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# On the generalized Riemann-Hilbert problem with irregular singularities

A. A. Bolibruch<sup>¶</sup>, S. Malek<sup>||</sup> and C. Mitschi<sup>\*\*</sup>

## Abstract

We study the generalized Riemann-Hilbert problem, which extends the classical Riemann-Hilbert problem to the case of irregular singularities. The problem is stated in terms of generalized monodromy data which include the monodromy representation, the Stokes matrices and the true Poincaré rank at each singular point. We give sufficient conditions for the existence of a linear differential system with such data. These conditions are in particular fulfilled when the monodromy representation is irreducible, as in the classical case. We solve the problem almost completely in dimension two and three. Our results have applications in differential Galois theory. We give sufficient conditions for a given linear algebraic group  $G$  to be the differential Galois group over  $\mathbb{C}(z)$  of a differential system with the smallest possible number of singularities, and with singularities all Fuchsian but one, at which the Poincaré rank is minimal.

There are many approaches to differential equations. One may focus on the existence and behaviour of the solutions, or on algebraic properties of their symmetries. One may also ask for the existence of differential equations that satisfy specific inverse problems such as the Riemann-Hilbert problem, the Birkhoff standard form problem or the inverse problem in differential Galois theory. This article is an attempt to relate the three problems through the statement and solutions of the generalized Riemann-Hilbert problem.

The classical Riemann-Hilbert problem asks for conditions under which a given representation

$$\chi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}, z_0) \longrightarrow \mathrm{GL}(p; \mathbb{C})$$

of the fundamental group of the Riemann sphere  $\mathbb{P}^1(\mathbb{C})$  punctured at each point of a finite subset  $\mathcal{D}$  not containing  $z_0$ , can be realized as the monodromy representation of

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a linear differential system with Fuchsian singularities only, all in  $\mathcal{D}$ . Let us recall that a point  $a \in \mathcal{D}$  is a *Fuchsian* singularity of a linear differential system  $dy/dz = B(z)y$ , where  $B$  is an  $n \times n$  matrix with coefficients in  $\mathbb{C}(z)$ , if  $a$  is a simple pole of  $B$  (*modulo* a Möbius transformation if  $a = \infty$ ). This problem is still open, although important results of A. Bolibruch ([13], [14], [15], [16]) have reduced it considerably. Several authors have given sufficient conditions either to solve the problem or to construct counterexamples. A. Bolibruch [13] and V. Kostov [34] have shown independently that the irreducibility of the representation  $\chi$  is a sufficient condition. In dimension two the problem always has a solution (*cf.*[1]) and in dimension three and four it has been completely elucidated ([1], [14], [26]). In [24], H. Esnault and E. Viehweg have extended the problem over curves of genus  $\geq 1$  and solved it for irreducible representations of the fundamental group. The Riemann-Hilbert problem is related to problems in many areas of mathematical physics and has become a trend of research over the last twenty years. There is extensive literature available on the subject, in particular on Painlevé equations and isomonodromic deformations. For recent results in this field we refer to [12], [18], [23], [29], [30], [27], [33], [50].

Closely related to the Riemann-Hilbert problem, the Birkhoff inverse problem asks the following. Consider a differential system  $zdy/dz = A(z)y$  where the matrix  $A(z) = z^r \sum_{n=0}^{\infty} A_n z^{-n}$  is meromorphic at infinity. Does there exist a differential system  $zdy/dz = B(z)y$ , where  $B(z)$  is a polynomial coefficient matrix, meromorphically equivalent to the given system and with a Poincaré rank at infinity not greater than the original one? In dimension two and three, the problem is known to have a positive answer, see [2], [31], but in higher dimension, although many sufficient conditions have been given, see [15], [5], [59], the problem remains open in general. The differential systems in Birkhoff standard form appear in complex algebraic geometry in the study of particular Frobenius manifolds, see [59] and references therein.

In the present paper we extend both the Riemann-Hilbert problem and the Birkhoff standard form problem to the case of an arbitrary number of irregular singularities. We define generalized monodromy data, consisting of the monodromy representation with respect to prescribed singularities and of further prescribed local data at each singularity. These data include the Poincaré rank and Stokes data. The generalized Riemann-Hilbert problem is the following: *Let singular points and generalized monodromy data be given in which all Poincaré ranks are minimal. Construct a system on  $\mathbb{P}^1(\mathbb{C})$  with these data.* We give sufficient conditions to solve this inverse problem and we show that they are in general fulfilled in dimension two and three.

We conclude the paper with applications to differential Galois theory, where we under suitable assumptions solve the inverse problem with a better control of the singularities. The global inverse problem in differential Galois theory over  $\mathbb{P}^1(\mathbb{C})$ , that is, over the differential field  $K = \mathbb{C}(z)$ , asks for the existence of a differential system  $dy/dz = B(z)y$  with coefficients in  $\mathbb{C}(z)$  and with a given linear algebraic group (over  $\mathbb{C}$ ) as its differential Galois group over  $\mathbb{C}(z)$ . It always has a solution. This was first proved by M. and C. Tretkoff [63], using a weak solution of the Riemann-Hilbert problem. Other proofs were given since, either analytic ([56], [57], [58]), or algebraic over a general field of constants ([28]). An algebraic and constructive proof was given in [47] for connected groups, and in

[49], [22] for large classes of non-connected groups. In the present paper we focus on the number and on the Poincaré rank of the singularities of a differential system with a given Galois group, and we show that under suitable conditions both are minimal.

The paper is organized as follows.

In Section 1 we define generalized monodromy data attached to a linear differential system over  $\mathbb{P}^1(\mathbb{C})$ .

In Section 2 we state the generalized Riemann-Hilbert problem and we show that it has a solution if a certain family of vector bundles with connections contains a holomorphically trivial bundle.

In Section 3 we give further sufficient conditions, in terms of the stability of a certain bundle, to solve the problem. These conditions are in particular fulfilled when the monodromy representation is irreducible and the data at one of the singularities are unramified. If all data are Fuchsian, we recover the irreducibility condition of Bolibruch and Kostov.

In Section 4 we look for the existence of possibly reduced systems with given generalized monodromy data, when the prescribed monodromy data are “non generic”. This should lead to a reduction of the problem to an equivalent problem in lower dimension.

In Section 5 the results for reduced systems enable us to solve the generalized Riemann-Hilbert problem completely in dimension two and three, assuming that not all the singularities are irregular with roots.

In Section 6 we apply our previous results to the inverse problem of differential Galois theory, which under suitable conditions can be solved with a minimal number of singularities and a minimal Poincaré rank at these.

## 1 Generalized Monodromy Data

Consider a system

$$(1) \quad \frac{dy}{dz} = B(z)y$$

of  $p$  linear differential equations with rational coefficients on the Riemann sphere  $\mathbb{P}^1(\mathbb{C})$ . Let  $\mathcal{D} = \{a_1, \dots, a_n\}$  be the set of singular points of (1), consisting of the poles of the matrix function  $B(z)$  and of a possible singular point at infinity (if the system obtained from (1) *via*  $z = 1/u$  has a singular point at the origin).

Consider the matrix differential form  $\omega = B(z)dz$ . In what follows we will rather write (1) in its invariant form

$$(2) \quad dy = \omega y.$$

in terms of which  $\mathcal{D}$  is a singular divisor of  $\omega$ .

To any system (2) there correspond what we will call *generalized monodromy data*, which we define below.

## 1.1 The monodromy representation

Let  $Y$  denote a fundamental solution of (2), holomorphic in a neighbourhood of a given non-singular point  $z_0 \in \mathbb{C}$ . Analytic continuation of  $Y$  along a loop  $\gamma$  in  $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$  yields a new fundamental solution  $\gamma^*(Y) = YG$  for some matrix  $G \in \text{GL}(p, \mathbb{C})$ . This defines the *monodromy representation*

$$(3) \quad \chi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}; z_0) \longrightarrow \text{GL}(p, \mathbb{C})$$

of the system, with respect to  $Y$ . Since the fundamental group of  $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$  is generated by the homotopy classes of all elementary loops  $\gamma_i$ , where  $\gamma_i$ ,  $i = 1, \dots, n$ , encloses the only singular point  $a_i$ , the monodromy representation of (2) is defined by the local monodromy matrices  $G_i$  corresponding to these loops. These matrices satisfy *a priori* the only relation  $G_1 \cdot \dots \cdot G_n = I$ .

## 1.2 The Poincaré rank

Let  $a \in \mathcal{D}$  be a given singular point  $a_i$  of (2) and  $G$  the corresponding monodromy matrix  $G_i$ . In the neighbourhood of  $a$  the coefficient matrix of (1) can be expanded as follows

$$(4) \quad B(z) = \frac{B_{-r-1}}{(z-a)^{r+1}} + \dots + \frac{B_{-1}}{z-a} + B_0 + \sum_{i=1}^{\infty} B_i(z-a)^i$$

where  $B_{-r-1} \neq 0$ .

**Definition 1** *The Poincaré rank of the system (2) at  $a$  is the integer  $r$  of (4). The true Poincaré rank of (2) at  $a$  is the smallest Poincaré rank of a local system in the meromorphic equivalence class of (2) at  $a$ .*

We recall that the singular point  $a$  is called *regular singular* if all solutions of (2) have an at most polynomial growth as  $z$  tends to  $a$  in some sector with vertex  $a$  (note that these are in general multivalued functions). In the opposite case the singular point is called *irregular*. The system (2) is called *Fuchsian* at  $a$  if  $r = 0$ , that is, if the coefficient form of the system has a simple pole at  $a$ , then called a Fuchsian singularity of (2). We recall that a Fuchsian singularity is in particular regular singular, and that the true Poincaré rank of a system at any regular singularity is  $r = 0$ .

Assume now that  $a$  is irregular. Then, in addition to the local monodromy matrix  $G$  and the Poincaré rank  $r$ , one can attach *local Stokes data* to the system at the singular point  $a$ . These are determined as follows. In a neighborhood of  $a$  it is well-known (*cf.* [7]) that there exists a formal fundamental solution  $\hat{Y}$  of (2) of the form

$$(5) \quad \hat{Y}(t) = \hat{F}(z)H(z)$$

where  $\hat{F}$  is a formal meromorphic matrix series in  $(z-a)$  (in general divergent) and

$$H(z) = (z-a)^{\tilde{J}} U e^{Q(z)},$$

where  $Q(z)$ ,  $U$ ,  $\tilde{J}$  are block-diagonal matrices with diagonal blocks  $Q_j(z)$ ,  $U_j$ ,  $\tilde{J}_j$  respectively,  $j = 1, \dots, N_Q$ , of the same size. We call these matrices superblocks, since they too are block-diagonal of the form

$$(Q) \quad Q_j(z) = \text{diag}(q_j(t)I_{s_j}, q_j(t\zeta_j)I_{s_j}, \dots, q_j(t\zeta_j^{p_j-1})I_{s_j}),$$

where  $q_j$  is a polynomial in  $t = (z - a)^{-1/p_j}$ , and  $\zeta_j = e^{2i\pi/p_j}$ , for some integer  $p_j$  not greater than the least common multiple of  $2, 3, \dots, p$ ,

$$(J) \quad \tilde{J}_j = \text{diag}(J_{s_j}, J_{s_j} + (1/p_j)I_{s_j}, \dots, J_{s_j} + ((p_j - 1)/p_j)I_{s_j}),$$

and the matrix  $U_j$  decomposes into blocks  $U_{lk}$  of the form

$$(U) \quad U_{lk} = [\zeta_j^{(l-1)(k-1)} I_{s_j}] \quad , \quad 1 \leq l, k \leq p_j.$$

The polynomial  $q_j(t)$  has no constant term and the integer  $s_j$  is the ‘‘multiplicity’’ with which  $q_j$  together with its analytic continuations around  $a$  occur on the diagonal of  $Q$ . As usual  $I_{s_j}$  denotes the  $s_j$ -dimensional identity matrix and  $J_{s_j}$  a constant  $s_j$ -dimensional matrix in canonical Jordan form whose eigenvalues  $(\rho_j^m)_{1 \leq m \leq s_j}$  satisfy for all  $m$  the condition

$$(\rho) \quad 0 \leq \text{Re } \rho_j^m < 1/p_j.$$

In the generic case, the form of the formal fundamental matrix has a simpler form. All superblocks  $H_j$  in the decomposition of  $H(z)$  are then usual blocks with  $p_j = 1$ ,  $U = I$ , that is,  $H$  decomposes into a direct sum of diagonal blocks

$$(6) \quad H_j = (z - a)^{J_{s_j}} e^{q_j(z)I_{s_j}}$$

where  $q_j(z)$  is a polynomial of degree non greater than  $r$  in  $1/(z - a)$  with no constant term and  $J_{s_j}$  is a matrix in Jordan normal form with eigenvalues  $\rho_j^m$  satisfying  $0 \leq \text{Re } \rho_j^m < 1/p_j$  for all  $m$ . This in particular occurs when the eigenvalues of the leading term  $B_{-r-1}$  in the expansion (4) are distinct.

We will refer to the generic case above as to the *unramified* case, or case of a *singularity without roots* (to the *ramified case* or case of a *singularity with roots* else). We will more precisely say that *the solution (5) is unramified* if  $Q$  is a polynomial in  $1/(z - a)$ , and that it is a *ramified solution* if  $Q$  is polynomial in  $1/t$  where  $t$  is a root of  $(z - a)$ . Note that  $r$  in the unramified case is the true Poincaré rank of (2). In the general (possibly ramified) case, the true Poincaré rank is the smallest integer greater or equal to the rational degree of  $Q$ , that is, to the Katz rank of (2) at  $a$ . Note that *via* a local meromorphic transformation it is always possible to reduce the Poincaré rank to the true Poincaré rank (for a review of general facts about the rank at an irregular singularity and rank reduction, we refer to [39] and [10]).

### 1.3 Stokes data

With notation as before, consider a formal fundamental solution

$$(7) \quad \hat{Y}(z) = \hat{F}(z)(z-a)^{\tilde{J}} U e^{Q(z)}$$

of (2) at  $a$ , where in particular  $Q$  is a diagonal polynomial matrix in a root of  $1/(z-a)$ , of fractional degree  $\rho \leq r$  in  $1/(z-a)$ , with no constant term, which we call the *exponential part* of  $\hat{Y}$ .

The *formal monodromy* (matrix) is defined as

$$\hat{G} = U^{-1} \exp(2i\pi \tilde{J}) U$$

or equivalently by  $\hat{Y}_{2i\pi} = \hat{Y} \hat{G}$  where  $\hat{Y}_{2i\pi}$  denotes the fundamental solution obtained from  $\hat{Y}$  by the change of sheet (on the Riemann surface of the logarithm) induced by meromorphic continuation around  $a$  one time in the positive (counterclockwise) direction.

Note that  $Q$  and  $\hat{G}$  are formal invariants of the system (2), depending on its formal meromorphic equivalence class only.

Let  $l_1 \prec \dots \prec l_N$  denote the *singular rays* of  $Q$ , that is, the rays from  $a$  (in an affine chart containing  $a$ ), labeled in ascending order with respect to the positive orientation of a circle centered at  $a$ , on which some  $e^{q_j - a}$  has maximal decay. The general theory of summability ensures that  $\hat{Y}$  is (multi)-summable along any non-singular ray  $l$  (cf. [46], [6], [3], [4]). If all the polynomials  $q_i - q_j$  have the same degree  $\rho$  (this is the case of one-level summability) this means that for any open sector  $S$  with vertex  $a$ , with opening  $> \pi/\rho$  and bisected by  $l$ , there is a unique analytic fundamental matrix  $Y_l$  called the *sum*, or  $\rho$ -*sum* in this case, of  $\hat{Y}$  along  $l$ , such that  $Y H^{-1}$  is Gevrey  $1/r$ -asymptotic to  $\hat{F}$  on this sector, that is, for any proper subsector  $S'$  of  $S$  there are constants  $A$  and  $C$  such that if we write

$$\hat{F}(z) = \sum_{k=-s}^{\infty} F_k (z-a)^k,$$

then for any  $m \geq -s$

$$|Y_j(z) H^{-1}(z) - \sum_{k=-s}^m F_k (z-a)^k| < C A^m (m!)^{\frac{1}{\rho}} (|z-a|^{m+1})$$

as  $z$  tends to  $a$  in  $S'$ .

