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STABILIZATION OF THE WAVE EQUATIONS WITH POTENTIAL AND INDEFINITE DAMPING

Kangsheng Liu*, Bopeng Rao[†] and Xu Zhang[‡]

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Abstract

By means of global Carleman-type estimate, we study the stabilization problem of the wave equations with potential and indefinite damping. The energy decay rate of the system is given explicitly. Also, we obtain an upper bound estimate on the negative damping to guarantee the energy of the system decays exponentially.

2000 Mathematics Subject Classification: Primary 93D15; Secondary 35B35, 93B07.

Key Words: Stabilization, energy decay rate, wave equation, indefinite damping, observability inequality, Carleman estimate.

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1 Introduction

Let us consider the following wave equation

$$\begin{cases} y_{tt} - \Delta y + [a(x) + b(x)]y_t + c(x)y = 0 & \text{in } (0, \infty) \times \Omega, \\ y = 0 & \text{on } (0, \infty) \times \Gamma, \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } \Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^2 boundary Γ , $a(\cdot) \geq 0$, $b(x) \leq 0$ and $c(x) \geq 0$ are given functions. We denote by

$$\mathcal{E}(t) = \frac{1}{2} \int_{\Omega} [|\nabla y(t, x)|^2 + y_t^2(t, x) + c(x)y^2(t, x)] dx \quad (1.2)$$

the energy of (1.1). It is easy to see that

$$\mathcal{E}(T) - \mathcal{E}(S) + \int_S^T \int_{\Omega} [a(x) + b(x)]y_t^2(t, x) dx dt = 0, \quad \forall T \geq S \geq 0. \quad (1.3)$$

The main purpose of this paper is to study the exponential decay rate for the energy of (1.1), i.e., we hope to find $\mu > 0$ and $M > 0$ such that

$$\mathcal{E}(t) \leq M\mathcal{E}(0)e^{-\mu t}, \quad \forall t \geq 0 \text{ and } \forall (y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega). \quad (1.4)$$

By (1.3), it is easy to see that if $a(\cdot) + b(\cdot) \leq 0$ a.e. in Ω , then (1.4) is impossible. Thus it is reasonable to assume that $a(\cdot) + b(\cdot) > 0$ in a subdomain G of Ω . Further, by [1], we know that in order to obtain the exponential decay for the energy of (1.1) we should assume G has a ‘‘good’’ position in Ω even if $b(\cdot) \equiv 0$ and $c(\cdot) \equiv 0$. Therefore, as usual, we assume that G is a neighborhood of the ‘‘star-shaped’’ part of the boundary Γ (see (H1) in Section 2).

In this work, we will allow $a(x) + b(x)$ to change its sign in Ω . This means that the damping in (1.1) is indefinite. As far as we know, this sort of stabilization problem was first posed in [3] for the one dimension wave equation

$$\begin{cases} y_{tt} - y_{xx} + d(x)y_t = 0 & \text{in } (0, \infty) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, \infty), \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } (0, 1). \end{cases} \quad (1.5)$$

In [3] it was conjectured that the energy of (1.5) decays exponentially if

$$\int_0^1 d(x) \sin^2(n\pi x) dx \geq \gamma > 0, \quad n = 1, 2, \dots \quad (1.6)$$

Later, it was found that ([4]) (1.6) is not enough to ensure the exponential stability of (1.5) when $|d(\cdot)|_{L^\infty(0,1)}$ is large. Thus, [5] studied the following equation with a small parameter ε :

$$\begin{cases} y_{tt} - y_{xx} + \varepsilon d(x)y_t = 0 & \text{in } (0, \infty) \times (0, 1), \\ y(t, 0) = y(t, 1) = 0, & t \in (0, \infty), \\ y(0) = y_0, \quad y_t(0) = y_1 & \text{in } (0, 1). \end{cases} \quad (1.7)$$

It was proved in [5] that when $d \in \text{BV}(0, 1)$ and (1.6) holds, there exists an $\varepsilon_0 = \varepsilon_0(d)$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the energy of (1.7) decays exponentially. This result was extended in [2] to the case that the first equation in (1.7) is replaced by

$$y_{tt} - y_{xx} + \varepsilon d(x)y_t + c(x)y = 0.$$

More recently, [8] gave an abstract treatment of these results under the condition that the norm of the (indefinite) damping operator is small enough and the condition that the damping operator is uniformly effective on some set of eigenvectors. We remark that the approaches developed in these works apply primarily to the one dimensional problems.

In this paper, we will work in any space dimension. Also, we do not need $|a(\cdot) + b(\cdot)|_{L^\infty(\Omega)}$ to be small. Instead, we will assume that $|b(\cdot)|_{L^\infty(\Omega)}$ is small compared to $|a(\cdot)|_{L^\infty(\Omega)}$. Basically, we will view (1.1) as a perturbation of the following equation

$$\begin{cases} u_{tt} - \Delta u + a(x)u_t + c(x)u = 0 & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } (0, \infty) \times \Gamma, \\ u(0) = u_0, \quad u_t(0) = u_1 & \text{in } \Omega. \end{cases} \quad (1.8)$$

Denote by

$$e(t) = \frac{1}{2} \int_{\Omega} [|\nabla u(t, x)|^2 + u_t^2(t, x) + c(x)u^2(t, x)] dx \quad (1.9)$$

the energy of (1.8). First, we shall find the decay rate of $e(t)$ under suitable conditions. Then, by the classical perturbation result ([9]), we can find an upper bound estimate on $|b(\cdot)|_{L^\infty(\Omega)}$ such that the energy of (1.1) decay exponentially. It is easy to see that the key of our method is to give an explicit decay rate of $e(t)$. We remark that, as far as we know, this explicit decay rate is not available in the literature although it is not hard to show that the energy of (1.8) does decay exponentially (under some conditions).

In order to estimate the decay rate of the energy of (1.8), we need an explicit observability inequality (*see* Theorem 2.1 in Section 2) for the following wave equation:

$$\begin{cases} w_{tt} - \Delta w + c(x)w = 0 & \text{in } Q, \\ w = 0 & \text{on } \Sigma, \\ w(0) = w_0, \quad w_t(0) = w_1 & \text{in } \Omega, \end{cases} \quad (1.10)$$

where $Q \triangleq (0, T) \times \Omega$ and $\Sigma \triangleq (0, T) \times \Gamma$, T is a fixed time duration. We will derive the desired observability inequality by means of the Carleman-type estimate developed in [7], [6] and [10].

The rest of this paper is organized as follows. In Section 2, we will state our main results. In Section 3, we show some preliminary results. Sections 4–5 are devoted to prove the main results.

2 Statement of the main results

To begin with, let us introduce some notations. For any $M \subset \mathbb{R}^n$ and $\varepsilon > 0$, we put $\mathcal{O}_\varepsilon(M) = \{y \in \mathbb{R}^n \mid |y - x| < \varepsilon, x \in M\}$. Fix a $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$, we put $\Gamma_0 \triangleq \{x \in \Gamma \mid (x - x_0) \cdot \nu(x) > 0\}$, where $\nu(x)$ denotes the outward unit normal to Ω at $x \in \Gamma$ and \cdot denotes the scalar product in \mathbb{R}^n .

Throughout this paper, we assume that

- (H1) $G = \Omega \cap \mathcal{O}_{\delta_0}(\Gamma_0)$ for some fixed $\delta_0 > 0$;
- (H2) $a(\cdot) \in L^\infty(\Omega; \mathbb{R}^+)$ and $a^0 \triangleq \inf_{x \in G} a(x) > 0$; and
- (H3) $T > 2 \max_{x \in \Omega} |x - x_0|$ and $c(\cdot) \in L^\infty(\Omega; \mathbb{R}^+)$.

