
\]

where each \( P_i \) is \( \mathcal{F} \)-projective and each 3-term subsequence is \( \mathcal{F} \)-exact. Then it is clear that each module \( M \) admits an \( \mathcal{F} \)-resolution and that any homomorphism of \( \mathbb{Z}_p[[G]] \)-modules \( M \to N \) is covered by a map of \( \mathcal{F} \)-resolutions which is unique up to chain homotopy.

We will be interested in constructing \( \mathcal{F} \)-resolutions of finite length of the trivial module \( \mathbb{Z}_p \) such that all modules in the resolution are finitely generated.

3.5.2. The construction of our \( \mathcal{F} \)-resolutions will rely on the following three results.

**Lemma 14.** Suppose \( G \) is a profinite group and \( H \) is a normal finite \( p \)-subgroup.

a) If \( P \) is \( \mathcal{F}(G/H) \)-projective, then considered as a \( \mathbb{Z}_p[[G]] \)-module via the canonical projection \( \pi : G \to G/H, P \) is \( \mathcal{F}(G) \)-projective.

b) If \( M' \to M \to M'' \) is a sequence of \( \mathbb{Z}_p[[G/H]] \)-modules which is \( \mathcal{F}(G/H) \)-exact, then considered as a sequence of \( \mathbb{Z}_p[[G]] \)-modules via the canonical projection \( \pi : G \to G/H, M' \to M \to M'' \) is \( \mathcal{F}(G) \)-exact.

**Proof.** a) This follows immediately from the following observation. If \( F \) is a finite \( p \)-subgroup of \( G/H \) and \( M \) is a \( \mathbb{Z}_p[F]- \)module then \( M \uparrow^G_{F/H} \), considered as \( \mathbb{Z}_p[[G]] \)-module via \( \pi \), is isomorphic to \( M \uparrow^G_{\pi^{-1}F} \).

b) If \( F \) is a finite \( p \)-subgroup of \( G \) and \( M \) is a \( \mathbb{Z}_p[F]- \)module then there is a natural isomorphism \( (M \uparrow^G_F) \otimes_{\mathbb{Z}_p[H]} \mathbb{Z}_p \cong (M \otimes_{\mathbb{Z}_p[F/H]} \mathbb{Z}_p) \uparrow^G_{F/H} \) of \( \mathbb{Z}_p[[G/H]] \)-modules. This implies that \( \otimes_{\mathbb{Z}_p[H]} \mathbb{Z}_p \) sends \( \mathcal{F}(G) \)-projectives to \( \mathcal{F}(G/H) \)-projectives which in turn implies (b). \( \square \)

**Lemma 15.** Suppose \( G \) is a profinite group and \( K \) is a closed subgroup of \( G \) which also contains only a finite number of conjugacy classes of finite subgroups,

a) If \( P \) is \( \mathcal{F}(K) \)-projective, then \( P \uparrow^G_K \) is \( \mathcal{F}(G) \)-projective.

b) Assume that there is a sequence of open normal subgroups \( U_n \) of \( G \) such that \( \bigcap_n U_n = \{1\} \). If \( M' \to M \to M'' \) is a sequence of \( \mathbb{Z}_p[[K]] \)-modules which is \( \mathcal{F}(K) \)-exact, then \( M' \uparrow^G_K \to M \uparrow^G_K \to M'' \uparrow^G_K \) is \( \mathcal{F}(G) \)-exact.

**Proof.** a) This is trivial.
b) It is enough to show that for each $F \in \mathcal{F}(G)$ and each $\mathbb{Z}_p[F]$-module $L$ the functor $\text{Hom}_{\mathbb{Z}_p[F]}(L, -)$ sends the induced sequence to an exact sequence of abelian groups. This is in fact a consequence of the Mackey decomposition formula for the restriction of an induced module. In the case of profinite groups this requires some care so that it seems appropriate to give some details.

To simplify notation we let $K_n = K/K \cap U_n$, $G_n = G/U_n$ and $F_n = F/F \cap U_n$. Note that if $F$ is finite then $F = F_n$ if $n$ is sufficiently large. Now let $M$ be any profinite $\mathbb{Z}_p[[K]]$-module. Then we can write $M = \lim_i M_i$ (where $i$ runs through some directed set $I$, not necessarily a countable sequence) with $M_i$ finite and acted on trivially by $K \cap U_{M(i)}$ for some increasing function $\lambda : I \to \mathbb{N}$. Then we have the classical Mackey decomposition formula for the $\mathbb{Z}_p[K_{\lambda(i)}]$-modules $M_i$ (where as usual $(-) \uparrow^G_F$ denotes the restriction of a $\mathbb{Z}_p[[G]]$-module to a $\mathbb{Z}_p[F]$-module)

$$M_i \uparrow^{G_{\lambda(i)}}_{K_{\lambda(i)}} \uparrow^{F_{\lambda(i)}} = \bigoplus_{g \in F_{\lambda(i)} \setminus G_{\lambda(i)}/K_{\lambda(i)}} (g M_i) \uparrow^{G_{\lambda(i)}}_{K_{\lambda(i)}} (g^{-1} F_{\lambda(i)})$$

and by passing to the limit we obtain

$$M \uparrow^G_F \cong \left( \mathbb{Z}_p[[G]] \otimes \mathbb{Z}_p[[K]] \right) M \uparrow^G_F \cong \lim_i M_i \uparrow^{G_{\lambda(i)}}_{K_{\lambda(i)}} \uparrow^{F_{\lambda(i)}} = \bigoplus_{g \in F_{\lambda(i)} \setminus G_{\lambda(i)}/K_{\lambda(i)}} (g M_i) \uparrow^{G_{\lambda(i)}}_{K_{\lambda(i)}} (g^{-1} F_{\lambda(i)}) \uparrow^{F_{\lambda(i)}} \uparrow^{g^{-1} F_{\lambda(i)}}.$$ 

Now we consider our $\mathcal{F}(K)$-split sequence $M' \to M \to M''$ and factor the first homomorphism via the kernel $\overline{M}$ of $M \to M''$ as $M' \to \overline{M} \to M$. It is enough to show that $M' \uparrow^G_F \to \overline{M} \uparrow^G_F \uparrow^G_F$ induces a surjection on $\text{Hom}_{\mathbb{Z}_p[F]}(L, -)$.

First we note that $p : M' \to \overline{M} \to 0$ is $\mathcal{F}(K)$-exact. This implies that for any finite $p$-subgroup $H$ of $K$ there exists an $H$-linear splitting $s : \overline{M} \to M'$, i.e. there exist increasing functions $\alpha : I \to I$, $\beta : I \to I$ and compatible families of $\mathbb{Z}_p[H]$-linear maps $p_i : (M')_{\alpha(i)} \to \overline{M}$ representing $p$ and $s_{\alpha(i)} : (M')_{\alpha(i)} \to (M')_{\alpha(i)}$ representing $s$ such that the composition $(\overline{M})_{\beta(i)} \to (M')_{\alpha(i)}$ is the map in the given system for $\overline{M}$. By explicitly choosing conjugations we may assume that we have such a splitting for all finite $p$-subgroups in the conjugacy class of $H$. And because we assume that there are only finitely many conjugacy classes of finite $p$-subgroups in $K$ we may assume that the functions $\alpha$ and $\beta$ work for all $H \in \mathcal{F}(K)$.

The Mackey decomposition formula for $(\overline{M})_{\beta(i)}$ and $(M')_{\alpha(i)}$ gives us therefore, for any choice of double coset representatives in $G_{\lambda \beta(i)}$ resp. $G_{\lambda \alpha(i)}$, well defined $\mathbb{Z}_p[F]$-linear maps

$$\tilde{\alpha}_{\beta(i)} : (\overline{M})_{\beta(i)} \uparrow^{G_{\lambda \beta(i)}}_{K_{\lambda \beta(i)}} \uparrow^{G_{\lambda \beta(i)}}_{F_{\lambda \beta(i)}} \to (M')_{\alpha(i)} \uparrow^{G_{\lambda \alpha(i)}}_{K_{\lambda \alpha(i)}} \uparrow^{G_{\lambda \alpha(i)}}_{F_{\lambda \alpha(i)}}.$$ 

Now the elementary theory of profinite sets (cf. Lemma 5.6.7 in [RZ]) tells us that the quotient map $G \to F \setminus G/K$ admits a continuous section and any such section gives us a compatible choice of double coset representatives in $G_n$ for all $n$. For any such choice the associated $\mathbb{Z}_p[F]$-linear maps $\tilde{\alpha}(i)$ are compatible with respect to $i$ so that they patch together and define a $\mathbb{Z}_p[F]$-linear map on the level of inverse limits. This shows that $M' \uparrow^G_F = \overline{M} \uparrow^G_F$ is split as a map of $\mathbb{Z}_p[F]$-modules and hence we are done. \[\square\]
Lemma 16. Suppose $G$ is a profinite group and $M$ is a $\mathbb{Z}_p[[G]]$-module which admits a finite projective resolution and which is projective as a $\mathbb{Z}_p$-module. Then $M$ is projective as a $\mathbb{Z}_p[F]$-module for every $F \in \mathcal{F}$.

Proof. By induction on the length of a finite projective resolution it is enough to show that for a finite $p$-group $F$ a short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of $\mathbb{Z}_p[F]$-modules splits if $M_1$ is projective as a $\mathbb{Z}_p[F]$-module and if the sequence splits as a sequence of $\mathbb{Z}_p$-modules. In fact, if $M_1$ is projective as a $\mathbb{Z}_p[F]$-module then it is factor in the induced module $M_1 \uparrow^F_{\mathbb{Z}_p}$. Because $F$ is finite the induced module is isomorphic to the coinduced module $\text{Hom}(\mathbb{Z}_p[F], M_1)$ and then a $\mathbb{Z}_p$-linear splitting $M_2 \to M_1$ determines a $\mathbb{Z}_p[F]$-linear map $M_2 \to \text{Hom}(\mathbb{Z}_p[F], M_1)$ which together with the retraction $\text{Hom}(\mathbb{Z}_p[F], M_1) \to M_1$ gives the desired splitting. □

3.5.3. Here is the promised answer to question (Q1) and the algebraic part of (Q3).

Proposition 17. Suppose $G$ is a virtually profinite $p$-group and $S$ is a closed normal subgroup which is a profinite $p$-group. Furthermore assume that

(1) $H^*(S; \mathbb{F}_p)$ is a finitely generated $\mathbb{F}_p$-algebra,

(2) all finite $p$-subgroups of $S$ are cyclic and there is a bound on their order,

(3) the trivial module $\mathbb{Z}_p$ admits a projective resolution of finite type and finite length for $G/S$ and all its closed subgroups.