Given a singular ray  $l_i$  of  $Q$ , let  $l_i^-$  and  $l_i^+$  be two rays such that  $l_i^- \prec l_i \prec l_i^+$  and such that  $l_i$  is the only singular ray contained in the oriented sector  $[l_i^-, l_i^+]$ . Let  $Y_i^-$  and  $Y_i^+$  denote the sums of  $\hat{Y}$  along  $l_i^-$  and  $l_i^+$  respectively. Comparing these solutions on a neighbourhood of  $l_i$  (they are both defined on sectors large enough to contain  $l_i$ ) we define the *Stokes matrix* with respect to the singular ray  $l_i$  to be the constant matrix  $C_i$  depending on  $l_i$  only, such that  $Y_i^- = Y_i^+ C_i$ . The Stokes matrices of (2) at  $a$  have the following properties, which we call *Stokes conditions*:

- For each  $i$  the matrix  $e^{Q(z)}C_i e^{-Q(z)}$  is asymptotic to the identity matrix  $I$ ,
- $C_1 \cdot \dots \cdot C_N \cdot \hat{G} = G$ .

Note that the first condition in particular implies that the Stokes matrices are unipotent. The second condition is often called the *cyclic relation*.

Thus, we have attached to each singular point  $a$  of the given system, the following data:

- the Poincaré rank at  $a$
- the proper monodromy matrix  $G$  (image by (3) of the elementary loop  $\gamma$  enclosing the singular point  $a$  only)
- Stokes data which consist of the exponential part  $Q$  of a formal fundamental solution (7), the formal monodromy  $\hat{G}$  and the Stokes matrices  $C_1, \dots, C_N$  corresponding to the respective singular directions  $l_1 \prec \dots \prec l_N$  of  $Q$ .

These data over all singular points of (2) constitute what we will call *generalized monodromy data*.

## 2 The Generalized Riemann-Hilbert Problem

We will now give the precise terms of the generalized Riemann-Hilbert problem, GRH-problem for short.

### 2.1 Statement of the problem

We first define the data for the inverse problem under consideration in terms of local invariants.

**Definition 2** A reduced datum  $\mathcal{M}$  consists of

- a finite subset  $\mathcal{D} = \{a_1, \dots, a_n\}$  of  $\mathbb{P}^1(\mathbb{C})$
- for some fixed  $z_0 \in \mathbb{C} \setminus \mathcal{D}$ , a representation

$$(8) \quad \chi : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}; z_0) \longrightarrow \mathrm{GL}(p, \mathbb{C})$$

of the fundamental group of  $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$  defined by matrices  $G_i = \chi(\gamma_i)$  for each elementary loop-class  $\gamma_i$  around  $a_i, i = 1, \dots, n$ ,

- for each  $a_i$  local Stokes data consisting of :
  - a non-negative integer  $r_i$ ,
  - a diagonal polynomial matrix  $Q_i$  in some root of  $1/z_i$  (where  $z_i$  denotes a local parameter at  $a_i$ ) with no constant term, with a block-diagonal decomposition  $Q_i = \mathrm{diag}(Q_{i,1}, \dots, Q_{i,N_{Q_i}})$  in blocks of the form (Q) above, and such that the fractional degree  $s_i$  of  $Q$  in  $1/z_i$  satisfies  $r_i = -[-s_i]$ ,
  - an invertible constant matrix  $\hat{G}_i$ , or equivalently a matrix  $\tilde{J}_i$  such that  $\hat{G}_i = U_i^{-1} \exp(2i\pi \tilde{J}_i) U_i$ , where  $U_i$  is decomposed in blocks of the form (U) and  $\tilde{J}_i$  is a



*Jordan, block-diagonal matrix with blocks of the form  $(J)$  and eigenvalues satisfying the condition  $(\rho)$ , and where the size of the blocks and superblocs is determined as before by the size of the corresponding block-decomposition of  $Q_i$ .*

*- a set of matrices  $C_i^1, \dots, C_i^{N_i}$  attached to the singular directions  $l_i^1 \prec \dots \prec l_i^{N_i}$  of  $Q_i$  and satisfying the Stokes conditions defined in Section 2.*

The GRH-problem asks for the existence of a system (2) with  $\mathcal{D}$  as its set of singular points and with  $\mathcal{M}$  as its corresponding set of generalized monodromy data, that is, a system (2) such that

- the representation  $\chi$  is the monodromy representation of (2) with respect to some fundamental solution defined in the neighbourhood of  $z_0$ ,
- the system (2) has Poincaré rank  $r_i$  at  $a_i$  for all  $i = 1, \dots, n$ ,
- at each  $a_i$  there is a formal fundamental solution  $\hat{Y}_i$  of the form (7) with  $Q_i$  as its exponential part,  $\hat{G}_i$  as its formal monodromy and the  $C_i^j$  as its Stokes matrices along the singular lines  $l_i^j$  of  $Q_i$ ,  $j = 1, \dots, N_i$ .

In the following cases, the GRH-problem reduces to classical problems.

**(RH)** If all Poincaré ranks  $r_i$  equal zero, then the data  $\mathcal{M}$  reduce to Representation (8), that is, the GRH-problem seeks a Fuchsian system of linear differential equations with given singular points and a given monodromy representation. This is the classical Riemann-Hilbert problem (Hilbert's 21st problem for Fuchsian equations).

**(BSF)** Consider the case of two singularities only, at  $a_1 = \infty$  and  $a_2 = 0$ , with  $\mathcal{M}$  data  $r_1 = r$ ,  $r_2 = 0$  and any Stokes data at  $a_1$ . This is the Birkhoff standard form problem.

**Remark** The definition of a reduced datum  $\mathcal{M}$  implies that a solution to the GRH-problem for  $\mathcal{M}$  has a *minimal* Poincaré rank  $r_i$  (equal to the true Poincaré rank) at each singular point  $a_i$ , as it is required for the classical Riemann-Hilbert problem.

Suppose now we are given a reduced datum  $\mathcal{M}$ . This section is devoted to the construction of a certain family  $\mathcal{E}$  of vector bundles with connections that realize the local data of  $\mathcal{M}$ . Once we have achieved the construction of  $\mathcal{E}$ , we naturally obtain the following result.

**Theorem 1** *The generalized Riemann-Hilbert problem has a solution for  $\mathcal{M}$  if at least one of the vector bundles in  $\mathcal{E}$  is holomorphically trivial.*

In the next sections the construction of the family  $\mathcal{E}$  will lead to more precise sufficient conditions for the problem.

## 2.2 Construction of $\mathcal{E}$

To solve the GRH-problem, we first apply well-known results of Malgrange and Sibuya ([44], [45], [60], see also [40]) which guarantee the existence, for each  $i = 1, \dots, n$ , of a local meromorphic system

$$(9) \quad dy = \omega_i y$$

of linear differential equations in a neighborhood of  $a_i$  with the given local Stokes data.

The GRH-problem then can be reformulated as follows.

*Let local systems of the form (9) be given in neighbourhoods  $O_1, \dots, O_n$  of  $a_1, \dots, a_n$  respectively, such that the local monodromies (with respect to suitable fundamental solutions) of these systems generate a representation (8). Does there exist a global system (2) with  $\{a_1, \dots, a_n\}$  as its set of singular points and with generalized monodromy data given by those of the local systems (9) ?*

A method of solution for the GRH-problem is the following.

### 2.2.1 Construction of the canonical pair $(\mathcal{F}, \nabla)$

Consider a covering of  $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$  by finitely many and sufficiently small discs  $U_{n+1}, \dots, U_N$  and connect each  $U_i$  to the base-point  $z_0$  via some path  $\eta_i$  in  $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$  from  $z_0$  to a given endpoint in  $U_i$ ,  $i = n + 1, \dots, N$ .

For each nonempty intersection  $U_i \cap U_j$  consider the loop  $\eta_i \circ \delta_{ij} \circ \eta_j^{-1}$ , where  $\delta_{ij}$  denotes a path in  $U_i \cup U_j$  connecting the endpoints in  $U_i$  and  $U_j$  of  $\eta_i$  and  $\eta_j$  respectively. We define the constant function

$$g_{ij} = \chi([\eta_i \circ \delta_{ij} \circ \eta_j^{-1}]) : U_i \cap U_j \longrightarrow \mathrm{GL}(p, \mathbb{C}).$$

It is not difficult to see that the functions  $g_{ij}$  define a gluing cocycle, hence a vector bundle  $\hat{\mathcal{F}}$  of rank  $p$  over  $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$  with these constant transition functions.

For  $i = n + 1, \dots, N$  consider the system of linear differential equations in  $U_i$

$$dy = \omega_i y, \quad \omega_i = 0.$$

This is a family of compatible local systems, in the following sense. On each nonempty intersection  $U_i \cap U_j$  we have

$$(10) \quad \omega_i = dg_{ij}g_{ij}^{-1} + g_{ij}\omega_jg_{ij}^{-1}$$

since the transition functions are constant. This defines a connection  $\hat{\nabla}$  on the vector bundle  $\hat{\mathcal{F}}$ , and the forms  $\omega_i$ ,  $i = n + 1, \dots, N$ , are as usual called the *local forms* of the connection.

If we consider another coordinate description of  $\hat{\mathcal{F}}$  by means of equivalent cocycles  $g'_{ij} = \Gamma_i^{-1}g_{ij}\Gamma_j$  where  $\Gamma_i$ ,  $i = n + 1, \dots, N$ , denotes a holomorphically invertible matrix function in  $U_i$ , then the corresponding local forms of the connection  $\hat{\nabla}$  are equal to

$$\omega'_i = d\Gamma_i\Gamma_i^{-1} + \Gamma_i\omega_i\Gamma_i^{-1}.$$

By construction, the connection  $\hat{\nabla}$  is holomorphic on  $\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}$  (since all  $\omega_i = 0$  are holomorphic,  $i = n + 1, \dots, N$ ) and it has the given monodromy representation (8).

We actually can extend  $(\hat{\mathcal{F}}, \hat{\nabla})$  to the whole Riemann sphere by means of the local systems  $\omega_i$  defined each in the neighbourhood  $O_i$  of  $a_i$ ,  $i = 1, \dots, n$ . This follows from the fact that the systems (9) have the monodromy prescribed by Representation (8). Thus, one can glue together on  $O_i \setminus \{a_i\}$  the local systems determined by (9) and by  $(\hat{\mathcal{F}}, \hat{\nabla})$  respectively.

In terms of cocycles one needs to do the following. Consider a nonempty intersection  $O_i \cap U_\alpha$  and choose a fundamental solution  $Y_i$  of (9) in this intersection. We may assume that  $Y_i$  has the following form

$$(11) \quad Y_i(z) = M_i(z)(z - a_i)^{E_i},$$

where the matrix  $M_i(z)$  is holomorphically invertible in  $O_i \cap U_\alpha$  and where  $E_i = (1/2\pi i) \log G_i$  and the eigenvalues  $\rho_i^m$  of the matrix  $E_i$  are normalized as follows

$$(12) \quad 0 \leq \operatorname{Re} \rho_i^m < 1.$$

This is a classical result which goes back to Poincaré. It can be found in [19], or in [1] Section 2.2, or in [25] Th. 11.11.

Let  $g_{i\alpha}(z) = M_i(z)(z - a_i)^{E_i}$ . For any other  $U_\beta$  that has a nonempty intersection with  $O_i$  consider a path starting from a point  $s$  in  $O_i \cap U_\alpha$  and ending in  $O_i \cap U_\beta$ , moving in  $O_i$  around  $a_i$  (less than one turn) in the counterclockwise direction. Let  $g_{i\beta}(z)$  denote the analytic continuation of  $g_{i\alpha}$  along this path. A simple verification shows that the set  $\{g_{\alpha\beta}, g_{i\alpha}\}_{\substack{1 \leq i \leq n \\ n+1 \leq \alpha \leq N}}$  defines a cocycle for the covering  $\{O_i, U_\alpha\}_{\substack{1 \leq i \leq n \\ n+1 \leq \alpha \leq N}}$ . Thus, one gets a vector bundle  $\mathcal{F}$  on the whole Riemann sphere.

It follows from the preceding construction that all the local systems  $dy = \omega_i y$ ,  $i = 1, \dots, N$ , including the systems (9) in the neighbourhoods  $O_i$  of  $a_i$ , are compatible. Indeed, for any  $i$  and  $\alpha$  such that  $O_i$  and  $U_\alpha$  have a nonempty intersection, we have

$$(13) \quad dg_{i\alpha} g_{i\alpha}^{-1} + g_{i\alpha} \omega_\alpha g_{i\alpha}^{-1} = dY_i Y_i^{-1} = \omega_i,$$

that is, we get a connection  $\nabla$  on the vector bundle  $\mathcal{F}$  with the given local forms  $\omega_i$ ,  $i = 1, \dots, N$ , and with the given monodromy (8). This pair  $(\mathcal{F}, \nabla)$  is the so-called *canonical extension* of  $(\hat{\mathcal{F}}, \hat{\nabla})$  in the sense of Deligne.

### 2.2.2 Construction of $\mathcal{E}$ by means of admissible matrices

If the vector bundle  $\mathcal{F}$  which we have constructed were holomorphically trivial, then on a holomorphic trivialization of the bundle, the connection  $\nabla$  would define a global system of linear differential equations (1) with the given generalized monodromy data. Thus, the inverse problem would be solved.

Indeed, the triviality of the bundle  $\mathcal{F}$  means that for every  $O_i, U_\alpha$  and  $U_\beta$  with  $O_i \cap U_\alpha \neq \emptyset$ ,  $U_\alpha \cap U_\beta \neq \emptyset$ , there exist holomorphically invertible matrix functions  $\Gamma_i, \Gamma_\alpha$  and  $\Gamma_\beta$  (on  $O_i, U_\alpha, U_\beta$  respectively) such that

$$\Gamma_\alpha = \Gamma_i g_{i\alpha}, \quad \Gamma_\beta = \Gamma_\alpha g_{\alpha\beta}$$

if the corresponding intersections are not empty. This implies that the forms

$$\omega'_i = d\Gamma_i\Gamma_i^{-1} + \Gamma_i\omega_i\Gamma_i^{-1}, \quad \omega'_\alpha = d\Gamma_\alpha\Gamma_\alpha^{-1} + \Gamma_\alpha\omega_\alpha\Gamma_\alpha^{-1}$$

coincide over the intersections of the corresponding pieces of the covering and thus define a global form  $\omega$ . The fundamental matrices of the new and original local systems are connected by gauge transformations  $Y'_i = \Gamma_i Y_i$ , which implies that the constructed system has the required generalized monodromy data.

Unfortunately, the bundle  $\mathcal{F}$  as a rule is not holomorphically trivial. But it turns out that this bundle is always meromorphically trivial. More precisely, for any choice of a point  $b$  in some  $O_l$  there exists a meromorphic trivialization of the bundle which is holomorphic outside of  $\{b\}$ . In terms of a cocycle involving  $O_l$ , as above, this means that the desired functions  $\Gamma_i$  will be holomorphically invertible for  $i \neq l$ , and  $\Gamma_l$  meromorphic only at  $b$  (and holomorphically invertible in  $O_l \setminus \{b\}$ ).

Choose  $b = a_l$  for some  $l$ ,  $1 \leq l \leq n$ . From the meromorphic trivialization  $\{\Gamma_i\}$  of the bundle  $\mathcal{F}$  we get a global system (2) with all the given generalized monodromy data except one, namely the Poincaré rank at  $a_l$  which may be greater than the given integer  $r_l$ , since the matrix  $\Gamma_l$  is meromorphic only at  $a_l$ . And for a number of inverse problems such as the classical Riemann-Hilbert problem or the problem of the standard Birkhoff form, we must realize this datum as well.

To achieve this, we shall replace the local systems (9) in the construction of  $(\mathcal{F}, \nabla)$  by new systems

$$(14) \quad dy = \omega'_i y$$

with

$$\omega'_i = d\Gamma_i\Gamma_i^{-1} + \Gamma_i\omega_i\Gamma_i^{-1},$$

via (families of) gauge transformations  $y'_i = \Gamma_i y$ , where  $\Gamma_i$  is holomorphically invertible in  $O_i \setminus \{a_i\}$  and meromorphic at  $a_i$ .