Let us fix

$$\delta_1 \in (0, \delta_0) \text{ and } \tilde{\delta} \in (\delta_1, \delta_0) \quad (2.1)$$

and denote

$$R_0 \triangleq \min_{x \in \Omega \setminus \mathcal{O}_{\delta_1}(\Gamma_0)} |x - x_0|, \quad R_1 \triangleq \max_{x \in \Omega \setminus \mathcal{O}_{\delta_1}(\Gamma_0)} |x - x_0|. \quad (2.2)$$

It is obvious that

$$0 < R_0 < R_1 < T/2. \quad (2.3)$$

On the other hand, it is well-known that one can find a function $\xi \in C^2(\overline{\Omega}; [0, 1])$ such that

$$\begin{cases} \xi \equiv 1 & \text{on } \Omega \setminus \mathcal{O}_{\tilde{\delta}}(\Gamma_0); \\ \xi \equiv 0 & \text{on } \Omega \cap \mathcal{O}_{\delta_1}(\Gamma_0) \end{cases} \quad (2.4)$$

and a function $\zeta \in C^1(\overline{\Omega}; [0, 1])$ such that

$$\begin{cases} \zeta \equiv 1 & \text{on } \Omega \cap \mathcal{O}_{\tilde{\delta}}(\Gamma_0); \\ \zeta \equiv 0 & \text{on } \Omega \setminus G. \end{cases} \quad (2.5)$$

We denote

$$K_0 \triangleq \max \left(\sup_{x \in \Omega} |\nabla \xi|^2, \sup_{x \in \Omega} |\Delta \xi|^2, \sup_{x \in \Omega} |\nabla \zeta|^2 \right). \quad (2.6)$$

We fix a

$$S \in (2R_1, T) \quad (2.7)$$

and denote

$$\begin{cases} a_0 \triangleq \frac{R_0}{2}, \quad \alpha \triangleq 1 - \frac{R_0^2}{4R_1^2}, \\ \varepsilon_0 \triangleq \frac{R_0}{4T\sqrt{\alpha}}, \quad \varepsilon_1 \triangleq \frac{R_0}{2T\sqrt{\alpha}}, \quad \varepsilon_2 \triangleq \frac{S}{2T}, \\ T_i \triangleq T/2 - \varepsilon_i T, \quad T'_i \triangleq T/2 + \varepsilon_i T, \quad (i = 0, 1, 2). \end{cases} \quad (2.8)$$

Further, we need the following notations.

$$\left\{ \begin{array}{l} C_0 \triangleq \sup\{|f|_{L^2(\Omega)}^2 \mid f \in H_0^1(\Omega) \text{ with } |\nabla f|_{L^2(\Omega)} = 1\} (\leq (\text{diam } \Omega)^2), \\ C_1 \triangleq 2(\varepsilon_0^2 - \varepsilon_1^2)^{-2} T^{-4} \max_{t \in [T_1, T_1']} \{(t - T_1)^2 (t - T_1')^2 + 2C_0(2t - T_1 - T_1')^2\}, \\ C_{21} \triangleq 2[5(n - 1 + \alpha)S + 2\alpha S^2 + 4R_1^2/\alpha], \\ C_{22} \triangleq 10S(\alpha^2 S^2 + 2R_1^2) + C_0 S^2 [8\alpha^2 + 4n + (n + \alpha - 1)^2], \\ C_{23} \triangleq 2S^2 [2(4\alpha + C_0 + C_0\alpha)R_1^2 + C_0(1 + \alpha)\alpha^2 S^2], \\ C_3 \triangleq (1 - \alpha)(T_1' - T_1)(T_0' - T_0)/C_1. \end{array} \right. \quad (2.9)$$

Finally, let us denote

$$\left\{ \begin{array}{l} \lambda_1 \triangleq \max \left(\frac{8\alpha^2 + 4n + (n + \alpha - 1)^2}{4(1 + \alpha)a_0^2}, \frac{|c(\cdot)|_{L^\infty(\Omega)}}{a_0 \sqrt{2(1 + \alpha)}} \right), \\ \lambda_{20} \triangleq \frac{1 + 2C_0 S^2 |c(\cdot)|_{L^\infty(\Omega)}^2}{C_3}, \quad \lambda_{21} \triangleq \frac{C_{21}}{C_3 a_0^2}, \quad \lambda_{22} \triangleq \frac{2C_{22}}{C_3 a_0^4}, \quad \lambda_{23} \triangleq \frac{6C_{23}}{C_3 a_0^6}. \end{array} \right. \quad (2.10)$$

The energy of (1.10) is defined by

$$E(t) \triangleq \frac{1}{2} \int_{\Omega} [|\nabla w(t, x)|^2 + w_t^2(t, x) + c(x)w^2(t, x)] dx. \quad (2.11)$$

A key estimate in this paper is the following.

Theorem 2.1 *Let (H1) and (H3) hold. Then there exists a constant $K > 0$ such that the weak solution $w(\cdot) \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ of (1.10) satisfies*

$$E(0) \leq K \int_0^T \int_G w_t^2 dx dt, \quad \forall (w_0, w_1) \in H_0^1(\Omega) \times L^2(\Omega). \quad (2.12)$$

Furthermore, the constant K is given by

$$K \triangleq K_0 e^{(R_1^2 - a_0^2)\lambda} \left[\frac{3T^5}{2T_2^2(T - T_2)^2} + \left(1 + \frac{8 + T^4 K_0}{T_2^2(T - T_2)^2}\right) T^3 \right] + (1 - \alpha)\lambda e^{a_0^2\lambda} (T_1' - T_1), \quad (2.13)$$

where $\lambda \triangleq \max(\lambda_{20}, \lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_1 + 1)$.

Remark 2.1 *We remark that the observability inequality like (2.12) is already known. In fact, in [10], the author actually obtained such an inequality even for any $c(\cdot) \in L^\infty(\Omega)$ (In this general case, of course the energy is defined by $E(t) = \frac{1}{2} \int_{\Omega} [|\nabla w(t, x)|^2 + w_t^2(t, x)] dx$). The novelty in Theorem 2.1 is the explicit expression (2.13) on the observability constant K in (2.12). We note that all the constants in K are computable.*

A proof of Theorem 2.1 will be given in Section 4. By Theorem 2.1, we can find the decay rate of the energy of (1.8) as follows.

Theorem 2.2 *Let (H1), (H2) and (H3) hold. Then the energy $e(t)$ of (1.8) defined by (1.9) satisfies*

$$e(t) \leq M_0 e(0) e^{-\mu_0 t}, \quad \forall t \geq 0 \text{ and } \forall (u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega), \quad (2.14)$$

where (recall (H3) for T)

$$M_0 = \frac{K_1}{K_1 - 1}, \quad \mu_0 = \frac{1}{T} \ln \frac{K_1}{K_1 - 1} \quad (2.15)$$

and (recall (H2) and (2.13) for a^0 and K respectively)

$$K_1 = 2K \left[\frac{1}{a^0} + |a(\cdot)|_{L^\infty(\Omega)} (e^T - 1) \right]. \quad (2.16)$$

A proof of Theorem 2.2 will be given in Section 5. Once Theorem 2.2 is known, then by Theorem 1.1 in Chapter 3 of [9] (pp. 76), it is easy to estimate the energy decay rate of (1.1) and give an explicit upper bound estimate on the negative damping $b(\cdot)$ to guarantee solutions of (1.1) decay exponentially. Our result reads

Theorem 2.3 *Let (H1), (H2) and (H3) hold. Assume that $b(\cdot) \in L^\infty(\Omega)$ satisfies $b(\cdot) \leq 0$ a.e. in Ω and (recall (2.15) for M_0 and μ_0)*

$$|b(\cdot)|_{L^\infty(\Omega)} < \frac{\mu_0}{M_0}. \quad (2.17)$$

Then the energy $\mathcal{E}(t)$ of (1.1) defined by (1.2) satisfies

$$\mathcal{E}(t) \leq M_0 \mathcal{E}(0) e^{-\mu t}, \quad \forall t \geq 0 \text{ and } \forall (y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega), \quad (2.18)$$

where

$$\mu = \mu_0 - M_0 |b(\cdot)|_{L^\infty(\Omega)}. \quad (2.19)$$

3 Preliminaries

In order to prove Theorem 2.1, we need the some preliminaries.

