



HAL
open science

Affine Kac-Moody algebras graded by affine root systems

Josiane Nervi-Gasparini

► **To cite this version:**

Josiane Nervi-Gasparini. Affine Kac-Moody algebras graded by affine root systems. 2001. hal-00129666

HAL Id: hal-00129666

<https://hal.science/hal-00129666>

Preprint submitted on 8 Feb 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Affine Kac-Moody Algebras Graded By Affine Root Systems

Josiane Nervi

*Institut de Recherche Mathématique Avancée, Université Louis Pasteur,
7, rue René Descartes, F-67084 Strasbourg Cedex, France
E-mail: nervi@math.u-strasbg.fr*

0. Introduction

The aim of this paper is to give the complete classification of all affine Kac-Moody algebras graded by affine root systems.

An affine Lie algebra $\hat{\mathfrak{g}}$ is said graded by the affine root system Σ if $\hat{\mathfrak{g}}$ contains an affine subalgebra \mathfrak{a} whose root system relatively to a Cartan subalgebra $\mathfrak{h}_{\mathfrak{a}}$ is equal to Σ and such that $\hat{\mathfrak{g}} = \bigoplus_{\lambda \in \Sigma \cup \{0\}} V_{\lambda}$ where $V_{\lambda} = \{x \in \hat{\mathfrak{g}}, [h, x] = \lambda(h)x \text{ for all } h \in \mathfrak{h}_{\mathfrak{a}}\}$.

The Lie algebra \mathfrak{a} is then called the grading subalgebra. This is slightly modification of the notion of grading by a finite root system which was introduced by Bermann and Moody ([BM]) and developed by Benkart-Zelmanov ([BZ]), Neher ([Ne]), Allison-Benkart-Gao ([ABG]).

In a preceding paper ([N]) we gave the complete classification of all simple finite dimensional Lie algebras graded by a finite root system.

Let us give now a description of the content of this paper.

In part 1 we construct a large class of affine subalgebras of a given affine algebra $\hat{\mathfrak{g}}$, the so-called admissible subalgebras. In the finite dimensional case, the admissible subalgebras were constructed by H. Rubenthaler ([Ru2]) and they played a crucial role in the classification of dual pairs in reductive Lie algebras ([Ru3]). These algebras are also important for the classification of the simple finite dimensional Lie algebras graded by finite root systems ([N]).

The admissible subalgebras are parametrized by some “nice” subsets Γ of a set Π of simple roots of the root system $\Delta = (\hat{\mathfrak{g}}, \mathfrak{h})$ where \mathfrak{h} is some cartan subalgebra of $\hat{\mathfrak{g}}$ (Definition 1.4.1.1.). Such “nice” pairs (Π, Γ) are also called admissible.

Roughly speaking if (Π, Γ) is admissible then it is possible to associate to Γ a set of sl_2 -triples (X_i, H_i, Y_i) $i = 0, 1, \dots, k$ ($k = |\Pi \setminus \Gamma|$). These sl_2 -triples arise classically from the regularity of some prehomogeneous vector spaces (see 1.1.) related to \mathbb{Z} -gradings of certain finite dimensional subalgebras of $\hat{\mathfrak{g}}$ (see 1.3.). If (Π, Γ) is admissible then the related sl_2 -triples (X_i, H_i, Y_i) ($i = 0, 1, \dots, k$) and the space $\mathfrak{h}_{\Gamma} = \{h \in \mathfrak{h}, \alpha(h) = 0 \text{ for all } \alpha \in \Gamma\}$ generate the admissible subalgebra $\hat{\mathfrak{g}}_{\Gamma}$ (Proposition 1.4.2.1.), which is affine. It is worthwhile to note that the pair (Π, \emptyset) is admissible, in that case the corresponding sl_2 -triples are the usual generators of $\hat{\mathfrak{g}}$, and therefore $\hat{\mathfrak{g}}_{\emptyset} = \hat{\mathfrak{g}}$.

From the definition, the classification of the admissible subalgebras relies on a Dynkin diagram combinatoric. This classification is given in Table A.

We also describe an important subfamily of the set of admissible subalgebras, the so-called C-admissible subalgebras. The root system of a C-admissible subalgebra $\hat{\mathfrak{g}}_{\Gamma}$ is studied

in section 1.5. In this section we prove also that $\hat{\mathfrak{g}}_\Gamma$ is an integrable $\hat{\mathfrak{g}}_\Gamma$ -module.

In part 2 we prove (section 2.2.) that any \mathbb{C} -admissible subalgebra $\hat{\mathfrak{g}}_\Gamma$ defines a grading of $\hat{\mathfrak{g}}$ relatively to its root system Δ_Γ (Proposition 2.2.1.). This gives a large family of gradings of $\hat{\mathfrak{g}}$.

In section 2.4. we prove that the grading defined by \mathbb{C} -admissible subalgebras can be characterized by the irreducibility of the representations (V_0, V_λ) for every simple root $\lambda \in \Delta_\Gamma$ (Theorem 2.4.3.).

After this, we study the converse problem and start with an arbitrary grading of $\hat{\mathfrak{g}}$ by an affine root system Σ .

In section 2.3. and 2.5. we associate to such a data a \mathbb{C} -admissible pair (Π, Γ) such that the corresponding \mathbb{C} -admissible subalgebra $\hat{\mathfrak{g}}_\Gamma$ is again graded by Σ . (Propositions 2.3.4., 2.3.7. and Theorem 2.5.5.). Moreover the grading of $\hat{\mathfrak{g}}_\Gamma$ by Σ is “maximal” in the sense of Definition 2.5.7. and we show that the classification of all gradings of $\hat{\mathfrak{g}}$ reduces to the classification of all maximal Σ -gradings of \mathbb{C} -admissible subalgebras (Theorem 2.5.10.).

In section 2.6. we first prove a technical lemma (Lemma 2.6.1.) which allows us, after some computations, to achieve the classification of the maximal gradations. The list of these maximal gradations is given in Table B.

If one takes the underlying finite dimensional subalgebra $\hat{\mathfrak{a}}$ of the grading subalgebra \mathfrak{a} then $\hat{\mathfrak{a}}$ gives rise to a finite grading (in the original sense of [BM]) of the algebra $\hat{\mathfrak{g}}$. We will study more precisely this aspect in a forthcoming paper. In another paper we will investigate the relationship between the grading subalgebras of $\hat{\mathfrak{g}}$ (especially the \mathbb{C} -admissible ones) and the dual pairs in $\hat{\mathfrak{g}}$ (in the spirit of [Ru3]).

1. \mathbb{C} -admissible subalgebras of affine Kac-Moody algebras

1.1. Prehomogeneous vector spaces of parabolic type

Let us recall here some basic facts from the theory of parabolic prehomogeneous vector spaces (details can be found in [S-K] and [Ru1]). If G is an algebraic group over \mathbb{C} and (ρ, V) a rational representation of G in a finite dimensional vector space, the triple (G, ρ, V) or briefly (G, V) , is called a prehomogeneous vector space if G has a Zariski open orbit in V . This is, of course, an infinitesimal condition; more precisely if \mathfrak{g} is the Lie algebra of G and if $(\mathfrak{g}, d\rho, V)$ is the corresponding derived representation of \mathfrak{g} , the condition of prehomogeneity is equivalent to the existence of an element $v_0 \in V$ such that the map

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & V \\ X & \longmapsto & d\rho(X)v_0 \end{array}$$

is surjective.

For that reason, a finite dimensional representation (\mathfrak{g}, V) verifying the above mentioned surjectivity condition will also be called a prehomogeneous vector space. A prehomogeneous

vector space (G, V) , where the group G is reductive is called regular if the isotropy subgroup of the open orbit is reductive. This is known to be equivalent to the fact that the complementary set of the open orbit is an hypersurface.

Let now $\mathfrak{g} = \bigoplus \mathfrak{g}_k$ be a \mathbb{Z} -gradation of a finite dimensional reductive Lie algebra. By a result of Vinberg ([Vi]), the representation $(\mathfrak{g}_0, \mathfrak{g}_i)$ ($i \neq 0$), is always a prehomogeneous vector space.

Let now \mathfrak{g} be a simple Lie algebra. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} and let R be the root system of the pair $(\mathfrak{g}, \mathfrak{h})$. We choose a basis Ψ of R and denote by θ a subset of Ψ . Let H_θ the unique element defined by the equations :

$$\begin{cases} \alpha(H_\theta) = 0 & \forall \alpha \in \theta \\ \alpha(H_\theta) = 2 & \forall \alpha \in \Psi \setminus \theta \end{cases}$$

Let $d_p(\theta) = \{X \in \mathfrak{g}, [H_\theta, X] = 2pX\}$. Then $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} d_p(\theta)$ is a \mathbb{Z} -gradation of \mathfrak{g} and the prehomogeneous vector spaces $(d_0(\theta), d_p(\theta))$ are said to be of parabolic type. Moreover, the representation $(d_0(\theta), d_1(\theta))$ is irreducible if and only if $|\Psi \setminus \theta| = 1$ (see for example [Ru1]). The following theorem will be used later.

Theorem 1.1.1. ([Ru1]) *Suppose that the prehomogeneous vector space of parabolic type $(d_0(\theta), d_1(\theta))$ is irreducible. Then it is regular if and only if there exist $Y \in d_{-1}(\theta)$ and $X \in d_1(\theta)$ such that (X, H_θ, Y) is a sl_2 -triple.*

If $[d_1(\theta), d_1(\theta)] = \{0\}$ then we will say that the prehomogeneous vector space of parabolic type is commutative (and this is equivalent to say that $d_p(\theta) = \{0\}$ for any $p \geq 2$).

The regular prehomogeneous vector spaces of parabolic type have been classified by H. Rubenthaler ([Ru1] table 2 p. 42).

1.2. Affine algebras

Let us first recall some facts about affine algebras and fix some notations (see [K] for details).

An affine generalized Cartan matrix $A = (a_{ij})_{i,j=0,\dots,k}$ is an indecomposable matrix of integers satisfying $a_{ii} = 2, a_{ij} \leq 0$ if $i \neq j$, $a_{ij} = 0$ implies $a_{ji} = 0$. Furthermore all the proper principal minors of A are positive and $\det A = 0$.

A realization of the matrix A is a triple $(\mathfrak{h}, \Pi, \check{\Pi})$, unique up to isomorphism, where \mathfrak{h} is a complex vector space, $\Pi = \{\alpha_0, \dots, \alpha_n\} \subset \mathfrak{h}^*$ and $\check{\Pi} = \{\check{\alpha}_0, \dots, \check{\alpha}_n\} \subset \mathfrak{h}$ satisfying the conditions :

- both sets Π and $\check{\Pi}$ are linearly independent
- $\langle \alpha_j, \check{\alpha}_i \rangle = a_{ij}$ ($i, j = 0, \dots, n$)
- $\dim \mathfrak{h} = n + 2$.

The affine Kac-Moody algebra $\hat{\mathfrak{g}}$ is the Lie algebra over \mathbb{C} generated by \mathfrak{h} and the symbols e_i, f_i with defining relations : $[\mathfrak{h}, \mathfrak{h}] = 0, [e_i, f_j] = \delta_{ij}\check{\alpha}_i, [h, e_i] = \alpha_i(h)e_i, [h, f_i] = -\alpha_i(h)f_i$, (for all i, j, h), $(ade_i)^{1-a_{ij}}(e_j) = 0 = (adf_i)^{1-a_{ij}}(f_j)$ ($i \neq j$). The last two relations are called Serre's relations and the quadruple $(\hat{\mathfrak{g}}, \mathfrak{h}, \Pi, \check{\Pi})$ is called the quadruple associated to the matrix A .

We have the canonical embedding $\mathfrak{h} \subset \hat{\mathfrak{g}}$ and linear independent Chevalley generators for the derived algebra \mathfrak{g}' of \mathfrak{g} .

The 1-dimensional center $\mathcal{Z}(\hat{\mathfrak{g}})$ of $\hat{\mathfrak{g}}$ lies in $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}' = \sum_{i=0}^n \mathbb{C}\alpha_i^\check$ and consists of the scalar multiples of the (canonical) central element $K = \sum_{i=0}^n a_i \alpha_i^\check$ such that ${}^t A({}^t(a_0, \dots, a_n)) = 0$ and the a_i^\check are positive relatively prime integers. Every non-zero ideal which is different of $\mathbb{C}K$ contains \mathfrak{g}' ([K] Proposition 1.7. b)).

Fix an element $d \in \mathfrak{h}$ defined modulo K by the following conditions : $\langle \alpha_i, d \rangle = 0$ for $i = 1, \dots, n$, $\langle \alpha_0, d \rangle = 1$. The element d is called the scaling element and we have $\mathfrak{h} = \mathfrak{h}' \oplus \mathbb{C}d$.

The root space decomposition is given by $\hat{\mathfrak{g}} = \bigoplus_{\alpha \in \mathfrak{h}^*} \hat{\mathfrak{g}}^\alpha$ where

$$\hat{\mathfrak{g}}^\alpha = \{x \in \hat{\mathfrak{g}}, [h, x] = \alpha(h)x, \text{ for all } h \in \mathfrak{h}\}$$

Put $Q = \sum_{i=0}^n \mathbb{Z}\alpha_i$ (resp. $Q_+ = \sum_{i=0}^n \mathbb{Z}_+\alpha_i$). The set $\Delta \subset Q$ of roots (resp. the set Δ_+ of positive roots) is defined by $\Delta = \{\alpha \in \mathfrak{h}^*, \alpha \neq 0, \hat{\mathfrak{g}}^\alpha \neq \{0\}\}$ (resp. $\Delta_+ = \Delta \cap Q_+$).

The Weyl group W is the subgroup of $\text{Aut}(\mathfrak{h}^*)$ generated by the fundamental reflections s_{α_j} ($\alpha_j \in \Pi$) defined by $s_{\alpha_j}(\lambda) = \lambda - \langle \lambda, \alpha_j^\check \rangle \alpha_j$, for all $\lambda \in \mathfrak{h}^*$.

The Weyl group W preserves the root system Δ and a real root (resp. an imaginary root) is an element of $\Delta^{Re} = W(\Pi)$ (resp. $\Delta^{Im} = \Delta \setminus \Delta^{Re}$). Let us consider $\alpha \in \Delta^{Re}$ such that $\alpha = w(\alpha_i)$ for some $\alpha_i \in \Pi$ and $w \in W$. Let us put $\check{\alpha} = w(\alpha_i^\check)$. One has $\langle \alpha, \check{\alpha} \rangle = 2$ and there exists $X_\alpha \in \hat{\mathfrak{g}}^\alpha$ and $Y_\alpha \in \hat{\mathfrak{g}}^{-\alpha}$ such that $(X_\alpha, \check{\alpha}, Y_\alpha)$ is a sl_2 -triple. For a real root α , we also have $\dim \hat{\mathfrak{g}}^\alpha = 1$ and $\Delta \cap \mathbb{Z}\alpha = \{\alpha, -\alpha\}$.

Let us denote by $\delta = \sum_{i=0}^n a_i \alpha_i$ the unique element of \mathfrak{h}^* such that $A({}^t(a_0, \dots, a_n)) = 0$ and the a_i are positive relatively prime integers; then $\Delta^{Im} = \{j\delta, j \in \mathbb{Z}^*\}$. The imaginary root spaces are finite dimensional too ([K] Corollary 8.3.).

Denote by $\mathring{\mathfrak{g}}$ the underlying simple subalgebra of $\hat{\mathfrak{g}}$ generated by the triples $(e_i, \alpha_i^\check, f_i)$, $i = 1, \dots, n$ and by $\mathring{\mathfrak{h}} = \sum_{i=1}^n \mathbb{C}\alpha_i^\check$ the Cartan subalgebra of $\mathring{\mathfrak{g}}$. One has $\mathfrak{h} = \mathring{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d$ and the root system $\Delta = (\mathring{\mathfrak{g}}, \mathring{\mathfrak{h}})$ can be completely described in terms of δ and the root system $\mathring{\Delta}$ of the pair $(\mathring{\mathfrak{g}}, \mathring{\mathfrak{h}})$ ([K] Proposition 6.3.). The set $\mathring{\Pi} = \{\alpha_1, \dots, \alpha_n\}$ is a basis of $\mathring{\Delta}$. The Cartan matrix of finite type \mathring{A} of $\mathring{\Pi}$ is obtained from the matrix A by deleting the 0^{th} row and column. The Dynkin diagram $S(\mathring{A})$ is obtained by deleting the 0^{th} vertex on the Dynkin diagram, $S(A)$, of the matrix A .

1.3. Construction of a \mathbb{Z} -gradation of $\hat{\mathfrak{g}}$

Let Γ be a proper subset (possibly empty) of the root basis Π . We set

$$\mathfrak{h}_\Gamma = \{h \in \mathfrak{h}, \alpha(h) = 0 \ \forall \alpha \in \Gamma\}$$

Remark 1.3.1. The 1-dimensional center of the Lie algebra $\hat{\mathfrak{g}}$ defined by

$$\mathcal{Z}(\hat{\mathfrak{g}}) = \mathbb{C}K = \{h \in \mathfrak{h}, \alpha_i(h) = 0 \ \forall i = 0, \dots, n\}$$

lies in \mathfrak{h}_Γ .

Remark 1.3.2. Since the roots in Γ are linearly independent we have

$$\dim \mathfrak{h}_\Gamma = \dim \mathfrak{h} - \text{card}\Gamma$$

Proposition 1.3.3.