**Definition 3** *Assume  $a_i$  is a singular point at which the formal solution  $\hat{Y}_i$  of (9) is unramified. This means that  $\hat{Y}_i$  has the form (5), (6). An admissible matrix is an integer-valued diagonal matrix  $\Lambda_i = \text{diag}(\Lambda_i^1, \dots, \Lambda_i^{N_Q})$ , that is, a diagonal matrix whose entries are integers, blocked in the same way as  $Q(z)$  and such that the matrix function*

$$(z - a_i)^{\Lambda_i} \tilde{J}(z - a_i)^{-\Lambda_i}$$

*is holomorphic at  $a_i$ .*

Note that any diagonal integer-valued matrix  $\Lambda_i$  whose diagonal elements form a non-increasing sequence is admissible and that the set of admissible matrices is infinite.

The matrix  $\hat{Y}_i$  can be written as follows

$$\hat{Y}_i(z) = \hat{F}(z)(z - a_i)^{-\Lambda_i}(z - a_i)^{\Lambda_i} H(z)$$

with  $H(z)$  as in (5). (For simplicity of notation, we will omit the index  $i$  when introducing new functions, although all calculations depend on  $a_i$ ). The formal matrix function  $\hat{F}'(z) = \hat{F}(z)(z - a_i)^{-\Lambda_i}$  is meromorphic at  $a_i$ . The proof of the following technical lemma proceeds as for Sauvage's lemma in [13].

**Lemma 1** *For any formal meromorphic matrix  $F(\tau) \in \text{GL}(p, \mathbb{C}[[\tau]][1/\tau])$  there exists a matrix  $\Gamma(\tau)$ , polynomial in  $1/\tau$  and holomorphically invertible outside of  $\tau = 0$ , such that*

$$\Gamma(\tau)F(\tau) = \tau^K F_0(\tau),$$

where  $K$  is a diagonal integer-valued matrix and  $F_0(\tau)$  is an invertible formal holomorphic matrix series in  $\tau$ .

If we apply this lemma to the matrix  $\hat{F}'(z)$  we get

$$\Gamma(z - a_i)\hat{F}'(z) = (z - a_i)^K \hat{F}_0(z).$$

Let us transform the local system (9) into (14) via the meromorphic gauge transformation  $y' = \Gamma_i y$  where  $\Gamma_i(z) = (z - a_i)^{-K} \Gamma(z - a_i)$ . We get a formal fundamental solution of the new system, of the form

$$(15) \quad \hat{Y}'_i(z) = \hat{F}_0(z)(z - a_i)^{\Lambda_i}(z - a_i)^{\tilde{J}} e^{Q(z)}.$$

This transformation does not increase the Poincaré rank  $r_i$ . Indeed, the form  $\omega'_i$  of (14) can be written as

$$(16) \quad \begin{aligned} \omega'_i = d\hat{Y}'_i(\hat{Y}'_i)^{-1} = d\hat{F}_0(\hat{F}_0)^{-1} + \hat{F}_0 \frac{1}{z - a_i} (\Lambda_i + \\ + (z - a_i)^{\Lambda_i} \tilde{J}(z - a_i)^{-\Lambda_i} + (z - a_i) \frac{dQ(z)}{dz}) (\hat{F}_0)^{-1} dz. \end{aligned}$$

The fact that the matrix  $\Lambda_i$  is admissible and the matrix  $\hat{F}_0$  invertible, and the fact that the degree of  $Q(z)$  is equal to  $r_i$  (with respect to  $1/(z - a_i)$ ) together guarantee that the Poincaré rank at  $a_i$  of the new local system remains equal to  $r_i$ .

Let us replace the initial local system (9) in  $O_i$  with the system (16), which we will write

$$(17) \quad dy = \omega^{\Lambda_i} y$$

to keep track of the admissible matrix  $\Lambda_i$  used in the construction. Let us extend the initial vector bundle  $(\hat{\mathcal{F}}, \hat{\nabla})$ , constructed from Representation (8), over the point  $a_i$  using this new system (instead of the initial one).

Assume that  $a_i$  is a regular singular point. Consider in this case an analytic fundamental solution  $Y_i(z) = M_i(z)(z - a_i)^{E_i}$  such that moreover the matrix  $E_i$  has an upper triangular form and the entries  $e_{kl}$  of  $E_i$  equal zero if  $\rho_i^k \neq \rho_i^l$ , where the complex numbers  $\rho_i^m$  denote the eigenvalues of  $E_i$ . Since  $a_i$  is regular singular, the matrix  $M_i(z)$  is meromorphic at

$a_i$ . Thus, we can follow the same procedure as in the case of an irregular point without roots, to construct a new system *via* an admissible matrix  $\Lambda_i$ , where admissibility here means that the matrix  $(z - a_i)^{\Lambda_i} E_i (z - a_i)^{-\Lambda_i}$  is holomorphic at  $a_i$ .

Now assume that  $a_i$  is an irregular singular point with roots. By an admissible matrix  $\Lambda_i$  we mean here a diagonal integer-valued matrix  $\Lambda_i = \text{diag}(\Lambda_i^1, \dots, \Lambda_i^{N_Q})$  blocked in the same way as  $Q(z)$  and such that the matrix function

$$(18) \quad (z - a_i)^{\Lambda_i^j} \tilde{J}_j (z - a_i)^{-\Lambda_i^j}$$

is holomorphic at  $a_i$  if the superblock  $Q_j$  has no ramification, and  $\Lambda_i^j$  is a scalar matrix if the superblock  $Q_j$  has ramification.

Let us proceed with the system (9) at an irregular singular point with roots (*i.e.* the formal solution is ramified) in the same way as in the unramified case. Again, we get a system (17) with the same local Stokes data as the initial one.

Choose a collection  $\Lambda = (\Lambda_1, \dots, \Lambda_n)$  of admissible matrices (in the above sense, depending on the type of the singularity  $a_i$ ) and consider the extension  $(\mathcal{F}^\Lambda, \nabla^\Lambda)$  of  $(\hat{\mathcal{F}}, \hat{\nabla})$  over the singular points  $a_i$  *via* the systems (17) obtained by means of the matrices  $\Lambda_i$ . Then, by construction, the extended connection  $\nabla^\Lambda$  has the given Poincaré ranks and generalized monodromy data. We get in this way an infinite set  $\mathcal{E}$  of vector bundles  $(\mathcal{F}^\Lambda, \nabla^\Lambda)$  with connections that have the prescribed generalized monodromy data.

It follows immediately from previous considerations that Theorem 1 holds for this family  $\mathcal{E}$  of vector bundles.

Note that the converse of Theorem 1 is not true, since  $\mathcal{E}$  does not contain all vector bundles with connections having the prescribed generalized monodromy data. The reason for this is that there are local systems (9) with the given data whose formal fundamental matrix cannot be written in the form (15) with an invertible matrix  $\hat{F}_0$  (this in particular occurs at any regular, but not Fuchsian singularity). This situation differs significantly from the Fuchsian case (where all Poincaré ranks equal zero) in which Theorem 1 gives necessary and sufficient conditions for the positive solvability of the Riemann-Hilbert problem (see [1], [13]).

### 3 Sufficient Conditions for the Generalized Riemann-Hilbert Problem

We keep the notation from Section 2. Consider a bundle  $(\mathcal{F}^\Lambda, \nabla^\Lambda)$  in  $\mathcal{E}$ . It follows from (5), (11) and (16) that

$$\text{tr } \omega^{\Lambda_i} = \text{tr} (\Lambda_i + T_i) \frac{dz}{z - a_i} + \text{tr} \left( \frac{dQ(z)}{z - a_i} \right) + \text{ a holomorphic form,}$$

where  $T_i = \tilde{J}$  in the irregular case and  $T_i = E_i$  if  $a_i$  is a regular singular point. The eigenvalues  $\beta_i^m$  of the matrix  $\Lambda_i + \tilde{J}$  in the irregular case (resp. of the matrix  $\Lambda_i + E_i$  in the regular singular case) are called *formal exponents* (resp. *exponents*) of the connection

$\nabla^{\Lambda_i}$  at  $a_i$ . Note that exponents play an important role in the study of singularities and were extensively used by A. Bolibruch in his work on the classical Riemann-Hilbert problem (cf. [1]). Exponents for differential systems were first introduced by A. H. M. Levelt [38] in the case of regular singularities. E. Corel ([20], [21]) has given a purely algebraic definition of exponents, for regular as well as for irregular singularities, and has proved an important refinement of the classical Fuchs inequality for the sum of all exponents.

The *degree*  $\deg \mathcal{F}^\Lambda$  of the bundle  $\mathcal{F}^\Lambda$  is by definition the sum

$$\deg \mathcal{F}^\Lambda = \sum_{i=1}^n \operatorname{res}_{a_i} \operatorname{tr} \omega^{\Lambda_i} = \sum_{i=1}^n \operatorname{tr} (\Lambda_i + T_i).$$

Let us recall that a bundle  $\mathcal{F}$  is called *stable* (respectively *semistable*) if for any proper subbundle  $\mathcal{F}'$  of  $\mathcal{F}$ , the *slope*  $\mu(\mathcal{F}') = \deg(\mathcal{F}')/\operatorname{rank}(\mathcal{F}')$  of  $\mathcal{F}'$  is less (resp. non greater) than the slope  $\mu(\mathcal{F})$  of  $\mathcal{F}$ .

A holomorphic bundle on the Riemann sphere is trivial if and only if it is a semistable bundle of degree 0. Indeed, each vector bundle  $\mathcal{F}$  on the Riemann sphere is holomorphically equivalent to a sum of line bundles

$$(19) \quad \mathcal{F} \cong \mathcal{O}(c_1) \oplus \dots \oplus \mathcal{O}(c_p),$$

where the ordered set of integers  $c_1 \geq \dots \geq c_p$  is called the *splitting type* of the bundle  $\mathcal{F}$ . If the bundle  $\mathcal{F}$  is semistable of degree zero, then  $c_1 + \dots + c_p = 0$  and  $c_i \leq 0$ ,  $i = 1, \dots, p$ . Thus,  $c_1 = \dots = c_p = 0$  and  $\mathcal{F}$  is holomorphically trivial.

In what follows we will need the notion of stability of a pair consisting of a vector bundle and a connection. A subbundle  $\mathcal{F}'$  of the bundle  $\mathcal{F}^\Lambda$  is said to be *stabilized by the connection*  $\nabla$  if the covariant derivative  $\nabla_{d/dz}$  maps local holomorphic sections of  $\mathcal{F}'$  into sections of the same subbundle. In the coordinate description  $\{O_i, U_\alpha\}$ ,  $\{g_{i\alpha}, g_{\alpha\beta}\}$ ,  $\{\omega^{\Lambda_i}, \omega_\alpha\}$  of the pair  $(\mathcal{F}^\Lambda, \nabla^\Lambda)$ , the existence of such a subbundle means the following. For all  $i = 1, \dots, n$ , there exist matrices  $\Gamma_i$ , each holomorphically invertible in the corresponding  $O_i$  and such that all systems (14) obtained from the systems (17) via the gauge transformations  $\Gamma_i$  have the form

$$(20) \quad \omega'_i = \begin{pmatrix} \omega_i^1 & * \\ 0 & \omega_i^2 \end{pmatrix}$$

with blocks  $\omega_i^1$  of the same size for all  $i$ . The local subsystems  $\omega_i^1$  define the restriction of the connection  $\nabla^\Lambda$  to a subbundle  $\mathcal{F}'$ . Each formal solution  $\hat{Y}'_i$  of such a system (14) can be chosen to have the same upper block-triangular structure

$$\hat{Y}'_i = \begin{pmatrix} \hat{Y}_i^1 & * \\ 0 & \hat{Y}_i^2 \end{pmatrix},$$

where the matrix  $\hat{Y}_i^1$  serves as a formal fundamental matrix for the subsystem  $\omega_i^1$ . Moreover, the matrix  $\hat{Y}'_i$  is connected to the initial matrix  $\hat{Y}_i$  by  $\hat{Y}'_i = \Gamma_i \hat{Y}_i S$  where  $S$  is a constant invertible matrix and  $\hat{Y}'_i$  has the same form (15) as the initial matrix  $\hat{Y}_i$ , namely

$$(21) \quad \hat{Y}'_i(z) = \hat{F}'_0(z)(z - a_i)^{\Lambda_i}(z - a_i)^{J'} e^{Q'(z)},$$

where  $\Lambda'_i = S^{-1}\Lambda_i S$ ,  $\tilde{J}' = S^{-1}\tilde{J}S$ ,  $Q'(z) = S^{-1}Q(z)S$ , where the matrices  $\Lambda'_i$  and  $Q'(z)$  are diagonal and obtained by suitable permutations of the diagonal elements of  $\Lambda_i$  and  $Q(z)$  respectively, and where the matrix  $\tilde{J}'$  is upper triangular and satisfies the following condition: the entry  $e_{kl}$  of the matrix equals zero if for corresponding eigenvalues of  $\tilde{J}'$  one has  $\rho_i^k \neq \rho_i^l$ . Moreover the invertible formal holomorphic matrix  $\tilde{F}'_0(z)$  has the same upper block triangular structure as the matrix  $Y'_i$ . The existence of such a fundamental matrix follows from results of [9] and [1].

Thus, the degree of the subbundle  $\mathcal{F}'$  can be determined in the same way as the degree of  $\mathcal{F}^\Lambda$ , just replacing the systems (17) by the subsystems  $\omega_i^1$  of (20).

Let us recall that a pair  $(\mathcal{F}, \nabla)$  consisting of a vector bundle  $\mathcal{F}$  and a connection  $\nabla$  on  $\mathcal{F}$  is said to be *stable* (resp. *semistable*) if for any proper subbundle  $\mathcal{F}'$  of  $\mathcal{F}$  stabilized by  $\nabla$  one has  $\mu(\mathcal{F}') < \mu(\mathcal{F})$  (resp.  $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$ ) (cf. [61]).

The main result of our paper is the following.

**Theorem 2** *Let  $\mathcal{M}$  be a reduced datum, in which at least one of the prescribed singularities is without roots. If there exists a collection  $\Lambda$  of admissible matrices  $\Lambda_1, \dots, \Lambda_n$  such that the corresponding pair  $(\mathcal{F}^\Lambda, \nabla^\Lambda)$  is stable, then the generalized Riemann-Hilbert problem for  $\mathcal{M}$  has a solution.*

Theorem 2 is analogous to Theorem 1 of [17] which was proved by A. Bolibruch in the case of the classical Riemann-Hilbert problem. The proof below follows the proof of [17] with some simplifications (Theorem 1 of [17] was proved for any compact Riemann surface).

### 3.1 Proof of Theorem 2

Without loss of generality we may assume that  $i = 1$  and  $a_1 = 0$ . We will denote a given stable pair  $(\mathcal{F}^\Lambda, \nabla^\Lambda)$  in  $\mathcal{E}$  simply by  $(\mathcal{F}, \nabla)$ .

**Part 1.** We first consider the case in which *all eigenvalues of the local monodromy matrices (formal and proper) are positive real numbers*. The real parts of all  $\rho_i^m$  then equal zero. The pair  $(\mathcal{F}, \nabla)$  being stable, one has  $\mu(\mathcal{F}') < \mu(\mathcal{F})$  for every (proper, nonzero) subbundle  $\mathcal{F}'$  stabilized by the connection  $\nabla$ . All  $\mu(\mathcal{F}')$  are rational numbers whose denominators are not greater than the rank of  $\mathcal{F}$ , hence the set of such numbers is finite and the number  $\mu_{max} = \max_{\mathcal{F}'}(\mu(\mathcal{F}') - \mu(\mathcal{F}))$  is well defined. The stability of the pair  $(\mathcal{F}, \nabla)$  implies  $\mu_{max} < 0$ .

Starting with the initial pair  $(\mathcal{F}, \nabla)$ , we construct a new pair  $(\bar{\mathcal{F}}, \bar{\nabla})$  as follows.

We replace the matrices  $\Lambda_i$  which were used to construct the initial bundle  $\mathcal{F}$ , with matrices  $\Lambda'_i = N\Lambda_i$  for some positive integer  $N$ . We choose  $N$  such that  $-N\mu_{max} \gg (R - 2 + n)p^3$ , where  $R$  is the sum over all singular points of the Poincaré ranks of  $\nabla$ , and where  $\gg$  means “sufficiently larger than” (the difference can be made as large as needed). The new matrices  $\Lambda'_i$  are clearly admissible and thus, the corresponding vector bundle  $\bar{\mathcal{F}}$  is equipped with a connection  $\bar{\nabla}$  which has the same generalized monodromy data (including the Poincaré ranks at the singularities) as the initial connection.