In what follows, we use the notations

$$f_i = f_i(x) \triangleq \frac{\partial f(x)}{\partial x_i}, \quad i = 1, 2, \dots, n; \quad \sum_i \triangleq \sum_{i=1}^n; \quad \sum_{i,j} \triangleq \sum_{i,j=1}^n.$$

The following lemma is a special case of LEMMA 1 at pp. 124 in [7].

Lemma 3.1 Let $\lambda > 0$, $\beta \in (0, 1)$, and

$$\begin{cases} \varphi = \varphi(t, s, x) = |x - x_0|^2 - \beta(t - T/2)^2 - \beta(s - T/2)^2, \\ \eta = \frac{\lambda}{2}\varphi, \theta = e^n, \Psi = (n - 1 + \beta)\lambda. \end{cases} \quad (3.1)$$

Let $v = v(t, s, x) \in C^2(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^n)$. Then

$$\begin{aligned} & \theta^2(v_{tt} + v_{ss} - \Delta v)^2 \\ & \geq 2(1 - \beta)\lambda\theta^2(v_t^2 + v_s^2 + \sum_i v_i^2) + \theta^2 B v^2 \\ & \quad + \left\{ 2\theta^2 \left[\eta_t(v_t^2 + v_s^2 - \sum_j v_j^2) - 2v_t(\eta_t v_t + \eta_s v_s - \sum_j \eta_j v_j) \right. \right. \\ & \quad \left. \left. + (\Psi - 2A)v_t v + (A_t - 2\eta_t A)v^2 \right] \right\}_t \\ & \quad + \left\{ 2\theta^2 \left[\eta_s(v_t^2 + v_s^2 - \sum_j v_j^2) - 2v_s(\eta_t v_t + \eta_s v_s - \sum_j \eta_j v_j) \right. \right. \\ & \quad \left. \left. + (\Psi - 2A)v_s v + (A_s - 2\eta_s A)v^2 \right] \right\}_s \\ & \quad - \sum_i \left\{ 2\theta^2 \left[\eta_i(v_t^2 + v_s^2 - \sum_j v_j^2) - 2v_i(\eta_t v_t + \eta_s v_s - \sum_j \eta_j v_j) \right. \right. \\ & \quad \left. \left. + (\Psi - 2A)v_i v + (A_i - 2\eta_i A)v^2 \right] \right\}_i, \end{aligned} \quad (3.2)$$

where

$$A = \lambda^2 [\beta^2(t - T/2)^2 + \beta^2(s - T/2)^2 - |x - x_0|^2] \quad (3.3)$$

and

$$\begin{aligned} B &= 4(1 + \beta)\lambda^3 [|x - x_0|^2 - \beta^2(t - T/2)^2 - \beta^2(s - T/2)^2] \\ & \quad - [8\beta^2 + 4n + (n - 1 + \beta)^2]\lambda^2. \end{aligned} \quad (3.4)$$

Next, we need the following result.

Lemma 3.2 Let T_0, T'_0, T_1 and T'_1 be given in (2.8), and $c(\cdot) \in L^\infty(\Omega; \mathbb{R}^+)$. Then for any weak solution $w(\cdot)$ of (1.10), it holds

$$\int_{T_0}^{T'_0} \int_{\Omega} [|\nabla w|^2 + c(x)w^2] dx dt \leq C_1 \int_{T_1}^{T'_1} \int_{\Omega} w_t^2 dx dt, \quad (3.5)$$

where C_1 is the constant in (2.9).

Proof: Denote $\phi(t) = (t - T_1)(T'_1 - t)$. Multiplying both sides of the first equation in (1.10) by $\phi^2 w$, integrating it in $(T_1, T'_1) \times \Omega$, using integration by parts, we get

$$\int_{T_1}^{T'_1} \int_{\Omega} \phi^2 [|\nabla w|^2 + c(x)w^2] dx dt = \int_{T_1}^{T'_1} \int_{\Omega} [\phi^2 w_t^2 + 2(\phi w)(\phi_t w_t)] dx dt. \quad (3.6)$$

However (recall (2.9) for C_0)

$$\begin{aligned} & \int_{T_1}^{T'_1} \int_{\Omega} [\phi^2 w_t^2 + 2(\phi w)(\phi_t w_t)] dx dt \\ & \leq \frac{1}{2C_0} \int_{T_1}^{T'_1} \int_{\Omega} \phi^2 w^2 dx dt + \int_{T_1}^{T'_1} \int_{\Omega} (\phi^2 + 2C_0 \phi_t^2) w_t^2 dx dt \\ & \leq \frac{1}{2} \int_{T_1}^{T'_1} \int_{\Omega} \phi^2 |\nabla w|^2 dx dt + \max_{t \in [T_1, T'_1]} (\phi^2 + 2C_0 \phi_t^2) \int_{T_1}^{T'_1} \int_{\Omega} w_t^2 dx dt. \end{aligned} \quad (3.7)$$

Thus, combining (3.6) and (3.7), we get

$$\int_{T_1}^{T'_1} \int_{\Omega} \phi^2 [|\nabla w|^2 + c(x)w^2] dx dt \leq 2 \max_{t \in [T_1, T'_1]} (\phi^2 + 2C_0\phi_t^2) \int_{T_1}^{T'_1} \int_{\Omega} w_t^2 dx dt. \quad (3.8)$$

On the other hand,

$$\begin{aligned} \int_{T_1}^{T'_1} \int_{\Omega} \phi^2 [|\nabla w|^2 + c(x)w^2] dx dt &\geq \int_{T_0}^{T'_0} \int_{\Omega} \phi^2 [|\nabla w|^2 + c(x)w^2] dx dt \\ &\geq (\varepsilon_0^2 - \varepsilon_1^2) T^4 \int_{T_0}^{T'_0} \int_{\Omega} [|\nabla w|^2 + cw^2] dx dt. \end{aligned} \quad (3.9)$$

Thus, combining (3.8)–(3.9), we obtain the desired estimate. \square

Finally, we need the following well-known result.

Lemma 3.3 *Let $c(\cdot) \in L^\infty(\Omega; \mathbb{R}^+)$. Then (recall (2.11) for $E(\cdot)$)*

$$E(t) = E(0), \quad \forall t \in [0, T]. \quad (3.10)$$

4 Proof of Theorem 2.1

This section is devoted to prove Theorem 2.1. The proof is divided into several steps.

Step 1. Let us introduce some notations and some transformations. Recall (2.8) for a_0 , α , ε_i , T_i and T'_i ($i = 0, 1, 2$). First of all, denote

$$\begin{cases} \tilde{\Omega} \triangleq \Omega \setminus \mathcal{O}_{\delta_1}(\Gamma_0), & \mathcal{Q} \triangleq (0, T) \times (0, T) \times \Omega, & \mathcal{S} \triangleq (0, T) \times (0, T) \times \Gamma, \\ \mathcal{Q}_i \triangleq (T_i, T'_i) \times (T_i, T'_i) \times \tilde{\Omega}, & \mathcal{S}_i \triangleq (T_i, T'_i) \times (T_i, T'_i) \times \partial\tilde{\Omega}, \end{cases} \quad (4.1)$$

where δ_1 is given in (2.1).