- If $\alpha_0 \notin \Gamma$ (i.e. $\Gamma \subset \overset{\circ}{\Pi}$) then $\mathfrak{h}_\Gamma = \overset{\circ}{\mathfrak{h}}_\Gamma \oplus \mathbb{C}K \oplus \mathbb{C}d$ where $\overset{\circ}{\mathfrak{h}}_\Gamma = \{h \in \overset{\circ}{\mathfrak{h}}, \alpha(h) = 0 \ \forall \alpha \in \Gamma\}$
- If $\alpha_0 \in \Gamma$ then $\mathfrak{h}_\Gamma = \{h - \alpha_0(h)d + \lambda K, \lambda \in \mathbb{C}, h \in \tilde{\mathfrak{h}}_\Gamma\}$ where $\tilde{\mathfrak{h}}_\Gamma = \{h \in \overset{\circ}{\mathfrak{h}}, \alpha(h) = 0 \ \forall \alpha \in \Gamma \setminus \{\alpha_0\}\}$

Proof : An element $H \in \mathfrak{h}$ can be written $H = h + \lambda K + \mu d$ ($h \in \overset{\circ}{\mathfrak{h}}, \lambda, \mu \in \mathbb{C}$)

- If $\alpha_0 \notin \Gamma$, one has

$$h \in \mathfrak{h}_\Gamma \iff \alpha(h) = \alpha(H) = 0 \ \forall \alpha \in \Gamma$$

- If $\alpha_0 \in \Gamma$, one has

$$h \in \mathfrak{h}_\Gamma \iff \begin{cases} \alpha(h) = \alpha(H) = 0 & \forall \alpha \in \Gamma \setminus \{\alpha_0\} \\ \mu = -\alpha_0(H) \end{cases}$$

□

For an element $\alpha = \sum_{\alpha_i \in \Pi} n_{\alpha_i} \alpha_i$ in the root lattice Q , **the integer** $ht_\Gamma(\alpha) = \sum_{\alpha_i \in \Pi \setminus \Gamma} n_{\alpha_i}$ **is called the Γ -height of α .**

$$\text{For any integer } p \text{ we set } \begin{cases} \Omega_p = \{\alpha \in \Delta \cup \{0\}, ht_\Gamma(\alpha) = p\} \\ d_p = \bigoplus_{\alpha \in \Omega_p} \hat{\mathfrak{g}}^\alpha & \text{if } \Omega_p \neq \emptyset \\ d_p = \{0\} & \text{if } \Omega_p = \emptyset \end{cases}$$

Remark 1.3.4.

- 1) Since $ht_\Gamma(\delta) \geq 1$ we have $\Omega_0 \cap \Delta^{Im} = \emptyset$
- 2) From the fact that for any integer j one has $ht_\Gamma(j\delta) = jht_\Gamma(\delta)$, we deduce that $|\Omega_p \cap \Delta^{Im}| \leq 1$ for any integer p .
- 3) The subalgebra d_0 is equal to $\mathfrak{h} \oplus \hat{\mathfrak{g}}^{\langle \Gamma \rangle}$ where $\langle \Gamma \rangle$ is the set of roots which are a linear combination of elements of Γ . Since $\Gamma \neq \Pi$, we have $\langle \Gamma \rangle \subset \Delta^{Re}$.

Proposition 1.3.5. For any integer p , $\dim d_p < \infty$.

Proof : Using Remark 1.2.4. 2) and since $\dim \hat{\mathfrak{g}}^\alpha < \infty$ for any $\alpha \in \Delta \cup \{0\}$, it is sufficient to prove that $|\Omega_p \cap \Delta^{Re}| < \infty$. Let α be a real root in Ω_p . Suppose that $\hat{\mathfrak{g}}$ is not of type $A_{2l}^{(2)}$, then $\alpha = j\delta + \gamma, j \in \mathbb{Z}, \gamma \in \overset{\circ}{\Delta}$ ([K] Proposition 6.3.). Hence we have $ht_\Gamma(\alpha) = jht_\Gamma(\delta) + ht_\Gamma(\gamma) = p$. Since the set $\{ht_\Gamma(\gamma), \gamma \in \overset{\circ}{\Delta}\}$ is finite, the result follows.

If $\hat{\mathfrak{g}}$ is of type $A_{2l}^{(2)}$ one can also have $\alpha = \frac{1}{2}(2j-1)\delta + \gamma$, $j \in \mathbb{Z}$, $\gamma \in \overset{\circ}{\Delta}$ and the proof is similar.

□

The proof of the following two propositions is straightforward.

Proposition 1.3.6. *The decomposition $\hat{\mathfrak{g}} = \bigoplus_{p \in \mathbb{Z}} d_p$ is a \mathbb{Z} -gradation of $\hat{\mathfrak{g}}$.*

Proposition 1.3.7. *The finite dimensional subalgebra d_0 is reductive and its center is equal to \mathfrak{h}_Γ . The Dynkin diagram of the semisimple algebra $d'_0 = [d_0, d_0]$ is obtained from the Dynkin diagram of $\hat{\mathfrak{g}}$ by removing the vertices of $\Pi \setminus \Gamma$.*

Let us assume from now on that $|\Pi \setminus \Gamma| \geq 2$.

In this section, we will decompose the d_0 -module d_1 (under the adjoint action) into a direct sum of irreducible d_0 -modules.

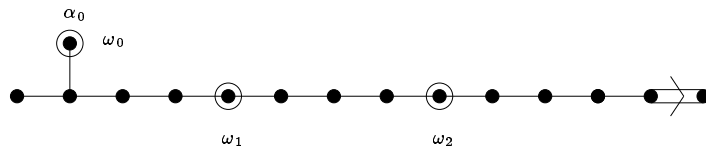
Let us make the convention which consists to circle the vertices $\omega_0, \dots, \omega_k$ of $\Pi \setminus \Gamma$ in the Dynkin diagram $S(A)$. We denote by Π_j the connected component of $\Gamma \cup \{\omega_j\}$ which contains ω_j . As $|\Pi \setminus \Gamma| \geq 2$ the set Π_j can be identified with a proper subdiagram of $S(A)$, hence it is of finite type. Denote by $\Delta_j = \langle \Pi_j \rangle$ the set of roots in Δ which are linear combinations of elements in Π_j . Let \mathfrak{g}_j be the associated simple algebra which can be viewed as a subalgebra of $\hat{\mathfrak{g}}$. The algebra $\mathfrak{h}_j = \sum_{\alpha \in \Pi_j} \mathbb{C}\alpha$ is a Cartan subalgebra of \mathfrak{g}_j , and Δ_j

is the set of roots of the pair $(\mathfrak{g}_j, \mathfrak{h}_j)$.

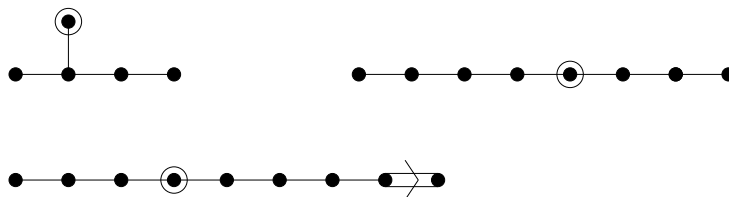
The subset $\Gamma_j = \Pi_j \setminus \{\omega_j\}$ of Π_j defines again a \mathbb{Z} -gradation of the finite dimensional algebra \mathfrak{g}_j as seen in section 1.1. To be more precise, let H_j be the unique element in \mathfrak{h}_j satisfying the equations :
$$\begin{cases} \alpha(H_j) = 0 & \forall \alpha \in \Gamma_j \\ \omega_j(H_j) = 2 \end{cases}$$

The spaces of this gradation are then defined by $d_p^j = \{x \in \mathfrak{g}_j, [H_j, x] = 2px\}$ and the representations (d_0^j, d_p^j) are known to be irreducible (see section 1.1.).

Let us illustrate the construction of the Π_j 's by the following example :



The algebra $\hat{\mathfrak{g}}$ is of type $B_{14}^{(1)}$. The connected components Π_j are of type D_5 , A_8 and B_9 respectively.



Proposition 1.3.8.

- We have $d_1 = \bigoplus_{j=0}^k d_1^j$
- For any j , the space d_1^j is an irreducible d_0 -module for the adjoint action.

Proof : Recall that $d_1 = \bigoplus_{\alpha \in \Omega_1} \hat{\mathfrak{g}}^\alpha$ where $\Omega_1 = \{\alpha \in \Delta, ht_\Gamma(\alpha) = 1\}$. We also have $d_1^j = \bigoplus_{\alpha \in \Omega_1^j} \hat{\mathfrak{g}}^\alpha$ where $\Omega_1^j = \{\alpha \in \Delta_j, \alpha = \omega_j \pmod{\Gamma_j}\}$. Since the support of a root is a connected subgraph of $S(A)$ one has $\Omega_1^j = \Omega_1 \cap \Delta_j$ and Ω_1 is the union of the disjoint sets Ω_1^j . Therefore $d_1 = \bigoplus_{j=0}^k d_1^j$. The space d_1^j is an irreducible d_0^j -module which is d_0 -stable. As $d_0^j \subset d_0$, the space d_1^j is also an irreducible d_0 -module. □

1.4. The admissible subalgebras

1.4.1. A construction of generalized Cartan matrices

The key point of our construction is the following choice of the subset Γ (or $\Pi \setminus \Gamma$). Using the same notations as in the preceding sections, we will choose the simple roots $\omega_0, \dots, \omega_k$ in $\Pi \setminus \Gamma$ such that (d_0^j, d_1^j) is regular (respectively regular and commutative) for any $j = 0, \dots, k$. This leads to the following definition.

Definition 1.4.1.1. *The pair (Π, Γ) is called admissible (respectively C-admissible) if (d_0^j, d_1^j) is regular (respectively regular and commutative) for any $j = 0, \dots, k$.*

The classification of admissible (resp. C-admissible) pairs (Π, Γ) is now easy. From above, such pairs are in one to one correspondence with weighted Dynkin diagrams such that each connected subdiagram with only one circled root (by the procedure explained in section 1.1.) lies in the table of regular (resp. regular commutative) irreducible parabolic prehomogeneous spaces (see the left column of Table A).

We assume now that (Π, Γ) is admissible.

Proposition 1.4.1.2.

- (i) For any $j = 0, \dots, k$, we have $H_j = \sum_{\alpha \in \Pi_j} x_\alpha^j \check{\alpha}$ where each x_α^j is a positive integer.
- (ii) The elements H_0, \dots, H_k are linearly independent in \mathfrak{h}_Γ .

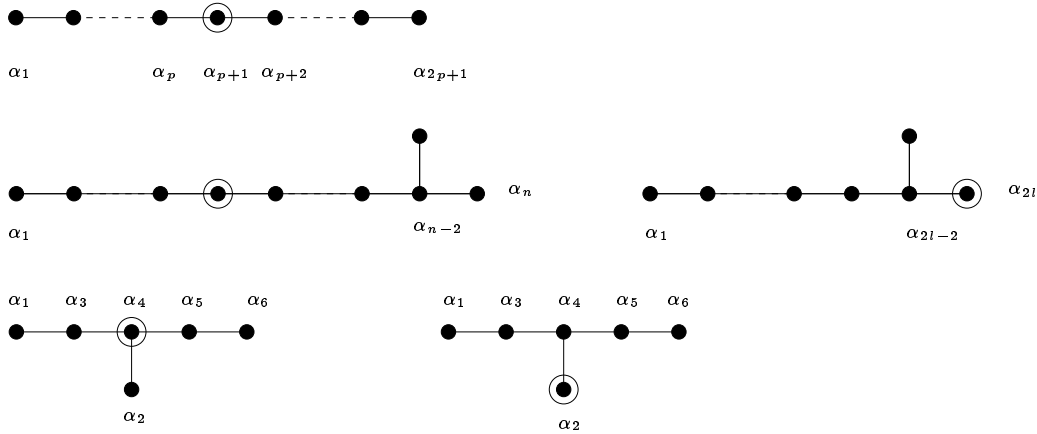
Proof : Let us fix j in $\{0, \dots, k\}$; let M_j denote the Cartan matrix of finite type associated to the quadruple $(\mathfrak{g}_j, \mathfrak{h}_j, \Pi_j, \Pi_j^\vee)$. Let us set $H_j = \sum_{\alpha \in \Pi_j} x_\alpha^j \check{\alpha}$. Since H_j verifies

$$\begin{cases} \alpha(H_j) &= 0 \quad \forall \alpha \in \Gamma_j \\ \omega_j(H_j) &= 2 \end{cases}$$

the column $X = {}^t((x_\alpha^j)_{\alpha \in \Pi_j})$ is the solution of the linear system ${}^tM_j X = B$ where B is a column whose coefficients are all zero except one (say the q^{th}) which is equal to 2. Using [K] Theorem 4.3. we deduce that $X > 0$ (i.e. $x_\alpha^j > 0 \forall \alpha \in \Pi_j$) and we have $X = ({}^tM_j)^{-1}B$, more precisely $X = 2\Lambda_q$ where Λ_q is the q^{th} column of the matrix $({}^tM_j)^{-1}$. These matrices $({}^tM_j)^{-1}$ are well known (they give the fundamental weights in terms of the simple roots and we find these relations in tables in [Bou 6]). It is easy to check that for every regular space (d_0^j, d_1^j) we have the expected result, namely that each x_α^j is a positive integer. More precisely :

- for the type B, C, E_7, E_8, F_4, G_2 , the coefficients of the matrix $2({}^tM_j)^{-1}$ are always integers.
- for the type A , the space (d_0^j, d_1^j) is regular if \mathfrak{g}_j is of type A_{2p+1} and ω_j is the central simple root in the Dynkin diagram (i.e. $q = p + 1$). Hence we have $X = 2\Lambda_{p+1} = {}^t(1, 2, \dots, p, p + 1, p, \dots, 2, 1)$.
- for the type D_n :
 - ◇ if ω_j is different from both α_n and α_{n-1} then $2\Lambda_q$ is integer.
 - ◇ if $\omega_j = \alpha_n$ (resp. $\omega_j = \alpha_{n-1}$) then n is even and $2\Lambda_n$ (resp. $2\Lambda_{n-1}$) is an integer.
- for the type E_6 it is sufficient to check that $2\Lambda_2$ and $2\Lambda_4$ are integers.

It is convenient to resume the cases A, D, E_6 by the following diagrams.



Let us prove now that $H_j = \sum_{\alpha \in \Pi_j} x_\alpha^j \alpha$ lies in \mathfrak{h}_Γ . Since for any $\gamma \in \Gamma \setminus \Gamma_j$ and any $\alpha \in \Pi_j$ one has $(\gamma, \alpha) = 0$ then $\gamma(H_j) = \sum_{\alpha \in \Pi_j} x_\alpha^j \gamma(\alpha) = 0$. For any $\gamma \in \Gamma_j$, one also has $\alpha(H_j) = 0$ by definition of the element H_j . Therefore $H_j \in \mathfrak{h}_\Gamma$.

Let us set now $H = \sum_{j=0}^k a_j H_j = 0$ for some (a_0, \dots, a_k) . For any j , the coefficient of ω_j in H is $a_j x_{\omega_j}^j$ (where $x_{\omega_j}^j$ is the coefficient of ω_j in H_j). Since the elements of Π are linearly independent we have $a_j x_{\omega_j}^j = 0$. Since $x_{\omega_j}^j > 0$ we have $a_j = 0$ for any j , and therefore the elements H_0, \dots, H_k are linearly independent. □

Remark 1.4.1.3. As $\dim \mathfrak{h}_\Gamma = \dim \mathfrak{h} - |\Gamma| = k + 2$, the set $\{H_0, \dots, H_k\}$ is not a basis of \mathfrak{h}_Γ .

Proposition 1.4.1.4. *The matrix $M = (\omega_j(H_i))_{ij}$ is a generalized Cartan matrix.*

Proof : For any (i, j) , put $m_{ij} = \omega_j(H_i)$. Then for all j we have $m_{jj} = \omega_j(H_j) = 2$. If $i \neq j$, we have

$$m_{ij} = \omega_j\left(\sum_{\gamma \in \Pi_i} x_\gamma^i \gamma\right) = \sum_{\gamma \in \Pi_i} x_\gamma^i \omega_j(\gamma) \quad (\star)$$

On the other hand, the constants in the preceding sum are positive integers (Proposition 1.4.1.2.) and the constants $\langle \omega_j, \gamma \rangle$ are nonpositive integers (since $\omega_j \notin \Pi_i$). Therefore the m_{ij} are nonpositive integers.

Suppose now that $m_{ij} = 0$ for $i \neq j$. For all $\gamma \in \Pi_i$, one has $\langle \omega_j, \gamma \rangle \leq 0$. As $x_\gamma^i > 0$ for all $\gamma \in \Pi_i$, the equality (\star) above implies that $\langle \omega_j, \gamma \rangle = 0$ for all $\gamma \in \Pi_i$. Then $\Pi_j \cap \Pi_i = \emptyset$ and furthermore $(\omega_i, \omega_j) = 0$. Hence $\langle \omega_i, \gamma \rangle = 0$ for all $\gamma \in \Pi_j$, therefore $m_{ji} = 0$. □

Theorem 1.4.1.5. *The generalized Cartan matrix M is of affine type.*

We will need the following lemma.

Lemma 1.4.1.6. *The set $\{H_0, \dots, H_k\}$ is a basis of the subspace $\mathfrak{h}' \cap \mathfrak{h}_\Gamma$.*

Proof : We remark first that $H_j = [X_j, Y_j]$ is an element of \mathfrak{h}' for any j . Using Remark 1.4.1.3., it is sufficient to prove that the space $\mathfrak{h}' \cap \mathfrak{h}_\Gamma$ is a proper subspace of \mathfrak{h}_Γ .