(4) There is a sequence of open subgroups $U_n$ of $G$ such that $\bigcap_n U_n = \{1\}$. Then the trivial $\mathbb{Z}_p[[G]]$-module $\mathbb{Z}_p$ admits an $\mathcal{F}$-resolution of finite length. Furthermore, all $\mathcal{F}$-projectives in this resolution can be chosen to be summands in finite direct sums of modules of the form $\mathbb{Z}_p \uparrow^G_F$ with $F \in \mathcal{F}$.

Remarks a) Assumption (1) implies by a recent result of Minh and Symonds [MS] that $S$ and hence also $G$ has only a finite number of conjugacy classes of finite $p$-subgroups. In particular the bound in assumption (2) is already a consequence of assumption (1). Furthermore recent work of Symonds suggests that finite length $\mathcal{F}$-resolutions should exist even if we skip condition (2).

b) Assumption (3) implies that $G/S$ has no elements of order $p$, in other words every finite $p$-subgroup of $G$ is already contained in $S$.

c) It is well-known that $H^*(S_n; \mathbb{F}_p)$ is a finitely generated algebra. As already mentioned in 1.2.4, $S_n$ has finite $p$-subgroups if and only if $n$ is divisible by $p - 1$. Furthermore, if $p$ is odd, or $p = 2$ and $n$ is odd, then all finite $p$-subgroups are cyclic and their conjugacy class is unique. The $p$-Sylow subgroup $S_n$ of $S_n$ is normal in $G_n$ of index $(p^n-1)n$ which is of order prime to $p$ if $n = k(p-1)$ with $k \not\equiv 0 \mod p$. Therefore, for such $n$ the assumptions of the proposition hold with $G = G_n$ and $S = S_n$.

If $n = k(p-1)$ with $k$ divisible by $p$ then we replace $G_n$ as in remark 2 of section 2.2 by $G_n(K) := S_n \times \text{Gal}(K/\mathbb{F}_p)$ where $K := \bigcup P \varphi P$. This has the effect that all $p$-torsion elements of $G_n(K)$ are already contained in $S_n$ resp. its normal Sylow subgroup $S_n$. In this case it is clear that the assumptions hold with $G = G_n(K)$ and $S = S_n$. In fact, $G/S$ is an extension of $\mathbb{Z}_p$ by a finite group of order prime to $p$. The somewhat artificially looking assumptions on the pair $(G, S)$ in Proposition 17 are mainly justified because we want to cover this case.
d) If \( n \) is even then \( S_n \) has finite 2-subgroups which are not cyclic. Thus for \( n \) even and \( p = 2 \) the proposition does not give any information. Nevertheless, if \( n = p = 2 \) the algebraic centralizer resolution mentioned in section 3.4 gives an explicit finite length \( \mathcal{F} \)-resolution of the trivial \( \mathbb{Z}_2[[S_2]] \)-module \( \mathbb{Z}_2 \).

**Proof.** The proof will be by induction over the order of the largest finite \( p \)-subgroup of \( G \). We distinguish the following cases.

**Case 1:** \( G \) has no non-trivial finite \( p \)-subgroups.

In this case it follows from Quillen’s \( F \)-isomorphism theorem \([Q] \) that \( H^*(S; \mathbb{F}_p) \) is a finite \( \mathbb{F}_p \)-algebra. Because \( S \) is a profinite \( p \)-group this implies \( cd_p(S) < \infty \) and then assumption (3) implies \( cd_p(G) < \infty \). Finally, (3) and the spectral sequence of the group extension \( 1 \to S \to G \to G/S \to 1 \) show that \( H^*(G, M) \) is a finite group for every finite discrete continuous \( \mathbb{Z}_p[[G]] \)-module \( M \) and this ensures the existence of a projective resolution of \( \mathbb{Z}_p \) of finite length in which all projectives are finitely generated (cf. Proposition 4.2.3 in \([SyW]\)).

**Case 2:** \( G \) has a normal finite \( p \)-subgroup \( F \).

By \([MS]\) \( H^*(S/F; \mathbb{F}_p) \) is still a finitely generated \( \mathbb{F}_p \)-algebra and hence \( (G/F, S/F) \) still satisfies our assumptions. Therefore, by induction hypothesis, the trivial \( G/F \)-module \( \mathbb{Z}_p \) admits an \( \mathcal{F}(G/F) \)-resolution of finite length such that all \( \mathcal{F}(G/F) \)-projectives are of the required form. Then Lemma 14 implies that the very same resolution is also an \( \mathcal{F}(G) \)-resolution of finite length for the trivial \( G \)-module \( \mathbb{Z}_p \), and all \( \mathcal{F}(G/F) \)-projectives are as required.

**Case 3:** The general case.

We may suppose that \( G \) does not contain any finite normal \( p \)-subgroups. In this case we consider the short exact sequence of \( \mathbb{Z}_p[[G]] \)-modules

\[
0 \to K \xrightarrow{f} \bigoplus_{(E)} \mathbb{Z}_p \xrightarrow{\varepsilon} \mathbb{Z}_p \to 0
\]

where the sum is over conjugacy classes of non-trivial elementary abelian \( p \)-subgroups of \( G \) (which by our assumption are all of order \( p \)), \( \varepsilon \) is the canonical augmentation and \( K \) is its kernel.

First we note that the exact sequence (5) is \( \mathcal{F} \)-exact. In fact, if \( F \) is a finite \( p \)-subgroup of \( G \) then it has a non-trivial central element of order \( p \) and \( F \) is contained in the normalizer of the elementary abelian \( p \)-subgroup \( E \) generated by this element of order \( p \). This means, that the action of \( F \) on \( G/N_G(E') \) has a fixed point. Such a fixed point determines a \( \mathbb{Z}_p[F] \)-linear splitting of the surjection in (5). In other words, the exact sequence splits upon restriction to every \( F \in \mathcal{F} \) and this is equivalent to saying that the sequence is \( \mathcal{F} \)-exact.

It will therefore be enough to construct \( \mathcal{F} \)-resolutions for \( \bigoplus_{(E)} \mathbb{Z}_p \xrightarrow{\varepsilon} \mathbb{Z}_p \) and for \( K \) where the \( \mathcal{F} \)-projectives have the required form. In fact, if we have two such resolutions we can lift \( f \) to a map of resolutions and then the resulting double complex will be an \( \mathcal{F} \)-resolution for \( \mathbb{Z}_p \) with all \( \mathcal{F} \)-projectives as required.

The pair \( (N_G(E), S \cap N_G(E)) \) satisfies the same assumptions as \((G, S)\): assumptions (2), (3) and (4) are obvious. For (1) we note that \( S \cap N_G(E) = N_S(E) \) agrees with the centralizer \( C_S(E) \) because \( E \) is of rank 1. In fact, because the \( p \)-rank of \( E \) is always 1, the \( p \)-group \( N_S(E)/C_S(E) \) injects into \( Aut(E) \cong \mathbb{Z}/(p-1) \) and hence it
is trivial. Now $C_S(E)$ satisfies (1) because it is given by a component of Lannes’ $T$-functor (by Theorem 2.6 of [He]) and Lannes’ $T$-functor takes unstable finitely generated algebras to unstable finitely generated algebras ([DW, Theorem 1.4]). Furthermore, by [MS] the groups $N_S(E)$ and hence also $N_G(E)$ have only a finite number of conjugacy classes of finite $p$-subgroups. Therefore we can apply case 2 and deduce that for each $E \in \mathcal{F}(G)$ the trivial $\mathbb{Z}_p[[N_G(E)]]$-module $\mathbb{Z}_p$ admits an $\mathcal{F}(N_G(E))$-resolution of finite length. Inducing this resolution gives, by Lemma 15, an $\mathcal{F}(G)$-resolution of $\mathbb{Z}_p \uparrow_{N_G(E)}^G$ with all $\mathcal{F}$-projectives as required.

Finally consider $K$. The group $G/S$ acts on the finite set of conjugacy classes of elementary abelian $p$-subgroups of $S$ and the stabilizer of such a subgroup $E$ is the image of $N_G(E)$ in $G/S$. In particular this image is of finite index in $G/S$ and this implies that for each $E \subset G$ we get an isomorphism of $\mathbb{Z}_p[[S]]$-modules

$$\mathbb{Z}_p \uparrow_{N_G(E)}^G \cong \bigoplus_{(E') < S} \mathbb{Z}_p \uparrow_{N_G(E')}^S$$

where the direct sum is taken over conjugacy classes of $E'$s in $S$ which become conjugate to $E$ in $G$. Furthermore, as before the normalizer $N_S(E')$ can be identified with the centralizer $C_S(E')$. Therefore we end up with an isomorphism of $\mathbb{Z}_p[[S]]$-modules

$$\bigoplus_{(E) < G} \mathbb{Z}_p \uparrow_{N_G(E)}^G \cong \bigoplus_{(E') < S} \mathbb{Z}_p \uparrow_{N_S(E')}^S$$

where the direct sums are taken over all conjugacy classes of $E'$s in $G$ resp. $S$.