Recall (2.2) for R_0 and R_1 . By (2.3) and (2.7), it is easy to show that

$$R_1^2 < a_0^2 + \alpha(S/2)^2. \quad (4.2)$$

For any $b > 0$, put

$$\begin{cases} \varphi = \varphi(t, s, x) = |x - x_0|^2 - \alpha(t - T/2)^2 - \alpha(s - T/2)^2, \\ \mathcal{Q}(b) = \{(t, s, x) \in (0, T) \times (0, T) \times \tilde{\Omega} \mid \varphi(t, s, x) > b^2\}. \end{cases} \quad (4.3)$$

By (4.2), it is easy to check that

$$\mathcal{Q}(a_0) \subset \mathcal{Q}_2. \quad (4.4)$$

By (4.3), (4.1) and (4.2), we have

$$\begin{aligned} \varphi(T_2, s, x) = \varphi(T'_2, s, x) &= |x - x_0|^2 - \alpha(S/2)^2 - \alpha(s - T/2)^2 \\ &\leq R_1^2 - \alpha(S/2)^2 < a_0^2, \quad \forall (s, x) \in (0, T) \times \tilde{\Omega}. \end{aligned} \quad (4.5)$$

Similarly,

$$\varphi(t, T_2, x) = \varphi(t, T'_2, x) < a_0^2, \quad \forall (t, x) \in (0, T) \times \tilde{\Omega}. \quad (4.6)$$

Also

$$\varphi(t, s, x) \geq R_0^2 - 2\alpha\varepsilon_1^2 T^2 = R_0^2/2 = 2a_0^2, \quad \forall (t, s, x) \in \mathcal{Q}_1. \quad (4.7)$$

Next, denote

$$y(t, x) \triangleq w_t(t, x), \quad (t, x) \in \mathcal{Q}, \quad (4.8)$$

where $w(\cdot) \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ is the weak solution of (1.10). Thus by (1.10), we see that $y(\cdot)$ solves

$$\begin{cases} y_{tt} - \Delta y + c(x)y = 0 & \text{in } \mathcal{Q}, \\ y = 0 & \text{on } \Sigma, \\ y(0) = w_1, \quad y_t(0) = \Delta w_0 - c(x)w_0, & \text{in } \Omega. \end{cases} \quad (4.9)$$

We need the following simple transformation. Put

$$z(t, s, x) \triangleq \int_s^t y(\tau, x) d\tau, \quad \forall (t, s, x) \in \mathcal{Q}. \quad (4.10)$$

Then $z(\cdot)$ satisfies

$$\begin{cases} z_{tt} + z_{ss} - \Delta z + c(x)z = 0 & \text{in } \mathcal{Q}, \\ z = 0 & \text{on } \mathcal{S}. \end{cases} \quad (4.11)$$

Put

$$v = v(t, s, x) = \xi(x)z(t, s, x), \quad (t, s, x) \in \mathcal{Q}, \quad (4.12)$$

where ξ is given by (2.4). Then by (4.11), it is easy to check that $v(\cdot)$ satisfies

$$\begin{cases} v_{tt} + v_{ss} - \Delta v = -c(x)v - (\Delta\xi)z - 2(\nabla\xi) \cdot (\nabla z) & \text{in } \mathcal{Q}, \\ v = 0 & \text{on } \mathcal{S}, \\ v \equiv 0 & \text{in } (0, T) \times (0, T) \times (\Omega \cap \mathcal{O}_{\delta_1}(\Gamma_0)). \end{cases} \quad (4.13)$$

Step 2. Let us use inequality (3.2) in Lemma 3.1 with φ given in (4.3) and v given by (4.12) respectively. Integrating (3.2) on \mathcal{Q}_2 (recall (4.1) for \mathcal{Q}_2), by (4.13) and using integration by parts, we get

$$\begin{aligned} & 2(1 - \alpha)\lambda \int_{\mathcal{Q}_2} \theta^2 (v_t^2 + v_s^2 + \sum_i v_i^2) dx dt ds + \int_{\mathcal{Q}_2} \theta^2 B v^2 dx dt ds \\ & \leq \int_{\mathcal{Q}_2} \theta^2 |c(x)v + (\Delta\xi)z + 2(\nabla\xi) \cdot (\nabla z)|^2 dx dt ds + I_1 + I_2, \end{aligned} \quad (4.14)$$

where θ and B are given in (3.1) and (3.4) respectively (with β replaced by α), I_1 and I_2 are given by

$$\begin{aligned} I_1 = & -2 \int_{T_2}^{T'_2} ds \int_{\tilde{\Omega}} dx \left\{ \theta^2 \left[\eta_t (v_t^2 + v_s^2 - \sum_j v_j^2) - 2v_t (\eta_t v_t + \eta_s v_s - \sum_j \eta_j v_j) \right. \right. \\ & \left. \left. + (\Psi - 2A)v_t v + (A_t - 2\eta_t A)v^2 \right] \right\} \Big|_{t=T_2}^{t=T'_2}, \end{aligned} \quad (4.15)$$

$$\begin{aligned}
I_2 = & -2 \int_{T_2}^{T'_2} dt \int_{\tilde{\Omega}} dx \left\{ \theta^2 \left[\eta_s (v_s^2 + v_t^2 - \sum_j v_j^2) - 2v_s (\eta_s v_s + \eta_t v_t - \sum_j \eta_j v_j) \right. \right. \\
& \left. \left. + (\Psi - 2A)v_s v + (A_s - 2\eta_s A)v^2 \right] \right\} \Big|_{s=T_2}^{s=T'_2}, \tag{4.16}
\end{aligned}$$

where η and Ψ are given in (3.1), and A is given by (3.3) (with β replaced by α).

By (3.1), (4.3) and (2.8), using Cauchy inequality, we have

$$\begin{aligned}
& \theta^2 \left[-\eta_t \sum_j v_j^2 + 2v_t \sum_j \eta_j v_j \right] \Big|_{t=T_2}^{t=T'_2} \\
= & \lambda e^{\lambda\varphi(T'_2, s, x)} \left[\frac{\alpha S}{2} \sum_j v_j^2(T'_2, s, x) + 2v_t(T'_2, s, x)(x - x_0) \cdot (\nabla v(T'_2, s, x)) \right] \\
& + \lambda e^{\lambda\varphi(T_2, s, x)} \left[\frac{\alpha S}{2} \sum_j v_j^2(T_2, s, x) - 2v_t(T_2, s, x)(x - x_0) \cdot (\nabla v(T_2, s, x)) \right] \\
\geq & \lambda e^{\lambda\varphi(T'_2, s, x)} \left[\frac{\alpha S}{2} \sum_j v_j^2(T'_2, s, x) - 2R_1 |v_t(T'_2, s, x)| |\nabla v(T'_2, s, x)| \right] \\
& + \lambda e^{\lambda\varphi(T_2, s, x)} \left[\frac{\alpha S}{2} \sum_j v_j^2(T_2, s, x) - 2R_1 |v_t(T_2, s, x)| |\nabla v(T_2, s, x)|^2 \right] \\
\geq & -\lambda \frac{2R_1^2}{\alpha S} \left[e^{\lambda\varphi(T'_2, s, x)} v_t^2(T'_2, s, x) + e^{\lambda\varphi(T_2, s, x)} v_t^2(T_2, s, x) \right]. \tag{4.17}
\end{aligned}$$