It is not difficult to see that the pair  $(\bar{\mathcal{F}}, \bar{\nabla})$  is stable, and that for any subbundle  $\bar{\mathcal{F}}' \subset \bar{\mathcal{F}}$  stabilized by the connection  $\bar{\nabla}$  one has

$$(22) \quad \mu(\bar{\mathcal{F}}') - \mu(\bar{\mathcal{F}}) \leq N\mu_{max} \ll -(R - 2 + n)p^3.$$

Indeed, if  $\mathcal{F}' \subset \mathcal{F}$  is a given (proper, nonzero) subbundle of  $\mathcal{F}$  that is stabilized by  $\nabla$ , then, as explained earlier, each local form  $\omega'_i$  of the connection  $\nabla$  (in the corresponding coordinate description of the bundle  $\mathcal{F}$ ) is of the form (20) at  $a_i$ , where the first diagonal block  $\omega'_1$  corresponds to  $\mathcal{F}'$  and the sum over all singular points  $a_i$  of the traces of  $\text{res}_{a_i} \omega'_1$  over all singular points  $a_i$  is equal to the degree of  $\mathcal{F}'$ . It follows from (21) that replacing  $\Lambda_i$  by  $\Lambda'_i$  multiplies the real parts of the traces by  $N$ . Thus, all degrees of all proper subbundles stabilized by  $\nabla$  are multiplied by  $N$ . (Note that the sum of the imaginary parts of the corresponding traces for every stabilized subbundle is zero; since we assumed that the eigenvalues of all local monodromy matrices are positive real, only the admissible matrices  $\Lambda_i$  will give a real input in the degrees.) The slopes of all stabilized subbundles (including the bundle  $\mathcal{F}$ ) are multiplied by  $N$ , hence

$$\mu(\bar{\mathcal{F}}') = N\mu(\mathcal{F}') < \mu(\bar{\mathcal{F}}) = N\mu(\mathcal{F}),$$

and the new pair is stable. If the entries  $\lambda_{i,k}$  and  $\lambda_{i,l}$  of the matrix  $\Lambda_i$  are distinct, then after multiplication by  $N$  their difference will be “sufficiently larger” than  $(R - 2 + n)p^3$ .

Since further in the proof we shall need “large” differences between entries of the matrix  $\Lambda_1$  at  $a_1 = 0$ , we have to modify the matrix  $\Lambda'_1$  once more to separate possible pairs of equal eigenvalues. In order to preserve the stability of the pair, replace the matrix  $\Lambda'_1 = \text{diag}(\lambda'_1, \dots, \lambda'_p)$  with an admissible matrix  $\Lambda''_1 = \text{diag}(\nu_1, \dots, \nu_p)$  such that  $\text{tr} \Lambda''_1 = \text{tr} \Lambda'_1$  and such that

$$|\nu_i - \lambda'_i| < (R - 2 + n)p^3,$$

$$(23) \quad (R - 2 + n)p^2 \leq \nu_i - \nu_j < (R - 2 + n)p^3 \quad \text{for } i < j$$

if  $\lambda'_i = \lambda'_j$ . To do this it is sufficient to replace every maximal chain of equal numbers  $\lambda'_{i_1} = \dots = \lambda'_{i_s}$ ,  $i_1 < \dots < i_s$  with the chain

$$\nu_{i_t} = \lambda'_{i_t} + (R - 2 + n)p^2([\![s/2]\!] - t + 1) \quad \text{for } t \leq [s/2],$$

$$\nu_{i_t} = \lambda'_{i_t} + (R - 2 + n)p^2([\![s+1/2]\!] - t) \quad \text{for } t > [s/2],$$

where  $[ ]$  stands for the integer part. Let  $\Lambda_1$  denote again this admissible matrix  $\Lambda''_1$ . Let  $\mathcal{G}$  denote the corresponding bundle and  $\nabla$  the corresponding logarithmic connection on  $\mathcal{G}$ . From the construction (in particular from (22)) it follows that the pair  $(\mathcal{G}, \nabla)$  is stable.

Assume that the bundle  $\mathcal{G}$  is not holomorphically trivial and consider a meromorphic trivialization of  $\mathcal{G}$ , holomorphic outside of  $a_1 = 0$ . As was explained in Section 2, the corresponding global system (2) constructed *via* this trivialization has the prescribed generalized monodromy data except at  $a_1 = 0$ , where the Poincaré rank may be greater than  $r_1$ . This means that the formal fundamental matrix  $\hat{Y}_1(z)$  of the system is of the form (15)

$$\hat{Y}_1(z) = \hat{F}(z)z^{\Lambda_1}z^{\hat{J}}e^{Q(z)},$$

where the matrix  $\hat{F}$  is a formal meromorphic series (and not formal holomorphic as it would be if  $\mathcal{G}$  was holomorphically trivial). Moreover, in view of the decomposition (19) one can choose a meromorphic trivialization of the bundle (that is, matrices  $\Gamma_i$ ) such that the matrix  $\Gamma_1(z)$  is of the form

$$\Gamma_1(z) = z^{-K} \Gamma_1^0(z),$$

where  $K$  is the integer-valued diagonal matrix  $K = \text{diag}(c_1, \dots, c_p)$ ,  $c_1 \geq \dots \geq c_p$ , and  $\Gamma_1^0(z)$  is holomorphically invertible in  $O_1$ . Thus, the formal fundamental matrix  $\hat{Y}_1$  is of the form

$$(24) \quad \hat{Y}_1(z) = z^{-K} \hat{F}^0(z) z^{\Lambda_1} z^{\tilde{J}} e^{Q(z)}$$

with an invertible formal holomorphic series  $\hat{F}^0(z)$ .

The following statement generalizes related results of [17] and plays a crucial role in the proof of Theorem 2.

**Lemma 2** *The following inequalities hold for the entries of the matrix  $K$  in the decomposition (24):*

$$c_j - c_{j+1} \leq R + n - 2, \quad j = 1, \dots, p - 1,$$

where  $R$  denotes the sum of the (prescribed) Poincaré ranks  $r_i$  of the initial connection  $\nabla$  at all singularities.

The proof of Lemma 2 is given in Section 3.2 below. It only uses the semistability of the pair  $(\mathcal{G}, \nabla)$  (which is weaker than its stability). In terms of vector bundles the previous lemma can be reformulated as follows.

**Lemma 3** *If a pair  $(\mathcal{G}, \nabla)$  is semistable, then the inequalities*

$$c_j - c_{j+1} \leq R + n - 2, \quad j = 1, \dots, p - 1$$

hold for the splitting type  $c_1 \geq \dots \geq c_p$  of  $\mathcal{G}$  and for the sum  $R$  of all Poincaré ranks of the connection  $\nabla$  at the  $n$  prescribed singular points.

We will moreover need the following technical lemma, which is given in [13] and [15].

**Lemma 4** *Let the matrix  $\hat{F}(z)$  (formal or analytic) be invertible in a neighborhood  $O_1$  of  $a_1 = 0$ . Then for any integer-valued diagonal matrix  $K = \text{diag}(k_1, \dots, k_p)$  there exists a matrix  $T(z)$ , polynomial in  $1/z$  and holomorphically invertible outside of  $\{a_1\}$ , such that*

$$(25) \quad T(z) z^K \hat{F}(z) = \hat{H}(z) z^D,$$

where the matrix  $\hat{H}(x)$  is invertible in  $O_1$  and  $D$  is a diagonal matrix obtained by a suitable permutation of the diagonal elements of  $K$ .

Apply this lemma to the factor  $z^{-K}\hat{F}^0(z)$  in the expression (24) of the fundamental matrix  $\hat{Y}_1(z)$ . The gauge transformation  $\hat{Y}_1^f(z) = T(z)\hat{Y}_1(z)$  (which is holomorphically invertible outside of zero) changes our system (2) into a system with the following formal fundamental matrix at  $a_1 = 0$

$$\hat{Y}_1^f(z) = \hat{H}(z)z^{D+\Lambda_1}z^{\tilde{J}}e^{Q(z)},$$

where the formal matrix series  $\hat{H}(z)$  is invertible. It follows from Lemmas 2 and 4 that the difference between any two diagonal elements of  $D$  is bounded by  $(R+n-2)(p-1)$ . Since by construction the matrix  $\Lambda_1$  satisfies Inequalities (23), the diagonal entries of  $D+\Lambda_1$  form a decreasing sequence; hence this matrix is admissible. From (16) we deduce that the final system has Poincaré rank  $r_1$  at  $a_1 = 0$ . And since  $T$  is holomorphically invertible outside of  $\{a_1\}$  we get that the final system has the required Poincaré ranks at all points. The theorem is proved (under the assumptions on the monodromy exponents made at the beginning of the proof).

**Part 2.** Now consider the case of *arbitrary eigenvalues of the local monodromy operators*. For any  $N \in \mathbb{Z}$ , we have

$$N\rho_i^j = N\lambda_i^j + N\rho_i^j = N\lambda_i^j + (N-1)\rho_i^j + \rho_i^j = \lambda_i^j + \rho_i^j + \alpha_i^j,$$

where  $\lambda_i^j = N\lambda_i^j + [\text{Re}(N-1)\rho_i^j]$  and  $0 \leq \text{Re}\alpha_i^j < 1$  (where  $[ ]$  as before stands for the integer part).

For each  $i = 1, \dots, n$ , consider (with the earlier notation of (5)) formal solutions  $\hat{Y}_i = \hat{F}_i H_i$ , where  $H_i = z^{\Lambda_i} z^{\tilde{J}} e^Q$ , and replace the matrix  $\Lambda_i$  with  $\Lambda'_i$  obtained as follows. Let  $\Lambda_i^j$  denote the blocks of  $\Lambda_i$  corresponding to the superblocks  $H_i^j$  of  $H_i$ . If  $H_i^j$  has no ramification, replace the block  $\Lambda_i^j$  with the diagonal matrix  ${}^N\Lambda_i^j$  with entries  $\lambda_i^j$ . If  $H_i^j$  is ramified, of size  $d$ , then replace the block  $\Lambda_i^j$  with  ${}^N\Lambda_i^j = N\Lambda_i^j + sI_d$ , where

$$s = \left\lceil \frac{\sum \text{Re}(N-1)\rho_i^j}{d} \right\rceil,$$

and where the sum is taken over all eigenvalues of the superblock  $\tilde{J}_j$ . The matrices  $\Lambda'_i$  are clearly admissible.

Let us prove that for sufficiently large  $N$  the corresponding pair  $(\mathcal{G}', \nabla')$  is stable and Inequalities (22) hold.

If for each  $i$  the formal solution at  $a_i$  is unramified, then one proceeds as before, since the sum of all  $\text{Re}\alpha_i^m$ ,  $0 \leq \text{Re}\alpha_i^m < 1$ , over all  $i, m$ , is bounded by  $pn-1$ . Thus, in the change of degrees carried out in Part 1. of the proof, one just has to replace the slopes  $\mu_l$  (obtained there after multiplying  $\Lambda_i$  by  $N$ ) of all subbundles stabilized by the connection, now by numbers  $\mu_l + t_l$ , where  $|t_l|$  is bounded by  $pn-1$ , which is  $\ll N\mu_{max}$  for sufficiently large  $N$ .

Let for some  $i$  the formal solution have a superblock  $H_j(z)$  with ramification. From the related result of [7] it follows that if our bundle has a subbundle stabilized by the connection and if the local form of the connection has the form (20), then the part of the

formal solution at  $a_i$  that corresponds to the superblock  $H_j$  appears entirely as a block of the formal solution of either the subsystem  $\omega_i^1$  or the corresponding quotient system  $\omega_i^2$  (in [7] this property is called *irreducibility of superblocks with roots*).

Thus, when we replace a block  $\Lambda_i^j$  with  ${}^N\Lambda_i^j$  instead of  ${}^l\Lambda_i^j$ , we actually replace all the final slopes  $\mu_l$  of subbundles stabilized by the connection, with numbers  $\mu_l + t_l^j$  where  $|t_l^j|$  is not greater than  $p(n+1) - 1$ , which again is  $\ll N\mu_{max}$ .

Therefore, the pair which we have constructed is stable and Inequalities (22) hold. The remainder of the proof for the general case is the same as in the special case considered before (of monodromy operators with positive eigenvalues).

### 3.2 Proof of Lemma 2

Assume that  $c_l - c_{l+1} > R + n - 2$  for some  $l$ , and let us show that this contradicts the fact that the pair  $(F, \nabla)$  is stable. In view of (24) the form  $\omega$ , for the system (2) constructed above, can be written in  $O_1$  as follows,

$$\omega = z^{-K} \theta z^K,$$

where the form  $\theta$  has a pole of order  $r_1 + 1$  since

$$\theta = -\frac{K}{z} + d \left( \hat{F}^0(z) z^{\Lambda_1} z^{\bar{J}} e^{Q(z)} \right) \left( \hat{F}^0(z) z^{\Lambda_1} z^{\bar{J}} e^{Q(z)} \right)^{-1}$$

following the calculation in (16).

The entries  $\omega_{mj}$  and  $\theta_{mj}$  of the matrix differential forms  $\omega$  and  $\theta$  respectively, for  $m \neq j$ , are connected as follows

$$\omega_{mj}(z) = \theta_{mj}(z) z^{-c_m + c_j},$$

and we have by assumption  $c_j - c_m > R + n - 2$  for  $m > l$ ,  $j \leq l$ . The orders of zero of the differential forms  $\omega_{mj}(z)$  at  $a_1 = 0$ , for  $m > l$ ,  $j \leq l$ , are therefore greater than  $R + n - r_1 - 3$ , whereas the sum of the orders of poles at the other singular points is not greater than  $R - r_1 + n - 1$  (respectively  $R - r_1 + n - 3$ ) if the point at infinity is non-singular (resp. singular). If the form  $\omega$  is holomorphic at infinity, then it has a zero of order two there. One gets in both cases that for each entry  $\omega_{mj}(z)$ ,  $m > l$ ,  $j \leq l$ , the degree of its singular divisor (the sum of orders of zeros minus the sum of orders of poles on the Riemann sphere) is greater than zero. Thus, all such entries  $\omega_{mj}$  equal zero identically and  $\omega$  has the form

$$(26) \quad \omega' = \begin{pmatrix} \omega^1 & * \\ 0 & \omega^2 \end{pmatrix},$$

where the form  $\omega^1$  has size  $l \times l$ .

This implies that there exists a constant invertible matrix  $S$  such that  $\hat{Y}_1(z)S$  has a form similar to (24)

$$\hat{Y}_0(z) = \hat{Y}_1(z)S = z^{-K} \hat{F}^0(z) z^{\Lambda_1} z^{\bar{J}'} e^{Q'(z)},$$

where

$$\hat{F}^0 = \begin{pmatrix} \hat{F}^1 & * \\ 0 & \hat{F}^2 \end{pmatrix},$$

and  $\hat{F}^1$  is of size  $l \times l$ . The vector bundle  $\mathcal{F}^1$  of rank  $l$  carrying the connection  $\nabla^1$  defined by the subsystem  $\omega^1$  is a subbundle of  $\mathcal{G}$  which is stabilized by  $\nabla$ . The degree of this subbundle is  $c_1 + \dots + c_l$ . It follows from the assumption  $c_l > c_{l+1}$  that

$$\mu(\mathcal{F}^1) = \frac{c_1 + \dots + c_l}{l} > \frac{c_1 + \dots + c_p}{p} = \mu(\mathcal{G}).$$

This contradicts the semistability of  $(\mathcal{G}, \nabla)$ , hence it proves the lemma.

### 3.3 Generic data

From Theorem 2 we deduce that for generic generalized monodromy data, as defined below, the GRH-problem always has a solution, up to a single local condition of non-ramification.