- If $\alpha_0 \notin \Gamma$ then $d \in \mathfrak{h}_\Gamma$ (Proposition 1.3.3.) and (d, H_0, \dots, H_k) is a basis of \mathfrak{h}_Γ , hence

$$\mathfrak{h}' \cap \mathfrak{h}_\Gamma = \sum_{j=0}^k \mathbb{C}H_j.$$

Observe that if $S(A)$ is of (circular) type $A_n^{(1)}$, we can always reorder the indices of the simple roots such that $\alpha_0 \notin \Gamma$.

- If $\alpha_0 \in \Gamma$ then, **we can assume that $S(A)$ is not of type $A_n^{(1)}$** , and we know that \mathfrak{h}_Γ is the set of the elements H in form of $H = h - \alpha_0(h)d + \lambda K$ where $\lambda \in \mathbb{C}$ and h varies in the orthogonal of $\Gamma \setminus \{\alpha_0\}$ in \mathfrak{h} (Proposition 1.3.3.). Let us show now that there exists an element in \mathfrak{h}_Γ which is not in \mathfrak{h}' .

Among the connected components defined in $S(A)$ as described in section 1.1., choose one that contains α_0 and denote it by Π_0 . Since $\alpha_0 \in \Gamma$, Π_0 contains more than one simple root and we will call ω_0 the unique vertex of $\Pi \setminus \Gamma$ which is in Π_0 . Put now $\Pi_0 = \{\alpha_0, \dots, \alpha_l = \omega_0, \dots, \alpha_p\}$. The simple algebra \mathfrak{g}_0 associated to Π_0 (and to the corresponding Cartan matrix denoted by M_0) contains the simple subalgebra $\tilde{\mathfrak{g}}_0$ corresponding to the connected set $\Pi_0 \setminus \{\alpha_0\}$. The invertible Cartan matrix \tilde{M}_0 associated to $\tilde{\mathfrak{g}}_0$ is obtained from M_0 by deleting the row and column corresponding to α_0 .

Let h be an element of the Cartan subalgebra of $\tilde{\mathfrak{g}}_0$ such that $h = x_1\alpha_1^\vee + \dots + x_p\alpha_p^\vee$. Set

$X = {}^t(x_1, \dots, x_p)$. The system $\begin{cases} \alpha_j(h) = 0 & \text{for all } j \neq l \\ \omega_0(h) = 1 \end{cases}$ is equivalent to the equation

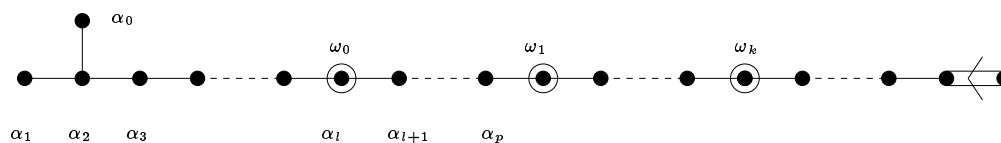
$\tilde{M}_0 X = {}^t(0, \dots, 0, 1, 0, \dots, 0)$. Since \tilde{M}_0 is of finite type, X is unique and $X > 0$ ([K] Theorem 4.3.).

This element h is obviously in the orthogonal of $\Gamma \setminus \{\alpha_0\}$ in \mathfrak{h} . Since the vertex α_0 is connected to only one vertex in $\Pi_0 \setminus \{\alpha_0\}$, one deduces that $\alpha_0(h) = \sum_{j=1}^p x_j \alpha_0(\alpha_j^\vee) < 0$.

Hence $h - \alpha_0(h)d$ is an element of \mathfrak{h}_Γ that does not lie in \mathfrak{h}' and $\mathfrak{h}_\Gamma \cap \mathfrak{h}'$ is a proper subspace of \mathfrak{h}_Γ . Then $\mathfrak{h}_\Gamma \cap \mathfrak{h}' = \sum_{j=0}^k \mathbb{C}H_j$.

□

We can illustrate this by the following example :



Let us prove now Theorem 1.4.1.5.

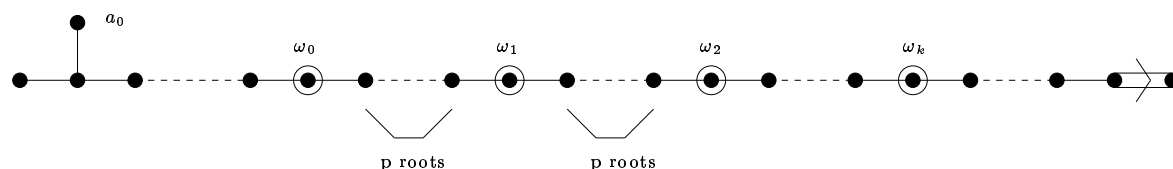
- The matrix M is not invertible. In fact, the non-zero canonical central element K which is obviously in $\mathfrak{h}' \cap \mathfrak{h}_\Gamma$, can be written $K = x_0H_0 + \dots + x_nH_n$ (Lemma 1.4.1.6.) and one has $M^t(x_0, \dots, x_n) = 0$.
 - The matrix M is indecomposable. In fact, recall that for any $j = 0, \dots, k$ the coefficients of the element H_j in the basis Π_j^\checkmark are positive (Proposition 1.4.1.2.). As $S(A)$ is a connected graph, one deduces that it is impossible to decompose the set $\{0, \dots, k\}$ into $I_1 \cup I_2$ with $\omega_j(H_i) = 0$ for all $j \in I_1$ and $i \in I_2$.
 - Let M' be a proper principal submatrix of M associated to a proper subset I' of $\{0, \dots, k\}$. Then M' corresponds to the C-admissible diagram of semisimple type constituted by the union of Π_j for $j \in I'$. Using [Ru2] Theorem 3.1., we deduce that $\det M' > 0$.
- We conclude by applying [K] Proposition 4.7. b).

□

We give in terms of Dynkin diagrams the generalized Cartan matrices associated to admissible (resp. C- admissible) pairs (Π, Γ) (see the right column of Table A). Note that this classification allows to give the type of the admissible (resp. C- admissible) algebras that we define in the following section 1.4.2.

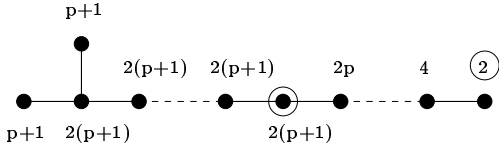
These Cartan matrices are obtained in the following way. We first calculate the coefficients of the elements H_j in the basis Π_j^\checkmark by procedure described in the proof of Proposition 1.4.1.2. Then it is easy to calculate the coefficients $m_{ij} = \omega_j(H_i)$ of the corresponding affine matrix M .

Let us illustrate by the following example. In an affine algebra $\hat{\mathfrak{g}}$ of type $B_n^{(1)}$, we consider the set Γ defined by the following diagram and such the pair (Π, Γ) is admissible.

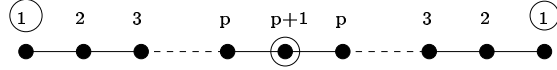


For any j , the numerical labels in the diagrams below are coefficients of H_j in the coroot basis Π_j^\checkmark .

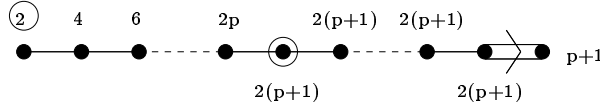
The connected component Π_0 is of type D_l .



The connected components Π_1, \dots, Π_{k-1} are of type A_{2p+1} .



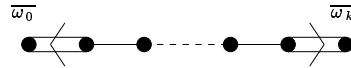
The connected component Π_k is of type B_l .



One remarks that only the circled coefficients are useful to calculate the $\omega_j(H_i)$'s. The Cartan matrix associated to the pair (Π, Γ) is then

$$M = \begin{pmatrix} 2 & -2 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & & \dots & -1 & 2 & -1 \\ 0 & & \dots & 0 & -2 & 2 \end{pmatrix}$$

and the Dynkin diagram is :



1.4.2. Construction of the admissible subalgebras

Let us consider the subalgebra $\hat{\mathfrak{g}}_\Gamma$ of $\hat{\mathfrak{g}}$, generated by \mathfrak{h}_Γ and the elements X_j, Y_j for $j = 0, \dots, k$. Put $\Psi = \{\bar{\omega}_0, \dots, \bar{\omega}_k\}$ where $\bar{\omega}_j$ is the restriction to \mathfrak{h}_Γ of the simple root ω_j and put $\Psi = \{H_0, \dots, H_k\}$

Proposition 1.4.2.1. *The quadruple $(\hat{\mathfrak{g}}_\Gamma, \mathfrak{h}_\Gamma, \Psi, \Psi)$ is the quadruple associated to the affine matrix M .*

Proof : The fact that the triple $(\mathfrak{h}_\Gamma, \Psi, \Psi)$ is a realization of M is straightforward. Furthermore one has clearly the relations :

- $[\mathfrak{h}_\Gamma, \mathfrak{h}_\Gamma] = \{0\}$
- As $[d_1^i, d_{-1}^j] = \{0\}$ for $i \neq j$, one has $[X_i, Y_j] = \delta_{ij}H_j$ for all i, j

· As $d_1^j = \bigoplus_{\alpha=\omega_j(\text{mod } \Gamma_j)} \hat{\mathfrak{g}}^\alpha$ (resp. $d_{-1}^j = \bigoplus_{\alpha=\omega_j(\text{mod } \Gamma_j)} \hat{\mathfrak{g}}^{-\alpha}$) one has $[h, X_j] = \overline{\omega_j}(h)X_j$ (resp. $[h, Y_j] = -\overline{\omega_j}(h)Y_j$ for all $h \in \mathfrak{h}_\Gamma$

Let us prove now that Serre's relations are verified.

Fix $i \in \{0, \dots, k\}$ and consider the sl_2 -triple (X_i, H_i, Y_i) .

- If $k \geq 1$ then for any $j \neq i$ the proper subset $\Pi_i \cup \Pi_j$ of $S(A)$ corresponds to a semisimple subalgebra $\tilde{\mathfrak{g}}$ of $\hat{\mathfrak{g}}$. This finite-dimensional subalgebra is a sl_2 -module under the action of (X_i, H_i, Y_i) . The element Y_j of $\tilde{\mathfrak{g}}$ satisfies the relations $[X_i, Y_j] = 0$, $[H_i, Y_j] = -\overline{\omega_j}(H_i)Y_j = -m_{ij}Y_j$. Hence Y_j generates an irreducible $(1 - m_{ij})$ -dimensional submodule with basis $\{v_m, v_m = (ad(Y_i))^m(Y_j)\}$ for $0 \leq m \leq 1 - m_{ij}$ and one has $(ad(Y_i))^{(1-m_{ij})}(Y_j) = 0$ ([Se]). The second relation, $(ad(X_i))^{(1-m_{ij})}(X_j) = 0$ ($i \neq j$), is proved the same way.

- If $k = 1$ then the argument does not hold because one has $\Pi_0 \cup \Pi_1 = S(A)$. Consider $\hat{\mathfrak{g}}$ as a sl_2 -module under the action of (X_0, H_0, Y_0) . Then Y_1 is a primitive element of weight $-m_{01}$. The vectors $v_m = \frac{1}{m!}(ad(Y_0))^m(Y_1)$, $m \geq 0$, are eigen vectors whose weights are $-m_{01} - 2m$ and one has the well known relations :

$$\begin{cases} ad(Y_0)(v_m) &= (m+1)v_{m+1} & (1) \\ ad(X_0)(v_m) &= (-m_{01} - m + 1)v_{m-1} & (2) \end{cases}$$

Since $Y_1 \in \bigoplus_{\beta} \hat{\mathfrak{g}}^{-\omega_1-\beta}$ and $Y_0 \in \bigoplus_{\beta} \hat{\mathfrak{g}}^{-\omega_0-\beta}$ where β varies in the set of positive linear combination of elements of Γ , we have $(ad(Y_0))^m(Y_1) \in \bigoplus_{\beta} \hat{\mathfrak{g}}^{-m\omega_0-\omega_1-\beta}$. For m great enough, $-m\omega_0 - \omega_1 - \beta$ is not a root, therefore $v_m = 0$. Put $N = \min\{m \geq 0, v_m = 0\}$. One has $N \geq 1$ and (1) implies that $v_m = 0$ for all $m \geq N$. Since $v_{N-1} \neq 0$, (2) implies that $-m_{01} - N + 1 = 0$. Hence $(ad(Y_0))^{1-m_{01}}(Y_1) = 0$. By similar arguments we prove the other Serre's relations and the theorem follows. □

Definition 1.4.2.2. The subalgebra $\hat{\mathfrak{g}}_\Gamma$ is called the admissible (resp. C-admissible) subalgebra associated to the admissible (resp. C-admissible) pair (Π, Γ) .

Denote by Δ_Γ the root system of the pair $(\hat{\mathfrak{g}}_\Gamma, \mathfrak{h}_\Gamma)$.

Remark 1.4.2.3. The set $\Psi = \{\overline{\omega_0}, \dots, \overline{\omega_k}\}$ is a root basis and $\Psi^\vee = \{H_0, \dots, H_k\}$ is the coroot basis.

Denote by δ_Γ the unique root of Δ_Γ such that $\delta_\Gamma = \sum_{j=0}^k b_j \overline{\omega_j}$ where the b_j are positive relatively prime integers satisfying $M^t(b_0, \dots, b_k) = 0$. One has $\Delta_\Gamma^{Im} = \{j\delta, j \in \mathbb{Z}^*\}$. We recall that $b_0 = 2$ if $\hat{\mathfrak{g}}_\Gamma$ is of type $A_{2l}^{(2)}$ and $b_0 = 1$ if not.

Remark 1.4.2.4. Since K is an element of \mathfrak{h}_Γ , the 1-dimensional center of $\hat{\mathfrak{g}}_\Gamma$ is $\mathcal{Z}(\hat{\mathfrak{g}}_\Gamma) = \mathbb{C}K$.

Remark 1.4.2.5. The subalgebra $\hat{\mathfrak{g}}_\Gamma$ depends upon the choice of the elements X_j, Y_j . Obviously all admissible subalgebras corresponding to the same subset Γ are isomorphic.

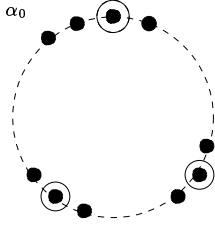
Remark 1.4.2.6. If the subset $\Gamma = \emptyset$ then one has $\mathfrak{h}_\Gamma = \mathfrak{h}$ and $\hat{\mathfrak{g}}_\Gamma = \hat{\mathfrak{g}}$.

1.5. The root system of the C-admissible subalgebras

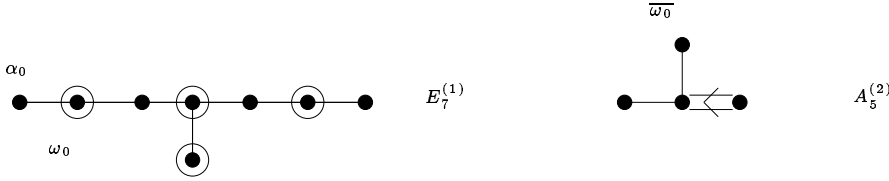
In this section we will assume that the pair (Π, Γ) is C-admissible.

Remark 1.5.1. The algebras of type $A_{2l}^{(2)}$ do not contain any C-admissible subalgebra (see Table A).

Except in the two following cases :



there exists in $\Pi \setminus \Gamma$ an unique root such the associated connected component of $S(A)$ (as defined in section 1.2.) contains α_0 . Such a root will be denoted by ω_0 (possibly $\omega_0 = \alpha_0$). The root $\bar{\omega}_0$ of Ψ corresponds to the so-called 0th vertex on the Dynkin diagram of $\hat{\mathfrak{g}}_\Gamma$. In the following example $\hat{\mathfrak{g}}$ is of type $E_7^{(1)}$, $\hat{\mathfrak{g}}_\Gamma$ is of type $A_5^{(2)}$ and one has $\omega_0 \neq \alpha_0$.



In the two exceptional cases before, one can consider that $\omega_0 = \alpha_0$ by reordering the indices.

Choose now in \mathfrak{h}_Γ an element denoted by d_Γ , unique up to a summand proportional to K and that verifies :
$$\begin{cases} \bar{\omega}_0(d_\Gamma) = 1 \\ \bar{\omega}_j(d_\Gamma) = 0 \quad \text{for all } j = 1, \dots, k \end{cases}$$
 The set $\{d_\Gamma, H_0, \dots, H_k\}$ is a basis of \mathfrak{h}_Γ ([K] §6.2. p. 81).

Remark 1.5.2. If $\omega_0 = \alpha_0$ then $d_\Gamma = d$.

Proposition 1.5.3. The restriction to \mathfrak{h}_Γ of the root δ of Δ , denoted by $\bar{\delta}$, is equal to $k\delta_\Gamma$ where k is a positive integer. Furthermore if $\omega_0 = \alpha_0$ then $\bar{\delta} = \delta_\Gamma$.