By the centralizer approximation theorem (Theorem 1.4 in [He]) the restriction maps induce a map

$$H^*(S; \mathbb{F}_p) \rightarrow \prod_{(E'). E < S} H^*(C_S(E); \mathbb{F}_p)$$

which has finite kernel and cokernel. In other words, the map induced by $\varepsilon$ in $\text{Ext}_{\mathbb{Z}_p[[S]]}(\cdot, \mathbb{F}_p)$ has finite kernel and cokernel and hence $\text{Ext}_{\mathbb{Z}_p[[S]]}^*(K, \mathbb{F}_p)$ is finite. Then the spectral sequence of the group extension $1 \rightarrow S \rightarrow G \rightarrow G/S \rightarrow 1$ implies that $\text{Ext}_{\mathbb{Z}_p[[G]]}^*(K, M)$ is finite for every finite discrete continuous $\mathbb{Z}_p[[G]]$-module $M$ and this ensures the existence of a resolution of $K$ of finite length by finitely generated projective $\mathbb{Z}_p[[G]]$-modules (by the obvious extension of Proposition 4.2.3 in [Sy-W]). By Lemma 16 we deduce that $K$ is projective as $\mathbb{Z}_p[F]$-module for every $F \in \mathcal{F}(G)$ and then it is clear that any (ordinary) projective resolution of $K$ is $\mathcal{F}$-projective. □

3.6. Resolutions for $L_{K(n)}S^0$ for $p$ odd and $n = p - 1$.

In this section we address the question whether the resolutions constructed in the previous section can be realized. For this we restrict attention to the case $n = p - 1$, the reason being that we do not know enough about the homotopy groups of the relevant spectra $E_h^{hf}$ for finite $p$-subgroups $F$ of $\mathbb{G}_n$ if $n = k(p - 1)$ with $k > 1$. 
3.6.1. Let us start by looking at the $\mathcal{F}$-resolution of the trivial $\mathbb{Z}_p[[G_n]]$-module constructed in the previous section. By 1.2.4 there is a unique conjugacy class of subgroups of $S_n$ of order $p$. We pick a representative and denote it by $E$ and its normalizer in $G_n$ simply by $N$. So the exact sequence of the proof of Proposition 17 takes the following form

$$0 \rightarrow K \xrightarrow{f} \mathbb{Z}_p \xrightarrow{\varepsilon} \mathbb{Z}_p \rightarrow 0.$$  

By 1.3.2 this sequence can be realized by the cofibration

$$L_{K(n)}S^0 \simeq E_n^hG_n \xrightarrow{\varepsilon} E_n^hN \rightarrow C$$

where $\varepsilon$ is induced by the inclusion of $N$ into $G_n$ and $C$ is the cofibre of $\varepsilon$. Of course, as before realizing means that the map $\text{Hom}_{cts}(\varepsilon, (E_n)_*)$ agrees with the map

$$\text{Hom}_{cts}(\mathbb{Z}_p, (E_n)_*) \cong (E_n)_* L_{K(n)}S^0 \xrightarrow{\varepsilon_*} (E_n)_* E_2^hN \cong \text{Hom}_{cts}(\mathbb{Z}_p, (E_n)_*)$$

induced by $\varepsilon$ in $(E_n)_*$. This map is injective and therefore $E_*C$ can be identified with $\text{Hom}_{cts}(K, (E_n)_*)$.

This suggests that we should start by constructing resolutions for $E_n^hN$ and $C$.

3.6.2. We begin with $C$.

**Lemma 18.** $C$ admits an $E_n$-resolution of finite length in which each spectrum is a summand of a finite wedge of $E_n$’s.

**Proof.** We know from section 3.5 that $K$ has a projective $\mathbb{Z}_p[[G_n]]$-resolution $P_* \rightarrow K$ of finite length $d$ in which each $P_i$ is finitely generated. Let $Q_*$ be the following exact complex of $\mathbb{Z}_p[[G_n]]$-modules

$$Q_* : 0 \rightarrow P_d \rightarrow \ldots \rightarrow P_0 \rightarrow \mathbb{Z}_p \xrightarrow{\varepsilon^G_n} \mathbb{Z}_p \rightarrow 0$$

obtained by splicing the resolution of $K$ and the exact sequence (6). By Proposition 3 there is a sequence of maps between spectra of the following form

$$Y_* : * \rightarrow L_{K(n)}S^0 \simeq E_2^hG_n \xrightarrow{\varepsilon_*} E_2^hN \rightarrow X_0 \rightarrow \ldots \rightarrow X_d \rightarrow *$$

where each $X_i$ is a summand in a finite wedge of $E_n$’s such that the complex $\text{Hom}_{cts}(Q_*, (E_n)_*)$ is isomorphic, as a complex of Morava modules to the complex $(E_n)_*(Y_*)$. Again by Proposition 3 the composite $L_{K(n)}S^0 \rightarrow E_2^hN \rightarrow X_0$ is null and we obtain a sequence of maps between spectra

$$X_* : * \rightarrow C \rightarrow X_0 \rightarrow \ldots \rightarrow X_d \rightarrow *$$

such that $\text{Hom}_{cts}(P_*, (E_n)_*)$ is isomorphic, as a complex of Morava modules to the complex $(E_n)_* X_*$. We claim that $X_*$ is, in fact, an $E_n$-resolution of $C$ in the sense of 2.2.1. In fact, it is clear that all $X_i$ for $i \geq 0$ are $E_n$-injective and $E_n$-exactness is seen as in the proof of Theorem 4. $\square$
3.6.3. The case of $E_2^{hN}$ is substantially more difficult and unlike in the case of $C$ the resolution will not be an $E_n$-resolution although it will still have some good properties. In this subsection we will make the algebraic resolution of $\mathbb{Z}_p \nabla^{S_n}_N$ explicit.

3.6.3.1. First we investigate the structure of the group $N = N_{G_n}(E)$. We choose a primitive root of unity $\omega \in \mathbb{W}_{F_n}^\times \subset \mathbb{S}_n$ and note that the element $X := \omega \frac{1}{\sqrt{p}} S \in \mathbb{D}_n$ satisfies $X^n = -p$.

**Lemma 19.** The subfield $\mathbb{Q}_p(X)$ of $\mathbb{D}_n$ is isomorphic to the cyclotomic extension $\mathbb{Q}_p(\zeta_p)$ generated by a primitive $p$-th root of unity $\zeta_p$.

**Proof (outline).** In fact, this is a straightforward consequence of local class field theory but for the convenience of the reader we outline an elementary and direct proof.

$R := \mathbb{Z}_p[X]/(X^n + p)$ is a local ring with maximal ideal generated by $X$. The powers of the maximal ideal induce as in section 1.2.3 a decreasing complete filtration $F_i$ with $i = \frac{1}{n}$, $k = 1, 2, \ldots$ of the group of units of $R$ via

$$F_i := \{ g \in R^\times | g \equiv 1 \mod X^{kn} \}$$

and with filtration quotients $g_{r_i} := F_i/F_{i+1}$ which are all canonically isomorphic to $F_p$. Furthermore the $p$-th power map induces, as in the case of the groups $S_n$, maps $g_{r_i} \to g_{r_{p(i)}}$ which via these isomorphisms are given by the identity if $i \neq \frac{1}{n}$ and by $\pi \to \pi^p - \pi$ if $i = \frac{1}{n}$ (with notation as in section 1.2.3). The completeness of the filtration shows now that $R$ contains a $p$-th root of unity, in fact for any $a \in R$ with $a^p \equiv a \mod (X)$ there is a $p$-th root of unity of the form $1 + aX \mod (X^2)$. This shows that $\mathbb{Q}_p(X)$ contains an isomorphic copy of the field $\mathbb{Q}_p(\zeta_p)$ and because both are extensions of degree $n = p - 1$ of $\mathbb{Q}$ they have to be isomorphic. $\square$

We choose a primitive $p$-th root of unity in $\mathbb{Q}_p(X)$ and we still denote it by $\zeta_p$ thus identifying $\mathbb{Q}_p(X)$ with $\mathbb{Q}_p(\zeta_p)$. We choose our subgroup $E \subset \mathbb{S}_n$ to be generated by $\zeta_p$. If $n = k(p - 1)$ then (cf. 1.2.4) the centralizer $C_{\mathbb{D}_n}(E)$ is a division algebra which is central over $\mathbb{Q}_p(\zeta_p)$ of dimension $k^2$. In particular, if $n = p - 1$, then $C_{\mathbb{D}_n}(E) = \mathbb{Q}_p(\zeta_p)$, and $C_{\mathbb{D}_n}(E) = \mathbb{Q}_p(\zeta_p)^\times$.

We need to know the structure of the group of units in $\mathbb{Q}_p(\zeta_p) = \mathbb{Q}_p(X)$. This is well known in algebraic number theory. For the convenience of the reader we give some details.

Consider the filtration by subgroups

$$(7) \quad 0 \subset F_2 \subset F_1 \subset \mathbb{Z}_p[\zeta_p]^\times \subset \mathbb{Q}_p(\zeta_p)^\times$$

where $F_2$ and $F_1$ are the filtration subgroups of $\mathbb{Z}_p[\zeta_p]^\times$ introduced above. The valuation $v$, normalized by $v(p) = 1$, is a split epimorphism $\mathbb{Q}_p(\zeta_p)^\times \twoheadrightarrow \frac{1}{n} \mathbb{Z} \cong \mathbb{Z}$ with kernel $\mathbb{Z}_p[\zeta_p]^\times$. Next, the description of the $p$-th power map given above shows that $F_2$ is a free $\mathbb{Z}_p$-module of rank $n$ and the torsion-subgroup of $\mathbb{Z}_p[\zeta_p]^\times$ identifies with $\mathbb{Z}_p[\zeta_p]^\times/F_2 \cong E \times \mathbb{Z}/n$. In particular, we see that the filtration (7) is split and we obtain an isomorphism

$$(7') \quad \mathbb{Q}_p(\zeta_p)^\times \cong \mathbb{Z} \times \mathbb{Z}/n \times \mathbb{Z}/p \times \mathbb{Z}_p^n .$$
Furthermore, the discussion has shown that we can choose generators as follows:
- The element $X$ satisfies $v(X) = \frac{1}{n}$ and can therefore be chosen as a generator of the factor $\mathbb{Z}$.
- The subgroup $\mathbb{Z}/n$ is the subgroup generated by $\omega^{\frac{n}{p-1}}$.
- $\mathbb{Z}/p \cong E$ is the subgroup generated by $\zeta_p$.
- The elements $\eta_j := 1 + X^j \in \mathbb{Z}_p[\mathbb{Z}/n] \cong \mathbb{Z}_p[X]^\times$, $j = 2, \ldots, n+1$, qualify as a system of topological generators of $\mathbb{Z}_p^n$.