**Definition 4** *If  $\mathcal{M}$  is a reduced datum, let  $\mathcal{M}_s$  denote the reduced datum consisting of  $\mathcal{M}$  and of a family of local systems (9) realizing  $\mathcal{M}$ . A datum  $\mathcal{M}_s$  is said to be generic if the pair  $(\mathcal{F}, \nabla)$ , or canonical extension, constructed in Section 2.2.1 from the systems (9), has no subbundle stabilized by  $\nabla$ .*

In terms of local systems, genericity means that it is impossible to transform the systems (9) in the form (20) by means of local holomorphic gauge transformations.

Any pair  $(\mathcal{F}^\Lambda, \nabla^\Lambda)$  constructed from a generic datum  $\mathcal{M}_s$  is by definition stable, and we obtain the following statement which generalizes a similar result of [14].

**Corollary 1** *Let  $\mathcal{M}_s$  be a generic datum. Then the generalized Riemann-Hilbert problem for  $\mathcal{M}$  has a solution if at least one of the prescribed singularities is without roots.*

If in particular the monodromy representation (8) is irreducible, then  $\mathcal{M}$  is clearly generic; thus we obtain the expected generalization of [13] and [34].

**Corollary 2** *Let  $\mathcal{M}$  be a reduced datum. Assume that the prescribed monodromy representation (8) is irreducible. Then the generalized Riemann-Hilbert problem for  $\mathcal{M}$  has a solution if at least one of the prescribed singularities is without roots.*

The conditions of Corollary (2) closely relate the GRH-problem to the Deligne-Simpson problem, which consists in constructing an irreducible monodromy representation for a suitable choice of conjugacy classes in  $\mathrm{GL}(n, \mathbb{C})$  for the images of the elementary loop-classes. We refer to [35] for an up-to-date survey by V. Kostov, of results on this problem.

## 4 Reducible Solutions of the Generalized Riemann-Hilbert Problem

We now consider a non-generic datum  $\mathcal{M}_s$  for which the generalized Riemann-Hilbert problem has a solution. One may ask whether it is possible to realize this datum by a *reducible* system (2) of differential equations of the form (26). The following statement is a generalization of the main result of [41] and answers the question.

**Theorem 3** *Let  $\mathcal{M}_s$  be a non-generic datum. Then, under the conditions of Theorem 2, the reduced datum  $\mathcal{M}$  can be realized by a reducible system of the form (26).*

**Proof** We proceed as for the proof of Theorem 2, and keep the same notation as before. Consider a vector bundle  $(F^\Lambda, \nabla^\Lambda)$  in  $\mathcal{E}$ , and assume that the pair  $(F^\Lambda, \nabla^\Lambda)$  is stable and holomorphically trivial, hence of degree zero.

The idea in the first step of the proof is the following. Starting with  $(F^\Lambda, \nabla^\Lambda)$  we construct a pair  $(F^{\tilde{\Lambda}}, \nabla^{\tilde{\Lambda}})$  in  $\mathcal{E}$  which has a subbundle  $\tilde{F}_1$  stabilized by the connection  $\nabla^{\tilde{\Lambda}}$ . The construction must be carried out in such a way that the pairs  $(\tilde{F}_1, \nabla^{\tilde{\Lambda}}|_{\tilde{F}_1})$  and  $(F^{\tilde{\Lambda}}/\tilde{F}_1, \nabla_q^{\tilde{\Lambda}})$ , where  $\nabla_q^{\tilde{\Lambda}}$  is the connection induced on the quotient bundle  $F^{\tilde{\Lambda}}/\tilde{F}_1$ , are stable, that they have degree zero and that the difference between any two entries of the matrix  $\tilde{\Lambda}_1$  is greater than  $(R - 2 + n)p$ . As before,  $R$  denotes the sum of all Poincaré ranks. For the construction of  $\nabla_q^{\tilde{\Lambda}}$ , see the beginning of Section 3.

The construction can be achieved as follows. Since the reduced datum  $\mathcal{M}_s$  is non-generic, the set  $\mathcal{F}$  of proper subbundles of  $F^\Lambda$  that are stabilized by the connection  $\nabla^\Lambda$  is non-empty. Consider a bundle  $F_1$  in  $\mathcal{F}$  of maximal rank with the property that

$$\deg(F_1) = \max_{F' \in \mathcal{F}} \deg(F').$$

It follows from the stability of  $(F^\Lambda, \nabla^\Lambda)$  that  $\deg(F_1) < 0$ . Consider any proper filtration  $F_{t_1} \subset F_1 \subset F_{t_2} \subset F^\Lambda$ , where  $F_{t_1}$  and  $F_{t_2}$  belong to  $\mathcal{F}$  (by a *proper* filtration we mean that all inclusions are strict). In the following, such a filtration will be called a *stabilized filtration*, and it satisfies the following inequalities:

$$\deg(F_{t_1}) \leq \deg(F_1) \quad , \quad \deg(F_{t_2}) \leq \deg(F_1) - 1 \quad , \quad \deg(F^\Lambda) = 0.$$

Let  $\lambda_i^j$  be an entry of a block  $\Lambda_i^l$  of  $\Lambda_i$  corresponding to the superblock  $\tilde{J}_l$  without roots of  $\tilde{J}$  (see (18)). As in the proof of Theorem 2, we have for any  $N \in \mathbb{Z}$

$$N(\lambda_i^j + \rho_i^j) = \tilde{\lambda}_i^j + \rho_i^j + \alpha_i^j$$

where  $\tilde{\lambda}_i^j \in \mathbb{Z}$  and  $0 \leq \operatorname{Re} \alpha_i^j < 1$ . If the block  $\Lambda_i^l$  corresponds to a factor of dimension  $m_{il}$  of a superblock  $\tilde{J}_l$  with roots, we replace this block by the block  $\Lambda_i^l = N\Lambda_i^l + sI_{m_{il}}$  where

$$s = \left\lceil \frac{\sum \operatorname{Re}(N - 1)\rho_i^j}{m_{il}} \right\rceil$$

and where the sum is taken over all eigenvalues of the superblock  $\tilde{J}_j$  of (18). To get a bundle of degree zero with large enough differences between any two exponents, we shall modify the integers  $\tilde{\lambda}_i^m$  as follows. We choose integers  $k_1^m$ ,  $1 \leq j \leq p$ , such that  $k_1^{l_1} - k_1^{l_2} > (R - 2 + n)p$  if for  $l_1 < l_2$  the entry  $e_{l_1, l_2}$  of  $\tilde{J}$  is non-zero, and such that  $k_1^j > 2np$  for  $1 \leq j \leq p$ . We define

$$\begin{aligned}\lambda_1^1 &= \tilde{\lambda}_1^1 + \sum_{i,j} \alpha_i^j + r + k_1^1 \\ \lambda_1^j &= \tilde{\lambda}_1^j + k_1^j \quad , \quad \lambda_1^p = \tilde{\lambda}_1^p - \sum_{j=1}^p k_1^j \\ \lambda_i^j &= \tilde{\lambda}_i^j \quad \text{for all other } (i, j),\end{aligned}$$

with

$$r = \sum m_{il} \left\{ \frac{\sum \operatorname{Re}(N-1)\rho_i^j}{m_{il}} \right\},$$

where the sum is taken over all superblocks  $\tilde{J}_i$  with roots, for  $2 \leq i \leq n$ , and  $\{ \}$  stands for the fractional part. The matrices  $\Lambda'_i$  with entries  $\lambda_i^j$  are clearly admissible, with the additional property that the difference between any two diagonal entries of  $\Lambda'_1$  is greater than  $(R - 2 + n)p$ .

Consider the pair  $(F^{\Lambda'}, \nabla^{\Lambda'})$  obtained for some  $N \gg 1$ . For any stabilized filtration  $F'_{t_1} \subset F'_1 \subset F'_{t_2} \subset F^{\Lambda'}$ , we get

$$\deg(F'_{t_1}) < \deg(F'_1) = N \deg(F_1) + c_{n,p,k} < 0,$$

$$\deg(F'_{t_2}) \leq N \deg(F_1) - N + 2np + \sum_{j=1}^p k_1^j < \deg(F'_1)$$

and

$$\deg(F^{\Lambda'}) = 0,$$

where  $c_{n,p,k}$  is a sum of terms involving  $\alpha_i^j$ ,  $m_{il} \left\{ \sum \operatorname{Re}(N-1)\rho_i^j / m_{il} \right\}$  and  $k_1^j$ . To construct the pair  $(F^{\tilde{\Lambda}}, \nabla^{\tilde{\Lambda}})$ , we shall modify the integers  $\lambda_i^j$  as follows

$$\begin{aligned}\tilde{\lambda}_1^1 &= \lambda_1^1 + N \deg(F_1) + c_{n,p,k} \\ \tilde{\lambda}_1^p &= \lambda_1^p - N \deg(F_1) - c_{n,p,k} \\ \tilde{\lambda}_i^j &= \lambda_i^j \quad \text{for the other } (i, j).\end{aligned}$$

Again, the matrices  $\tilde{\Lambda}_i$  with entries  $\tilde{\lambda}_i^j$  are admissible with the additional property that the difference between any two diagonal entries of  $\tilde{\Lambda}_1$  is greater than  $(R - 2 + n)p$ . The pair  $(F^{\tilde{\Lambda}}, \nabla^{\tilde{\Lambda}})$  moreover satisfies the following property. For any stabilized filtration  $\tilde{F}_{t_1} \subset \tilde{F}_1 \subset \tilde{F}_{t_2} \subset F^{\tilde{\Lambda}}$  we have

$$\deg(\tilde{F}_{t_1}) < 0 \quad , \quad \deg(\tilde{F}_1) = 0 \quad , \quad \deg(\tilde{F}_{t_2}) < 0 \quad , \quad \deg(F^{\tilde{\Lambda}}) = 0.$$

This says that the pairs  $(\tilde{F}_1, \nabla^{\tilde{\Lambda}}|_{\tilde{F}_1})$  and  $(F^{\tilde{\Lambda}}/\tilde{F}_1, \nabla_q^{\tilde{\Lambda}})$  are stable and of degree zero.

In the second part of the proof, consider a meromorphic trivialization of the bundle  $(F^{\tilde{\Lambda}}, \nabla^{\tilde{\Lambda}})$  holomorphic outside of the point  $a_1 = 0$ , that induces a meromorphic trivialization of  $\tilde{F}_1$ . The corresponding global system (2) constructed from this trivialization has the prescribed generalized monodromy data, except at  $a_1 = 0$ , where the Poincaré rank may be greater than  $r_1$ . We can choose a formal fundamental matrix  $\hat{Y}_1(z)$  of (2) of the form

$$\hat{Y}_1(z) = \begin{pmatrix} \hat{Y}_1^1(z) & * \\ 0 & \hat{Y}_2^1(z) \end{pmatrix} = \hat{F}(z) z^{\tilde{\Lambda}_1} z^{\tilde{J}} e^{Q(z)}, \quad \hat{F} = \begin{pmatrix} \hat{F}_1 & * \\ 0 & \hat{F}_2 \end{pmatrix},$$

where  $\hat{F}$  is formal meromorphic and where the matrix  $\hat{Y}_1^1(z)$  is chosen to be the formal fundamental matrix of a subsystem  $\omega_1$  that represents the restriction of  $\nabla^{\tilde{\Lambda}}$  on  $\tilde{F}_1$ , and  $\hat{Y}_2^1(z)$  the formal fundamental matrix of a quotient-system  $\omega_2$  that represents the connection  $\nabla_q^{\tilde{\Lambda}}$  on  $F^{\tilde{\Lambda}}/\tilde{F}_1$ .

The bundles  $\tilde{F}_1$  and  $F^{\tilde{\Lambda}}/\tilde{F}_1$  are holomorphically equivalent to sums of line bundles

$$(27) \quad \tilde{F}_1 \cong \mathcal{O}(c_1) \oplus \cdots \oplus \mathcal{O}(c_{p_1}) \quad , \quad F^{\tilde{\Lambda}}/\tilde{F}_1 \cong \mathcal{O}(c_{p_1+1}) \oplus \cdots \oplus \mathcal{O}(c_p)$$

where  $c_1 \geq \dots \geq c_{p_1}$  and  $c_{p_1+1} \geq \dots \geq c_p$ .

Since they are of degree zero, the trace of the matrices  $K_1 = \text{diag}(c_1, \dots, c_{p_1})$  and  $K_2 = \text{diag}(c_{p_1+1}, \dots, c_p)$  is equal to zero. By construction, the bundles  $(\tilde{F}_1, \nabla^{\tilde{\Lambda}}|_{\tilde{F}_1})$  and  $(F^{\tilde{\Lambda}}/\tilde{F}_1, \nabla_q^{\tilde{\Lambda}})$  are stable pairs. By Lemma 2, we get the following estimates

$$(28) \quad |c_j - c_k| \leq (R + n - 2)p$$

for  $1 \leq j, k \leq p$ . It follows from the holomorphic equivalence (27) that there is a gauge transformation  $\hat{Y}_1^b(z) = \Gamma^b(z) \hat{Y}_1(z)$ , holomorphic outside of zero, of the form

$$\Gamma^b(z) = \begin{pmatrix} \Gamma_1^b(z) & 0 \\ 0 & \Gamma_2^b(z) \end{pmatrix},$$

that transforms the system (2) into a system with the following formal fundamental solution at zero

$$\hat{Y}_1^b(z) = \begin{pmatrix} z^{-K_1} \hat{F}_1^0(z) & * \\ 0 & z^{-K_2} \hat{F}_2^0(z) \end{pmatrix} z^{\tilde{\Lambda}_1} z^{\tilde{J}} e^{Q(z)},$$

where  $\hat{F}_j^0$ ,  $j = 1, 2$ , is formal-holomorphically invertible. From Lemma 4 we get a gauge transformation  $\hat{Y}_1^k(z) = \Gamma(z) \hat{Y}_1^b(z)$ , holomorphic outside of zero, that transforms the latter system into a system with the following formal fundamental solution at zero,

$$\hat{Y}_1^k(z) = \begin{pmatrix} \hat{H}_1(z) & \hat{H}_3(z) \\ 0 & \hat{H}_2(z) \end{pmatrix} z^{\tilde{K} + \tilde{\Lambda}_1} z^{\tilde{J}} e^{Q(z)},$$

where  $\tilde{K}$  is obtained after a suitable permutation of the diagonal elements of  $\text{diag}(K_1, K_2)$ , and where  $\hat{H}_1$ ,  $\hat{H}_2$  are formal-holomorphically invertible, and  $\hat{H}_3$  is formal meromorphic. Moreover, the estimates on the  $c_j$  and  $\lambda_1^j$  imply that  $\tilde{K} + \tilde{\Lambda}_1$  is admissible.

We now need the following lemma, which is given in its analytic version in [14].



**Lemma 5** Consider a formal meromorphic matrix  $F(z) = \begin{pmatrix} F_1(z) \\ F_2(z) \end{pmatrix}$  where  $F_2(z)$  is formal-holomorphically invertible. There exists a meromorphic matrix  $\Gamma(z)$  at 0, which is holomorphically invertible outside of zero, such that

$$\Gamma(z)F(z) = \begin{pmatrix} \tilde{F}_1(z) \\ F_2(z) \end{pmatrix},$$

where  $\tilde{F}_1$  is formal holomorphic.

Using this lemma, we gauge-transform the system into a system  $dy = \tilde{\omega}y$  that has  $\hat{Y}_1^k(z)$  as formal fundamental solution, and where  $\tilde{H}_3$  is formal holomorphic.

Formula (16) shows that the latter system has Poincaré rank  $r_1$  at zero, and the required Poincaré ranks at all other points. Moreover, the form of the fundamental matrix  $\hat{Y}_1^k(z)$  tells us that this system is *reducible*, which means that the coefficient matrix  $\tilde{\omega}$  is upper block-triangular,

$$\tilde{\omega} = \begin{pmatrix} \tilde{\omega}_1 & * \\ 0 & \tilde{\omega}_2 \end{pmatrix}.$$

□

As an application of the preceding results, we will consider the problem of reducibility for a special type of systems (2) which we will call *formally*.

**Definition 5** A differential system (2) is called *formally Fuchsian* on  $\mathbb{P}^1(\mathbb{C})$  if its formal fundamental solution (15) at each singular point  $a_i$ ,  $1 \leq i \leq n$ , is

$$\hat{Y}_i(z) = \hat{F}_0(z)(z - a_i)^{\Lambda_i}(z - a_i)^{\tilde{J}}e^{Q(z)}$$

where  $\Lambda_i$  is admissible and  $\hat{F}_0$  is formal-holomorphically invertible.