Similarly

$$\begin{aligned}
& \theta^2 \left[\eta_t (v_t^2 + v_s^2) - 2v_t (\eta_t v_t + \eta_s v_s) + (\Psi - 2A)v_t v + (A_t - 2\eta_t A)v^2 \right] \Big|_{T_2}^{T'_2} \\
= & \theta^2 \left[-\eta_t v_t^2 + \eta_t v_s^2 - 2\eta_s v_t v_s + (\Psi - 2A)v_t v + (A_t - 2\eta_t A)v^2 \right] \Big|_{T_2}^{T'_2} \\
= & \lambda e^{\lambda\varphi(T'_2, s, x)} \left\{ \frac{\alpha S}{2} v_t^2(T'_2, s, x) - \frac{\alpha S}{2} v_s^2(T'_2, s, x) + \alpha(2s - T)v_t(T'_2, s, x)v_s(T'_2, s, x) \right. \\
& + \left[n - 1 + \alpha - 2\lambda \left(\frac{\alpha^2 S^2}{4} + \alpha^2 (s - T/2)^2 - |x - x_0|^2 \right) \right] v_t(T'_2, s, x)v(T'_2, s, x) \\
& \left. + \lambda \alpha S \left[\alpha + \lambda \left(\frac{\alpha^2 S^2}{4} + \alpha^2 (s - T/2)^2 - |x - x_0|^2 \right) \right] v^2(T'_2, s, x) \right\} \\
& + \lambda e^{\lambda\varphi(T_2, s, x)} \left\{ \frac{\alpha S}{2} v_t^2(T_2, s, x) - \frac{\alpha S}{2} v_s^2(T_2, s, x) - \alpha(2s - T)v_t(T_2, s, x)v_s(T_2, s, x) \right. \\
& - \left[n - 1 + \alpha - 2\lambda \left(\frac{\alpha^2 S^2}{4} + \alpha^2 (s - T/2)^2 - |x - x_0|^2 \right) \right] v_t(T_2, s, x)v(T_2, s, x) \\
& \left. + \lambda \alpha S \left[\alpha + \lambda \left(\frac{\alpha^2 S^2}{4} + \alpha^2 (s - T/2)^2 - |x - x_0|^2 \right) \right] v^2(T_2, s, x) \right\}. \tag{4.18}
\end{aligned}$$

However, for any $(s, x) \in (T_2, T'_2) \times \tilde{\Omega}$, we have

$$\alpha(2s - T)v_t(T'_2, s, x)v_s(T'_2, s, x) \geq -\frac{\alpha S}{2} \left(v_t^2(T'_2, s, x) + v_s^2(T'_2, s, x) \right), \tag{4.19}$$

and

$$-\alpha(2s - T)v_t(T_2, s, x)v_s(T_2, s, x) \geq -\frac{\alpha S}{2} \left(v_t^2(T_2, s, x) + v_s^2(T_2, s, x) \right), \tag{4.20}$$

and

$$\begin{aligned}
& \left[n - 1 + \alpha - 2\lambda \left(\frac{\alpha^2 S^2}{4} + \alpha^2 (s - T/2)^2 - |x - x_0|^2 \right) \right] v_t(T'_2, s, x) v(T'_2, s, x) \\
& \geq - \left[n - 1 + \alpha + \lambda (\alpha^2 S^2 + 2R_1^2) \right] |v_t(T'_2, s, x) v(T'_2, s, x)| \\
& \geq - \frac{1}{2} \left[n - 1 + \alpha + \lambda (\alpha^2 S^2 + 2R_1^2) \right] \left(v_t^2(T'_2, s, x) + v^2(T'_2, s, x) \right),
\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}
& - \left[n - 1 + \alpha - 2\lambda \left(\frac{\alpha^2 S^2}{4} + \alpha^2 (s - T/2)^2 - |x - x_0|^2 \right) \right] v_t(T_2, s, x) v(T_2, s, x) \\
& \geq - \frac{1}{2} \left[n - 1 + \alpha + \lambda (\alpha^2 S^2 + 2R_1^2) \right] \left(v_t^2(T_2, s, x) + v^2(T_2, s, x) \right).
\end{aligned} \tag{4.22}$$

Now, by (4.18)–(4.22), we get

$$\begin{aligned}
& \theta^2 \left[\eta_t (v_t^2 + v_s^2) - 2v_t (\eta_t v_t + \eta_s v_s) + (\Psi - 2A) v_t v + (A_t - 2\eta_t A) v^2 \right] \Big|_{T_2}^{T'_2} \\
& \geq - \frac{\lambda}{2} e^{\lambda \varphi(T'_2, s, x)} \left\{ \left[n - 1 + \alpha + \lambda (\alpha^2 S^2 + 2R_1^2) \right] v_t^2(T'_2, s, x) + 2\alpha S v_s^2(T'_2, s, x) \right. \\
& \quad \left. + \left[n - 1 + \alpha + \lambda (\alpha^2 S^2 + 2R_1^2) + 2\lambda^2 \alpha S R_1^2 \right] v^2(T'_2, s, x) \right\} \\
& \quad - \frac{\lambda}{2} e^{\lambda \varphi(T_2, s, x)} \left\{ \left[n - 1 + \alpha + \lambda (\alpha^2 S^2 + 2R_1^2) \right] v_t^2(T_2, s, x) + 2\alpha S v_s^2(T_2, s, x) \right. \\
& \quad \left. + \left[n - 1 + \alpha + \lambda (\alpha^2 S^2 + 2R_1^2) + 2\lambda^2 \alpha S R_1^2 \right] v^2(T_2, s, x) \right\}.
\end{aligned} \tag{4.23}$$

Then, by (4.5), and (4.15)–(4.17) and (4.23), we get

$$\begin{aligned}
I_1 & \leq \lambda e^{a_0^2 \lambda} \left\{ \left[n - 1 + \alpha + \frac{4R_1^2}{\alpha S} + \lambda (\alpha^2 S^2 + 2R_1^2) \right] \int_{T_2}^{T'_2} ds \int_{\tilde{\Omega}} \left(v_t^2(T'_2, s, x) \right. \right. \\
& \quad \left. \left. + v_t^2(T_2, s, x) \right) dx + 2\alpha S \int_{T_2}^{T'_2} ds \int_{\tilde{\Omega}} \left(v_s^2(T'_2, s, x) + v_s^2(T_2, s, x) \right) dx \right. \\
& \quad \left. + \left[n - 1 + \alpha + \lambda (\alpha^2 S^2 + 2R_1^2) + 2\lambda^2 \alpha S R_1^2 \right] \int_{T_2}^{T'_2} ds \int_{\tilde{\Omega}} \left(v^2(T'_2, s, x) \right. \right. \\
& \quad \left. \left. + v^2(T_2, s, x) \right) dx \right\}.
\end{aligned} \tag{4.24}$$

However, by (4.8), (4.10) and (4.12), and using Lemma 3.3, we see that (recall (2.11) for $E(\cdot)$)

$$\begin{aligned}
& \int_{T_2}^{T'_2} ds \int_{\tilde{\Omega}} \left(v_t^2(T'_2, s, x) + v_t^2(T_2, s, x) \right) dx \leq \int_{T_2}^{T'_2} \int_{\Omega} \left(y^2(T'_2, x) + y^2(T_2, x) \right) dx ds \\
& = S \int_{\Omega} \left(w_t^2(T'_2, x) + w_t^2(T_2, x) \right) dx \leq 2S(E(T'_2) + E(T_2)) = 4SE(0).
\end{aligned} \tag{4.25}$$