Proof : Recall that δ_Γ is also defined by the relations :
$$\begin{cases} \delta_\Gamma | \sum_{j=0}^k \mathbb{C}H_j = 0 \\ \langle \delta_\Gamma, d_\Gamma \rangle = b_0 \end{cases} \quad \text{where}$$

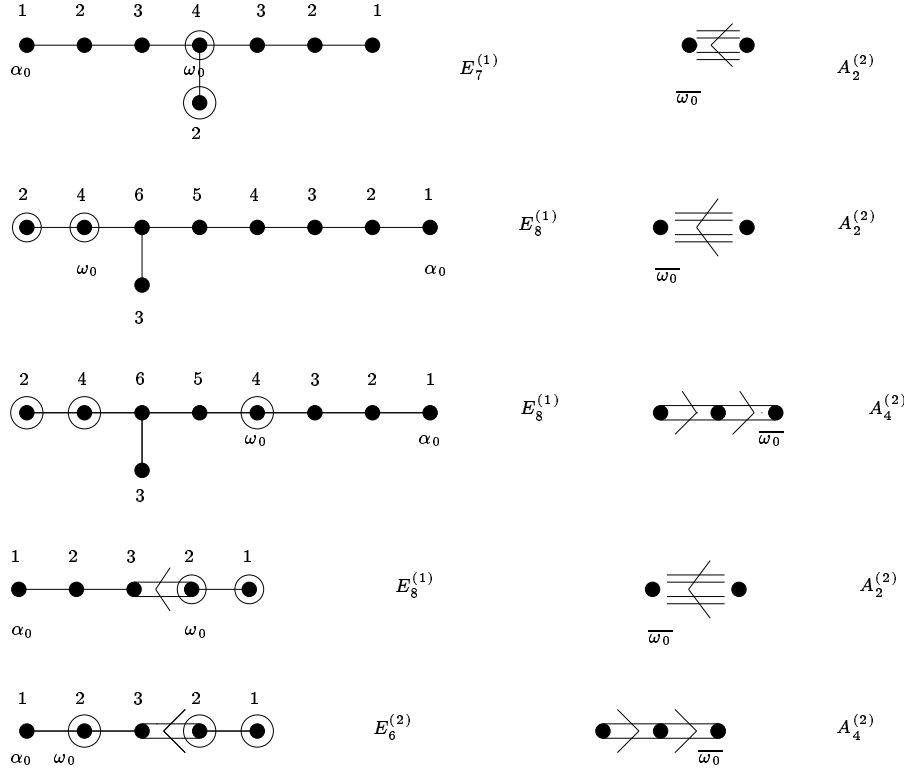
$b_0 = 1$ if $\hat{\mathfrak{g}}_\Gamma$ is of type $A_{2l}^{(2)}$ and $b_0 = 2$ if not.

One has $\delta = \sum_{i=0}^n a_{\alpha_i} \alpha_i$ where $a_{\alpha_0} = 1$ (see Remark 1.5.1.). Then $\bar{\delta} = \sum_{j=0}^k a_{\omega_j} \bar{\omega}_j$. As $H_j \in \mathfrak{h}'$

for any j , one has $\delta(H_j) = 0$ and $\delta(d_\Gamma) = \sum_{j=0}^k a_{\omega_j} \bar{\omega}_j(d_\Gamma) = a_{\omega_0}$. Hence $\bar{\delta} = \frac{a_{\omega_0}}{b_0} \delta_\Gamma$.

- If $\hat{\mathfrak{g}}_\Gamma$ is not of type $A_{2l}^{(2)}$ then $b_0 = 1$ and the result follows.

- If $\hat{\mathfrak{g}}_\Gamma$ is of type $A_{2l}^{(2)}$ then $b_0 = 2$. One checks that for each case where the C-admissible algebra $\hat{\mathfrak{g}}_\Gamma$ is of type $A_{2l}^{(2)}$ the number a_{ω_0} is an even integer. From Table A the list of such (Π, Γ) is given by the following diagrams (where the numerical labels are the coefficients of the root δ).



In any case where $\omega_0 = \alpha_0$ we have $a_{\omega_0} = a_{\alpha_0} = 1$ and $\bar{\delta} = \delta_\Gamma$.

Corollary 1.5.4. $\overline{\Delta^{Im}} \subset \Delta_\Gamma^{Im}$.

The following proposition is standard.

Proposition 1.5.5. $\Delta_\Gamma \subset \overline{\Delta}$.

Proposition 1.5.6. *The algebra $\hat{\mathfrak{g}}$ is an integrable $\hat{\mathfrak{g}}_\Gamma$ -module.*

Proof : One recall that a $\hat{\mathfrak{g}}_\Gamma$ -module V is called \mathfrak{h}_Γ -diagonalizable if $V = \bigoplus_{\lambda \in \mathfrak{h}_\Gamma^*} V_\lambda$ where

$V_\lambda = \{v \in V, h(v) = \lambda(h)v, \text{ for } h \in \mathfrak{h}_\Gamma\}$ and it is called integrable if all X_j, Y_j for $j = 0, \dots, k$ are locally nilpotent on V .

The $\hat{\mathfrak{g}}_\Gamma$ -module $\hat{\mathfrak{g}}$ is clearly \mathfrak{h}_Γ -diagonalizable since one has $\hat{\mathfrak{g}} = \bigoplus_{\bar{\alpha} \in \overline{\Delta} \cup \{0\}} V_{\bar{\alpha}}$ where

$$V_{\bar{\alpha}} = \bigoplus_{\bar{\beta} = \bar{\alpha}} \hat{\mathfrak{g}}^{\bar{\beta}} = \{x \in \hat{\mathfrak{g}}, [h, x] = \bar{\alpha}(h)x, h \in \mathfrak{h}_\Gamma\}$$

Since $\hat{\mathfrak{g}}^\alpha \subset V_{\bar{\alpha}}$ then $\bar{\alpha}$ is a weight of the $\hat{\mathfrak{g}}_\Gamma$ -module $\hat{\mathfrak{g}}$.

Let us prove now that for any $j = 0, \dots, k$, adX_j and adY_j are locally nilpotent on $\hat{\mathfrak{g}}$. It is sufficient to prove they are locally nilpotent on \mathfrak{h} and on every e_i and f_i ($i = 0, \dots, n$). ([K] Lemme 3.4.).

Fix $j \in \{0, \dots, k\}$. Let h be an element of \mathfrak{h} . Since $\mathfrak{h} = \mathfrak{h}_\Gamma \oplus \sum_{\alpha \in \Gamma} \mathbb{C}\check{\alpha}$ we can write $h = H + h_j + k_j$ where $H \in \mathfrak{h}_\Gamma$, $h_j \in \mathfrak{h}(\Gamma_j) = \sum_{\alpha \in \Gamma_j} \mathbb{C}\check{\alpha}$ and $k_j \in \mathfrak{h}(\Gamma \setminus \Gamma_j) = \sum_{\alpha \in \Gamma \setminus \Gamma_j} \mathbb{C}\check{\alpha}$. Note that $[\mathfrak{h}(\Gamma \setminus \Gamma_j), d_1^j] = \{0\}$, hence $[X_j, k_j] = 0$. One also has $[X_j, H] = -\overline{\omega_j}(H)X_j$, hence $(adX_j)^2(H) = 0$. Since the pair (Π, Γ) is C-admissible, we have $[X_j, [X_j, h_j]] \in [d_1^j, d_1^j] = \{0\}$ hence $(adX_j)^2(h_j) = 0$. So we have $(adX_j)^2(h) = 0$ for all $h \in \mathfrak{h}$. The same proof shows that $(adY_j)^2(h) = 0$.

Let i be $\in \{0, \dots, n\}$, and consider the corresponding $\alpha_i \in \Pi$, we have $e_i \in \hat{\mathfrak{g}}^{\alpha_i}$.

- If $\alpha_i \in \Gamma_j$ then $e_i \in d_0^j$ and $(adX_j)^2(e_i) \in [d_1^j, d_1^j] = \{0\}$.
- If $\alpha_i \in \Gamma \setminus \Gamma_j$ then $(adX_j)(e_i) = 0$.
- If $\alpha_i \notin \Gamma$ (i.e. $\overline{\alpha_i} \neq 0$) then there exists l such that $\alpha_i = \omega_l$ and for some integer N , one has

$$(adX_j)^N(e_i) \in V_{N\overline{\omega_j} + \overline{\omega_l}} = \bigoplus_{\beta = N\omega_j + \omega_l \pmod{\Gamma}} \hat{\mathfrak{g}}^\beta$$

For N great enough one has $\hat{\mathfrak{g}}^\beta = \{0\}$ and the result follows.

For $f_i \in \hat{\mathfrak{g}}^{-\alpha_i}$ let us consider the following cases :

- If $\alpha_i \in \Gamma_j$ then $(adX_j)^2(f_i) \in [d_1^j, d_1^j] = \{0\}$.
- If $\alpha_i \in \Gamma \setminus \Gamma_j$ then $(adX_j)(f_i) = 0$.
- If $\alpha_i = \omega_l$ ($l \neq j$) then $(adX_j)(f_i) = 0$ because $[d_1^j, d_{-1}^l] = \{0\}$.
- If $\alpha_i = \omega_j$ then $(adX_j)(f_i) \in d_0^j$, so $(adX_j)^3(f_i) = 0$.

The same proof shows that adY_j is locally nilpotent on e_i, f_i for $i = 0, \dots, n$. □

Remark 1.5.7. A slightly modification of the preceding proof shows that actually $\hat{\mathfrak{g}}$ is an integrable $\hat{\mathfrak{g}}_\Gamma$ -module even if (Π, Γ) is admissible.

Theoreme 1.5.8. *The root system Δ_Γ of the C-admissible algebra $\hat{\mathfrak{g}}_\Gamma$ is $\overline{\Delta} \setminus \{0\}$.*

Proof : Since $\hat{\mathfrak{g}}$ is an integrable $\hat{\mathfrak{g}}_\Gamma$ -module we can define the automorphisms of $\hat{\mathfrak{g}}$

$$r_j^{ad} = \exp(adY_j)\exp(-adX_j)\exp(adY_j)$$

Denote by r_j the fundamental reflection corresponding to the simple root $\overline{\omega_j} \in \Psi$. Recall that

$$r_j^{ad}(V_{\overline{\alpha}}) = V_{r_j(\overline{\alpha})}, \quad r_j^{ad} \in \text{Aut}(\hat{\mathfrak{g}}_\Gamma), \quad r_j^{ad}|_{\mathfrak{h}_\Gamma} = r_j \quad (\star)$$

(see [K] Lemma 3.8.).

Let w be an element of the Weyl group W_Γ of $\hat{\mathfrak{g}}_\Gamma$ generated by the r_j . If $w = r_{i_1} \dots r_{i_p}$ then put $w^{ad} = r_{i_1}^{ad} \dots r_{i_p}^{ad} \in \text{Aut}(\hat{\mathfrak{g}})$. The fact that $w(\overline{\alpha})$ is a weight of $\hat{\mathfrak{g}}$ for any $\overline{\alpha} \in \overline{\Delta} \cup \{0\}$ follows from the first assertion of (\star) and $w(\overline{\alpha}) \in \overline{\Delta} \cup \{0\}$.

Let α be an element of Δ such that $\overline{\alpha} \neq 0$ and $\overline{\alpha}$ is not proportional to some root of Δ_Γ . Consider the \mathbb{R} -subspace $\mathfrak{h}_{\Gamma, \mathbb{R}}$ of \mathfrak{h}_Γ generated by the elements $d_\Gamma, H_0, \dots, H_k$. Since $\overline{\omega_j}(d_\Gamma)$

and $\overline{\omega_j}(H_i)$ are integers for all i, j then $\overline{\alpha}|\mathfrak{h}_{\Gamma, \mathbb{R}} \in \mathfrak{h}_{\Gamma, \mathbb{R}}^*$. Denote by $L_{\overline{\omega_j}}$ (resp. $L_{\delta_\Gamma}, L_{\overline{\alpha}}$) the kernel of $\overline{\omega_j}|\mathfrak{h}_{\Gamma, \mathbb{R}}$ (resp. $\delta_\Gamma|\mathfrak{h}_{\Gamma, \mathbb{R}}, \overline{\alpha}|\mathfrak{h}_{\Gamma, \mathbb{R}}$).

We have obviously $L_{\overline{\alpha}} \neq L_{\overline{\omega_j}}, L_{\overline{\alpha}} \neq L_{\delta_\Gamma}$ and $L_{\overline{\alpha}}$ is not contained in the union of the kernels L_{δ_Γ} and $L_{\overline{\omega_j}}, (j = 0, \dots, k)$.

Let H be an element of $L_{\overline{\alpha}}$ which is not in this union. We can choose H such that $\delta_\Gamma(H) > 0$ so, H is in the Tits cone ([K] Proposition 5.8.). Hence there exists $w \in W_\Gamma$ such that $\overline{\omega_j}(w(H)) > 0$ for any j . Put $\overline{\beta} = w(\overline{\alpha})$; we have $L_{\overline{\beta}} = w(L_{\overline{\alpha}})$, $w(H) \in L_{\overline{\beta}}, \overline{\beta} \neq 0$ and $\overline{\beta}$ is

not proportional to some root of Δ_Γ . Set $\overline{\beta} = \sum_{j=0}^k n_j \overline{\omega_j}$ where the coefficients n_j are integers

of the same sign. We have $\overline{\beta}(w(H)) = \sum_{j=0}^k n_j \overline{\omega_j}(w(H)) = 0$. It is impossible since $\overline{\beta} \neq 0$.

Therefore either $\overline{\beta} = 0$ or $\overline{\beta} = k\nu$ with $\nu \in \Delta_\Gamma$ and the same is true for $\overline{\alpha}$ ($\overline{\alpha} = 0$ or $\overline{\alpha} = k\mu, \mu \in \Delta_\Gamma$).

- If μ is imaginary then $\overline{\alpha} \in \Delta_\Gamma^{Im}$.

- If μ is real then there exists $w \in W_\Gamma$ and j such that $w(\overline{\alpha}) = k\overline{\omega_j}$. Since $[d_1^i, d_1^j] = \{0\}$, we have $k = \pm 1$. Hence $\overline{\alpha} \in \Delta_\Gamma^{Re}$.

Therefore one has $\overline{\Delta} \subset \Delta_\Gamma \cup \{0\}$ and Proposition 1.5.5. implies the final result. □

2. Affine algebras graded by affine root systems

2.1. Definition

The definition of a Lie algebra graded by a finite root system has been introduced by S. Berman and M. Moody (see also the papers by Benkart, Zelmanov [BZ], Allison, Benkart and Gao [ABG] and Neher [N]). In this work we study more particularly affine algebras graded by affine root systems. This needs a slightly extension of the definition given in [ABG].

Definition 2.1.1. *Let Σ be an affine root system. An affine Lie algebra $\hat{\mathfrak{g}}$ over \mathbb{C} is graded by Σ or Σ -graded if*

(i) $\hat{\mathfrak{g}}$ contains as a subalgebra an affine algebra \mathfrak{a} whose root system relatively to a Cartan subalgebra $\mathfrak{h}_\mathfrak{a} = \mathfrak{a}^0$ is equal to Σ

(ii) $\hat{\mathfrak{g}} = \bigoplus_{\tilde{\alpha} \in \Sigma \cup \{0\}} V_{\tilde{\alpha}}$ where $V_{\tilde{\alpha}} = \{x \in \hat{\mathfrak{g}}, [h, X] = \tilde{\alpha}(h)x, \text{ for all } h \in \mathfrak{h}_\mathfrak{a}\}$.

The affine subalgebra \mathfrak{a} is called the grading subalgebra relatively to Σ .

Remark 2.1.2. For any $\tilde{\alpha} \in \Sigma \cup \{0\}$ the root space $\mathfrak{a}^{\tilde{\alpha}}$ is included in $V_{\tilde{\alpha}}$ and one has $V_{\tilde{\alpha}} \neq \{0\}$.

Remark 2.1.3. As $\sum_{\tilde{\alpha} \in \Sigma} ([V_{\tilde{\alpha}}, V_{-\tilde{\alpha}}] \oplus V_{\tilde{\alpha}})$ is a non zero ideal of $\hat{\mathfrak{g}}$ that contains $[\mathfrak{a}, \mathfrak{a}]$, we have $[\hat{\mathfrak{g}}, \hat{\mathfrak{g}}] = \sum_{\tilde{\alpha} \in \Sigma} ([V_{\tilde{\alpha}}, V_{-\tilde{\alpha}}] \oplus V_{\tilde{\alpha}})$ ([K] Proposition 1.7.b.).

Therefore a general definition of an arbitrary Lie algebra graded by an affine root system should include the fact that the derived algebra is generated by the spaces $V_{\tilde{\alpha}}$ ($\tilde{\alpha} \in \Sigma$). The preceding computation shows that this is always true if $\hat{\mathfrak{g}}$ is affine.

2.2. The grading defined by a C-admissible subalgebra

Proposition 2.2.1. *The root system Δ_Γ of any C-admissible subalgebra $\hat{\mathfrak{g}}_\Gamma$ of an affine algebra $\hat{\mathfrak{g}}$ graduates $\hat{\mathfrak{g}}$ as in definition 2.1.1. and $\hat{\mathfrak{g}}_\Gamma$ is the corresponding grading algebra.*

Proof : For any C-admissible pair (Π, Γ) in $\Delta = (\hat{\mathfrak{g}}, \mathfrak{h})$ we have defined

$$\mathfrak{h}_\Gamma = \{h \in \mathfrak{h}, \alpha(h) = 0, \forall \alpha \in \Gamma\}$$

and we have $\Delta_\Gamma = (\hat{\mathfrak{g}}_\Gamma, \mathfrak{h}_\Gamma) = \overline{\Delta} \setminus \{0\}$ where $\overline{\Delta} = \{\bar{\alpha} = \alpha|_{\mathfrak{h}_\Gamma} \text{ for all } \alpha \in \Delta\}$ (Theorem 1.5.8.). Set

$$V_{\bar{\alpha}} = \bigoplus_{\gamma|_{\mathfrak{h}_\Gamma} = \bar{\alpha}} \hat{\mathfrak{g}}^\gamma \quad \text{for all } \bar{\alpha} \in \Delta_\Gamma \cup \{0\}$$

Since $\hat{\mathfrak{g}}$ is an integrable \mathfrak{g}_Γ -module whose weights are $\Delta_\Gamma \cup \{0\}$, (Proposition 1.5.6.) the proof of the proposition is then straightforward.

2.3. Converse problem : first results

In this section we establish a link with the grading affine subalgebras of $\hat{\mathfrak{g}}$ and the C-admissible subalgebras.