The following statement is a generalization of the main results of [41], [42].

**Proposition 1** Consider a differential system (2) which is formally Fuchsian on  $\mathbb{P}^1(\mathbb{C})$ . Assume that the generalized monodromy data of (2) define a non-generic datum  $\mathcal{M}_s$  and that at least one of the singularities is without roots. Then the reduced datum  $\mathcal{M}$  can be realized by a reducible system of the form (26).

**Proof** Consider the holomorphic trivial bundle  $(F^\Lambda, \nabla^\Lambda)$  constructed from the given system (2). By construction, the degree  $\deg(F^\Lambda)$  of the bundle  $F^\Lambda$  is equal to zero. Moreover, for each subbundle  $F'$  of  $F^\Lambda$  that is stabilized by the connection  $\nabla^\Lambda$ , the inequality  $\deg(F') \leq 0$  holds (see the beginning of Section 3). If the pair  $(F^\Lambda, \nabla^\Lambda)$  is stable, then the result follows from Theorem 3. If the pair is unstable, there exists a proper subbundle  $F_1$  stabilized by the connection  $\nabla$  and such that  $\deg(F_1) = 0$ . In Section 3 we have seen that a holomorphic bundle on  $\mathbb{P}^1(\mathbb{C})$  is trivial if and only if it is a semi-stable bundle of degree zero. From the holomorphic triviality of  $F^\Lambda$  we deduce that  $F^\Lambda$  is semi-stable, which implies that  $F_1$  and  $F^\Lambda/F_1$  are semi-stable too, of degree zero. Therefore,  $F_1$  and  $F^\Lambda/F_1$  are trivial bundles. To construct the reducible system we apply the second part of the proof of Theorem 3 to the bundle  $(F^\Lambda, \nabla^\Lambda)$ . Indeed, both bundles  $F_1$  and  $F^\Lambda/F_1$  have a trivial splitting type,  $c_1 = \dots = c_{p_1} = 0$  and  $c_{p_1+1} = \dots = c_p = 0$ , hence the estimates (28) hold. □

## 5 The generalized Riemann-Hilbert problem in dimension two and three

In this section, we solve the generalized Riemann-Hilbert problem in dimension two and three. We keep the notation from previous sections. As usual, a formal solution (5) is said to be convergent (divergent otherwise) if the asymptotic factor  $\hat{F}$  in (5) is convergent in a neighbourhood of the singularity. We prove the following result.

**Theorem 4** *In dimension two or three, consider a reduced datum  $\mathcal{M}_s$ . The generalized Riemann-Hilbert problem for  $\mathcal{M}$  has a solution if we assume that at least one of the prescribed singularities is without roots, and that the formal fundamental solution of at least one of the local systems (9) is divergent.*

**Remark** If all the local systems (9),  $1 \leq i \leq n$ , have a convergent fundamental solution, then the generalized Riemann-Hilbert problem reduces to the classical Riemann-Hilbert problem. Indeed, a finite number of gauge transformations of the form  $y = e^{q(z-a_i)}u$  (modulo a Möbius transform if  $a_i = \infty$ ) where  $q(t) \in \frac{1}{t}\mathbb{C}[\frac{1}{t}]$ , will reduce the datum  $\mathcal{M}$  to a datum of Fuchsian singularities only. This is due to the fact that there is no Stokes phenomenon at the irregular singularities in this case, hence the exponential part  $Q$  is a scalar matrix at each  $a_i$ . The classical problem always has a solution in dimension two (*cf.* [1]). In dimension three, a complete classification of the counterexamples for the classical problem was given in [1] and [26]. Thus, the GRH-problem always has a solution in dimension two and is completely elucidated in dimension three, if we except the case where all data are those of irregular singularities with roots.

In dimension two the result is a generalization of Theorem 1 of [31]. In dimension three, it is a generalization of the main result of [2].

### 5.1 Proof of Theorem 4 in dimension two

Choose a set  $\Lambda = (\Lambda_1, \dots, \Lambda_n)$  of admissible matrices and consider the corresponding pair  $(F^\Lambda, \nabla^\Lambda)$  constructed in Section 2.2.2.

If the bundle  $F^\Lambda$  has no proper subbundle stabilized by the connection  $\nabla^\Lambda$ , then the pair  $(F^\Lambda, \nabla^\Lambda)$  is stable and by Theorem 2 the GRH-problem for  $\mathcal{M}$  has a solution.

If the bundle  $F^\Lambda$  is the direct sum of two proper subbundles  $F_1$  and  $F_2$  that are stabilized by  $\nabla^\Lambda$ , then, starting with the pairs  $(F_k, \nabla^\Lambda|_{F_k})$ ,  $k = 1, 2$ , one can easily construct two global differential systems

$$dy = \omega_1 y \quad , \quad dy = \omega_2 y$$

on  $\mathbb{P}^1(\mathbb{C})$  with the generalized monodromy data of the bundles  $(F_1, \nabla^\Lambda|_{F_1})$  and  $(F_2, \nabla^\Lambda|_{F_2})$  respectively. It is easy then to see that the differential system

$$dy = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} y$$

has the prescribed generalized monodromy data  $\mathcal{M}$ .

Now assume that the bundle  $F^\Lambda$  has a unique proper subbundle  $F_1$  that is stabilized by  $\nabla^\Lambda$ . As in (20), (21), starting with the local differential system  $dy = \omega^{\Lambda_{i_1}} y$ , we construct a local differential system  $dy = \omega'_{i_1} y$  with a formal fundamental solution of the form

$$\hat{Y}_{i_1}(z) = \begin{pmatrix} \hat{Y}_{i_1}^1 & * \\ 0 & \hat{Y}_{i_1}^2 \end{pmatrix} = \hat{F}_0(z)(z - a_{i_1})^{\Lambda_{i_1}}(z - a_{i_1})^{J'} e^{Q'(z)},$$

where  $\hat{Y}_{i_1}^1$  is a formal fundamental solution of a local system defining the restriction of the connection  $\nabla^\Lambda$  to the subbundle  $F_1$ . Via a basis change  $\hat{Y}_{i_1} S$ ,  $S \in \text{GL}(2, \mathbb{C})$ , and a suitable permutation of the diagonal elements of  $\Lambda_{i_1}$  (preserving admissibility) we may assume that  $J'$  has two specified forms which we detail below.

*Case 1.* The matrix  $J'$  has the form

$$J' = \begin{pmatrix} \rho_{i_1}^1 & 1 \\ 0 & \rho_{i_1}^2 \end{pmatrix}.$$

In this case, the matrix  $Q'(z)$  is a scalar matrix of the form  $Q'(z) = q'(z)I_2$ . From the classical theory (*cf.* [8], p.262), we know that the gauge transformation  $u = \exp(-q'(z))I_2 y$  changes the system  $dy = \omega'_{i_1} y$  into a local system with a regular singularity at  $a_{i_1}$ . This contradicts our assumptions, hence case 1 does not occur.

*Case 2.* The matrix  $J'$  is of the form

$$J' = \begin{pmatrix} \rho_{i_1}^1 & 0 \\ 0 & \rho_{i_1}^2 \end{pmatrix}.$$

In this case, it is possible to construct a stable pair  $(F^{\tilde{\Lambda}}, \nabla^{\tilde{\Lambda}})$  with the prescribed generalized monodromy data. Thus, by Theorem 2, the GRH-problem is solved. Indeed, starting with the set of admissible matrices  $\Lambda$ , we construct a new set of admissible matrices  $\tilde{\Lambda} = (\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_n)$  in the following way. Let  $b$  be a positive integer. Let  $\tilde{\lambda}_{i_1}^1 = \lambda_{i_1}^1 - b$ ,  $\tilde{\lambda}_{i_1}^2 = \lambda_{i_1}^2 + b$ , and  $\tilde{\lambda}_i^j = \lambda_i^j$  for all other  $i, j$ . For sufficiently large  $b$  we get the inequality

$$\sum_{i=1}^n \tilde{\lambda}_i^1 + \rho_i^1 < \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^2 \tilde{\lambda}_i^j + \rho_i^j = \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^2 \lambda_i^j + \rho_i^j.$$

Let  $\tilde{\Lambda}_i = \text{diag}(\tilde{\lambda}_i^1, \tilde{\lambda}_i^2)$ , for  $1 \leq i \leq n$ , and consider the extension  $(F^{\tilde{\Lambda}}, \nabla^{\tilde{\Lambda}})$ . The above inequality implies that the latter pair is stable.

## 5.2 Proof of Theorem 4 in dimension three

The proof in dimension three follows the same lines as in dimension two. We will only give the last part of the proof.

We first assume that the bundle  $F^\Lambda$  has a unique proper subbundle  $F_1$  that is stabilized by  $\nabla^\Lambda$ . As in dimension two, and with the same notation, we shall construct a stable pair  $(F^{\tilde{\Lambda}}, \nabla^{\tilde{\Lambda}})$  with the given data, hence solve the GRH-problem in each case. To construct this pair we consider the following cases.

*Case 1.* The matrix  $J'$  has the form

$$J' = \begin{pmatrix} \rho_{i_1}^1 & 0 & 0 \\ 0 & \rho_{i_1}^2 & 1 \\ 0 & 0 & \rho_{i_1}^3 \end{pmatrix} \quad \text{or} \quad J' = \begin{pmatrix} \rho_{i_1}^1 & 0 & 0 \\ 0 & \rho_{i_1}^2 & 0 \\ 0 & 0 & \rho_{i_1}^3 \end{pmatrix}.$$

Starting with the set of admissible matrices  $\Lambda$  we construct a set of admissible matrices  $\tilde{\Lambda} = (\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_n)$  in the following way. Let  $b$  be a positive integer. Let  $\tilde{\lambda}_{i_1}^1 = \lambda_{i_1}^1 - 2b$ ,  $\tilde{\lambda}_{i_1}^2 = \lambda_{i_1}^2 + b$ ,  $\tilde{\lambda}_{i_1}^3 = \lambda_{i_1}^3 + b$  and  $\tilde{\lambda}_i^j = \lambda_i^j$  for all other  $i, j$ . For sufficiently large  $b$  we get the following inequality

$$\sum_{i=1}^n \sum_{j=1}^2 \tilde{\lambda}_i^j + \rho_i^j < \frac{2}{3} \sum_{i=1}^n \sum_{j=1}^3 \tilde{\lambda}_i^j + \rho_i^j = \frac{2}{3} \sum_{i=1}^n \sum_{j=1}^3 \lambda_i^j + \rho_i^j.$$

Let  $\tilde{\Lambda}_i = \text{diag}(\tilde{\lambda}_i^1, \tilde{\lambda}_i^2, \tilde{\lambda}_i^3)$  for  $1 \leq i \leq n$ , and consider the extension  $(F^{\tilde{\Lambda}}, \nabla^{\tilde{\Lambda}})$ . It follows from the above inequality that this pair is stable, hence, by Theorem 2, the GRH-problem is solved in this case.

*Case 2.* The matrix  $J'$  has the form

$$J' = U^{-1} \begin{pmatrix} \rho_{i_1}^1 & 1 & 0 \\ 0 & \rho_{i_1}^2 & 0 \\ 0 & 0 & \rho_{i_1}^3 \end{pmatrix} U$$

where  $U$  is the matrix described in Section 1.2. From the set of admissible matrices  $\Lambda$  we again construct a set of admissible matrices  $\tilde{\Lambda} = (\tilde{\Lambda}_1, \dots, \tilde{\Lambda}_n)$  in the following way. Let  $b$  be a positive integer, and let  $\tilde{\lambda}_{i_1}^1 = \lambda_{i_1}^1 - b$ ,  $\tilde{\lambda}_{i_1}^2 = \lambda_{i_1}^2 - b$ ,  $\tilde{\lambda}_{i_1}^3 = \lambda_{i_1}^3 + 2b$  and  $\tilde{\lambda}_i^j = \lambda_i^j$  for all other  $i, j$ . For sufficiently large  $b$  we get the inequality

$$\sum_{i=1}^n \sum_{j=1}^2 \tilde{\lambda}_i^j + \rho_i^j < \frac{2}{3} \sum_{i=1}^n \sum_{j=1}^3 \tilde{\lambda}_i^j + \rho_i^j = \frac{2}{3} \sum_{i=1}^n \sum_{j=1}^3 \lambda_i^j + \rho_i^j$$

Consider the extension  $(F^{\tilde{\Lambda}}, \nabla^{\tilde{\Lambda}})$  obtained with  $\tilde{\Lambda}_i = \text{diag}(\tilde{\lambda}_i^1, \tilde{\lambda}_i^2, \tilde{\lambda}_i^3)$  for  $1 \leq i \leq n$ . It is a stable pair in view of the above inequality. Thus, by Theorem 2, the GRH-problem has a solution in this case.

*Case 3.* The matrix  $J'$  is a Jordan block

$$J' = \begin{pmatrix} \rho_{i_1}^1 & 1 & 0 \\ 0 & \rho_{i_1}^2 & 1 \\ 0 & 0 & \rho_{i_1}^3 \end{pmatrix}.$$

In this case, the matrix  $Q'(z)$  is a scalar matrix of the form  $Q'(z) = q'(z)I_3$  and the gauge transformation  $u = \exp(-q'(z))I_3 y$  changes the system  $dy = \omega'_{i_1} y$  into a system with a regular singularity at  $a_{i_1}$ . This contradicts our assumptions, hence this case will not occur.

Now assume that the bundle  $F^\Lambda$  has a stabilized filtration  $F_1 \subset F_2 \subset F^\Lambda$  where  $F_1$  has dimension 1 and  $F_2$  dimension 2. As in (20), (21), starting with the local differential

system  $dy = \omega^{\Lambda_i} y$  we construct a local differential system  $dy = \omega'_i y$  which has a formal fundamental solution of the form

$$\hat{Y}_i(z) = \begin{pmatrix} \hat{Y}_i^1 & * & * \\ 0 & \hat{Y}_i^2 & * \\ 0 & 0 & \hat{Y}_i^3 \end{pmatrix} = \hat{F}_0(z)(z - a_i)^{\Lambda_i}(z - a_i)^{J'_i} e^{Q'(z)}$$

where  $\hat{Y}_i^1$  is a formal fundamental solution of a local system which defines the restriction of the connection  $\nabla^\Lambda$  to the subbundle  $F_1$ , and  $\hat{Y}_i^2$  is a formal solution of a local system which defines the connection constructed from  $\nabla^\Lambda$  on the quotient bundle  $F_2/F_1$ . We notice that the matrices  $J'_i$ ,  $1 \leq i \leq n$  are simultaneously upper-triangular.

The following lemma can be proved in the same way as Proposition 3.1.3 of [64].

**Lemma 6** *There exist invertible upper triangular matrices  $S_i$ ,  $1 \leq i \leq n$ , and integers  $\varphi_i^j$ ,  $1 \leq i \leq n$ ,  $j = 1, 2, 3$ , such that*

- a) *if, for  $l_1 < l_2$ , the entry  $e_{l_1, l_2}$  of  $S_i^{-1} J'_i S_i$  is non-zero, then  $\varphi_i^{l_1} \geq \varphi_i^{l_2}$ ,*  
b) *for  $t = 1, 2, 3$ , one has*

$$\sum_{i=1}^n \varphi_i^t + \rho_i^t = 0.$$

In the following, let  $\tilde{\Lambda}_i = \text{diag}(\varphi_i^1, \varphi_i^2, \varphi_i^3)$  and consider the pair  $(F^{\tilde{\Lambda}}, \nabla^{\tilde{\Lambda}})$ . There is a stabilized filtration  $\tilde{F}_1 \subset \tilde{F}_2 \subset F^{\tilde{\Lambda}}$  such that  $\text{rank}(\tilde{F}_1) = 1$  and  $\text{rank}(\tilde{F}_2) = 2$ . The bundles  $\tilde{F}_1$ ,  $\tilde{F}_2/\tilde{F}_1$ ,  $F^{\tilde{\Lambda}}/\tilde{F}_2$  are equivalent to line bundles,

$$\tilde{F}_1 \cong \mathcal{O}(c_1) \quad , \quad \tilde{F}_2/\tilde{F}_1 \cong \mathcal{O}(c_2) \quad , \quad F^{\tilde{\Lambda}}/\tilde{F}_2 \cong \mathcal{O}(c_3).$$

By the above lemma, these bundles are of degree zero, that is,  $c_1 = c_2 = c_3 = 0$ . To construct a global system on  $\mathbb{P}^1(\mathbb{C})$  with the given data, we use the second part of the proof of Theorem 3 applied to the pair  $(F^{\tilde{\Lambda}}, \nabla^{\tilde{\Lambda}})$ . Indeed, each of the bundles  $\tilde{F}_1$ ,  $\tilde{F}_2/\tilde{F}_1$ ,  $F^{\tilde{\Lambda}}/\tilde{F}_2$  has a trivial splitting type, which implies that all the estimates (28) hold. This ends the proof of Theorem 4 in dimension three.