Similarly,

$$\begin{aligned}
& \int_{T_2}^{T'_2} ds \int_{\tilde{\Omega}} \left(v_s^2(T'_2, s, x) + v_s^2(T_2, s, x) \right) dx \\
& \leq 2 \int_{T_2}^{T'_2} dt \int_{\Omega} w_t^2(t, x) dx \leq 4 \int_{T_2}^{T'_2} E(t) dt = 4SE(0)
\end{aligned} \tag{4.26}$$

and

$$\begin{aligned}
& \int_{T_2}^{T'_2} ds \int_{\Omega} \left(v^2(T'_2, s, x) + v^2(T_2, s, x) \right) dx \\
& \leq \int_{T_2}^{T'_2} ds \int_{\Omega} \left(\left| \int_s^{T'_2} y(\tau, x) d\tau \right|^2 + \left| \int_{T_2}^s y(\tau, x) d\tau \right|^2 \right) dx \\
& = \int_{T_2}^{T'_2} ds \int_{\Omega} \left(\left| \int_s^{T'_2} w_{\tau}(\tau, x) d\tau \right|^2 + \left| \int_{T_2}^s w_{\tau}(\tau, x) d\tau \right|^2 \right) dx \\
& = \int_{T_2}^{T'_2} \int_{\Omega} \left(|w(T'_2, x) - w(s, x)|^2 + |w(T_2, x) - w(s, x)|^2 \right) dx ds \\
& \leq 2 \int_{T_2}^{T'_2} \int_{\Omega} \left(w^2(T'_2, x) + w^2(T_2, x) + 2w^2(s, x) \right) dx ds \leq 16SE(0).
\end{aligned} \tag{4.27}$$

Now, combining (4.24)–(4.27), we obtain

$$I_1 \leq 4S\lambda e^{a_0^2\lambda} \left[5(n-1+\alpha) + 2\alpha S + \frac{4R_1^2}{\alpha S} + 5\lambda(\alpha^2 S^2 + 2R_1^2) + 8\lambda^2 \alpha S R_1^2 \right] E(0). \tag{4.28}$$

Similarly, we have

$$I_2 \leq 4S\lambda e^{a_0^2\lambda} \left[5(n-1+\alpha) + 2\alpha S + \frac{4R_1^2}{\alpha S} + 5\lambda(\alpha^2 S^2 + 2R_1^2) + 8\lambda^2 \alpha S R_1^2 \right] E(0). \tag{4.29}$$

On the other hand, by (2.4), (2.6) and (4.3), we have

$$\begin{aligned}
& \int_{\mathcal{Q}_2} \theta^2 |c(x)v + (\Delta\xi)z + 2(\nabla\xi) \cdot (\nabla z)|^2 dx dt ds \\
& \leq 2 \int_{\mathcal{Q}_2} \theta^2 |(\Delta\xi)z + 2(\nabla\xi) \cdot (\nabla z)|^2 dx dt ds + 2|c(\cdot)|_{L^\infty(\Omega)}^2 \int_{\mathcal{Q}_2} \theta^2 v^2 dx dt ds \\
& \leq 4K_0 \int_{T_2}^{T'_2} \int_{T_2}^{T'_2} \int_{\Omega \cap \mathcal{O}_{\delta}(\Gamma_0)} \theta^2 (z^2 + 4|\nabla z|^2) dx dt ds \\
& \quad + 2|c(\cdot)|_{L^\infty(\Omega)}^2 \int_{\mathcal{Q}(a_0)} \theta^2 v^2 dx dt ds + 2|c(\cdot)|_{L^\infty(\Omega)}^2 \int_{\mathcal{Q}_2 \setminus \mathcal{Q}(a_0)} \theta^2 v^2 dx dt ds \\
& \leq 4K_0 e^{R_1^2\lambda} \int_{T_2}^{T'_2} \int_{T_2}^{T'_2} \int_{\Omega \cap \mathcal{O}_{\delta}(\Gamma_0)} (z^2 + 4|\nabla z|^2) dx dt ds \\
& \quad + 2|c(\cdot)|_{L^\infty(\Omega)}^2 \int_{\mathcal{Q}(a_0)} \theta^2 v^2 dx dt ds + 2|c(\cdot)|_{L^\infty(\Omega)}^2 e^{a_0^2\lambda} \int_{\mathcal{Q}_2} v^2 dx dt ds.
\end{aligned} \tag{4.30}$$

Further, by (3.4), we get

$$B(= B(t, s, x)) > 4(1+\alpha)a_0^2\lambda^3 - [8\alpha^2 + 4n + (n+\alpha-1)^2]\lambda^2, \quad \forall (t, s, x) \in \mathcal{Q}(a_0) \tag{4.31}$$

and

$$|B| \leq 2(1+\alpha)(2R_1^2 + \alpha^2 S^2)\lambda^3 + [8\alpha^2 + 4n + (n+\alpha-1)^2]\lambda^2, \quad \forall (t, s, x) \in \mathcal{Q}_2. \tag{4.32}$$

Taking $\lambda > \lambda_1 + 1$, where λ_1 is given in (2.10), we get

$$B > 4(1+\alpha)a_0^2\lambda^2(\lambda - \lambda_1) > 2|c(\cdot)|_{L^\infty(\Omega)}^2, \quad \forall (t, s, x) \in \mathcal{Q}(a_0). \tag{4.33}$$

Thus, by (4.32) and (4.33), for any $\lambda > \lambda_1 + 1$, we have

$$\begin{aligned}
& \int_{\mathcal{Q}_2} \theta^2 B v^2 dx dt ds = \left(\int_{\mathcal{Q}(a_0)} + \int_{\mathcal{Q}_2 \setminus \mathcal{Q}(a_0)} \right) \theta^2 B v^2 dx dt ds \\
& \geq \int_{\mathcal{Q}(a_0)} \theta^2 B v^2 dx dt ds - e^{a_0^2 \lambda} \int_{\mathcal{Q}_2 \setminus \mathcal{Q}(a_0)} |B| v^2 dx dt ds \\
& \geq 2|c(\cdot)|_{L^\infty(\Omega)}^2 \int_{\mathcal{Q}(a_0)} \theta^2 v^2 dx dt ds - e^{a_0^2 \lambda} \left\{ 2(1 + \alpha) (2R_1^2 + \alpha^2 S^2) \lambda^3 \right. \\
& \quad \left. + [8\alpha^2 + 4n + (n + \alpha - 1)^2] \lambda^2 \right\} \int_{\mathcal{Q}_2} v^2 dx dt ds.
\end{aligned} \tag{4.34}$$

Now, combining (4.14), (4.30) and (4.34), we arrive at

$$\begin{aligned}
& 2(1 - \alpha) \lambda \int_{\mathcal{Q}_2} \theta^2 (v_t^2 + v_s^2 + \sum_i v_i^2) dx dt ds \\
& \leq 4K_0 e^{R_1^2 \lambda} \int_{T_2}^{T'_2} \int_{T_2}^{T'_2} \int_{\Omega \cap \mathcal{O}_{\bar{s}}(\Gamma_0)} (z^2 + 4|\nabla z|^2) dx dt ds + I_1 + I_2 \\
& \quad + e^{a_0^2 \lambda} \left\{ 2(1 + \alpha) (2R_1^2 + \alpha^2 S^2) \lambda^3 + [8\alpha^2 + 4n + (n + \alpha - 1)^2] \lambda^2 \right. \\
& \quad \left. + 2|c(\cdot)|_{L^\infty(\Omega)}^2 \right\} \int_{\mathcal{Q}_2} v^2 dx dt ds, \quad \forall \lambda > \lambda_1 + 1.
\end{aligned} \tag{4.35}$$

However, by (4.8), (4.10) and (4.12), and using Lemma 3.3, we get

$$\begin{aligned}
& \int_{\mathcal{Q}_2} v^2 dx dt ds \leq \int_{\mathcal{Q}_2} z^2 dx dt ds = \int_{\mathcal{Q}_2} (w(t, x) - w(s, x))^2 dx dt ds \\
& \leq \int_{T_2}^{T'_2} \int_{T_2}^{T'_2} \int_{\Omega} (w(t, x) - w(s, x))^2 dx dt ds \\
& = 2 \int_{T_2}^{T'_2} \int_{T_2}^{T'_2} \int_{\Omega} w^2(t, x) dx dt ds - 2 \int_{\Omega} \left(\int_{T_2}^{T'_2} w(t, x) dt \right)^2 dx \\
& \leq 2 \int_{T_2}^{T'_2} \int_{T_2}^{T'_2} \int_{\Omega} w^2(t, x) dx dt ds \leq 2C_0 S \int_{T_2}^{T'_2} \int_{\Omega} |\nabla w(t, x)|^2 dx dt \\
& \leq 4C_0 S \int_{T_2}^{T'_2} E(t) dt = 4C_0 S^2 E(0),
\end{aligned} \tag{4.36}$$

where C_0 is the constant in (2.9).