Let $\hat{\mathfrak{g}}$ be an affine algebra graded by an affine root system Σ . Denote by $\mathcal{B} = \{\tilde{\omega}_0, \dots, \tilde{\omega}_k\}$ ($k \geq 1$) a root basis of Σ and by \mathfrak{a} the grading subalgebra. The elements of the Cartan subalgebra $\mathfrak{h}_\mathfrak{a}$ of \mathfrak{a} are semi-simple since $\hat{\mathfrak{g}}$ is $\mathfrak{h}_\mathfrak{a}$ -diagonalizable (condition (ii) of Definition 2.1.1.). One deduces by applying the conjugacy theorem of Kac-Peterson ([K-P]) that there exists a Cartan subalgebra \mathfrak{h} of $\hat{\mathfrak{g}}$ containing $\mathfrak{h}_\mathfrak{a}$.

$$\begin{aligned} \text{Put } \Delta = (\hat{\mathfrak{g}}, \mathfrak{h}) \text{ and } \rho : \mathfrak{h}^* &\longrightarrow \mathfrak{h}_\mathfrak{a}^* \\ \alpha &\longmapsto \alpha|_{\mathfrak{h}_\mathfrak{a}} \end{aligned}$$

Remark 2.3.1. For any $\tilde{\alpha} \in \Sigma \cup \{0\}$, the subspace $V_{\tilde{\alpha}}$ of the Σ -gradation is equal to $\bigoplus_{\rho(\gamma) = \tilde{\alpha}} \hat{\mathfrak{g}}^\gamma$ and one has $V_0 = \mathcal{Z}_{\hat{\mathfrak{g}}}(\mathfrak{h}_\mathfrak{a}) = \mathfrak{h} \oplus \bigoplus_{\rho(\gamma) = 0} \hat{\mathfrak{g}}^\gamma$.

Lemma 2.3.2. *The map ρ is a surjection from $\Delta \cup \{0\}$ onto $\Sigma \cup \{0\}$.*

Proof : Since $\hat{\mathfrak{g}} = \bigoplus_{\tilde{\alpha} \in \Sigma \cup \{0\}} V_{\tilde{\alpha}}$ then for any $\gamma \in \Delta$, there exists $\tilde{\alpha} \in \Sigma \cup \{0\}$ such that $\hat{\mathfrak{g}}^\gamma \subset V_{\tilde{\alpha}}$ and $\rho(\gamma) = \tilde{\alpha}$.
Let $\tilde{\alpha}$ be an element of Σ , since $V_{\tilde{\alpha}} \neq \{0\}$ (Remark 2.1.2.), there exists $\gamma \in \Delta$ such that $\hat{\mathfrak{g}}^\gamma \subset V_{\tilde{\alpha}}$ and $\rho(\gamma) = \tilde{\alpha}$. □

Denote by δ (resp. $\delta_\mathfrak{a}$) the unique imaginary root of Δ (resp. Σ) whose coefficients in a given basis Π (resp. \mathcal{B}) are positive relatively prime integers.

Lemma 2.3.3. *The restriction to $\mathfrak{h}_\mathfrak{a}$ of the root δ is proportional to $\delta_\mathfrak{a}$ and one can choose a root basis Π of Δ such that the coefficient is a positive integer.*

Proof : Let Π be a basis of Δ . Lemma 2.3.2. implies that $\rho(\delta)$ is a root of Σ or is zero; for any $j \in \mathbb{Z}$, one has $\rho(j\delta) = j\rho(\delta)$ and it is an element of $\Sigma \cup \{0\}$ too. Suppose that

$\rho(\delta) = 0$. Recall that any real root can be written as $\alpha = j\delta + \gamma$ ($\gamma \in \overset{\circ}{\Delta}, j \in \mathbb{Z}$) (possibly $\alpha = \frac{1}{2}((2j-1)\delta + \gamma$ if $\hat{\mathfrak{g}}$ is of type $A_2^{(2)}$). Since $|\rho(\overset{\circ}{\Delta})| < \infty$ the hypothesis $\rho(\delta) = 0$ implies $|\rho(\Delta)| < \infty$. This conclusion is not true since $\rho(\Delta \cup \{0\}) = \Sigma \cup \{0\}$. Hence $\rho(\delta)$ and $j\rho(\delta)$ for any j are roots of Σ . We deduce that there exists $k \in \mathbb{Z}^*$ such that $\rho(\delta) = k\delta_a$.

Suppose now that $k < 0$. One can choose the basis $\Pi' = -\Pi$ and for this basis and the corresponding element δ , one has $\rho(\delta) > 0$. (Note that this is just choosing a conjugacy class of basis under the Weyl group of Δ).

From now on we choose a basis Π such that $\rho(\delta) > 0$.

Define the non empty set

$$\mathcal{P} = \{\gamma \in \Delta, \rho(\gamma) \in \Sigma_+ \cup \{0\}\}$$

We have clearly $\mathcal{P} \cup (-\mathcal{P}) = \Delta$ and \mathcal{P} is a closed set.

Proposition 2.3.4. *There exists a basis Π of Δ such that $\Delta_+(\Pi) \subset \mathcal{P}$ (where $\Delta_+(\Pi)$ is the set of positive roots relatively to Π).*

Recall that $\Delta_+^{Im}(\Pi) = \{j\delta, j \in \mathbb{N}^*\}$. The choice of Π implies that $\Delta_+^{Im}(\Pi) \subset \mathcal{P}$.

Any real root can be written as $\alpha = j\delta + \gamma$ ($\gamma \in \overset{\circ}{\Delta}, j \in \mathbb{Z}$) (possibly $\alpha = \frac{1}{2}((2j-1)\delta + \gamma$ if $\hat{\mathfrak{g}}$ is of type $A_2^{(2)}$) and $\Delta_+^{Re}(\Pi) = \{\alpha \in \Delta^{Re}(\Pi), j > 0\} \cup \overset{\circ}{\Delta}_+(\overset{\circ}{\Pi})$ ([K] Proposition 6.3.e)). Set

$$\mathcal{S} = \mathcal{P} \cap (-\mathcal{P}) = \{\alpha \in \Delta, \rho(\alpha) = 0\}$$

One has

$$\{\alpha \in \Delta_+(\Pi), \rho(\alpha) < 0\} = \Delta_+(\Pi) \cap ((-\mathcal{P}) \setminus \mathcal{S})$$

This set does not contain any positive imaginary root. Let α be a positive real root of Δ .

If $\alpha = j\delta + \gamma$ ($\gamma \in \overset{\circ}{\Delta}, j \in \mathbb{N}$) then $\rho(\alpha) = jk\delta_a + \rho(\gamma)$. Since $|\rho(\overset{\circ}{\Delta})| < \infty$, then for j great enough one has $\rho(\alpha) > 0$. Hence $|\{\alpha = j\delta + \gamma, j \geq 0, \rho(\alpha) < 0\}| < \infty$.

If $\hat{\mathfrak{g}}$ is of type $A_{2l}^{(2)}$, a positive real root could be written as $\alpha = \frac{1}{2}((2j-1)\delta + \gamma)$ with $2j-1 > 0$, and the conclusion is the same. Hence we always have $|\Delta_+(\Pi) \cap ((-\mathcal{P}) \setminus \mathcal{S})| < \infty$. If the previous set is empty then one has obviously $\Delta_+(\Pi) \subset \mathcal{P}$.

Suppose now that $\Delta_+(\Pi)$ is not a subset of \mathcal{P} , then there exists a root $\alpha_i \in \Pi$ such that $\rho(\alpha_i) < 0$. Consider the reflection $s_{-\alpha_i}$ ([K] p. 35) and set $\Pi' = s_{-\alpha_i}(\Pi)$. The set Π' is a root basis of Δ ([K] Proposition 5.9.) and $s_{-\alpha_i}$ preserves the root δ . We clearly have $\rho(-\alpha_i) > 0$ so $-\alpha_i \notin (-\mathcal{P}) \setminus \mathcal{S}$ (1). On the other hand one has $s_{-\alpha_i}(\alpha_i) = -\alpha_i \in \Pi'$ hence $-\alpha_i \in \Delta_+(\Pi')$ (2).

Let us prove now that $\Delta_+(\Pi') \cap ((-\mathcal{P}) \setminus \mathcal{S})$ is a proper subset of $\Delta_+(\Pi) \cap ((-\mathcal{P}) \setminus \mathcal{S})$. Note that que $\Delta_+(\Pi') \cap ((-\mathcal{P}) \setminus \mathcal{S}) \subset \Delta_+(\Pi') \setminus \{-\alpha_i\}$ (cf (1) et (2)). We know that the set $\Delta_+(\Pi') \setminus \{-\alpha_i\}$ is $s_{-\alpha_i}$ -invariant ([K] Lemme 3.7.). Hence for any $\beta \in \Delta_+(\Pi') \setminus \{-\alpha_i\}$, one has

$$\beta = (s_{-\alpha_i})^2(\beta) \in s_{-\alpha_i}(\Delta_+(\Pi'))$$

Since $\Pi = s_{-\alpha_i}(\Pi')$ we have $s_{-\alpha_i}(\Delta_+(\Pi')) \subset \Delta_+(\Pi)$. Therefore

$$\Delta_+(\Pi') \cap ((-\mathcal{P}) \setminus \mathcal{S}) \subset \Delta_+(\Pi) \cap ((-\mathcal{P}) \setminus \mathcal{S})$$

Furthermore one has $\alpha_i \in \Delta_+(\Pi) \cap ((-\mathcal{P}) \setminus \mathcal{S})$ and $\alpha_i \notin \Delta_+(\Pi')$. Hence the inclusion is strict.

After a finite number of such steps we get a root basis Θ such that $\Delta_+(\Theta) \cap ((-\mathcal{P}) \setminus \mathcal{S}) = \emptyset$ and the proposition follows. \square

Fix once for all a basis Π that verifies $\Delta_+(\Pi) \subset \mathcal{P}$.

The following corollary is straightforward.

Corollary 2.3.5. *The map ρ is a surjection from $\Delta_+ \cup \{0\}$ onto $\Sigma_+ \cup \{0\}$ and $\rho^{-1}(\Sigma_+) \subset \Delta_+$.*

Remark 2.3.6. If the set $\mathcal{S} = \mathcal{P} \cap (-\mathcal{P})$ is empty (this means that $\Delta_- \cap \mathcal{P} = \emptyset$ or $\Delta_+ \cap (-\mathcal{P}) = \emptyset$) then ρ is a surjection from Δ_+ onto Σ_+ .

Proposition 2.3.7. *If the set $\mathcal{S} = \mathcal{P} \cap (-\mathcal{P})$ is not empty then there exists a non empty subset Γ of Π , such that $\Delta_+ \cup \langle \Gamma \rangle^- = \mathcal{P}$. This condition is equivalent to $\langle \Gamma \rangle = \mathcal{P} \cap (-\mathcal{P})$.*

Proof : The condition $\mathcal{S} \neq \emptyset$ implies that there exists a simple root in $(-\mathcal{P})$ (Remark 2.3.6.). Set $\Gamma = \Pi \cap (-\mathcal{P}) = \{\gamma \in \Pi, \rho(\gamma) = 0\}$. Γ is a proper subset of Π and $\langle \Gamma \rangle$ is a finite root system.

Let us prove now that $\langle \Gamma \rangle^- = \Delta_- \cap \mathcal{P}$. Let α be an element of $\langle \Gamma \rangle^-$, write $\alpha = -\beta_1 - \dots - \beta_n$ ($\beta_j \in \Gamma$). Let us prove by induction on n that $\langle \Gamma \rangle^- \subset \Delta_- \cap \mathcal{P}$. For $n = 1$, one has $\alpha = -\beta_1$; since $\beta_1 \in \Gamma \subset (-\mathcal{P})$ then $-\beta_1 \in \Delta_- \cap \mathcal{P}$. Let n be a positive integer, suppose that if $\alpha = -\beta_1 - \dots - \beta_n$, $\beta_j \in \Gamma$ then $\alpha \in \Delta_- \cap \mathcal{P}$. Let α be a root such that $\alpha = -\beta_1 - \dots - \beta_n - \beta_{n+1}$ ($\beta_j \in \Gamma$). There exists a permutation σ such that $\alpha = -\beta_{\sigma(1)} - \dots - \beta_{\sigma(n)} - \beta_{\sigma(n+1)}$ and the partial sums are roots. Set $\gamma = -\beta_{\sigma(1)} - \dots - \beta_{\sigma(n)}$. One has $\gamma \in \Delta_- \cap \mathcal{P}$ and $-\beta_{\sigma(n+1)} \in \Delta_- \cap \mathcal{P}$. Since $\alpha = \gamma - \beta_{\sigma(n+1)} \in \Delta_-$ and \mathcal{P} is closed then $\alpha \in \Delta_- \cap \mathcal{P}$.

Let us prove now the converse inclusion by induction too. Let α be an element of $\Delta_- \cap \mathcal{P}$, one can write $\alpha = -\beta_1 - \dots - \beta_n$ ($\beta_j \in \Pi$). If $n = 1$ then $\alpha = -\beta_1 \in \mathcal{P}$ and $\beta_1 \in \Pi \cap (-\mathcal{P}) = \Gamma$. Let n be a positive integer, suppose that if $\alpha = -\beta_1 - \dots - \beta_n$ ($\beta_j \in \Pi$) and if $\alpha \in \Delta_- \cap \mathcal{P}$, then $\alpha \in \langle \Gamma \rangle^-$. Let α be an element of $\Delta_- \cap \mathcal{P}$ such that $\alpha = -\beta_1 - \dots - \beta_n - \beta_{n+1}$ ($\beta_j \in \Pi$). There exists a permutation σ such that $\alpha = -\beta_{\sigma(1)} - \dots - \beta_{\sigma(n)} - \beta_{\sigma(n+1)}$ and the partial sums are roots. Set $\gamma = -\beta_{\sigma(1)} - \dots - \beta_{\sigma(n)}$. One has $\gamma \in \Delta_-$ and $\gamma = \alpha + \beta_{\sigma(n+1)}$; since \mathcal{P} is closed then $\gamma \in \mathcal{P}$. We deduce that $\gamma \in \langle \Gamma \rangle^-$. Furthermore $\alpha - \gamma = -\beta_{\sigma(n+1)}$ and $-\gamma, \alpha \in \mathcal{P}$, we deduce that $\beta_{\sigma(n+1)} \in (-\mathcal{P}) \cap \Pi = \Gamma$. \square

Corollary 2.3.8. *The space $V_0 = \mathfrak{h} \oplus \hat{\mathfrak{g}}^{\langle \Gamma \rangle}$ is finite-dimensional.*

Proposition 2.3.9. *The space $V_{\tilde{\alpha}}$ is finite-dimensional for all $\tilde{\alpha} \in \Sigma$.*

Proof : Let $\tilde{\alpha}$ be a root of Σ . We can assume $\tilde{\alpha}$ is a positive root. Recall that $V_{\tilde{\alpha}} = \bigoplus_{\rho(\gamma)=\tilde{\alpha}} \hat{\mathfrak{g}}^\gamma$; let us prove that $|\rho^{-1}(\{\tilde{\alpha}\})| < \infty$. Put $\tilde{\alpha} = \sum_{j=0}^k m_j \tilde{\omega}_j$ ($m_j \in \mathbb{N}$). Denote by $\omega_0, \dots, \omega_q$ the roots of $\Pi \setminus \Gamma$. One has $\rho(\omega_l) > 0$ for all $l = 0, \dots, q$. Set $\rho(\omega_l) = \sum_{j=0}^k p_j^l \tilde{\omega}_j$

where p_j^l are non negative integers. Let γ be a root of Δ such that $\rho(\gamma) = \tilde{\alpha}$, we know that $\gamma \in \Delta_+$ (Corollary 2.3.5.) and one can write $\gamma = \sum_{l=0}^q n_l \omega_l + \sum_{\beta_t \in \Gamma} s_t \beta_t$ with $n_l \geq 0$ and $s_t \geq 0$.

One has

$$\rho(\gamma) = \sum_{l=0}^q n_l \rho(\omega_l) = \sum_{l=0}^q n_l \left(\sum_{j=0}^k p_j^l \tilde{\omega}_j \right) = \sum_{j=0}^k \left(\sum_{l=0}^q n_l p_j^l \right) \tilde{\omega}_j = \sum_{j=0}^k m_j \tilde{\omega}_j$$

Hence one has for any j

$$\sum_{l=0}^q n_l p_j^l = m_j$$

The set of the q -uplets (n_1, \dots, n_q) which are solutions of the preceding system of equations is finite. And for each q -uplet (n_1, \dots, n_q) , the number of the roots in Δ whose coefficients relatively to the simple roots in a non empty subset of Π (here $\Pi \setminus \Gamma$) are fixed, is finite. One deduces that $|\rho^{-1}(\{\tilde{\alpha}\})| < \infty$.