## 6 The global inverse problem in differential Galois theory

In this section we show how the global inverse problem in differential Galois theory is related to the generalized Riemann-Hilbert problem. We first recall some results of differential Galois theory, and we refer to the book of M. Singer and M. van der Put [52] for an extensive exposition of the theory.

### 6.1 Differential Galois groups

Consider a linear differential system

$$(29) \quad y' = By$$

where  $B$  is a  $p \times p$  matrix with entries in a differential field  $K$ , whose subfield  $C$  of constants is algebraically closed and of characteristic zero. A *Picard-Vessiot extension* of  $K$  with respect to (29) is a differential extension of  $K$  with no new constants, containing the entries of a fundamental solution of (29) and generated by these entries over  $K$ . Such extensions always exist, and they are isomorphic. The *Galois group* of (29) over  $K$  is the group  $G$  of all differential  $K$ -automorphisms of a Picard-Vessiot extension of  $K$  with respect to (29). A representation of this group in  $\mathrm{GL}(p, C)$  is given by any fundamental solution of (29) generating a Picard-Vessiot extension of  $K$  for (29), and  $G$  is then an algebraic subgroup of  $\mathrm{GL}(p, C)$ . Systems which are  $K$ -equivalent have isomorphic Galois groups over  $K$ .

The inverse problem in differential Galois theory is the following: *Given a differential field  $K$  as before, and a linear algebraic group  $G$  defined over the field  $C$  of constants of  $K$ , is it possible to realize  $G$  as a differential Galois group over  $K$ ?*

Over the field  $C(z)$  of rational functions or the field  $\mathbb{C}(\{z\})$  of convergent Laurent series, the problem is completely solved (*cf.* [63], [56], [57], [58], [28], see also [36], [37], [62], [47], [48], [49], [11], [22], and [52] chap. 11). Any group can be realized as a differential Galois group over  $C(z)$ , but not necessarily as a Galois group over  $\mathbb{C}(\{z\})$  as we will see now.

## 6.2 The local inverse problem

In this section, the differential base field is  $\mathbb{C}(\{z\})$ .

### 6.2.1 Ramis's solution

The inverse problem over  $\mathbb{C}(\{z\})$ , also called the *local Galois inverse problem*, was solved by J.-P. Ramis ([55], [56], [57], [58], *cf.* [48], see also [52], chap.11), who has proved that a linear algebraic group  $G$  is a Galois group over  $\mathbb{C}(\{z\})$  if and only if it has a *local Galois structure*, which he defined as follows. It is a triple  $\mathcal{L} = (T, a, \mathcal{N})$ , where

- (a)  $T$  is a torus of  $G$  and  $a \in G$  normalizes  $T$
- (b) the image of  $a$  generates the finite group  $G/G^0$  (with the usual notation  $G^0$  for the identity component of  $G$ )
- (c)  $\mathcal{N}$  is a Lie subalgebra of dimension  $\leq 1$  of the Lie algebra  $\mathcal{G}$  of  $G$ , which commutes with  $a$  and with  $T$
- (d)  $\mathcal{G} = \mathcal{T} + \mathcal{N} + \mathcal{Q}(T)$ , where  $\mathcal{T}$  denotes the Lie algebra of  $T$ , and  $\mathcal{Q}(T)$  the critical subalgebra for  $T$ , defined as the Lie subalgebra of  $\mathcal{G}$  generated by the rootspaces of  $\mathcal{G}$  under the adjoint action of  $T$ .

In [48] it was shown that local Galois groups also are characterized by the condition (i)  $G/G^0$  is cyclic, (ii)  $\dim(R_u/(R_u, G^0)) \leq 0$ , and (iii)  $G/G^0$  acts trivially on  $R_u/(R_u, G^0)$ , where  $R_u$  is the unipotent radical of  $G$ .

A *reduced* local Galois structure on  $G$  is a local Galois structure  $\mathcal{L}' = (T', a', \mathcal{N}')$  as before, satisfying the additional conditions that

1.  $T'$  is a maximal torus
2.  $a'$  is of finite order in  $G$  (and semisimple)

3.  $\mathcal{N}'$  is either (0) or the Lie algebra of a subgroup isomorphic to  $\mathbb{C}$ , and  $\mathcal{N}' \cap (\mathcal{T}' + \mathcal{Q}(T')) = \emptyset$ .

The density theorem of Ramis ([53], [54], *cf.* [39], see also [52] Th. 11.13) states that the differential Galois group  $G$  of a linear differential system (29) over  $\mathbb{C}(\{z\})$  is topologically generated by the formal monodromy, the Stokes matrices (as defined in Section 2) and the *exponential torus*, that is, the torus  $T_e$  of  $K$ -differential automorphisms of the field  $K(e^{q_1}, \dots, e^{q_p})$  where the  $q_i$ 's (see Section 1) are the diagonal entries of the exponential part  $Q$  in (5).

The system (29) *via* its local Stokes data described in Section 2, gives rise to a local Galois structure on  $G$ . In short,  $T$  is the sum of the exponential torus  $T_e$  and the monodromy torus  $T_m$  (generated by the semisimple part of the formal monodromy) whereas the infinitesimal Stokes matrices (inverse images by the exponential map of the Stokes matrices) can be developed in  $\mathcal{G}$  as sums of rootspace elements under the action of the exponential torus to produce generators of the critical subalgebra  $\mathcal{Q}(T_e)$ . The Lie algebra  $\mathcal{N}$  arises from the unipotent part of the formal monodromy and  $a$  from its finite part (*cf.* [55], [52]).

Given any local Galois structure  $\mathcal{L} = (T, a, \mathcal{N})$  on  $G$ , in particular one induced by a system (29), there is a natural way to associate to  $\mathcal{L}$  a reduced local Galois structure  $\mathcal{L}' = (T', a', \mathcal{N}')$ , where  $T'$  contains  $T$ , and  $a'$  equals  $a$  modulo  $G^0$ .

### 6.2.2 The Poincaré rank for a local Galois structure

Our aim here is to realize local Galois data with a minimal Poincaré rank.

**Definition 6** *A local Galois datum is a pair  $(G, \mathcal{L})$  where  $G$  is a linear algebraic group endowed with a reduced local Galois structure  $\mathcal{L}$ . A differential system (29) over  $\mathbb{C}(\{z\})$  is said to realize  $(G, \mathcal{L})$  if  $G$  is the differential Galois group of (29) over  $\mathbb{C}(\{z\})$  and if the system (29) induces the local Galois structure  $\mathcal{L}$  on  $G$ .*

*The Poincaré rank  $r_{\mathcal{L}}$  of a local Galois datum  $(G, \mathcal{L})$  is the smallest possible Poincaré rank of a differential system realizing these data. For a given group  $G$  with local Galois structures, let  $r(G)$  denote the minimal Poincaré rank  $r_{\mathcal{L}}$ , over all possible local data  $(G, \mathcal{L})$ .*

To determine the Poincaré rank  $r_{\mathcal{L}}$  of a given local datum  $(G, \mathcal{L})$  we will use the construction of Ramis in his proof of the local inverse problem ([56] Section 2.1, *cf.* [52] pp. 273-74, 279-82).

We will carry out this construction in such a way that the Katz rank  $\rho_{\mathcal{L}}$  of the system (29) realizing the data, that is, the fractional degree in  $z$  of the exponential part  $Q$  (of a formal fundamental solution of (29) of the form (5), Section 1.2) is minimal. We determine  $\rho_{\mathcal{L}}$ , and hence  $r_{\mathcal{L}}$ , explicitly in terms of  $\mathcal{L}$ .

Let  $\mathcal{L} = (T, a, \mathcal{N})$  be the given reduced local structure on  $G$ , where  $a \in G$  acts by conjugation on the maximal torus  $T$  as an automorphism of order  $\nu \in \mathbb{N}^*$ . Let  $G$  be given with a faithful representation  $G \subset \mathrm{GL}(n, \mathbb{C})$  such that  $T$  is a diagonal subgroup, and let

$\chi_i$ ,  $i = 1, \dots, s$ , denote the corresponding distinct diagonal weights of  $T$ , which generate the (abelian) dual group  $\check{T}$  of  $T$  as a  $\mathbb{Z}$ -module.

Consider the  $\mathbb{Q}$ -vector space  $E = \check{T} \oplus_{\mathbb{Z}} \mathbb{Q}$ , and the  $\mathbb{Q}$ -automorphism  $\delta$  of  $E$  of order  $\nu$  induced by the conjugation by  $a$  on  $T$ . The decomposition  $\delta^\nu - id = \prod_{\nu'|\nu} \Phi_{\nu'}(\delta)$  where  $\Phi_{\nu'}$  denotes the  $\nu'$ -th cyclotomic polynomial, yields a decomposition  $E = \bigoplus_{k=1}^m E_k$  of  $E$  into a direct sum of  $\delta$ -invariant  $\mathbb{Q}$ -subspaces  $E_k$ , each of dimension  $\nu_k$  for some divisor  $\nu_k$  of  $\nu$ , and such that  $\Phi_{\nu_k}$  is the minimal polynomial of  $\delta$  on  $E_k$ .

Let  $\mathcal{F} = \bigoplus_{\lambda \in \mathbb{Q}, \lambda < 0} \mathbb{C}[z^\lambda]$  denote the ring of polynomials in (non-negative) fractional powers of  $1/z$ . For each  $k$  one can realize  $E_k$  as the  $\mathbb{Q}$ -span of an isomorphic image of some lattice  $\bigoplus_{i=1}^{\varphi(\nu_k)} \mathbb{Z}p_i$ ,  $p_i \in \mathcal{F}$ , in the following way. For any given arbitrary integer  $\mu_k \geq 1$  prime to  $\nu_k$  we can choose  $p_1 = z^{-\mu_k/\nu_k}$  and  $p_j = m^{j-1}(p_1)$ ,  $j = 1, \dots, \varphi(\nu_k)$ , where  $m$  denotes the monodromy operator on  $\mathcal{F}$ . This defines an isomorphism which clearly commutes with  $\delta$  and  $m$ . The family  $\underline{p}$  of all such polynomials  $p_i \in \mathcal{F}$  for all  $k$ , is  $m$ -invariant and  $\mathbb{Z}$ -independent, and the above isomorphisms glue together in a global isomorphism  $\psi : \mathcal{P} \rightarrow \mathcal{E}$  from the  $\mathbb{Q}$ -span  $\mathcal{P}$  of  $\underline{p}$  to  $E$ . Let  $c_i \in \mathcal{P}$ , for each  $i = 1, \dots, s$ , denote the inverse image of  $\chi_i$ .

The roots of the adjoint action of  $T$  on  $\mathcal{G}$  are elements of  $\check{T}$  and each non-zero root is actually of the form  $\chi_i - \chi_j$  for some  $i, j \in \{1, \dots, s\}$ ,  $i \neq j$  ([52], p.280, proof of lemma 11.16). For each non-zero root  $\alpha$ , let  $\mathcal{G}_\alpha$  denote the corresponding rootspace.

Note that since  $\mu_k$  and  $\nu_k$  are relatively prime for all  $k$ , the correspondence  $c_i \leftrightarrow \chi_i$  does not *a priori* depend on a precise choice of the  $\mu_k$ . For each  $k$  let now  $\mu_k \geq 1$  be the smallest integer prime to  $\nu_k$ , and such that  $\mu_k \geq \dim(\mathcal{G}_\alpha)$  for all  $\alpha = \chi_i - \chi_j$  such that the corresponding polynomial  $c_i - c_j$  is of degree  $\mu_k/\nu_k$ .

Let  $\rho_{\mathcal{L}}$  denote the largest of the fractional degrees  $\mu_k/\nu_k$ , for all divisors  $\nu_k$  of  $\nu$  occurring in the decomposition of  $E$ .

We have obtained the following result.

**Proposition 2** *Any local datum  $(G, \mathcal{L})$  can be realized by a system (29) whose Katz degree is equal to  $\rho_{\mathcal{L}}$ .*

**Proof** Let  $(q_1, \dots, q_n)$  denote the family of polynomials  $q_i \in \mathcal{F}$  corresponding to the complete family of diagonal weights of  $T$ , taking into account their multiplicity and ordering in the given representation. In the construction of Ramis, this family produces the exponential part  $Q = \text{diag}(q_1, \dots, q_n)$  of the desired system, of degree  $\max_k \{\mu_k/\nu_k\}$ . The above choice of the  $\mu_k$  makes it possible, in this construction, to define sufficiently many Stokes operators to generate the critical algebra  $\mathcal{Q}(\mathcal{T})$ , hence to solve the local inverse problem with a minimal Katz degree. (The degree of any polynomial  $q_i - q_j$  corresponding to a given root  $\alpha$  is large enough to define sufficiently many Stokes rays).  $\square$

Conversely, any system (29) inducing local Stokes data  $(G, \mathcal{L})$  can easily be seen to have a Katz degree greater or equal to  $\rho_{\mathcal{L}}$ . In view of the previous construction, this implies the following result.

**Corollary 3** *With notation as above, we have  $r_{\mathcal{L}} = -[-\rho_{\mathcal{L}}]$ .*



### 6.3 The global inverse problem

In this section we apply our results on the generalized Riemann-Hilbert problem to solve the global differential Galois problem with a better control of the singularities.

Let  $G$  be a given linear algebraic group over  $\mathbb{C}$ . We know that  $G$  is the differential Galois group of some linear differential system over  $\mathbb{C}(z)$ . Moreover, results of J.-P. Ramis also tell us that  $G$  can be realized as the Galois group of systems with a certain type and number of singularities.

Our aim is to prove the existence, under certain conditions, of a system (realizing  $G$  as its Galois group) with a minimal number of singularities, and with the smallest possible Poincaré rank at these.

#### 6.3.1 Number of singularities

We first recall the main results of J.-P. Ramis about the singularities of a system solution of the global inverse problem, for which we refer to ([55], Theorems 7.1.4 and 7.2.11, Proposition 7.1.6) and ([52], Theorem 11.21).

Let  $G$  be a complex algebraic group. Consider the subgroup  $L(G)$  of  $G$  generated by all (maximal) tori of  $G$  and let  $V(G)$  denote the quotient  $G/L(G)$ . Let  $s = s(G)$  (resp.  $\bar{s} = \bar{s}(G)$ ) denote the least positive integer  $\geq 2$  such that  $G$  (resp.  $V(G)$ ) can be topologically generated by  $s - 1$  (resp.  $\bar{s} - 1$ ) elements. By a *family of generators* of  $G$  we mean a family of elements of  $G$  generating  $G$  topologically, and by a *minimal* family of generators, a family of  $s(G) - 1$  such elements. By *generating* we will always, if not specified otherwise, mean “generating topologically”.

Tretkoff’s original result, based on the Riemann-Hilbert correspondence, states that any linear algebraic group  $G$  over  $\mathbb{C}$  can be realized as the Galois group over  $\mathbb{C}(z)$  of a differential system with  $s(G)$  possible singularities, all regular singular, and Fuchsian but possibly one. This was generalized by Ramis who proved that a given linear algebraic group  $G$  topologically generated by closed subgroups  $G_1, \dots, G_{m-1}$ ,  $m \geq 2$ , each endowed with a local Galois structure, is the Galois group over  $\mathbb{C}(z)$  of a system (29) with no more than  $m$  singularities; these belong to a subset  $\{a_1, \dots, a_m\}$  of  $\mathbb{P}^1(\mathbb{C})$  such that the local Galois group of (29) at each  $a_i$  is  $G_i$ ,  $i = 1, \dots, m - 1$ , and  $a_m$ , if singular, is Fuchsian. Moreover there exists such a system for which all but one of its regular singularities are Fuchsian. A more precise result states that any group  $G$  is the Galois group of a differential system with  $\bar{s}(G)$  possible singularities, all Fuchsian but one, possibly irregular. But we know nothing *a priori* about the Poincaré rank at the irregular singularity.