We will need to understand the action of the Galois group on $\mathbb{Z}_p[\mathbb{Z}/n]^\times$. It is clear that the three factors $F_2$, $E$ and $\mathbb{Z}/n$ are invariant with respect to the action of the Galois group and that the action on $\mathbb{Z}/n \subset \mathbb{Z}_p^\times$ is trivial. Furthermore, the Galois group can be canonically identified with the group $\text{Aut}(E)$ of automorphisms of $E$, in other words the action on $E$ is the tautological action. To understand the action on $F_2$ we observe that the Galois automorphisms can be realized by the conjugation action (in $\mathbb{D}_n$) of the quotient of the cyclic group generated by $\tau := \omega^{\frac{n}{p-1}}$ by the subgroup generated by $\tau^n$. In fact, from $X = \omega^{\frac{n}{p-1}} S$ we get

$$\tau_\ast (\eta_j) = 1 + \omega^{\frac{1}{p-1} \cdot \frac{n}{p-1}} X^j,$$

which shows that $\eta_j$ is an eigenvector in $gr_1$ with eigenvalue $\omega^{\frac{1}{p-1} \cdot \frac{n}{p-1}}$. To obtain a genuine eigenvector with eigenvalue $\omega^{\frac{1}{p-1} \cdot \frac{n}{p-1}}$ we note that

$$e_j := \frac{1}{n} \sum_{i=1}^n \omega^{\frac{i (i-n)}{p-1}} \tau^i$$

is an idempotent in $\mathbb{Z}_p[\mathbb{Z}/n]$ which satisfies $\tau e_j = \omega^{\frac{1}{p-1} \cdot \frac{n}{p-1}} e_j$. If we replace $\eta_j$ by

$$\eta'_j := (e_j) \ast \eta_j$$

then the elements $\eta'_j$, $j = 2, \ldots, n+1$, generate one-dimensional characters, say $\chi(j)$, and as a Galois-module $F_2$ is isomorphic to the direct sum of all the different one-dimensional characters of $\text{Aut}(E)$ over $\mathbb{Z}_p$.

We note that if we rename $\chi(n+1)$ as $\chi(1)$ then we get $\chi(j) = \chi(1)^{\otimes j}$ for $j = 2, \ldots, n+1$. In the sequel $\chi(j)$ will we be, for any integer $j \in \mathbb{Z}$, be interpreted as the tensor product $\chi(1)^{\otimes j}$.

After these preparations we will now describe $N_{\mathbb{D}_n}(E)$. By abuse of notation we continue to denote the image of elements of $\mathbb{D}_n^\times$ in $\mathbb{G}_n \cong \mathbb{D}_n^\times / < S^n >$ by the same name.

**Proposition 20.** Let $n = p - 1$ and $p$ be odd,

a) There is an exact sequence

$$1 \to C_{\mathbb{G}_n}(E) \to N_{\mathbb{G}_n}(E) \to \text{Aut}(E) \to 1.$$

b) There is an isomorphism

$$C_{\mathbb{G}_n}(E) \cong H \times E \times \mathbb{Z}_p^n.$$
where $H \cong \mathbb{Z}/2n \times \mathbb{Z}/\mathbb{Z}/2$ is generated by $X$ and $\tau^nX^2$, $E$ is generated by $\zeta_n$, and the elements $\eta_j = 1 + X^j$, $j = 1, \ldots, n$, are a system of topological generators of $\mathbb{Z}_n^n$. 

c) The action of $\text{Aut}(E)$ leaves $H$ and $\mathbb{Z}_n^n$ invariant: the action of $\text{Aut}(E)$ on $E$ is the tautological action while $\mathbb{Z}_n^n \cong \bigoplus_{i=1}^n \chi(i)$ and $\chi(i)$ is generated by $\eta_i$ for $i = 2, \ldots, n$ and by $\eta_{n+1}$ for $i = 1$.

d) The subgroup generated by $\zeta_n$, $H$ and $\tau$ is a finite subgroup $F$ of $N$ of order $pn^3$ which contains $H \times E$ as a normal subgroup with quotient $\text{Aut}(E)$.

Proof. a) We have seen above that there is an exact sequence

$$1 \to C_{\mathbb{D}_n}(E) \to N_{\mathbb{D}_n}(E) \to \text{Aut}(E) \to 1$$

where $\omega_\mathbb{D}_n^{(n-1)}$ projects to a generator of $\text{Aut}(E)$. In particular all of the automorphisms of the group $E$ are realized by conjugation in $\mathbb{D}_n$ and hence also in the central quotient $\mathbb{G}_n \cong \mathbb{D}_n^\times / \langle S^n \rangle$. This shows (a).

b) We claim that the epimorphism $\mathbb{D}_n \to \mathbb{G}_n$ induces an epimorphism $C_{\mathbb{D}_n}(E) \to C_{\mathbb{G}_n}(E)$ with kernel the central subgroup generated by $S^n$. In fact, if we lift an element $g \in C_{\mathbb{G}_n}(E)$ to an element $\tilde{g} \in \mathbb{D}_n$, then $\tilde{g}z^{-1} = \zeta_nz$ with $z \in <S^n>$. But then we obtain

$$1 = \tilde{g}z^{-1} = (\tilde{g}z)^p = (\zeta_nz)^p = z^p$$

and hence $z = 1$, in other words $\tilde{g} \in C_{\mathbb{D}_n}(E)$. Part (b) follows now from the observation that $S^n = \omega_\mathbb{D}_n^{(n-1)} X^n$, i.e. in terms of the decomposition (7') of $C_{\mathbb{D}_n}(E)$, the element $S^n$ has components $(n, \mathbf{Z})$ in $\mathbb{Z} \times \mathbb{Z}/n$ and trivial components in $\mathbb{Z}_n \times \mathbb{Z}_n$.

(c) and (d) are clear from the discussion above. □

Remark The subgroup $H$ is intrinsically defined as the subgroup of the abelian profinite group $C_{\mathbb{G}_n}(E)$ generated by elements of finite order prime to $p$. As such it is necessarily invariant with respect to the action of $\text{Aut}(E)$. The splitting $H \cong \mathbb{Z}/2n \times \mathbb{Z}/\mathbb{Z}/2$, however, is visibly not invariant with respect to this action.

3.6.3.2. In the next step we will construct an explicit $F(N)$-resolution of the trivial $\mathbb{Z}_n[[N]]$-module $\mathbb{Z}_p$. If we set $\overline{N} := N/(H \times E)$ then we have $\overline{N} = \mathbb{Z}_n^n \times \mathbb{Z}/n$ where the semidirect product is taken with respect to the $\mathbb{Z}/n$-module structure on $\mathbb{Z}_n^n$ given by the direct sum of the $n$ different characters of $\mathbb{Z}/n$.

Proposition 21.

a) The trivial $\mathbb{Z}_n[[N]]$-module $\mathbb{Z}_p$ admits a projective resolution

$$0 \to P_n \to \ldots \to P_1 \to P_0 \to \mathbb{Z}_p$$

with

$$P_r \cong \bigoplus_{(i_1, \ldots, i_r)} \chi(i_1 + \ldots + i_r) \mathbb{Z}_n^n$$

and where the direct sum is taken over all sequences $(i_1, \ldots, i_r)$ with $n - 1 \geq i_1 > i_2 > \ldots > i_r \geq 0$ (For $r = 0$ there is a unique summand 1 in $\overline{N}$ corresponding to the empty sequence).
b) Considered as a complex of $\mathbb{Z}_p[[N]]$-modules, via the projection $N \to \overline{N}$, this complex is an $\mathcal{F}(N)$-resolution in which all modules are summands in a finite direct sum of modules $\mathbb{Z}_p \leftarrow \mathbb{Z}_p^N$.

Proof. a) We start by observing that

$$H^*(\mathbb{Z}_p^n; \mathbb{Z}_p) \cong \Lambda \left( \bigoplus_{i=0}^{n-2} \chi(i) \right)$$

as a module over $\mathbb{Z}_p[\mathbb{Z}/n]$. The resolution is now constructed step by step as in the proof of Theorem 6.

The identity of $\mathbb{Z}_p$ considered as a map of $\mathbb{Z}_p[\mathbb{Z}/n]$-modules extends to a $\mathbb{Z}_p[[N]]$-linear homomorphisms $P_0 := \mathbb{Z}_p \leftarrow \overline{N} \to \mathbb{Z}_p$. Let $N_0$ be its kernel. Then we can compute $H_*(\mathbb{Z}_p^n, N_0/(p))$ and we find

$$H_0(\mathbb{Z}_p^n; N_0/(p)) \cong \bigoplus_{i=0}^{n-1} \chi(i) \otimes_{\mathbb{Z}_p} F_p,$$

as a module over $\mathbb{Z}_p[\mathbb{Z}/n]$. The $\mathbb{Z}_p[\mathbb{Z}/n]$-module $\bigoplus_{i=0}^{n-1} \chi(i)$ is projective and hence we can lift the resulting homomorphism $\bigoplus_{i=0}^{n-1} \chi(i) \to H_0(\mathbb{Z}_p^n, N_0/(p))$ to $N_0$ and by Nakayama’s Lemma the $\mathbb{Z}_p[[N]]$-linear extension $P_1 := \bigoplus_{i=0}^{n-1} \chi(i) \leftarrow \overline{N} \to N_0$ is onto. Then we repeat the game with the kernel $N_1$ of this epimorphism, and so on.

When we finally arrive at $N_{n-1}$ we see that

$$H_0(\mathbb{Z}_p^n; N_{n-1}/(p)) \cong \chi\left( \sum_{i=0}^{n-1} i \right) \otimes_{\mathbb{Z}_p} F_p, \quad H_i(\mathbb{Z}_p^n, N_{n-1}/(p)) = 0 \text{ if } i > 0.$$

Then we can construct in the same manner a map $P_n := \chi\left( \sum_{i=0}^{n-1} i \right) \leftarrow \overline{N} \to N_{n-1}$ which by Nakayama’s lemma is now even an isomorphism.

b) This is an immediate consequence of Lemma 14. □

3.6.4. Now we turn towards the problem of realizing the resolution $P_\bullet$ constructed in Proposition 21b, or rather the induced resolution $P_{\bullet} \leftarrow \mathbb{Z}_p^n$ of the induced module $\mathbb{Z}_p \leftarrow \mathbb{Z}_p^n$, by a sequence of maps between spectra.