On the other hand, by (4.1), (4.7) and (4.10), we get

$$\begin{aligned}
& \int_{\mathcal{Q}_2} \theta^2 (v_t^2 + v_s^2 + \sum_j v_j^2) dx dt ds \geq \int_{\mathcal{Q}_1} \theta^2 (v_t^2 + v_s^2) dx dt ds \\
& \geq e^{2a_0^2 \lambda} \int_{\mathcal{Q}_1} (v_t^2 + v_s^2) dx dt ds \geq e^{2a_0^2 \lambda} \int_{T_1}^{T'_1} \int_{T_1}^{T'_1} \int_{\Omega \setminus G} (v_t^2 + v_s^2) dx dt ds \\
& = e^{2a_0^2 \lambda} \int_{T_1}^{T'_1} \int_{T_1}^{T'_1} \int_{\Omega \setminus G} (z_t^2 + z_s^2) dx dt ds = 2e^{2a_0^2 \lambda} (T'_1 - T_1) \int_{T_1}^{T'_1} \int_{\Omega \setminus G} y^2(t, x) dx dt.
\end{aligned} \tag{4.37}$$

Consequently, combining (4.35), (4.28)–(4.29) and (4.36)–(4.37), we end up with

$$\begin{aligned}
& (1 - \alpha) \lambda e^{a_0^2 \lambda} (T'_1 - T_1) \int_{T_1}^{T'_1} \int_{\Omega \setminus G} y^2(t, x) dx dt \\
& \leq K_0 e^{(R_1^2 - a_0^2) \lambda} \int_{T_2}^{T'_2} \int_{T_2}^{T'_2} \int_{\Omega \cap \mathcal{O}_{\bar{s}}(\Gamma_0)} (z^2 + 4|\nabla z|^2) dx dt ds + C_2 E(0),
\end{aligned} \tag{4.38}$$

where

$$C_2 = C_2(\lambda) \triangleq 2C_0 S^2 |c(\cdot)|_{L^\infty(\Omega)}^2 + C_{21}\lambda + C_{22}\lambda^2 + C_{23}\lambda^3, \quad (4.39)$$

where C_{21} , C_{22} and C_{23} are given in (2.9).

Step 3. We now estimate “ $\int_{T_2}^{T_2'} \int_{T_2}^{T_2'} \int_{\Omega \cap \mathcal{O}_{\delta}(\Gamma_0)} |\nabla z|^2 dx dt ds$ ”. Denote

$$h = h(t, s) \triangleq t(T-t)s(T-s). \quad (4.40)$$

By (4.11), and noting that $c(\cdot) \geq 0$, we get (recall (2.5) and (2.6) for ζ and K_0 respectively)

$$\begin{aligned} 0 &= \int_{\mathcal{Q}} \zeta^2 h z [z_{tt} + z_{ss} - \Delta z + c(x)z] dx dt ds \\ &= - \int_{\mathcal{Q}} \zeta^2 [z_t(h_t z + h z_t) + z_s(h_s z + h z_s)] dx dt ds + \int_{\mathcal{Q}} \zeta^2 h c z^2 dx dt ds \\ &\quad + \int_{\mathcal{Q}} \zeta^2 h |\nabla z|^2 dx dt ds + 2 \int_{\mathcal{Q}} \zeta h (\nabla z) \cdot (\nabla \zeta) z dx dt ds \\ &\geq - \int_{\mathcal{Q}} \zeta^2 [z_t(h_t z + h z_t) + z_s(h_s z + h z_s)] dx dt ds \\ &\quad + \frac{1}{2} \int_{\mathcal{Q}} \zeta^2 h |\nabla z|^2 dx dt ds - \frac{T^4}{8} K_0 \int_0^T \int_0^T \int_G z^2 dx dt ds \\ &\geq - \int_{\mathcal{Q}} \zeta^2 [z_t(h_t z + h z_t) + z_s(h_s z + h z_s)] dx dt ds \\ &\quad + \frac{T_2^2 (T - T_2)^2}{2} \int_{T_2}^{T_2'} \int_{T_2}^{T_2'} \int_{\Omega \cap \mathcal{O}_{\delta}(\Gamma_0)} |\nabla z|^2 dx dt ds - \frac{T^2}{8} K_0 \int_0^T \int_0^T \int_G z^2 dx dt ds. \end{aligned} \quad (4.41)$$

Thus we get

$$\begin{aligned} &\int_{T_2}^{T_2'} \int_{T_2}^{T_2'} \int_{\Omega \cap \mathcal{O}_{\delta}(\Gamma_0)} |\nabla z|^2 dx dt ds \\ &\leq \frac{2}{T_2^2 (T - T_2)^2} \left\{ \int_0^T \int_0^T \int_G |z_t(h_t z + h z_t) + z_s(h_s z + h z_s)| dx dt ds \right. \\ &\quad \left. + \frac{T^4}{8} K_0 \int_0^T \int_0^T \int_G z^2 dx dt ds \right\}. \end{aligned} \quad (4.42)$$

Note that

$$\begin{aligned} &\int_0^T \int_0^T \int_G |z_t(h_t z + h z_t) + z_s(h_s z + h z_s)| dx dt ds \\ &\leq 2 \int_0^T \int_0^T \int_G (|h_t z_t z| + h z_t^2) dx dt ds \\ &\leq \int_0^T \int_0^T \int_G z^2 dx dt ds + \int_0^T \int_0^T \int_G (h_t^2 + 2h) z_t^2 dx dt ds \\ &\leq \int_0^T \int_0^T \int_G z^2 dx dt ds + \frac{3T^4}{16} \int_0^T \int_0^T \int_G z_t^2 dx dt ds. \end{aligned} \quad (4.43)$$

Combining (4.42)–(4.43), we get

$$\begin{aligned} &\int_{T_2}^{T_2'} \int_{T_2}^{T_2'} \int_{\Omega \cap \mathcal{O}_{\delta}(\Gamma_0)} |\nabla z|^2 dx dt ds \\ &\leq \frac{1}{T_2^2 (T - T_2)^2} \left\{ \frac{3T^4}{8} \int_0^T \int_0^T \int_G z_t^2 dx dt ds + \left(2 + \frac{T^4}{4} K_0\right) \int_0^T \int_0^T \int_G z^2 dx dt ds \right\}. \end{aligned} \quad (4.44)$$

Hence by (4.38) and (4.44), we end up with

$$\begin{aligned}
& (1 - \alpha)\lambda e^{a_0^2\lambda}(T'_1 - T_1) \int_{T_1}^{T'_1} \int_{\Omega \setminus G} y^2(t, x) dx dt \\
& \leq K_0 e^{(R_1^2 - a_0^2)\lambda} \left[\frac{3T^4}{2T_2^2(T - T_2)^2} \int_0^T \int_0^T \int_G z_t^2 dx dt ds \right. \\
& \quad \left. + \left(1 + \frac{8 + T^4 K_0}{T_2^2(T - T_2)^2}\right) \int_0^T \int_0^T \int_G z^2 dx dt ds \right] + C_2 E(0).
\end{aligned} \tag{4.45}$$