#### 6.3.2 Poincaré ranks

In this section we apply our results on the generalized Riemann-Hilbert problem to refine the results of Ramis, taking into account the Poincaré rank at the singularities.

Throughout this section,  $G$  is given with a faithful representation  $G \subset \mathrm{GL}(n, \mathbb{C})$ . We will use the following notation. For any family  $\underline{M} = (M_1, \dots, M_r)$  of elements of  $G$ , let  $\chi_{\underline{M}}$

denote the representation

$$\chi_{\underline{M}} : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \mathcal{D}, z_0) \longrightarrow \mathrm{GL}(n, \mathbb{C})$$

of the fundamental group of  $\mathbb{P}^1(\mathbb{C})$  punctured at a set  $\mathcal{D} = \{a_1, \dots, a_{r+1}\}$  of arbitrarily chosen points of  $\mathbb{P}^1(\mathbb{C})$  not containing the base-point  $z_0$ , such that  $\chi_{\underline{M}}(\gamma_i) = M_i$ , for all  $i = 1, \dots, r$ , where  $\gamma_i$  denotes the class of an elementary loop around  $a_i$ .

The following result immediately follows from the Bolibruch-Kostov irreducibility condition on the monodromy for the classical Riemann-Hilbert problem.

**Proposition 3** *If the representation  $G \subset \mathrm{GL}(n, \mathbb{C})$  is irreducible, then  $G$  is the Galois group of a differential system with no more than  $s(G)$  singularities, all Fuchsian.*

We will now combine the above mentioned results of differential Galois theory with results of the previous sections to get more precise results on the singularities of a system realizing a given group  $G$  as its Galois group.

**Theorem 5** *Let  $G$  be a linear algebraic group over  $\mathbb{C}$ , topologically generated by closed subgroups  $G_1, \dots, G_{m-1}$ ,  $m \geq 2$ , each endowed with a reduced local Galois structure  $\mathcal{L}_i$ , and let  $\mathcal{D} = \{a_1, \dots, a_m\}$  be an arbitrary set of  $m$  points of  $\mathbb{P}^1(\mathbb{C})$ . Let  $\mathcal{M}_s$  be a reduced datum on  $\mathcal{D}$  realized by  $m$  local systems (9) with Galois group  $G_i$  and Poincaré rank  $r_i$  at  $a_i$ ,  $i = 1, \dots, m-1$ , and Fuchsian at  $a_m$ . If  $\mathcal{M}$  fulfils the conditions of Theorem 2, in particular if  $\mathcal{M}_s$  is generic, then  $G$  is the Galois group of a global system with Poincaré rank  $r_i$  at each  $a_i$ ,  $i = 1, \dots, m-1$ , and which is Fuchsian at  $a_m$ .*

**Proof** The datum  $\mathcal{M}$  is Fuchsian at  $a_m$ , hence without roots. Apply Theorem 2. A system (29) with generalized monodromy data  $\mathcal{M}$  has  $G_i$  as its local differential Galois group at  $a_i$ ,  $i = 1, \dots, m-1$ , and since the  $G_i$  together generate  $G$ , the global Galois group of (29) is  $G$ . If  $\mathcal{M}_s$  is generic, apply Corollary 1.  $\square$

From Theorem 5 and from the monodromy criterion of Corollary 2 we deduce the following results.

**Corollary 4** *Let  $G$  be a linear algebraic group over  $\mathbb{C}$ , topologically generated by closed subgroups  $G_1, \dots, G_{m-1}$ ,  $m \geq 2$ , each endowed with a reduced local Galois structure  $\mathcal{L}_i$ , and let  $\mathcal{D} = \{a_1, \dots, a_m\}$  be an arbitrary set of  $m$  points of  $\mathbb{P}^1(\mathbb{C})$ . If there exists a family  $\underline{M} = (M_1, \dots, M_{m-1})$  of elements of  $G$  such that*

- (i)  $M_i$ ,  $i = 1, \dots, m-1$ , is the monodromy matrix of a local system (9) at  $a_i$  realizing the local Galois data  $(G_i, \mathcal{L}_i)$  with true Poincaré rank  $r_i$ ,*
- (ii) the representation  $\chi_{\underline{M}}$  is irreducible,*

*then  $G$  is the Galois group over  $\mathbb{C}(z)$  of a system (29) with singularities all in  $\mathcal{D}$ , whose Poincaré rank at each  $a_i$  is  $r_i$ ,  $i = 1, \dots, m-1$ , and which is Fuchsian at  $a_m$ .*

If a group with a local Galois structure is *connected*, then it has local structures of the form  $\mathcal{L} = (\mathrm{id}, T, \mathcal{N})$ . It is in then possible (cf. [56], 4.2.2) to realize  $\mathcal{L}$  with a system for which the formal and the topological monodromy matrices coincide and are equal to  $\exp(u)$ , for any generator  $u$  of the Lie algebra  $\mathcal{N}$ . We obtain the following result in this case.

**Corollary 5** *Let  $G$  be a linear algebraic group over  $\mathbb{C}$ . Assume that  $G$  is topologically generated by closed connected subgroups  $G_1, \dots, G_{m-1}$ ,  $m \geq 2$ , each endowed with a reduced local Galois structure  $\mathcal{L}_i = (\text{id}, T_i, \mathcal{N}_i)$  and let  $\mathcal{D} = \{a_1, \dots, a_m\}$  be an arbitrary set of  $m$  points of  $\mathbb{P}^1(\mathbb{C})$ . Let  $u_i$ , for  $i = 1, \dots, m-1$ , be a generator of the Lie algebra  $\mathcal{N}_i$ , and  $r_i$  the true Poincaré rank of a local system (9) realizing  $(G_i, \mathcal{L}_i)$  with the monodromy matrix  $M_i = \exp(u_i)$ . If the representation  $\chi_{\underline{M}}$ , where  $\underline{M} = (M_1, \dots, M_{m-1})$ , is irreducible, then  $G$  is the Galois group over  $\mathbb{C}(z)$  of a system (29) with singularities all in  $\mathcal{D}$ , whose Poincaré rank at each  $a_i$  is  $r_i$ ,  $i = 1, \dots, m-1$ , and which is Fuchsian at  $a_m$ .*

**Remark :** Corollaries 4 and 5 in particular hold if, for some or all  $i$ , the local systems in the statements realize the *minimal* Poincaré rank  $r_{\mathcal{L}_i}$  at  $a_i$ .

We now wish to realize a given group globally with  $\bar{s}(G)$  singularities, all Fuchsian but possibly one, with a minimal Poincaré rank at the irregular singularity.

We will use yet another characterization of a local Galois group  $\Gamma$  (cf. [52], Th. 11.13), namely that  $V(\Gamma) = \Gamma/L(\Gamma)$  be topologically generated by one element, where  $L(G)$  as before denotes the subgroup generated by the tori of  $G$ .

It follows from this criterion that for any  $\alpha \in V(G)$  the inverse image  $G_\alpha = pr^{-1}(\langle \alpha \rangle)$ , by the projection  $pr : G \rightarrow V(G)$ , of the closed subgroup topologically generated by  $\alpha$ , has a local Galois structure.

**Notation** *Let  $\mathcal{A}$  denote the set of all elements  $\alpha \in V(G)$  which belong to a minimal family of generators of  $V(G)$ , and let  $r(G) = \min_{\alpha \in \mathcal{A}}(r(G_\alpha))$  denote the minimal possible Poincaré rank of a system realizing the local Galois data  $(G_\alpha, \mathcal{L}_\alpha)$ ,  $\alpha \in \mathcal{A}$ .*

*Let us write  $\bar{s}$  for  $\bar{s}(G)$ , and  $\bar{X}$  for the class of an element  $X$  of  $G$  in the quotient  $V(G)$ .*

With this notation, we obtain the following result.

**Theorem 6** *Let  $\mathcal{D} = \{a_1, \dots, a_{\bar{s}}\}$  be an arbitrary set of  $\bar{s}$  points of  $\mathbb{P}^1(\mathbb{C})$ . Let  $M_1$  denote the monodromy matrix of a system of Poincaré rank  $r(G)$  at  $a_1$  and realizing  $G_\alpha$ , for some  $\alpha \in \mathcal{A}$ , as its local Galois group. Consider a subset  $\{M_2, \dots, M_{\bar{s}-1}\} \subset G$  such that  $(\alpha, \overline{M_2}, \dots, \overline{M_{\bar{s}-1}})$  is a minimal family of generators of  $V(G)$ . Let  $\mathcal{M}$  be a reduced datum on  $\mathcal{D}$  which includes the representation  $\chi_{\underline{M}}$  for  $\underline{M} = (M_1, M_2, \dots, M_{\bar{s}-1})$ , Fuchsian data at  $a_2, \dots, a_{\bar{s}}$ , and the Poincaré rank  $r(G)$  at  $a_1$ . If  $\mathcal{M}$  fulfils the conditions of Theorem 2, then  $G$  is the Galois group over  $\mathbb{C}(z)$  of a system with Fuchsian singularities at  $a_2, \dots, a_{\bar{s}}$  and Poincaré rank  $r(G)$  at  $a_1$ .*

**Proof** The group  $G$  is generated by the closed subgroups  $G_\alpha$  and  $G_2, \dots, G_{\bar{s}-1}$ , where  $G_i$ ,  $i = 2, \dots, \bar{s}-1$ , is generated by  $M_i$ . It is then possible to define generalized monodromy data which include  $\chi_{\underline{M}}$  and a true Poincaré rank equal to  $r(G)$  at  $a_1$ . Apply Theorem 5 to conclude.  $\square$

The irreducibility condition on the monodromy in particular implies the following result.

**Corollary 6** *If for some  $\alpha \in \mathcal{A}$  such that  $r(G_\alpha) = r(G)$  and for a family  $\underline{M} = (M_2, \dots, M_{\bar{s}-1})$  of elements of  $G$  such that  $(\alpha, \overline{M_2}, \dots, \overline{M_{\bar{s}-1}})$  is a minimal family of generators of  $V(G)$  the representation  $\chi_{\underline{M}}$  is irreducible, then  $G$  is the Galois group over  $\mathbb{C}(z)$  of a system with no more than  $\bar{s}(G)$  singularities, all Fuchsian but one, possibly irregular, at which the system has Poincaré rank  $r(G)$ .*

**Proof** The group  $G$  is generated by the closed subgroup  $G_\alpha$  and  $G_2, \dots, G_{\bar{s}-1}$ , where  $G_i$ ,  $i = 1, \dots, \bar{s} - 1$ , is generated by  $M_i$ . Consider a subset  $\mathcal{D} = \{a_1, \dots, a_{\bar{s}}\}$  of  $\mathbb{P}^1(\mathbb{C})$  and let  $M_1$  denote the monodromy matrix of a system of minimal Poincaré rank  $r(G)$  at  $a_1$  realizing  $G_\alpha$  as its local Galois group. The representation  $\chi_{\underline{M}'}$ , where  $\underline{M}' = (M_1, \dots, M_{\bar{s}-1})$ , is irreducible, and we can apply Corollary 4 to conclude.  $\square$

We can restate this result with a weaker condition on the monodromy representation, but with less control on the Poincaré rank.

**Corollary 7** *Consider a subset  $\mathcal{D} = \{a_1, \dots, a_{\bar{s}}\}$  of  $\mathbb{P}^1(\mathbb{C})$ . Assume that for some  $\alpha \in \mathcal{A}$  and a family  $\underline{M} = (M_1, \dots, M_{\bar{s}-1})$  of elements of  $G$  one has*

- (i)  $\overline{M_1} = \alpha$ ,
- (ii)  $(\overline{M_1}, \dots, \overline{M_{\bar{s}-1}})$  is a minimal family of generators of  $V(G)$ ,
- (iii) the representation  $\chi_{\underline{M}}$  is irreducible.

*Let  $r_1$  be the true Poincaré rank at  $a_1$  of a system realizing  $G_\alpha$  as its local Galois group at  $a_1$  and with monodromy matrix  $M_1$ . Then  $G$  is the Galois group over  $\mathbb{C}(z)$  of a system with no more than  $\bar{s}(G)$  singularities, all Fuchsian but one, possibly irregular, at which the system has Poincaré rank  $r_1$ .*

**Proof** The existence of a system (9) with the monodromy matrix  $M_1 \in G_\alpha$  at  $a_1$  follows from Ramis's construction for the solution of the local inverse problem (*cf.* [52], Section 11). We may assume that the Poincaré rank of (9) at  $a_1$  is minimal (equal to the true Poincaré rank). This local system at  $a_1$ , together with the representation  $\chi_{\underline{M}}$  and Fuchsian data at  $a_2, \dots, a_{\bar{s}}$ , defines a reduced datum  $\mathcal{M}$  which by Corollary 2 can be realized with a global system (29) since  $\chi_{\underline{M}}$  is irreducible and one point at least,  $a_{\bar{s}}$ , is without roots. The Galois group of (29) is clearly  $G$ , and its Poincaré rank at  $a_1$  is  $r_1$  by the generalized Riemann-Hilbert problem.  $\square$

In dimension two and three we can say more. To apply the results for the GRH-problem to this case, we need to assume that the generalized monodromy data are, at the irregular singularity, those of a local system with a divergent fundamental solution.

We first recall a characterization of Galois groups over the differential field  $\mathbb{C}((z))$  of formal Laurent series, or *formal Galois groups*. Ramis has characterized such groups by a *formal local Galois structure*  $(T, a, \mathcal{N})$  which only differs from the above mentioned local Galois structure by the condition (d), which in the formal case is replaced by

$$(d') \quad \mathcal{G} = \mathcal{N} + \mathcal{T}$$

where  $\mathcal{T}$  denotes the Lie algebra of the torus  $T$ . In ([52], Th; 11.2) formal local Galois groups  $\Gamma$  are characterized by the simpler, equivalent condition that  $\Gamma$  contain a normal subgroup  $T$  such that  $T$  is a torus and  $\Gamma/T$  be topologically generated by one element.

If we apply Theorem 4 of Section 5 in dimension two and three we obtain the following result.

**Theorem 7** *Let  $G$  be a linear algebraic subgroup of  $\mathrm{GL}(p, \mathbb{C})$ ,  $p = 2, 3$ , and assume that*

- (i)  $p=2$ , or*
- (ii)  $p=3$  and for some  $\alpha \in \mathcal{A}$  such that  $r(G_\alpha) = r(G)$ , the quotient  $G_\alpha/T$  of  $G_\alpha$  by any torus  $T$ , normal in  $G_\alpha$ , is not topologically generated by one element.*

*Then  $G$  is the Galois group over  $\mathbb{C}(z)$  of a linear differential system with no more than  $\bar{s}(G)$  singularities, all Fuchsian but one, irregular of Poincaré rank  $r(G)$ .*

**Proof** The assumption for the subgroup  $G_\alpha$  not to be a formal Galois group means that any of its local Galois structures will realize it as the Galois group over  $\mathbb{C}(\{z\})$  (the field of convergent Laurent series) of a local system with a divergent fundamental solution. Otherwise, the Stokes matrices would be trivial, hence the formal solutions would be convergent. Moreover, since  $\bar{s}(G) \geq 2$  we can assume, for  $p = 2$  and  $3$ , that the data at the other singularities are all Fuchsian, and apply the results of Corollaries 4 and 5 to conclude.  $\square$

**Note** Andrey Andreevich Bolibruch<sup>1</sup> passed away on November 11, 2003, during the completion of this paper. We dedicate it to the memory of our late coauthor, colleague, and wonderful friend.

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<sup>1</sup>Andrey A. Bolibruch was born on January 30, 1950, in Moscow. In 1989 he exhibited a famous counterexample to the Riemann-Hilbert problem, which for 70 years had been considered as positively solved by Plemelj in 1908. Bolibruch devoted his career not only to outstanding research activity, but with equal energy to many demanding tasks in favour of the mathematical community. One should in particular mention his invaluable efforts to promote scientific cooperation between Russia and France, a country with which he had special mathematical and cultural affinities. Those who met Andrey Andreevich Bolibruch will forever remember his bright and radiant personality.

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