□

Recall that since ρ is a surjection from $\Delta \cup \{0\}$ onto $\Sigma \cup \{0\}$, the set $\Gamma = \{\alpha \in \Pi, \rho(\alpha) = 0\}$ is a proper subset (possibly empty) of Π . Hence we can define as in Part 1. a \mathbb{Z} -gradation in $\hat{\mathfrak{g}}$ relating Γ . One has

$$\begin{aligned} d_0 &= \mathfrak{h} \oplus \hat{\mathfrak{g}}^{\langle \Gamma \rangle} = V_0 \\ d_1 &= \bigoplus_{\substack{\gamma = \omega \pmod{\Gamma} \\ \omega \in \Pi \setminus \Gamma}} \hat{\mathfrak{g}}^\gamma \end{aligned}$$

Proposition 2.3.10. *Every root in \mathcal{B} has a preimage under ρ in $\Pi \setminus \Gamma$ and any preimage α in Δ of a root of \mathcal{B} can be written $\alpha = \omega \pmod{\Gamma}$ with $\omega \in \Pi \setminus \Gamma$.*

Proof : Let $\tilde{\omega} \in \mathcal{B}$. Let γ be a preimage of $\tilde{\omega}$ in Δ , one has $\gamma \in \Delta_+$ (Corollary 2.3.5.). Put $\gamma = \sum_{l=0}^q n_l \omega_l + \beta$ where β is a linear combination of elements of Γ . If we denote the height of any $\tilde{\alpha} \in \Sigma$ relatively to the basis \mathcal{B} by $h_{\mathcal{B}}(\tilde{\alpha})$, one has

$$h_{\mathcal{B}}(\rho(\gamma)) = \sum_{l=0}^q n_l h_{\mathcal{B}}(\rho(\omega_l))$$

Since $\rho(\omega_l) > 0$ for all l , one has

$$h_{\mathcal{B}}(\rho(\gamma)) = 1 \implies \begin{cases} \exists! l_0, n_{l_0} = 1, h_{\mathcal{B}}(\rho(\omega_{l_0})) = 1 \\ l \neq l_0 \implies n_l = 0 \end{cases}$$

Hence $\gamma = \omega_{l_0} \pmod{\Gamma}$ and $\rho(\omega_{l_0}) = \tilde{\omega}$. We can see that the root ω_{l_0} depends on the choice of the preimage γ .

Corollary 2.3.11. *One has $\text{card}(\Pi \setminus \Gamma) \geq \text{card}(\mathcal{B}) \geq 2$ and $\bigoplus_{j=0}^k V_{\tilde{\omega}_j} \subset d_1$.*

Proposition 2.3.12. *The map ρ is a surjection from $\Pi \setminus \Gamma$ onto \mathcal{B} .*

Proof : From Proposition 2.3.10. it is enough to prove that for any $\omega \in \Pi \setminus \Gamma$ one has $\rho(\omega) \in \mathcal{B}$. Suppose that there exists $\omega \in \Pi \setminus \Gamma$ such that $\rho(\omega) \in \Sigma_+ \setminus \mathcal{B}$. One can write

$$\rho(\omega) = \tilde{\omega}_{j_1} + \dots + \tilde{\omega}_{j_p} \quad (\star)$$

where $\tilde{\omega}_l \in \mathcal{B}$ and $p \geq 2$. We can also assume that the partial sums are roots.

◇ Suppose first that $\rho(\omega)$ is a real root. Let $(X_{\rho(\omega)}, \rho(\omega), Y_{\rho(\omega)})$ be the corresponding sl_2 -triplet. Since $\hat{\mathfrak{g}}$ is \mathfrak{a} -graded, any $Z \in V_{\rho(\omega)}$ verifies

$$2Z = [\rho(\omega), Z] = [[X_{\rho(\omega)}, Y_{\rho(\omega)}], Z] = [X_{\rho(\omega)}, [Y_{\rho(\omega)}, Z]]$$

We deduce that $V_{\rho(\omega)} = [V_0, X_{\rho(\omega)}]$. From (\star) we deduce that

$$X_{\rho(\omega)} = [X_{\tilde{\omega}_{j_p}} \dots [X_{\tilde{\omega}_{j_2}}, X_{\tilde{\omega}_{j_1}}] \dots]$$

with $X_{\tilde{\omega}_{j_m}} \in \mathfrak{a}^{\tilde{\omega}_{j_m}} \subset V_{\tilde{\omega}_{j_m}} \subset d_1$ (Corollary 2.3.11.). Hence we have $X_{\rho(\omega)} \in d_p$ ($p \geq 2$). Furthermore

$$\hat{\mathfrak{g}}^\omega \subset [V_0, X_{\rho(\omega)}] \subset [d_0, d_p] \subset d_p$$

This is a contradiction to the fact that we have $\hat{\mathfrak{g}}^\omega \subset d_1$. Hence **if $\omega \in \Pi \setminus \Gamma$ and if $\rho(\omega)$ is real, then $\rho(\omega) \in \mathcal{B}$.**

◇ Suppose now that $\rho(\omega)$ is an imaginary root. Since $|\mathcal{B}| \geq 2$, there exists at least two simple roots in $\Pi \setminus \Gamma$ whose restrictions to $\mathfrak{h}_\mathfrak{a}$ are simple roots of \mathcal{B} ; hence one has $|\Pi \setminus \Gamma| \geq 3$. Let ω_{l_0} be one of these roots. We set $\rho(\omega_{l_0}) = \tilde{\omega} \in \mathcal{B}$. It is possible to choose ω_{l_0} and $\tilde{\omega}$ such that there exists a sequence $(\alpha_1, \dots, \alpha_s)$ in Γ (possibly empty) such that the sum $\xi = \omega + \alpha_1 + \dots + \alpha_s + \omega_{l_0}$ is a positive root of Δ . Since $|\Pi \setminus \Gamma| \geq 3$, the support of ξ is a proper subset of Π , hence ξ is a real root of Δ . One has $\rho(\xi) = \rho(\omega) + \tilde{\omega} \in \Sigma_+$ and since $\rho(\omega)$ is an imaginary root, $\rho(\xi)$ is real. Since $\rho(\omega)$ is imaginary, $\text{supp}(\rho(\omega)) = \mathcal{B}$ and $\rho(\xi)$ is the sum of p simple roots ($p \geq 3$). As in the first case, we can write

$$\hat{\mathfrak{g}}^\omega = [\hat{\mathfrak{g}}^\xi, \hat{\mathfrak{g}}^{-\alpha_1 - \dots - \alpha_s - \omega_{l_0}}] \subset [V_{\rho(\xi)}, V_{-\tilde{\omega}}] = [[V_0, X_{\rho(\xi)}], V_{-\tilde{\omega}}] \subset d_{p-1}$$

Since $p - 1 \geq 2$ we conclude as in the first case. □

Let $\omega \in \Pi \setminus \Gamma$. Let Π_ω be the connected component of $\Gamma \cup \{\omega\}$ containing ω . Let $\mathfrak{g}_\omega = \bigoplus_{p \in \mathbb{Z}} d_p(\omega)$ be the \mathbb{Z} -gradation of the corresponding finite dimensional subalgebra \mathfrak{g}_ω corresponding to Π_ω (see section 1.1.)

Corollary 2.3.13. *Let $\tilde{\omega}$ an element of \mathcal{B} . Let $\omega_1, \dots, \omega_q$ be the corresponding preimages of $\tilde{\omega}$ in $\Pi \setminus \Gamma$. Then $V_{\tilde{\omega}} = \bigoplus_{l=1}^q d_1(\omega_l)$. Moreover if $\mathcal{B} = \{\tilde{\omega}_0, \dots, \tilde{\omega}_k\}$ then $d_1 = \bigoplus_{j=0}^k V_{\tilde{\omega}_j}$.*

Corollary 2.3.14. *The algebra $\hat{\mathfrak{g}}$ and the grading algebra \mathfrak{a} have the same one-dimensional center $\mathbb{C}K$.*

Proof: Let us denote by $K_{\mathfrak{a}}$ the standard generator of the one-dimensional center of the affine algebra \mathfrak{a} . $K_{\mathfrak{a}}$ is an element of the orthogonal of $\langle \tilde{\omega}_0, \dots, \tilde{\omega}_k \rangle$ in $\mathfrak{h}_{\mathfrak{a}}$. For any $\alpha \in \Gamma$, one has $\alpha(K_{\mathfrak{a}}) = \rho(\alpha)(K_{\mathfrak{a}}) = 0$ and for any $\omega_j \in \Pi \setminus \Gamma$, one has $\omega_j(K_{\mathfrak{a}}) = \rho(\omega_j)(K_{\mathfrak{a}}) = 0$. Hence $K_{\mathfrak{a}} \in \mathbb{C}K$. □

2.4. The irreducible case

In this section we prove that under a condition of irreducibility for some representations attached to the gradation, the subset Γ defined in the preceding gives rise to a C-admissible subalgebra which is exactly the grading subalgebra.

Recall that $\mathcal{B} = \{\tilde{\omega}_0, \dots, \tilde{\omega}_k\}$ is a fixed basis of Σ . The proof of the following proposition is straightforward.

Proposition 2.4.1. *The following conditions are equivalent*

- (i) *The map ρ is a bijection from $\Pi \setminus \Gamma$ onto \mathcal{B}*
- (ii) $\mathfrak{h}_{\mathfrak{a}} = \mathfrak{h}_{\Gamma}$
- (iii) *The representation $(V_0, V_{\tilde{\omega}_j})$ is irreducible for any $j = 0, \dots, k$*

Proposition 2.4.2. *If the representation $(V_0, V_{\tilde{\omega}_j})$ is irreducible for any $j = 0, \dots, k$ then the pair $\Pi \setminus \Gamma$ is C-admissible and the grading subalgebra \mathfrak{a} , is the associated C-admissible subalgebra.*

Proof: Let $\tilde{\omega}_j \in \mathcal{B}$. Denote by ω_j its unique preimage in $\Pi \setminus \Gamma$, one has

$$\tilde{\omega}_j \in [\mathfrak{a}^{\tilde{\omega}_j}, \mathfrak{a}^{-\tilde{\omega}_j}] \subset [V_{\tilde{\omega}_j}, V_{-\tilde{\omega}_j}] \subset [d_1(\omega_j), d_{-1}(\omega_j)] \subset d_0(\omega_j)$$

One deduces that $\tilde{\omega}_j \in \mathfrak{h}(\Pi_{\omega_j})$. Furthermore $\omega_j(\tilde{\omega}_j) = \tilde{\omega}_j(\tilde{\omega}_j) = 2$ and $\alpha(\tilde{\omega}_j) = 0$ for all $\alpha \in \Pi_{\omega_j} \setminus \{\omega_j\}$ because $\rho(\alpha) = 0$ by definition of Γ . Hence $\tilde{\omega}_j = H_j$ the grading element of the connected component of $S(A)$ relating to ω_j . For any j the triple $(X_{\tilde{\omega}_j}, H_j, Y_{\tilde{\omega}_j})$ is a natural sl_2 -triple. Hence the pair (Π, Γ) is admissible.

Suppose that there exists $j \in \{0, \dots, k\}$ and $p \geq 2$ such that $d_p(\omega_j) \neq \{0\}$. Then there exists a root γ of $(\mathfrak{g}_j, \mathfrak{h}_j)$ which can be written $\gamma = p\omega_j \pmod{\Gamma_j}$. One has $\rho(\gamma) = p\tilde{\omega}_j \in \Sigma_+$. Since $\tilde{\omega}_j$ is not an imaginary root then it is not possible and the pair (Π, Γ) is C-admissible. The C-admissible subalgebra generated by $\mathfrak{h}_{\Gamma} = \mathfrak{h}_{\mathfrak{a}}$ and the elements $X_{\tilde{\omega}_j}, Y_{\tilde{\omega}_j}$ is obviously the affine algebra \mathfrak{a} . □

We can summarize the preceding results as follows

Theorem 2.4.3. *Let $\hat{\mathfrak{g}}$ be an affine algebra and Σ be an affine root system. Denote a basis of Σ by $\mathcal{B} = \{\tilde{\omega}_0, \dots, \tilde{\omega}_k\}$. The following conditions are equivalent*

- (i) *Σ graduates $\hat{\mathfrak{g}}$ and the representations $(V_0, V_{\tilde{\omega}_j})$ are irreducible.*
- (ii) *There exist a root system Δ of $\hat{\mathfrak{g}}$, a basis Π of Δ , and a subset Γ of Π such that*

$|\Pi \setminus \Gamma| \geq 2$, the pair (Π, Γ) is C -admissible and the corresponding C -admissible subalgebra whose root system is Σ is the grading subalgebra.

In the following we give an example of a Σ -gradation such that one at least among the representations $(V_0, V_{\tilde{\omega}_j})$ is not irreducible.

Example 2.4.4. Consider the affine algebra of type $A_3^{(1)}$

$$\hat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes sl_4(\mathbb{C}) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

where $\mathbb{C}[t, t^{-1}]$ is the algebra of Laurent polynomials in t and $sl_4(\mathbb{C})$ the simple Lie algebra of complex matrices of trace zero. (For details about concrete constructions of this affine algebra see [K] chap. 7). In the underlying subalgebra $\mathring{\mathfrak{g}} = sl_4(\mathbb{C})$, consider $\mathring{\mathfrak{h}}$ the Cartan subalgebra of the diagonal matrices. Denote by $(\alpha_1, \alpha_2, \alpha_3)$ the standard basis of $\mathring{\Delta} = (\mathring{\mathfrak{g}}, \mathring{\mathfrak{h}})$.

The subalgebra $\mathfrak{h} = \mathring{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}d$ is a Cartan subalgebra of $\hat{\mathfrak{g}}$. Let Δ be the root system of the pair $(\hat{\mathfrak{g}}, \mathfrak{h})$. Let δ be the root of Δ defined by
$$\begin{cases} \delta | \mathring{\mathfrak{h}} \oplus \mathbb{C}K & = 0 \\ \langle \delta, d \rangle & = 1 \end{cases}$$

Put $\alpha_0 = \delta - \alpha_1 - \alpha_2 - \alpha_3 = \delta - \theta$, the sequence $(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ is a basis of Δ and the coroot of α_0 is $\alpha_0^\vee = K - \theta^\vee$. Denote by $(X_{\alpha_i}, \alpha_i^\vee, Y_{\alpha_i})_{i=0, \dots, 3}$ the standard sl_2 -triples. Consider

the subalgebra $\mathring{\mathfrak{a}}$ of $sl_4(\mathbb{C})$ generated by the two sl_2 -triples $(X_{\alpha_1} + X_{\alpha_3}, \alpha_1^\vee + \alpha_3^\vee, Y_{\alpha_1} + Y_{\alpha_3})$ and $(X_{\alpha_2}, \alpha_2^\vee, Y_{\alpha_2})$. The algebra $\mathring{\mathfrak{a}}$ is the simple subalgebra (of type C_2) of the matrices
$$\begin{pmatrix} a & c & -d & e \\ c' & -a + f & b & d \\ -d' & b' & a - f & c \\ e' & d' & c' & -a \end{pmatrix}$$
. Denote by $\mathfrak{h}_\mathring{\mathfrak{a}}$ the Cartan subalgebra generated by $\alpha_1^\vee + \alpha_3^\vee$

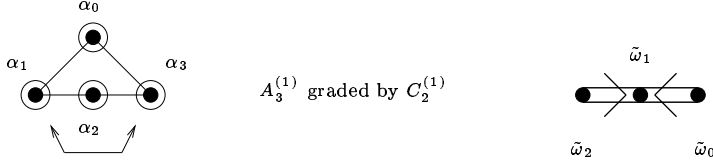
and α_2^\vee . Put $\tilde{\omega}_1 = \alpha_1 | \mathfrak{h}_\mathring{\mathfrak{a}} = \alpha_3 | \mathfrak{h}_\mathring{\mathfrak{a}}$ and $\tilde{\omega}_2 = \alpha_2 | \mathfrak{h}_\mathring{\mathfrak{a}}$; $(\tilde{\omega}_1, \tilde{\omega}_2)$ is a basis of $\mathring{\Sigma} = (\mathring{\mathfrak{a}}, \mathfrak{h}_\mathring{\mathfrak{a}})$. (For details about these calculations see [N] example 5.5.). Put $\mathfrak{a} = \mathbb{C}[t, t^{-1}] \otimes \mathring{\mathfrak{a}} \oplus \mathbb{C}K \oplus \mathbb{C}d$, then \mathfrak{a} is an affine subalgebra of $\hat{\mathfrak{g}}$ of type $C_2^{(1)}$ and $\mathfrak{h}_\mathfrak{a} = \mathfrak{h}_\mathring{\mathfrak{a}} \oplus \mathbb{C}K \oplus \mathbb{C}d$ is a Cartan subalgebra of \mathfrak{a} . Denote

by Σ the root system of $(\mathfrak{a}, \mathfrak{h}_\mathfrak{a})$ and define $\delta_\mathfrak{a}$ in $\mathfrak{h}_\mathfrak{a}^*$ by the relations
$$\begin{cases} \delta_\mathfrak{a} | \mathfrak{h}_\mathring{\mathfrak{a}} \oplus \mathbb{C}K & = 0 \\ \langle \delta_\mathfrak{a}, d \rangle & = 1 \end{cases}$$

Put $\tilde{\omega}_0 = \delta_\mathfrak{a} - (2\tilde{\omega}_1 + \tilde{\omega}_2)$; $(\tilde{\omega}_0, \tilde{\omega}_1, \tilde{\omega}_2)$ is a basis of Σ . Define as precedently $\rho : \mathfrak{h}^* \longrightarrow \mathfrak{h}_\mathfrak{a}^*$

We have $\rho(\alpha_1) = \rho(\alpha_3) = \tilde{\omega}_1$, $\rho(\alpha_2) = \tilde{\omega}_2$, $\rho(\delta) = \delta_\mathfrak{a}$, $\rho(\alpha_0) = \tilde{\omega}_0$. It is easy to verify that $\rho(\Delta) = \Sigma$. The algebra $\hat{\mathfrak{g}}$ is Σ -graded and for any $\tilde{\alpha} \in \Sigma \cup \{0\}$, we have
$$V_{\tilde{\alpha}} = \bigoplus_{\substack{\gamma \in \Delta \cup \{0\} \\ \rho(\gamma) = \tilde{\alpha}}} \hat{\mathfrak{g}}^\gamma.$$

Furthermore the set $\Gamma = \{\alpha \in \Delta, \rho(\alpha) = 0\}$ is empty, hence the C -admissible subalgebra associated to the C -admissible pair (Π, \emptyset) is $\hat{\mathfrak{g}}$ itself and \mathfrak{a} is a proper subalgebra of $\hat{\mathfrak{g}}$. One has $V_{\tilde{\omega}_1} = d_1(\alpha_1) \oplus d_1(\alpha_3)$ and the representation $(V_0, V_{\tilde{\omega}_1})$ is not irreducible. We can illustrate this example by the following diagrams :



In this example, as in the sequel of the paper, we have joined by an arrow the simple roots having the same restriction.