However, by (4.8) and (4.10), we see that

$$\begin{aligned}
& \int_0^T \int_0^T \int_G z^2 dx dt ds = \int_0^T \int_0^T \int_G \left(\int_s^t w_\tau(\tau, x) d\tau \right)^2 dx dt ds \\
& \leq \int_0^T \int_0^T \int_G |t - s| \left| \int_s^t w_\tau^2(\tau, x) d\tau \right| dx dt ds \\
& \leq T \int_0^T \int_0^T \int_G \int_0^T w_\tau^2(\tau, x) d\tau dx dt ds \leq T^3 \int_0^T \int_G w_t^2 dx dt.
\end{aligned} \tag{4.46}$$

Consequently by (4.45)–(4.46) and (4.8), we get

$$\begin{aligned}
& (1 - \alpha)\lambda e^{a_0^2\lambda}(T'_1 - T_1) \int_{T_1}^{T'_1} \int_{\Omega \setminus G} w_t^2(t, x) dx dt \\
& \leq K_0 e^{(R_1^2 - a_0^2)\lambda} \left[\frac{3T^5}{2T_2^2(T - T_2)^2} + \left(1 + \frac{8 + T^4 K_0}{T_2^2(T - T_2)^2}\right) T^3 \right] \int_0^T \int_G w_t^2 dx dt + C_2 E(0).
\end{aligned} \tag{4.47}$$

Thus

$$\begin{aligned}
& (1 - \alpha)\lambda e^{a_0^2\lambda}(T'_1 - T_1) \int_{T_1}^{T'_1} \int_{\Omega} w_t^2(t, x) dx dt \\
& \leq \left\{ K_0 e^{(R_1^2 - a_0^2)\lambda} \left[\frac{3T^5}{2T_2^2(T - T_2)^2} + \left(1 + \frac{8 + T^4 K_0}{T_2^2(T - T_2)^2}\right) T^3 \right] \right. \\
& \quad \left. + (1 - \alpha)\lambda e^{a_0^2\lambda}(T'_1 - T_1) \right\} \int_0^T \int_G w_t^2 dx dt + C_2 E(0).
\end{aligned} \tag{4.48}$$

Step 4. Let us complete the proof. By Lemma 3.2, we have (recall (2.8) for T_0 and T'_0)

$$\int_{T_0}^{T'_0} \int_{\Omega} [|\nabla w|^2 + c(x)w^2] dx dt \leq C_1 \int_{T_1}^{T'_1} \int_{\Omega} w_t^2 dx dt, \tag{4.49}$$

where $C_1 (\geq 1)$ is given in (2.9). Hence, by (4.49) and (4.8), we get

$$\begin{aligned}
& \int_{T_1}^{T'_1} \int_{\Omega} w_t^2 dx dt \geq \frac{1}{2} \int_{T_0}^{T'_0} \int_{\Omega} w_t^2 dx dt + \frac{1}{2C_1} \int_{T_0}^{T'_0} \int_{\Omega} [|\nabla w|^2 + c(x)w^2] dx dt \\
& \geq \frac{1}{C_1} \int_{T_0}^{T'_0} E(t) dt = \frac{T'_0 - T_0}{C_1} E(0).
\end{aligned} \tag{4.50}$$

Combining (4.48) and (4.50), we arrive at (recall (2.9) for C_3)

$$\begin{aligned}
& (C_3 \lambda e^{a_0^2\lambda} - C_2) E(0) \leq \left\{ K_0 e^{(R_1^2 - a_0^2)\lambda} \left[\frac{3T^5}{2T_2^2(T - T_2)^2} + \left(1 + \frac{8 + T^4 K_0}{T_2^2(T - T_2)^2}\right) T^3 \right] \right. \\
& \quad \left. + (1 - \alpha)\lambda e^{a_0^2\lambda}(T'_1 - T_1) \right\} \int_0^T \int_G w_t^2 dx dt.
\end{aligned} \tag{4.51}$$

However, by (4.39) and the elementary inequality $e^x > 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$ ($\forall x > 0$), we have

$$\begin{aligned} C_3 \lambda e^{a_0^2 \lambda} - C_2 &> C_3 \lambda - 2C_0 S^2 |c(\cdot)|_{L^\infty(\Omega)}^2 + (C_3 a_0^2 \lambda - C_{21}) \lambda \\ &+ \left(\frac{C_3 a_0^4}{2} \lambda - C_{22} \right) \lambda^2 + \left(\frac{C_3 a_0^6}{6} \lambda - C_{23} \right) \lambda^3. \end{aligned} \quad (4.52)$$

Now, we take

$$\lambda \triangleq \max(\lambda_{20}, \lambda_{21}, \lambda_{22}, \lambda_{23}, \lambda_1 + 1), \quad (4.53)$$

where $\lambda_1, \lambda_{20}, \lambda_{21}, \lambda_{22}, \lambda_{23}$ and λ_3 are given in (2.10). Then one gets

$$C_3 \lambda e^{a_0^2 \lambda} - C_2 > 1. \quad (4.54)$$

Finally, the desired result follows from (4.51) and (4.54) immediately. \square

5 Proof of Theorem 2.2

This section is devoted to give a proof of Theorem 2.2. Let p and q solve

$$\begin{cases} p_{tt} - \Delta p + c(x)p = 0 & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(0) = u_0, \quad p_t(0) = u_1 & \text{in } \Omega \end{cases} \quad (5.1)$$

and

$$\begin{cases} q_{tt} - \Delta q + c(x)q = -a(x)u_t & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(0) = q_t(0) = 0 & \text{in } \Omega \end{cases} \quad (5.2)$$

respectively. Then it is easy to see that

$$u = p + q. \quad (5.3)$$

Applying Theorem 2.1 to equation (5.1), we get

$$e(0) \leq K \int_0^T \int_G p_t^2 dx dt. \quad (5.4)$$

By (5.3)–(5.4) and (H2), we have

$$e(0) \leq 2K \left[\frac{1}{a^0} \int_Q a(x) u_t^2 dx dt + \int_Q q_t^2 dx dt \right]. \quad (5.5)$$

However, concerning (5.2), using the usual energy estimate, one gets

$$\int_Q q_t^2 dx dt \leq |a(\cdot)|_{L^\infty(\Omega)} (e^T - 1) \int_Q a(x) u_t^2 dx dt. \quad (5.6)$$

Thus, we get

$$e(0) \leq K_1 \int_Q a(x) u_t^2 dx dt, \quad (5.7)$$

where K_1 is the constant in (2.16).

On the other hand, similar to (1.3), concerning the solution $u(\cdot)$ of (1.8), we have

$$e(T) - e(0) + \int_Q a(x) u_t^2(t, x) dx dt = 0. \quad (5.8)$$

Combining (5.7) and (5.8), we get

$$e(T) \leq \left(1 - \frac{1}{K_1}\right) e(0). \quad (5.9)$$

Noting that (1.8) is time-invariance, therefore we arrive at

$$e(nT) \leq \left(1 - \frac{1}{K_1}\right)^n e(0), \quad \forall n \in \mathbb{N}. \quad (5.10)$$

Now, for any $t \in [T, \infty)$, we choose an integer $n \in \mathbb{N}$ such that $nT \leq t < (n+1)T$. From (5.10) and noting that $e(t)$ is decreasing, we get

$$e(t) \leq \left(1 - \frac{1}{K_1}\right)^{\frac{t}{T}-1} e(0), \quad \forall t \geq 0,$$

which is exactly the desired result. □

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