Let $F$ be the subgroup of $N$ of order $pn^3$ described in Proposition 20. The characters $\chi(i)$ of the last section can be considered as characters of $F$ via the canonical projection $F \to F/(H \times E) = \text{Aut}(E) \cong \mathbb{Z}/n$. We will also need to consider the group $F_1 := F \cap S_n$, which is of order $pn^2$ and is generated by $\zeta_n$ and $\tau$, and the group $F_2$ which is cyclic of order $pn$ and is generated by $\zeta_n$ and $\tau^n$. (Of course, these groups are very different from the filtration subgroups $F_i$ that we considered in section 3.6.3. Those will not be used in the sequel so that there is no danger of confusion!)
Lemma 22. For any \( i \in \mathbb{Z} \) there is an isomorphism of Morava modules

\[
(E_n)_* \left( \Sigma^{2pni} E_n^{hF} \right) \cong \text{Hom}_{cts}(\chi(i) \uparrow_{\mathbb{Z}_p}^{\mathbb{G}_n}, (E_n)_*) .
\]

Proof. For \( i = 0 \) this is nothing but 1.3.2. More generally, for every \( k \in \mathbb{Z} \) there is an isomorphism of Morava-modules

\[
(E_n)_* \left( \Sigma^k E_n^{hF} \right) \rightarrow \text{Hom}_{cts}(\mathbb{Z}_p \uparrow_{\mathbb{F}_p}^{\mathbb{G}_n}, (E_n)_* (S^k)) .
\]

It is therefore enough to show that \((E_n)_* (S^{2pni})\) is isomorphic as an \((E_n)_*[F]\)-module to \((E_n)_* \otimes \chi(-i)\), or equivalently that there is an \((E_n)_*\)-module generator of \((E_n)_*\) of degree \(-2pni\) on which \(F\) acts via \(\chi(-i)\).

We recall that \(F\) is generated by \(\tau, \zeta_p\) and \(X = \omega \frac{d}{dx} S\). We need some information about the action of these elements on \((E_n)_*\). For this we recall that the action of an element \(g \in \mathbb{S}_n\) is determined as follows (cf. [DH1]): if we lift \(g\) to a power series \(\overline{g}(x) \in (E_n)_0[[x]]\) then there is a unique continuous ring homomorphism \(g_* : (E_n)_0 \rightarrow (E_n)_0\) and a unique *-isomorphism \(h \in (E_n)_0[[x]]\) from the formal group law \(g_* (\overline{\Gamma}_n)\) to the formal group law \(H\) defined by \(H(x, y) = \overline{g}^{-1} \overline{\Gamma}_n (\overline{g}(x), \overline{g}(y))\).

The action of \(g\) on \(u\) is then given by \(g_* (u) = \overline{g}'(0) h'(0) u\).

In particular, if \(g(x) = ax\) with \(a \in \mathbb{F}_p^\times\) and if the Teichmüller lift of \(a\) will, by abuse of notation, still be denoted by \(a\) then we can take as lift \(\overline{g}(x) = ax\). Then the \([p]\)-series of the formal group law \(H\) satisfies

\[
[p] H(x) = a^{-1} (\overline{p} \overline{\Gamma}_n (ax)) = a^{-1} (\overline{p} \overline{\Gamma}_n (ax))
\]

\[
= a^{-1} (p(ax) + \overline{p} u_1 (ax)^p + \overline{p} u_2 (ax)^{p^2} + \overline{p} u_{n-1} (ax)^{p^{n-1}})
\]

\[
= px + H u_1 a^{p-1} x^p + H \cdots + H u_{n-1} a^{p^{n-2}} x^{p^{n-1}} + H x^{p^n} .
\]

This shows that the *-isomorphism \(h\) is the identity, i.e. \(h(x) = x\), and that \(g_*\) is given by

\[
(8) \quad g_* (u_i) = a^{p^{i-1}} u_i, \quad g_* (u) = au .
\]

In the case of \(\tau\) we have \(a = \omega \frac{d}{dx} \frac{1}{x^{p^{n-1}}}\), so that

\[
(9) \quad \tau_* (u) = \omega \frac{d}{dx} \frac{1}{x^{p^{n-1}}} u .
\]

The action of \(\zeta_p\) is more difficult. However, we will only need to know that \(\zeta_p\) fixes \(u\) modulo the maximal ideal \(m = (p, u_1, \ldots, u_{n-1}) \subset (E_n)_*\), i.e.

\[
(10) \quad \zeta_p (u) \equiv u \mod m
\]

which holds for any element \(g\) in the \(p\)-Sylow subgroup \(S_n\) of \(S_n\).

The universal deformation \(\overline{\Gamma}_n\) is already defined over \(\mathbb{Z}_p[[u_1, \ldots, u_{n-1}]\) and therefore the subgroup of \(\mathbb{G}_n\) given by Galois automorphisms of \(\mathbb{F}_q\) acts trivially on the \(u_i\) and on \(u\). Together with (8) this implies

\[
(11) \quad X_* (u) = \omega \frac{d}{dx} u .
\]
The action of $\tau$ and of $\zeta_p$ on $(E_n)_*$ is $\mathbb{W}_q$-linear while the action of $X$ is only $\mathbb{Z}_p$-linear and satisfies $X_*(wX) = w^n X_*(x)$ if $w \in \mathbb{W}_q$ and $x \in (E_n)_*$. Now consider the element

$$\Delta' := \prod_{g \in F_2} g_*(u).$$

This is clearly fixed by the subgroup $F_2$. Furthermore by (9) and (10) we have

$$\Delta' = \prod_{i=1}^n \tau_*^{n_i} (u^p) = \prod_{i=1}^n \omega^{\frac{n_i}{p^{n_i}}} u^p = \omega^{\frac{n_1}{p^{n_1}}} u^p = \omega^{p^{n_1} \Delta'} \mod m$$

so that $\Delta'$ is an $(E_n)_*$-generator of degree $-2pn$. Now $F_2$ is normal in $F$ and the quotient $F/F_2$ is isomorphic to $\mathbb{Z}/n \times \mathbb{Z}/n$ with generators $\tau$ and $X$. This quotient acts on the $F_2$-fixed points and by (11) we find for any $\lambda \in \mathbb{W}_q$

$$X_*(\lambda \Delta') = \lambda^n X_*(\Delta') = \lambda^n \prod_{g \in F_2} g_*(u) = \lambda^n \prod_{g \in F_2} g_*(\omega^{\frac{n_1}{p^{n_1}}} u^p) = \lambda^n \omega^{\frac{n_1}{p^{n_1}} \Delta'}. $$

In particular we get $X_*(\omega^{-\frac{n_1}{p^{n_1}}} \Delta') = \omega^{-\frac{n_1}{p^{n_1}} \Delta'}$, and thus

$$\Delta'' := \omega^{-\frac{n_1}{p^{n_1}} \Delta'}$$

is still an $(E_n)_*$-generator in $(E_n)_{-2pn}$ which is fixed by $\tau^n$, $\zeta_p$, and $X$, and which satisfies

$$\tau_*(\Delta'') = \omega^{-\frac{n_1}{p^{n_1}} \tau_*(\Delta')} = \omega^{-\frac{n_1}{p^{n_1}}} \prod_{g \in F_2} g_*(\tau_*(u))$$

$$= \omega^{-\frac{n_1}{p^{n_1}}} \prod_{g \in F_2} g_*(\omega^{\frac{n_1}{p^{n_1}}} \omega^{\frac{n_1}{p^{n_1}}} \Delta') = \omega^{-\frac{n_1}{p^{n_1}}} \omega^{\frac{n_1}{p^{n_1}}} \Delta''. $$

Therefore, for any $i \in \mathbb{Z}$, the class $(\Delta'')^i$ is an $(E_n)_*$-generator in $(E_n)_{-2pn}$ on which $F$ acts via $\chi(-i)$ (with the convention adopted in the discussion before Proposition 20).

We will need some partial information about the homotopy groups of the spectra $\Sigma^{2p^n i} E_n^{hG}$, $i \in \mathbb{Z}$, where $G$ runs through various subgroups of $F$ which contain the central subgroup $Z \subset F$ generated by $\tau^n$.

**Lemma 23.** Suppose $G$ is a subgroup of $F$ which contains $Z$.

a) If $G$ is of order prime to $p$, then for any $i \in \mathbb{Z}$ the homotopy fixed point spectral sequence converging to $\pi_* (\Sigma^{2p^n i} E_n^{hG})$ satisfies $E^{s,t}_2 = 0$ if $s > 0$. The spectral sequence collapses at $E_2$ and $\pi_* (E_n^{hG}) \cong (E_n)_G$ is concentrated in degrees divisible by $2n$.

b) If $G$ contains an element of order $p$ then for any $i \in \mathbb{Z}$ the homotopy fixed point spectral sequence converging to $\pi_* (\Sigma^{2p^n i} E_n^{hG})$ satisfies

$$\pi_q (\Sigma^{2p^n i} E_n^{hG}) \cong \begin{cases} 
0 & \text{if } q \text{ is even, } 0 < q < 2n \\
E^{0,0}_2 & \text{if } q = 0 \\
E^{s(q),s(q)+q}_2 & \text{if } q \text{ is odd, } q \not\equiv \{1, 3, \ldots, 2p-3\} \mod 2pn \\
E^{s(q),s(q)+q}_2 & \text{if } q \text{ is odd, } q \equiv \{1, 3, \ldots, 2p-3\} \mod 2p^2n 
\end{cases}$$

where $s(2q+1) = 2(p - 2 - q) + 1$. 


Proof. a) If $G$ contains no elements of order $p$ then it is clear that $E_2^{s,t} = 0$ if $s > 0$ and the spectral sequence collapses. Furthermore, $G$ contains the central subgroup $Z$ and this subgroup acts trivially on $(E_n)_0$ (by (8)) and because of $(\tau^n)_\ast (u) = \omega_{\frac{2n}{p}}^t$ (again by (8)) we see that $E_0^{0,t} = (E_n)_0^G$ is trivial if $t \neq 0 \mod 2n$.

b) Because $G$ always contains $Z$ and because $E$ is normal in $F$, the assumption implies that $G$ contains $F_2 = Z \times E$. Because the index of $F_2$ in $G$ is prime to $p$ the homotopy fixed point spectrum $E_n^{hG}$ is a direct summand in $E_n^{hF_2}$ and it is therefore enough to discuss the case $G = F_2$.