2.5. The general case

Let $\hat{\mathfrak{g}}$ be an affine algebra graded by an affine root system, Σ . The notations are the same as precedently. The following definition is useful :

Definition 2.5.1. *The two different roots α and β of $\Pi \setminus \Gamma$ are said Γ -connected if $(\alpha, \beta) \neq 0$ or if there exists in $S(A)$ a string whose extremities are α and β and whose vertices (except α and β) are all in Γ .*

Proposition 2.5.2. *Let α and β in $\Pi \setminus \Gamma$ be two roots which are Γ -connected, one has $\rho(\alpha) \neq \rho(\beta)$ and $(\rho(\alpha), \rho(\beta)) \neq 0$*

Proof : There exists a sequence, possibly empty, $(\alpha_1, \dots, \alpha_l)$ in Γ such that $\gamma = \alpha + \alpha_1 + \dots + \alpha_l + \beta$ is a root of Δ ; hence $\rho(\gamma) = \rho(\alpha) + \rho(\beta)$ is a root of Σ . Since $\rho(\alpha) \in \mathcal{B}$, then $2\rho(\alpha)$ is not a root and the result follows. □

Proposition 2.5.3. *Let $\tilde{\omega}$ an element of \mathcal{B} . Denote the preimages of $\tilde{\omega}$ in $\Pi \setminus \Gamma$ by $\omega_1, \dots, \omega_q$. One has*

$$\tilde{\omega}^\vee = H_1 + \dots + H_q$$

where H_j is the semi-simple element in $\mathfrak{h}(\Pi_{\omega_j})$ that graduates the simple Lie algebra \mathfrak{g}_j associated to the simple root ω_j .

Proof : One has

$$\tilde{\omega}^\vee \in [\mathfrak{a}^{\tilde{\omega}}, \mathfrak{a}^{-\tilde{\omega}}] \subset [V_{\tilde{\omega}}, V_{-\tilde{\omega}}] = \left[\bigoplus_{l=1}^q d_1(\omega_l), \bigoplus_{l=1}^q d_{-1}(\omega_l) \right] \subset \bigoplus_{l=1}^q d_0(\omega_l)$$

and the last sum is direct since, from Proposition 2.5.2., the roots ω_l are not Γ -connected in $\Pi \setminus \Gamma$. Hence the decomposition $\tilde{\omega}^\vee = h_1 + \dots + h_q$ is unique. One has $\alpha(\tilde{\omega}^\vee) = \alpha(h_l) = 0$ for all $\alpha \in \Gamma_l$, and $\omega_l(\tilde{\omega}^\vee) = \omega_l(h_l) = \tilde{\omega}(\tilde{\omega}^\vee) = 2$. We deduce that $h_l = H_l$ for all $l = 1, \dots, q$. □

Corollary 2.5.4. *If the set Γ is empty then*

$$\tilde{\omega}^\vee = \omega_1^\vee + \dots + \omega_q^\vee$$

where ω_l^\vee is the coroot in \mathfrak{h} of the simple root ω_l .

Theorem 2.5.5. *The pair (Π, Γ) is C -admissible and one can choose the C -admissible subalgebra $\hat{\mathfrak{g}}_\Gamma$ such that $\hat{\mathfrak{g}}_\Gamma$ contains the grading subalgebra \mathfrak{a} .*

Proof : Let $\tilde{\omega}$ be an element of \mathcal{B} and $\omega_1, \dots, \omega_q$ its preimages in $\Pi \setminus \Gamma$. Since

$$\mathfrak{a}^{\tilde{\omega}} \subset V_{\tilde{\omega}} = \bigoplus_{l=1}^q d_1(\omega_l) \quad \text{and} \quad \mathfrak{a}^{-\tilde{\omega}} \subset V_{-\tilde{\omega}} = \bigoplus_{l=1}^q d_{-1}(\omega_l)$$

we can write

$$X_{\tilde{\omega}} = X_1 + \dots + X_q \quad \text{and} \quad Y_{\tilde{\omega}} = Y_1 + \dots + Y_q$$

where $X_l \in d_1(\omega_l)$ and $Y_l \in d_{-1}(\omega_l)$. One has, from Proposition 2.5.3.

$$\tilde{\omega}^\vee = [X_{\tilde{\omega}}, Y_{\tilde{\omega}}] = \sum_{l=1}^q [X_l, Y_l] = \sum_{l=1}^q H_l$$

The decomposition is unique, so we have $[X_l, Y_l] = H_l$ for all l and the space $(d_0(\omega_l), d_1(\omega_l))$ is regular (section 1.1.). Since ρ is a surjection from $\Pi \setminus \Gamma$ onto \mathcal{B} , it is clear that the same is true for any $j = 0, \dots, \text{card}(\Pi \setminus \Gamma)$; hence the pair (Π, Γ) is admissible.

Suppose now there exists $j \in \{0, \dots, \text{card}(\Pi \setminus \Gamma)\}$ and $p \geq 2$ such that $d_p(\omega_j) \neq \{0\}$. This means there exists a root $\sigma \in \Delta_+$ that verifies $\sigma = p\omega_j \pmod{\Gamma}$, hence $\rho(\sigma) = p\rho(\omega_j) \in \Sigma_+$. This is impossible since $\rho(\omega_j)$ is a real root of Σ . The pair (Π, Γ) is therefore C-admissible. Consider now the affine subalgebra $\hat{\mathfrak{g}}_\Gamma$ generated by $\mathfrak{h}_\Gamma = \{h \in \mathfrak{h}, \alpha(h) = 0 \ \forall \alpha \in \Gamma\}$ and the sl_2 -triples (X_j, H_j, Y_j) . The algebra $\hat{\mathfrak{g}}_\Gamma$ contains \mathfrak{h}_α and the generators $X_{\tilde{\omega}}, Y_{\tilde{\omega}}$ ($\tilde{\omega} \in \mathcal{B}$). Hence $\hat{\mathfrak{g}}_\Gamma$ contains \mathfrak{a} . □

Proposition 2.5.6. *The root system Σ graduates $\hat{\mathfrak{g}}_\Gamma$ and the corresponding grading subalgebra is \mathfrak{a} . Moreover if we apply the preceding procedure to this grading, the associated C-admissible subalgebra is $\hat{\mathfrak{g}}_\Gamma$ itself.*

Proof : One has

$$\mathfrak{a} \subset \hat{\mathfrak{g}}_\Gamma \subset \hat{\mathfrak{g}} \quad \text{and} \quad \mathfrak{h}_\alpha \subset \mathfrak{h}_\Gamma \subset \mathfrak{h}$$

Let \mathcal{R}_Γ be the root system of the pair $(\hat{\mathfrak{g}}_\Gamma, \mathfrak{h}_\Gamma)$ and put $\bar{\rho} : \begin{array}{ccc} \mathfrak{h}^* & \longrightarrow & \mathfrak{h}_\Gamma^* \\ \alpha & \longmapsto & \alpha|_{\mathfrak{h}_\Gamma} = \bar{\alpha} \end{array}$. One knows

that $\mathcal{R}_\Gamma = \overline{\Delta} \setminus \{0\}$ and $\overline{\Pi} \setminus \overline{\Gamma}$ is a basis of \mathcal{R}_Γ (Theorem 1.5.9.). Consider $\rho_\Gamma : \begin{array}{ccc} \mathfrak{h}_\Gamma^* & \longrightarrow & \mathfrak{h}_\alpha^* \\ \bar{\alpha} & \longmapsto & \bar{\alpha}|_{\mathfrak{h}_\alpha} \end{array}$

One has $\rho = \rho_\Gamma \circ \bar{\rho}$. Since ρ is a surjection from $\Delta_+ \cup \{0\}$ onto $\Sigma_+ \cup \{0\}$ and since $\overline{\Pi} \setminus \overline{\Gamma}$ is a basis of \mathcal{R}_Γ one has $\rho_\Gamma^{-1}(\{0\}) \cap \mathcal{R}_\Gamma^+ = \emptyset$ and ρ_Γ is clearly a surjection from \mathcal{R}_Γ^+ onto Σ_+ . Put $V_{\bar{\alpha}} = \bigoplus_{\rho_\Gamma(\bar{\alpha}) = \bar{\alpha}} \hat{\mathfrak{g}}_\Gamma^{\bar{\alpha}}$ for any $\bar{\alpha} \in \Sigma \cup \{0\}$. Note that $V_0 = \hat{\mathfrak{g}}_\Gamma^0 = \mathfrak{h}_\Gamma$ and $\hat{\mathfrak{g}}_\Gamma = \bigoplus_{\bar{\alpha} \in \Sigma \cup \{0\}} V_{\bar{\alpha}}$.

Hence $\hat{\mathfrak{g}}_\Gamma$ is Σ -graded. In this new gradation we can construct a C-admissible subalgebra of $\hat{\mathfrak{g}}_\Gamma$ that contains \mathfrak{a} by the procedure of Theorem 2.5.5. Since the set $\{\bar{\alpha} \in \overline{\Pi} \setminus \overline{\Gamma}, \rho_\Gamma(\bar{\alpha}) = 0\}$ is empty, then this subalgebra is $\hat{\mathfrak{g}}_\Gamma$ itself. □

Definition 2.5.7. *If the C-admissible subalgebra of $\hat{\mathfrak{g}}$ constructed by the procedure of Theorem 2.5.5. is the algebra $\hat{\mathfrak{g}}$ itself (i.e. the set Γ is empty), then the Σ -gradation is said to be maximal in $\hat{\mathfrak{g}}$.*

Proposition 2.5.8. *Let $\hat{\mathfrak{g}}$ be an affine algebra graded by a root system Σ_1 which is associated to a grading subalgebra $\hat{\mathfrak{g}}_1$. Let Σ_2 be a grading affine root system in $\hat{\mathfrak{g}}_1$. Then Σ_2 graduates $\hat{\mathfrak{g}}$.*

Proof : Denote by $\hat{\mathfrak{g}}_2$ the grading subalgebra in $\hat{\mathfrak{g}}_1$ which is associated to Σ_2 . One has

$$\hat{\mathfrak{g}}_2 \subset \hat{\mathfrak{g}}_1 \subset \hat{\mathfrak{g}}$$

Let \mathfrak{h}_2 be a Cartan subalgebra of $\hat{\mathfrak{g}}_2$, by applying the conjugacy theorem of Cartan subalgebras ([K-P] Theorem 2) one can find a Cartan subalgebra \mathfrak{h}_1 (resp. \mathfrak{h}) of $\hat{\mathfrak{g}}_1$ (resp. $\hat{\mathfrak{g}}$) such that

$$\mathfrak{h}_2 \subset \mathfrak{h}_1 \subset \mathfrak{h}$$

In the corresponding root systems $\Delta, \Sigma_1, \Sigma_2$, one can choose basis Π, Π_1, Π_2 such that

$$\begin{aligned} \bar{\rho} : \mathfrak{h}^* &\longrightarrow \mathfrak{h}_1^* && \text{is a surjection from } \Delta_+ \cup \{0\} \text{ onto } \Sigma_1^+ \cup \{0\} \text{ (Corollary 2.3.5.) and} \\ \alpha &\longmapsto \alpha|_{\mathfrak{h}_1} = \bar{\alpha} \end{aligned}$$

$$\begin{aligned} \tilde{\rho} : \mathfrak{h}_1^* &\longrightarrow \mathfrak{h}_2^* && \text{is a surjection from } \Sigma_1^+ \cup \{0\} \text{ onto } \Sigma_2^+ \cup \{0\} \text{ (Corollary 2.3.5.). The} \\ \bar{\alpha} &\longmapsto \bar{\alpha}|_{\mathfrak{h}_2} = \tilde{\alpha} \end{aligned}$$

spaces of the Σ_1 -gradation are denoted by $V_{\bar{\alpha}}$ and one has $\hat{\mathfrak{g}} = \bigoplus_{\bar{\alpha} \in \Sigma_1 \cup \{0\}} V_{\bar{\alpha}}$. For any $\tilde{\alpha} \in \Sigma_2$,

let us set :

$$U_{\tilde{\alpha}} = \bigoplus_{\bar{\alpha}|_{\mathfrak{h}_2} = \tilde{\alpha}} V_{\bar{\alpha}}$$

It is easy to prove that $\hat{\mathfrak{g}}$ is Σ_2 -graded and that the spaces of the gradation are the $U_{\tilde{\alpha}}$'s, $\tilde{\alpha} \in \Sigma_2 \cup \{0\}$.

Remark 2.5.9. If Σ_2 is a maximal grading root system in $\hat{\mathfrak{g}}_1$, then $U_0 = V_0 = \mathfrak{h} \oplus \hat{\mathfrak{g}}^{<\Gamma>}$ where $\Gamma = \{\alpha \in \Pi, \alpha|_{\mathfrak{h}_1} = 0\}$.

□

The following theorem is now straightforward.

Theorem 2.5.10. *Let $\hat{\mathfrak{g}}$ an affine algebra and Σ be an affine root system. The two following conditions are equivalent :*

- (i) $\hat{\mathfrak{g}}$ is Σ -graded
- (ii) there exists a root system Δ of $\hat{\mathfrak{g}}$, a basis Π of Δ , a subset Γ of Π such that
 - the pair (Π, Γ) is C-admissible
 - the C-admissible subalgebra $\hat{\mathfrak{g}}_{\Gamma}$, associated to the pair (Π, Γ) is graded by Σ and Σ is maximal in $\hat{\mathfrak{g}}_{\Gamma}$.

It is now clear that the study of the Σ -gradations of the affine algebras can be reduced to the study of the maximal Σ -gradations of its C-admissible subalgebras. Recall that maximality is equivalent to say that the subset Γ of Π precendently defined is empty.

We can assume without loss of generality that the homomorphism ρ is not a bijection from Π onto $\mathcal{B} = \{\tilde{\omega}_0, \dots, \tilde{\omega}_k\}$ the basis of Σ that we have fixed once for all. Recall that it is equivalent to say that the representations $(V_0, V_{\tilde{\omega}_j})$ are not all irreducible (Proposition 2.4.1.). Such maximal gradations will be called non trivial (abbreviated *n.t.m.*).

For the trivial maximal gradations the grading subalgebra is the algebra $\hat{\mathfrak{g}}$ itself (Theorem 2.4.3.).

2.6. Investigation of the possible non trivial maximal gradations

In this section we assume that the gradation of the affine algebra $\hat{\mathfrak{g}}$ by the root system Σ is maximal non trivial. Let $\Pi^i = \rho^{-1}(\{\tilde{\omega}_i\}) \cap \Pi$ and set $H^i = \tilde{\omega}_i = \sum_{\gamma \in \Pi^i} \gamma$ for all $i = 0 \dots k$.

The following lemma plays a crucial role for the classification.

Lemma 2.6.1. *Let α and β be two different roots in Π^i , one has*

- a) $(\alpha, \beta) = 0$
- b) $\alpha(H^j) = \beta(H^j)$ for all $j = 0, \dots, k$
- c) let γ be a root in $\Pi \setminus \{\alpha\}$ such that $(\alpha, \gamma) \neq 0$, then $\gamma \notin \Pi^i$ and there exists a simple root μ such that $\rho(\mu) = \rho(\gamma)$ and $(\mu, \beta) \neq 0$. If $(\gamma, \beta) = 0$ then $\mu \neq \gamma$.

Proof : a) suppose that $(\alpha, \beta) \neq 0$ then $\alpha + \beta$ is a root of Δ , hence $\rho(\alpha + \beta) = 2\tilde{\omega}_i$ is a root of Σ . This is obviously not true since $\tilde{\omega}_i$ is a real root.

b) Since $H^j \in \mathfrak{h}_a$ one has $\alpha(H^j) = \beta(H^j) = \tilde{\omega}_i(H^j)$.

c) Let γ be a root in $\Pi \setminus \{\alpha\}$ such that $(\alpha, \gamma) \neq 0$, then a) implies that $\gamma \notin \Pi^i$. Let us set $\rho(\gamma) = \tilde{\omega}_j$ where $j \neq i$. Since $\Gamma = \emptyset$ one has $H^j = \gamma^\vee + \sum_{\xi \in \Pi^j \setminus \{\gamma\}} \xi^\vee$ (Corollary 2.5.4.) and

$d_1(\alpha) = \hat{\mathfrak{g}}^\alpha$ for $\alpha \in \Pi$. One has by Theorem 2.5.5.:

$$X_{\tilde{\omega}_i} = X_\alpha + X_\beta + \sum_{\lambda \in \Pi^i \setminus \{\alpha, \beta\}} X_\lambda$$

$$[H^j, X_{\tilde{\omega}_i}] = \tilde{\omega}_i(H^j)X_{\tilde{\omega}_i} = \alpha(H^j)X_{\tilde{\omega}_i}.$$

$$\begin{aligned} [H^j, X_{\tilde{\omega}_i}] &= [\gamma^\vee + \sum_{\xi \in \Pi^j \setminus \{\gamma\}} \xi^\vee, X_\alpha + X_\beta + \sum_{\lambda \in \Pi^i \setminus \{\alpha, \beta\}} X_\lambda] \\ &= \alpha(\gamma^\vee)X_\alpha + \beta(\gamma^\vee)X_\beta + \sum_{\xi \in \Pi^j \setminus \{\gamma\}} \alpha(\xi^\vee)X_\alpha + \sum_{\xi \in \Pi^j \setminus \{\gamma\}} \beta(\xi^\vee)X_\beta + \sum_{\xi, \lambda} \lambda(\xi^\vee)X_\lambda \end{aligned}$$

Note that $\alpha(\gamma^\vee) < 0$ and $\alpha(\xi^\vee) \leq 0$ for all $\xi \in \Pi^j$. Hence the coefficient of X_α in the previous sum (i.e. $\alpha(\gamma^\vee) + \sum_{\xi \in \Pi^j \setminus \{\gamma\}} \alpha(\xi^\vee)$) is not zero.