In the case of $G = F_1$ the homotopy fixed point spectrum sequence has been analyzed by Hopkins and Miller. Their account remains unpublished. A summary of this analysis is given in section 2 of [N]. If $p = 3$ a rather detailed discussion which includes the case of $F_2$ can be found in [GHM] and [GHMRL]. The approach used in these papers generalizes without much problems to the case of any $p > 2$. In the following we will describe the $E_2$-term and the differentials of this spectral sequence.

First of all, let $\rho$ be the $(p-1)$-dimensional $\mathbb{Z}_p[F_2]$-module which restricted to $E$ is the reduced regular representation and on which the central element $\tau^n$ acts by multiplication by $\omega_{\frac{2n}{p}}^t$. Then, as a graded $\mathbb{Z}_p[F_2]$-algebra, $(E_n)_0$, is isomorphic to the completion of $\mathbb{S}_\ast (\rho)[N^{-1}]$ at its maximal ideal, where $\mathbb{S}_\ast (\rho)$ is the graded symmetric algebra on $\rho$ with $\rho$ in degree $-2$, and we have inverted $N := \prod_{\rho \in F_2} g_\rho(e) \in S_\ast (\rho)$ where $e$ is a suitable generator of $\rho$ (cf. [GHMRL, Lemma 3.2]). In fact, the isomorphism identifies $N$ up to a scalar, with the element $\Delta''$ of the proof of Lemma 22. This isomorphism can be used to calculate the $E_2$-term of the homotopy fixed point spectral sequence as follows (cf. section 2 in [N], or Theorem 3.7 in [GHMRL] if $p = 3$):

- The invariants $E_2^{0,t}$ are trivial unless $t \equiv 0 \mod 2n$ and periodic of period $2pm$ with periodicity generator $\Delta'' \in (E_n)_{2pm}$. In the sequel we let $\Delta := (\Delta'')^{-1} \in (E_n)_{2pm}$.
- Multiplication with $p$ annihilates $E_2^{s,r}$ if $s > 0$. Furthermore, there are elements $\alpha \in E_2^{1,2p-2}$ and $\beta \in E_2^{2,2p-2}$ such that, as a module over $F_q[\Delta^{\pm 1}]$,
  - $E_2^{2k,s}$, for $k > 0$, is free with generator $\beta^k$,
  - $E_2^{2k+1,s}$, for $k \geq 0$, is free with generator $\alpha \beta^k$.

The elements $\alpha$ and $\beta$ are infinite cycles and represent the images of the elements $\alpha_1 \in \pi_{2p-3}(S^0)$, $\beta_1 \in \pi_{2(p^2 - p - 1)}(S^0)$ with respect to the localization map $S^0 \to L_{K(n)} S^0$.

The only non-trivial differentials in this spectral sequence are $d_{2p-1}$ and $d_{2n^2+1}$. They are forced by Toda’s relations $\alpha_1 \beta^p_1 = 0$ and $\beta_1^{p+1} = 0$ and are determined by

\[ d_{2p-1}(\Delta^n) = c \alpha \beta^{p-1}, \quad d_{2n^2+1}(\Delta^n \alpha) = c' \beta^{n^2+1}, \]

i.e. $d_{2p-1}(\Delta) = -c\Delta^{1-n} \alpha \beta^{p-1}$, and $d_{2n^2+1}(\Delta \alpha) = c' \Delta^{-n^2-2} \beta^{n^2+1}$ where $c, c'$ are suitable units in $F_q$.

Then we end up with the following result which is more precise than Lemma 23 above.

Proposition 24 (cf. [N, Proposition 2.1, 2.2]).

a) $\pi_\ast (E_n^{hF_2})$ is periodic of period $2p^2n$ and with periodicity generator $\Delta^p$.

b) $E_\ast^{s,r}$ is trivial if $s$ is even and $s > 2n^2$, or if $s$ is odd and $s > 2n - 1$.

\[ \text{Note that the element } \Delta \text{ in [N] corresponds to } \Delta^{p-1} = \Delta^n \text{ in this paper.} \]
c) If $2n^2 \geq s = 2k > 0$ then $E_{\infty}^s$ is a free module over $F_q[\Delta^{+p}]$ with generator $\beta^k$, of total degree $2k(p^2 - p - 1)$.

d) If $2n - 1 \geq s = 2k + 1 > 0$ then $E_{\infty}^s$ is a free module over $F_q[\Delta^{+p}]$ with generators $\Delta^l \beta^k \alpha$, $2 \leq l \leq p$ of total degree $2pm + 2k(p^2 - p - 1) + 2p - 3$.

e) $E_{\infty}^0 = 0$ if $t \not\equiv 0 \mod 2n$. 

Furthermore, if $s = 2k + 1 \leq 2n - 1$ and $t \not\equiv 2pm + 2k(p^2 - p) + 2(p - 1) \mod 2p^2n$ then $E_2^t \cong E_{\infty}^s$. And finally, $E_{\infty}^s = E_{\infty}^{s'}$ unless $t = 2pm$ and $t' \not\equiv 0 \mod p$. ☐

After these preparations we can continue with the proof of part (b) of Lemma 23. First we investigate which of the generators in part (c) and (d) of Proposition 24 can contribute to $\pi_q$ for $q$ as in Lemma 23. We distinguish two cases according to the parity of $q$.

1) If $0 \leq q = 2q' < 2n$ then it is enough to show that there is no $k$ with $0 < k \leq n^2$ such that

$$2k(p^2 - p - 1) \equiv 2q' \mod 2p^2n.$$ 

Calculating modulo $2pm$ gives

$$-2k \equiv 2q' \mod 2pn$$ 

and because of $0 < 2k \leq 2n^2$ and $0 \leq 2q' < 2n$ this is clearly impossible. In particular, we see that $\pi_q = 0$ if $q$ is even and $0 < q < 2n$ and $\pi_q \approx E_2^{0,1}$.

2) If $q = 2q' + 1$ then we have to consider the congruence

$$(12) \quad 2pn + 2k(p^2 - p - 1) + 2p - 3 \equiv 2q' + 1 \mod 2p^2n.$$ 

Reducing mod $2pn$ gives

$$-2k + 2p - 3 \equiv 2q' + 1 \mod 2pn.$$ 

and thus $q' \equiv p - 2 - k \mod pn$. In view of $0 \leq k \leq p - 2$ this implies that modulo $pn$ we have $q' \in \{0, 1, \ldots, p - 2\}$, i.e. $q \in \{1, 3, \ldots, 2p - 3\}$ modulo $2p$.

In particular, for any given $q$ we see that $k$ (with $0 \leq k \leq n - 1$) is uniquely determined by (12) and $\Delta^l \beta^k \alpha$ is in filtration $2k + 1 = 2(p - 2 - q' + 1)$ as required.

It remains to show that $l \not\equiv 1 \mod p$ if $q \in \{1, 3, \ldots, 2p - 3\}$ modulo $2p^2n$. Calculating modulo $2p^2n$ gives

$$2pn + 2k(p^2 - p - 1) + 2p - 3 - (2q' + 1) = 2pn(l + k) + 2(-k + p - 2 - q') \equiv 0 \mod p.$$ 

and this cannot be $0$ modulo $2p^2n$ if $l \equiv 1 \mod 2p^2n$ and $0 \leq k \leq p - 2$. Therefore the total degree of the elements $\Delta^l \beta^k \alpha$, for $0 \leq k \leq p - 2$, cannot be equal to $q$, if $l \equiv 1 \mod p$, $q$ is odd and $q \equiv \{1, 3, \ldots, 2p - 3\}$ mod $2p^2n$. ☐

We can finally state and prove the following realization result.
Theorem 25. There is a resolution of length \( n \) (in the sense of 3.3.1)

\[ * \rightarrow E_n^{hN} := X_{n-1} \rightarrow X_0 \rightarrow \ldots \rightarrow X_n \rightarrow * \]

such that the complex \( E_\bullet(X_\bullet) \) is isomorphic, as a complex of Morava modules, to the complex \( \text{Hom}_{cts}(P_\bullet \mid_{G_n^1}, (E_n)_\bullet) \) where \( P_\bullet \) is the complex of Proposition 21. Furthermore, for \( r > 0 \) we have

\[ X_r \simeq \bigvee_{(i_1, \ldots, i_r)} \Sigma^{2\hat{a}^2n(i_1 + \ldots + i_r)} E_n^{hF} \]

where \( F \) is the finite subgroup of \( N \) of Proposition 20d and where the wedge is taken over all sequences \((i_1, \ldots, i_r)\) with \( n - 1 \geq i_1 > i_2 > \ldots > i_r \geq 0 \) (For \( r = 0 \) there is a unique summand \( E_n^{hF} \) corresponding to the empty sequence).

Proof. Because of \( \chi(pi) \cong \chi(i) \) Lemma 22 implies that the spectra \( X_r \) realize the Morava modules \( \text{Hom}_{cts}(P_r \mid_{G_n^1}, (E_n)_\bullet) \). So it remains to realize the maps and show that the resulting sequence of maps of spectra can be refined into a resolution in the sense of 3.3.1.

It follows easily from Proposition 24 that \( \Sigma^{2\hat{a}^2n} E_n^{hF} \) is, for each \( i \), a direct wedge summand in \( E_n^{hF} \). Therefore, in order to realize the maps it is enough to show that the \( (E_n)_\bullet \)-Hurewicz homomorphism

\[ (13) \quad \pi_0(F(E_n^{hF_1}, E_n^{hF_2})) \rightarrow \text{Hom}(E_n, (E_n)(E_n^{hF_1})) \]

is an isomorphism (onto the group of degree preserving homomorphisms). By Proposition 3 this happens iff the canonical map

\[ (14) \quad \pi_0(E_n[[G_n/F_2]])^{hF_2} \rightarrow (E_n)_0[[G_n/F_2]]^{F_2} \]

is an isomorphism. Furthermore, if \( U_\infty \) is a decreasing sequence of open subgroups of \( G_n \) with \( F_2 = \bigcap_j U_j \) then

\[ E_n[[G_n/F_2]] \simeq \text{holim}_j E_n[[G_n/U_j]] \]

By Proposition 2 we have

\[ E_n[[G_n/U_j]]^{hF_2} \simeq \prod_{F_2 \backslash G_n/\U_j} E_n^{hF_{x,j}} \]

where \( F_{x,j} \) is the isotropy of the coset \( xU_j \) with respect to the action of \( F_2 \) on \( G_n/U_j \). Because \( Z \subset F_2 \) is central each \( F_{x,j} \) contains \( Z \). Then Lemma 23 implies that the canonical maps \( \pi_0(E_n^{hF_{x,j}}) \rightarrow (E_n)_0^{F_{x,j}} \) are isomorphisms for all \( x \) and \( j \) and therefore

\[ \pi_0(E_n[[G_n/U_j]]^{hF_2}) \rightarrow (E_n)_0[[G_n/U_j]]^{F_2} \]

is an isomorphism for each \( j \). Furthermore, it is clear that \( (E_n)_1[[G_n/U_j]] = 0 \) for all \( j \), and by the remark on the Mittag-Leffler condition following Proposition 2 the relevant \( \lim^1 \)-terms for the homotopy groups of \( \text{holim}_j (E_n)[[G_n/U_j]]^{hF_2} \) are also trivial. Therefore we obtain the desired isomorphism (14) by passing to the limit.
We have now proved that all maps $X_r \to X_{r+1}$ can be (uniquely) realized and the compositions of two successive maps are trivial.

It remains to construct the factorizations $X_r \to C_r \to X_{r+1}$, $0 \leq r \leq n - 1$, such that $C_{r-1} \to X_r \to C_r$ is a cofibration. This will be done inductively. We note that these factorizations will realize the splitting of the exact complex of Morava modules $E_\ast(X_\ast)$ into the usual short exact sequences. In particular, this will show that $C_n \simeq X_n$ so that the resolution will be automatically of length $n$.

For $r = 0$ (where we take $C_{-1} = X_{-1}$) this is just a consequence of the fact that the composition $X_{-1} \to X_0 \to X_1$ is null. Now suppose that we have already constructed

$$C_{r-1} \to X_r \to C_r,$$  

$0 \leq r \leq k < n - 1$.

We need to show that the composition

$$C_k \to X_{k+1} \to X_{k+2}$$

is null so that we can factor it through the cofibre $C_{k+1}$ of the map $C_k \to X_{k+1}$. For this it is enough to show that the induced map $[C_k, X_{k+2}] \to [X_k, X_{k+2}]$ is injective, which will be implied by $[\Sigma C_{k-1}, X_{k+2}] = \pi_1(F(C_{k-1}, X_{k+2})) = 0$.

Now the inductively already constructed part of the resolution

$$\ast \to X_{-1} \to X_0 \to \ldots \to X_{k-1} \to C_{k-1} \to \ast$$

can be viewed as a tower of (co)fibrations for $C_{k-1}$ which we can use to compute $\pi_1(F(C_{k-1}, X_{k+2}))$. In fact, there is an Adams type spectral sequence associated to this tower which has the form

$$E_1^{p,q} \Rightarrow \pi_{q-p}(F(C_{k-1}, X_{k+2}))$$

with

$$E_1^{p,q} \cong \begin{cases} 
\pi_q(F(X_{k-1-p}, X_{k+2})) & \text{if } 0 \leq p \leq k \\
0 & \text{if } p > k
\end{cases}$$

so that it is enough to show that $E_2^{p,p+1} = 0$ for all $p$.

Now Proposition 2 (including the remark on the Mittag Leffler condition following it) together with Lemma 23 implies already $E_1^{p,q} = 0$ if $0 \leq p \leq k < n$ and $q = p+1$ is even.

If $0 \leq p \leq k$ and $q = p+1$ is odd we let $Y_r := \bigvee_{(i_1, \ldots, i_r)} \Sigma^{2p^2 \ast \sum i_1 \ldots + i_r} E_n$ so that $X_r = (Y_r)^{hF}$. We claim that there are natural isomorphisms

$$E_1^{p,q} \cong \pi_q(F(X_{k-1-p}, (Y_{k+2})^{hF})) \cong \pi_q(F(X_{k-1-p}, Y_{k+2})^{hF})$$

$$\cong H^{s(q), s(q)+q}(F, \text{Hom}_{(E_n), \ast}[\mathbb{G}_n][E_n], (X_{k-1-p}), (E_n), (Y_{k+2}))$$

$$\cong H^{s(q), s(q)+q}(F, \text{Hom}_{(E_n), \ast}(P_{k-1-p}, (E_n), \pi_\ast(Y_{k+2})))$$

where $s(q)$ is as in Lemma 23. If we accept these isomorphism for the moment then we can finish off the proof because $P_{\ast} \uparrow_{\mathbb{G}_n} F$ is $F(\mathbb{G}_n)$-exact and therefore $E_2^{p,q} = 0$ for all $q$ with $q$ odd and $1 \leq q \leq 2p - 3$, in particular for all $q$ with $q = p+1$ odd and $0 \leq p \leq k \leq n$. 

It remains to justify the chain of isomorphism which identify $E^n_{i,h}$. The first two of the claimed isomorphisms are obvious and the last one holds because by 1.3.2 $(E_n)_*(Y_r)$ is a coinduced module, i.e.

$$(E_n)_*(Y_r) \cong \text{Hom}(E_n)_*[[G_n]](Z_p[[G_n]], \pi_*(Y_r))$$.

For the third isomorphism it suffices to show that the homotopy fixed point spectral sequence gives us an isomorphism

$$\pi_q(F(E_{n,h}^F, E_{n,h}^F)) \cong H^q((q)_0 \cup q(F, \text{Hom}(E_n)_*[[G_n]]((E_n)_*, (E_n)_*)), (E_n)_*).$$

In fact, if we replace $E_{n,h}^F$ in the source by $E_{n,h}^U$ where $U$ is an open subgroup of $G$, then this follows from Proposition 2 and Proposition 3 together with Lemma 23. Finally we write $E_{n,h}^U \simeq L_{K(n)}\text{holim}_j(E_{n,h}^U)$ and pass to the limit once more using the remark on the Mittag-Leffler condition following Proposition 2. □

3.6.5. The resolution of $L_{K(n)}S^0$.

It remains to construct a resolution of $L_{K(n)}S^0$. For this we start with an $F$-resolution $Q_*$ of finite length of the trivial $Z_p[[G_n]]$-module $Z_p$ as given by Proposition 17 and Proposition 21. The modules $Q_r$ in this resolution are finitely generated projectives if $r > n$, and of the form $P_r \cap G_N$ with $P_r$ as in Proposition 21 and $P_r'$ finitely generated projective, if $r \leq n$. We can realize these modules by spectra $Z_r$ which are summands in a finite wedge of $E_n$'s if $r > n$ and of the form $X_r \vee Z_r'$ with $X_r$ as in Theorem 25 and $Z_r'$ a summand in a finite wedge of $E_n$'s. Then the strategy of the proof of Theorem 25 can be applied to this situation. The $Z_p[[G_n]]$-linear maps $Q_r \rightarrow Q_{r-1}$ can be uniquely realized by maps $X_{r-1} \rightarrow X_r$ of spectra and the resulting sequence of spectra is a resolution of finite length in the sense of 3.3.1.

In fact, the factorizations $C_{r-1} \rightarrow Z_r \rightarrow C_r$, $0 \leq r \leq n$, can be constructed just as in the proof of Theorem 25 by using again that $Q_* \rightarrow Z_p$ is $F(G_n)$-exact. For $r > n$ the factorizations are easy because the spectra $Z_r$ are $E_n$-injective and because ordinary exactness of $Q_* \rightarrow Z_p$ implies that the sequence $L_{K(n)}S^0 \rightarrow Z_*$ is $E_n$-exact. Therefore we obtain the following result.

**Theorem 26.** Suppose $p$ is odd and $n = p - 1$ and suppose $Q_* \rightarrow Z_p$ is an $F(G_n)$-resolution of length $m$ of the trivial $Z_p[[G_n]]$-module $Z_p$ such that $Q_r$ is a finitely generated projective $Z_p[[G_n]]$-module if $r > n$ while $Q_r \cong P_r \cap G_N \otimes P_r'$ with $Q_r'$ finitely generated projective and $P_r$ as in Theorem 25 resp. Proposition 21 if $r \leq n$.

a) Then there is a resolution of length $m$ (in the sense of 3.3.1)

$$* \rightarrow L_{K(n)}S^0 := Z_{-1} \rightarrow Z_0 \rightarrow \ldots \rightarrow Z_n \rightarrow \ldots \rightarrow Z_m \rightarrow *$$

in which $Z_r$ is a direct summand in a finite wedge of $E_n$'s if $r > n$ while

$$Z_r \simeq \bigvee_{(i_1, \ldots, i_r)} \Sigma^{-2p^2n(i_1 + \ldots + i_r)} E_{n,h}^F \vee Z_r'$$

and $Z_r'$ is a direct summand in a finite wedge of $E_n$'s if $r \leq n$.

b) The complex $(E_\bullet(X_\bullet))$ is isomorphic, as a complex of Morava modules, to $(E_\bullet)_* \rightarrow \text{Hom}_{cts}(Q_*(E_\bullet)_*)$. □
3.6.6. Proof of Theorem 10 and Theorem 11.

By Theorem 26 it is enough to prove Theorem 10 and for this we only need to show that the kernel $K$ of the augmentation $Z_3 \xrightarrow{G} Z_3$ admits a projective resolution of the form

$$0 \rightarrow Z_3 \xrightarrow{G_2} Z_3 \xrightarrow{(Z_3 \oplus \lambda_2)} Z_3 \xrightarrow{\lambda_2 \oplus \chi} Z_3 \xrightarrow{\chi} Z_3 \xrightarrow{K} 0.$$

In fact, because of the isomorphism $G_2 \cong G_2 \times Z_3$ (cf. [GHMR1]) it is enough to show that the kernel $K^1$ of the augmentation $Z_3 \xrightarrow{G_1} Z_3$ with $N^1 = N_{G_2}(E)$ admits a projective resolution of the form

$$0 \rightarrow Z_3 \xrightarrow{G_2} Z_3 \xrightarrow{\lambda_2} Z_3 \xrightarrow{\chi} Z_3 \xrightarrow{K^1} 0.$$

From [He] we know that $\text{Ext}^i_{Z_3[[S^3_1]]}(K^1, F_3)$ can be identified with the cokernel of the map

$$H^i(S^1_2; F_3) \rightarrow \prod_{i=0}^1 H^i(\omega^i C_{S^1_2}(E) \omega^{-i}; F_3)$$

given by the inclusions $\omega^i C_{S^1_2}(E) \omega^{-i} \rightarrow S^1_2$, $i = 0, 1$. Furthermore, this map is $\mathbb{Z}_3[SD_{16}]$-linear and from the explicit description of this map in Theorem 4.4 of [GHMR1] it is straightforward to see that, as $\mathbb{Z}_3[SD_{16}]$-modules, we obtain

$$\text{Ext}^i_{Z_3[[S^3_1]]}(K^1, F_3) \cong \begin{cases} 
\chi \downarrow_{SD_{16}} \otimes \mathbb{Z}_3 F_3 & \text{if } i = 0 \\
\lambda_2 \downarrow_{SD_{16}} \otimes \mathbb{Z}_3 F_3 & \text{if } i = 1 \\
Z_3 \downarrow_{SD_{16}} \otimes \mathbb{Z}_3 F_3 & \text{if } i = 2 \\
0 & \text{if } i > 2.
\end{cases}$$

The resolution of $K^1$ is then constructed as in the proof of Theorem 6. □

**Appendix: Splitting $E_n$ with respect to $F_{n(q-1)}$**

A.1. Let $q = p^n$ and $F := F_{n(q-1)} = C_{q-1} \rtimes \text{Gal}(F_q/F_p)$ be as in section 2.3. The homotopy groups of $E_n$ can be described as

$$\pi_*(E_n) \cong \mathbb{W}_{F_q}[v_1, \ldots, v_{n-1}][u^\pm 1] \sim,$$

with $v_i := u_i u_i^{-1}$, i.e. $u_i = v_i u_i^{-1}$, and where $\sim$ means completion with respect to the ideal $(u_1, \ldots, u_{n-1})$. The elements $v_i$ are invariant with respect to the action of $F$ and the action of a generator $\omega$ of the cyclic subgroup $C_{q-1}$ on $u$ is given by multiplication with $\omega$, i.e. $\omega(u) = \omega^i u$ (cf. formula (8) in the proof of Lemma 22), while the Galois group acts only on the coefficients via the lifting of Frobenius.

In particular, the homotopy groups of the homotopy fixed points are given by the appropriate completion

$$\pi_*(E_n^{HF}) \cong \mathbb{Z}_p[v_1, \ldots, v_{n-1}][v_n^\pm 1] \sim.$$
with $v_n = u^{-q}$. Furthermore we have an isomorphism of $\pi_\ast(E_n^bF)[F]$-modules

$$\pi_\ast(E_n) \cong \bigoplus_{i=0}^{q-1} \pi_\ast(E_n^bF) \otimes \mathbb{W}_{\mathbb{F}_q} u^i$$

and the action on the modules $\lambda_i := \mathbb{W}_{\mathbb{F}_q} u^i$ is as described above. We are interested in the decomposition of $E_n$ obtained from a splitting of the group algebra $\mathbb{Z}_p[F]$ as a module over itself. The decomposition suggested by (15) is close but not quite equal to the decomposition obtained from such a splitting.

A.2. From now on we will restrict attention to the case $n = 2$ and $p$ odd which has actually been used in section 2.3 and section 3.3.

If $i$ is divisible by $p + 1$ then $\omega^i$ belongs to $\mathbb{Z}_p$ and therefore the action of $\omega$ on $\lambda_i$ commutes with the action of Frobenius. Therefore, for such $i$, $\lambda_i$ splits into 2 one-dimensional pieces $\lambda_{i, +}$ and $\lambda_{i, -}$, with Frobenius acting trivially on $\lambda_{i, +}$ and by multiplication by $-1$ on $\lambda_{i, -}$.

We claim that there is a direct sum decomposition of $\mathbb{Z}_p[F]$ into a direct sum of $\mathbb{Z}_p[F]$-modules

$$\mathbb{Z}_p[F] \cong \bigoplus_{i \neq 0(p-1)} \lambda_i \oplus \bigoplus_{i=0(p-1)} (\lambda_{i, +} \oplus \lambda_{i, -}).$$

We leave it to the reader to check that the modules on the right hand side of this isomorphism are all indecomposable and that the only repetition in the decomposition comes from the isomorphisms $\lambda_i \cong \lambda_{pi}$, $i \neq 0$ mod $p + 1$, which are induced by Frobenius.

We give an outline of a direct construction of the decomposition: our identification of the roots of unity of $\mathbb{W}_{\mathbb{F}_q}$ with $C_{q-1}$ specifies a character $\chi_1$ of $C_{q-1}$ defined over $\mathbb{W}_{\mathbb{F}_q}$. Let $\chi_i = \chi_1^\otimes i$. Then the elements

$$e_i = \frac{1}{(q-1)} \sum_{g \in C_{q-1}} \chi_i(g^{-1})g$$

belong to $\mathbb{W}_{\mathbb{F}_q}[C_{q-1}]$ and they are easily checked to be orthogonal idempotents which sum up to the element $1 \in \mathbb{W}_{\mathbb{F}_q}[C_{q-1}]$. The elements $e_i$ form a basis of $\mathbb{W}_{\mathbb{F}_q}[C_{q-1}]$ as a $\mathbb{W}_{\mathbb{F}_q}$-module and each $e_i$ generates a one-dimensional representation over $\mathbb{W}_{\mathbb{F}_q}$ on which $C_{q-1}$ acts via $\chi_i$.

For $i \equiv 0 \bmod (p + 1)$ the element $e_i$ lives already in $\mathbb{Z}_p[C_{q-1}]$, while for $i \neq 0 \bmod (p + 1)$ the elements $e_i + e_{pi}$ and $\omega^{i+1}(e_i - e_{pi})$ belong to $\mathbb{Z}_p[C_{q-1}]$ and together they form a basis of $\mathbb{Z}_p[C_{q-1}]$ as a $\mathbb{Z}_p$-module. Furthermore, the $\mathbb{Z}_p$-module generated by $e_i$, $i \equiv 0 \bmod (p + 1)$, is a $\mathbb{Z}_p[C_{q-1}]$-module on which $C_{q-1}$ acts via $\chi_i$, and the $\mathbb{Z}_p$-submodule generated by $\delta_i := e_i + e_{pi}$ and $\varepsilon_i := \omega^{i+1}(e_i - e_{pi})$ is also a $\mathbb{Z}_p[C_{q-1}]$-module which is isomorphic to $\lambda_i$ restricted to $C_{q-1}$.

Now consider the isomorphism of $\mathbb{Z}_p[C_{q-1}]$-modules

$$\mathbb{Z}_p[F] \cong \mathbb{Z}_p[C_{q-1}] \oplus \mathbb{Z}_p[C_{q-1}] \sigma$$
where $\sigma$ is Frobenius considered as an element of $F$. Then there is a $\mathbb{Z}_p\sigma$-basis of $\mathbb{Z}_p[F]$ given by $e_i \pm e_i$, if $i \equiv 0 \mod (p+1)$, and $\delta_i \pm \delta_i \sigma$, $\varepsilon_i \pm \varepsilon_i \sigma$ if $i \not\equiv 0 \mod (p+1)$. In $\mathbb{W}_p[F]$ we have $\sigma e_i = e_i \sigma$, in particular $\sigma e_i = e_i \sigma$ if $i \equiv 0 \mod (p+1)$, and therefore $\sigma(e_i \pm e_i) = \pm(e_i \pm e_i \sigma)$ if $i \equiv 0 \mod (p+1)$ i.e. $(e_i \pm e_i \sigma)$ generates a direct summand isomorphic to $\lambda_i$. Furthermore we have $\sigma \delta_i = \delta_i \sigma$, $\varepsilon_i = -\varepsilon_i \sigma$ which shows that the $\mathbb{Z}_p$-submodule generated by $\delta_i + \delta_i \sigma$ and $\varepsilon_i + \varepsilon_i \sigma$ is a $\mathbb{Z}_p[F]$-module, and likewise the $\mathbb{Z}_p$-submodule generated by $\delta_i - \delta_i \sigma$ and $\varepsilon_i - \varepsilon_i \sigma$ is a $\mathbb{Z}_p[F]$-module. Both of these modules are isomorphic to $\lambda_i$ (or $\lambda_i$).

A.3. Consequently, by the elementary theory of stable splittings we find that $E_n$ splits into a direct sum of $E_n^{HF}$-module spectra whose homotopy groups are given by

$$\text{Hom}_{\mathbb{Z}_p[F]}(\lambda_i, \bigoplus_{j=0}^{q-1} \mathbb{W}_p u^j) \otimes \pi_*(E_n^{HF}) \cong (\mathbb{W}_p u^j)_{\pm} \otimes \pi_*(E_n^{HF})$$

resp.

$$\text{Hom}_{\mathbb{Z}_p[F]}(\lambda_i, \bigoplus_{j=0}^{q-1} \mathbb{W}_p u^j) \otimes \pi_*(E_n^{HF}) \cong (\mathbb{Z}_p u^i \oplus \mathbb{Z}_p u^{mi}) \otimes \pi_*(E_n^{HF}) .$$

where $(\mathbb{W}_p u^j)_{\pm}$ is the $\pm$-eigenspace of the action of Frobenius on $\mathbb{W}_p u^j$. In the first case, the corresponding summand of $E_n$ can be identified as $E_n^{HF}$-module spectrum with $\Sigma^{2i} E_n^{HF}$, in the second case with $\Sigma^{2i} E_n^{HF} \vee \Sigma^{2mi} E_n^{HF}$ and the multiplicity of this spectrum in a splitting of $E_n$ (constructed via the action of $F$) is 2.

References