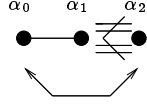
- If $\beta(\gamma^\vee) \neq 0$ (i.e. $(\beta, \gamma) \neq 0$) then we can set $\mu = \gamma$.

- If $\beta(\gamma^\vee) = 0$ (i.e. $(\beta, \gamma) = 0$) then $ad(H^j)$ acts diagonally on $X_{\tilde{\omega}_i}$ and there exists $\xi \in \Pi^j$ such that $\beta(\xi^\vee) \neq 0$. Then we can set $\mu = \xi$.

□

Proposition 2.6.2. *The algebras $A_1^{(1)}, A_2^{(1)}, A_2^{(2)}$ and $D_4^{(3)}$ do not admit any n.t.m. Σ -gradation.*

Proof : The result is straightforward for the three first algebras by applying Lemma 2.6.1. a). For $D_4^{(3)}$, note that the only possibility can be illustrated by the following arrow diagram :

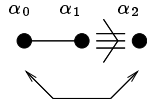


Put $\Pi^0 = \{\alpha_0, \alpha_2\}$ and $\Pi^1 = \{\alpha_1\}$. One has $\alpha_0(H^1) = \alpha_0(\alpha_1) = -1$ and $\alpha_2(H^1) = -4$. This is not possible by Lemma 2.6.1. b). □

Proposition 2.6.3. *If the algebra $G_2^{(1)}$ admits a n.t.m. Σ -gradation, then Σ is necessarily of type $A_2^{(2)}$.*

If the algebra $C_2^{(1)}$ admits a n.t.m. Σ -gradation, then Σ is necessarily of type $A_1^{(1)}$.

Proof : For $G_2^{(1)}$ the only possibility is illustrated by the following diagram



with $\alpha_1(H^0) = \alpha_1(\alpha_0 + \alpha_2) = -4$ and $\alpha_0(H^1) = \alpha_2(H^1) = -1$. Hence the generalized Cartan matrix of Σ is of type $A_2^{(2)}$.

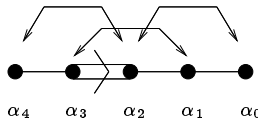
For the case $C_2^{(1)}$ the proof is similar. □

Proposition 2.6.4. *If the algebra $\hat{\mathfrak{g}}$ is of type $F_4^{(1)}, E_6^{(2)}, A_{2l}^{(2)}, D_{l+1}^{(2)}$ or $C_l^{(1)}$ and if $\rho^{-1}(\{\rho(\alpha_0)\}) \cap \Pi = \{\alpha_0\}$ then for all $i = 0, \dots, n$, then $\rho^{-1}(\{\rho(\alpha_i)\}) \cap \Pi = \alpha_i$ and the gradation is trivial.*

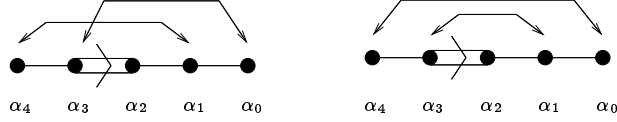
Proof : Note that the Dynkin diagrams of such algebras are linear (i.e. non circular) and without branching point. Let j_0 the smallest index ≥ 1 such that $|\rho^{-1}(\{\rho(\alpha_{j_0})\}) \cap \Pi| \geq 2$. Denote by α_{j_0+k} a root in this set. One has $k \geq 2$ (Lemma 2.6.1.a) and $\rho(\alpha_{j_0-1}) = \alpha_{j_0+k-1}$ (or α_{j_0+k+1} if this root exists) (Lemma 2.6.1.c)). It follows that $|\rho^{-1}(\{\rho(\alpha_{j_0-1})\}) \cap \Pi| \geq 2$ and this is not possible from to the choice of j_0 . □

Proposition 2.6.5. *The algebra $E_6^{(2)}$ does not admit any n.t.m. Σ -gradation. If the algebra $F_4^{(1)}$ admits a n.t.m. Σ -gradation then Σ is of type $D_4^{(3)}$.*

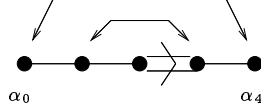
Proof : Suppose that $\hat{\mathfrak{g}}$ (of type $E_6^{(2)}$) admits a n.t.m. Σ -gradation, then one has $|\rho^{-1}(\{\rho(\alpha_0)\}) \cap \Pi| \geq 2$. If $\rho(\alpha_0) = \rho(\alpha_2)$, by applying Lemma 2.6.1.c), one has $\rho(\alpha_1) = \rho(\alpha_3)$ and $\rho(\alpha_4) = \rho(\alpha_2)$. Hence $\Pi^0 = \{\alpha_0, \alpha_2, \alpha_4\}$, $\Pi^1 = \{\alpha_1, \alpha_3\}$, $\alpha_0(H^1) = \alpha_0(\alpha_1) = -1$ and $\alpha_2(H^1) = \alpha_2(\alpha_1 + \alpha_3) = -2$; this is not possible by Lemma 2.6.1.b). This case is illustrated by the following diagram :



The other two possibilities for $E_6^{(2)}$ (illustrated by the following diagrams) are treated similarly.



For $F_4^{(1)}$, the only possibility is illustrated by this diagram and the calculus of the $\tilde{\omega}_j(H^i)$'s gives a generalized Cartan matrix of type $D_4^{(3)}$.



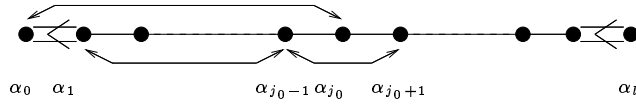
□

The other possibilities for *n.t.m.* Σ -gradation are given by the following propositions. All these proofs are very similar and use essentially Lemma 2.6.1. For this reason we only indicate an outline of proof for two of these.

Proposition 2.6.6. *The algebra $A_{2l}^{(2)}$ does not admit any *n.t.m.* Σ -gradation.*

Proof : The case $l = 1$ has been studied in Proposition 2.6.2. The case $l = 1$ is straightforward by Lemma 2.6.1.b).

Let us assume $l \geq 3$ and choose notations such that $\alpha_0 \in \Pi^0$ and $\alpha_1 \in \Pi^1$. Denote by j_0 the smallest non zero integer such that $\rho(\alpha_0) = \rho(\alpha_{j_0})$. We clearly have $j_0 \geq 2$. If $j_0 < l$ then $\rho(\alpha_1) = \rho(\alpha_{j_0-1}) = \rho(\alpha_{j_0+1})$ by Lemma 2.6.1.c), one has $\alpha_0(H^1) = -1$ and $\alpha_{j_0}(H^1) = -2$. This is not possible by Lemma 2.6.1.b) then $j_0 = l$. It follows that $\rho(\alpha_1) = \rho(\alpha_{l-1})$. This is not possible by Lemma 2.6.1.b) and the fact that $\alpha_1(H^0) = -2$ and $\alpha_{l-1}(H^0) = -1$. The following diagram illustrates this case.

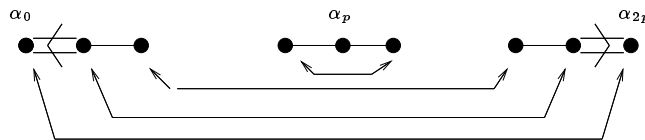


□

Proposition 2.6.7.

- If $l + 1$ is even then the algebra $D_{l+1}^{(2)}$ does not admit any *n.t.m.* Σ -gradation
- If $l + 1$ is odd and if the algebra $D_{l+1}^{(2)}$ admits a *n.t.m.* Σ -gradation then Σ is of type $A_{2p}^{(2)}$ with $p = \frac{l}{2}$.

Proof : Note that $l \geq 2$. By Lemma 2.6.1.b), we prove that if $|\Pi^0| \geq 2$ then $\Pi^0 = \{\alpha_0, \alpha_l\}$. We prove then by induction that for all $j = 0, \dots, [\frac{l}{2}]$, one has $\rho^{-1}(\{\rho(\alpha_j)\}) = \{\alpha_j, \alpha_{l-j}\}$. If $l + 1 = 2p$ then $\rho(\alpha_p) = \rho(\alpha_{p-1})$. This is impossible by Lemma 2.6.1.a). The case $l + 1 = 2p + 1$ is illustrated by the following diagram :



The computation of the $\tilde{\omega}_j(H^i)$'s gives the type of the Cartan matrix of the possible

grading root system : $A_{2p}^{(2)}$.

□

Proposition 2.6.8. *If the algebra $C_l^{(1)}$ admits a n.t.m. Σ -gradation then Σ is of type $A_1^{(1)}$ or $C_p^{(1)}$ with $l = 2pm$.*

Proposition 2.6.9. *The algebra $E_8^{(1)}$ does not admit any n.t.m. Σ -gradation. If the algebra $E_7^{(1)}$ admits a n.t.m. Σ -gradation then Σ is of type $E_6^{(2)}$. If the algebra $E_6^{(1)}$ admits a n.t.m. Σ -gradation then Σ is of type $D_4^{(3)}$.*

Proposition 2.6.10. *If the algebra $B_l^{(1)}$ admits a n.t.m. Σ -gradation then Σ is of type $D_l^{(2)}$ or $A_{2p}^{(2)}$ with $l = 2p + 1$.*

Proposition 2.6.11. *If the algebra $D_l^{(1)}$ admits a n.t.m. Σ -gradation then Σ is of type $B_{l-1}^{(1)}$ or $A_{2p+1}^{(2)}$ or $A_{2p}^{(2)}$ with $l = 2p + 2$ in these two last cases.*

Proposition 2.6.12. *If the algebra $A_{2l-1}^{(2)}$ admits a n.t.m. Σ -gradation then Σ is of type $A_{2l-2}^{(2)}$.*

Let us now study the “circular” case $A_l^{(1)}$

Proposition 2.6.13. *If the algebra $A_l^{(1)}$ ($l \geq 3$) admits a n.t.m. Σ -gradation and if $\rho^{-1}(\{\rho(\alpha_0)\}) \cap \Pi = \{\alpha_0\}$ then for all $j \in \{0, \dots, l\}$ one has $\rho^{-1}(\{\rho(\alpha_j)\}) \cap \Pi = \{\alpha_j, \alpha_{l+1-j}\}$ (one sets $\alpha_{l+1} = \alpha_0$). The integer l is odd and if we set $l = 2p + 1$, then Σ is of type $C_{p+1}^{(1)}$.*

Remark 2.6.14. Because of the circularity of the diagram, one can replace α_0 with any other simple root and change the indices. The result is the same.

In the following we introduce the hypothesis :

($\star\star$) It does not exist no simple root α_j such that $\rho^{-1}(\{\rho(\alpha_j)\}) \cap \Pi = \{\alpha_j\}$

Proposition 2.6.15. *If the algebra $A_l^{(1)}$ admits a n.t.m. Σ -gradation and if the hypothesis ($\star\star$) is verified, then Σ is of type $A_p^{(1)}$ with $l = mp - 1$ or $C_p^{(1)}$ with $l = 2mp + 1$.*

All the preceding propositions are summarized in **Table B**.

2.7. The effective non trivial maximal gradations

We prove in this section that the possible gradations given in section 2.6. are effective gradations.

Let $\hat{\mathfrak{g}}$ be an affine algebra and use the standard notations from the preceding sections. Consider a partition of a basis Π as in one of the previous diagrams. That it is to say that $\Pi = \bigsqcup_{i=0}^k \Pi^i$ where Π^i is either a simple root or a set of orthogonal simple roots connected together by arrows. Let us set $\alpha_0 \in \Pi^0$. Put

$$H^i = \sum_{\alpha \in \Pi^i} \check{\alpha}, \quad X_i = \sum_{\alpha \in \Pi^i} X_\alpha, \quad Y_i = \sum_{\alpha \in \Pi^i} Y_\alpha$$

where $(X_\alpha, \check{\alpha}, Y_\alpha)$ is a sl_2 -triple. The elements H^0, \dots, H^k are linearly independent. For any i , and any $\alpha \in \Pi^i$, one has $\alpha(H^i) = 2$ and (X_i, H^i, Y_i) is a sl_2 -triple. Since the subset Π^i

verifies Lemma 2.6.1., one has $\alpha(H^j) = \beta(H^j)$ for all α, β in Π^i and for any j .

The matrix $M = (m_{ij})_{i,j=0,\dots,k}$ where $m_{ij} = \alpha(H^i)$ for any $\alpha \in \Pi^j$ is a **generalized Cartan matrix of affine type** (cf section 2.6.). Set $\overset{\circ}{M} = (m_{ij})_{i,j=1,\dots,k}$ the associated invertible matrix.

Denote by K' the element of \mathfrak{h} defined by $K' = \sum_{j=0}^k a_j \check{H}^j$ where a_j are relatively prime integers such that ${}^t M^t(a_0, \dots, a_k) = 0$ ([K] §6.1. §6.2 p. 79-80).

The following lemma is straightforward.

Lemma 2.7.1. *One has $\alpha(K') = 0$ for all $\alpha \in \Pi$ and K' is an element of the one-dimensional center $\mathbb{C}K$ of $\hat{\mathfrak{g}}$.*

Remark 2.7.2. Since $a_0 = 1$ ([K] p. 79), the elements K', H^1, \dots, H^k are linearly independent.

Since $\overset{\circ}{A}$ and $\overset{\circ}{M}$ are invertible, the following proposition is easy to prove :

Proposition 2.7.3. *There exists in \mathfrak{h} an element d' unique modulo K such that $\alpha(d') = 1$ for all $\alpha \in \Pi^0$ and $\alpha(d') = 0$ for all $\alpha \in \Pi \setminus \Pi^0$. The elements d', K', H^1, \dots, H^k are linearly independent.*

Consider now the subalgebra \mathfrak{c} of \mathfrak{h} generated by d', H^0, \dots, H^k (or by d', K', H^1, \dots, H^k).

Denote by $\overline{\omega}_j$ the restriction to \mathfrak{c} of any root in Π^j .

Lemma 2.7.4. *The elements $(\overline{\omega}_0, \dots, \overline{\omega}_k)$ of \mathfrak{c}^* are linearly independent.*

Proof : The sequence of coefficients (x_0, \dots, x_k) such that $\sum_{j=0}^k x_j \overline{\omega}_j = 0$ is solution of the linear system

$$\begin{cases} \sum_{j=0}^k x_j \overline{\omega}_j(d') & = 0 \\ \sum_{j=0}^k x_j \overline{\omega}_j(H^i) & = 0 \quad \forall i = 0, \dots, k \end{cases}$$

□

The following proposition is now straightforward.

Proposition 2.7.5. *The triple $(\mathfrak{c}, (\overline{\omega}_j)_{j=0,\dots,k}, (H^j)_{j=0,\dots,k})$ is a realization of the generalized Cartan matrix M .*

Proposition 2.7.6. *The subalgebra \mathfrak{a} , generated by \mathfrak{c} and the elements X_j, Y_j , is an affine subalgebra whose Cartan matrix is M .*

Proof : It is enough to prove that Serre's relations are verified (i.e. $(adY_i)^{1-m_{ij}}(Y_j) = 0$ and $(adX_i)^{1-m_{ij}}(X_j) = 0$ for $i \neq j$). Let i, j be two different integers.

- If the partition of Π contains more than two subsets (i.e. the order of $M \geq 3$) then we can consider the finite dimensional semi-simple algebra, \mathfrak{l} , generated by the sl_2 -triples $(X_\alpha, \check{\alpha}, Y_\alpha)$ for $\alpha \in \Pi^i$ and $(X_\beta, \check{\beta}, Y_\beta)$ for $\beta \in \Pi^j$. The Lie algebra \mathfrak{l} is a sl_2 -module under the action of (X_i, H^i, Y_i) . The element Y_j is primitive of weight $-m_{ij}$, it generates an irreducible $(1 - m_{ij})$ -dimensional submodule and one has $ad(Y_i)^{1-m_{ij}}(Y_j) = 0$ ([Se]). The

second equality is proved similarly.

- If the partition contains exactly two subsets (i.e. $\Pi = \Pi^0 \cup \Pi^1$), the previous argument is not correct. Let us prove that $(adY_0)^{1-m_{01}}(Y_1) = 0$. The Lie algebra $\hat{\mathfrak{g}}$ is a sl_2 -module under the action of (X_0, H^0, Y_0) . The element Y_1 is primitive of weight $-m_{01}$. One has $(adY_0)^j(Y_1) = \sum_{\beta \in \Pi^1} (adY_0)^j(Y_\beta)$ for any integer j . Each term of this sum is in $\sum_{\alpha_1, \dots, \alpha_j \in \Pi^0} \hat{\mathfrak{g}}^{\beta + \alpha_1 + \dots + \alpha_j}$. Since $\Pi^0 \cap \Pi^1 = \emptyset$, one has $(adY_0)^j(Y_\beta) = 0$ for β fixed in Π^1 and j great enough. Put $N = \text{Inf}\{j, (adY_0)^j(Y_1) = 0\}$, one has $N \geq 1$. Put $v_j = \frac{1}{j!}(adY_0)^j(Y_1)$, $j \geq 0$. The following relations are verified $\begin{cases} (adY_0)(v_j) = (j+1)v_{j+1} & (1) \\ (adX_0)(v_j) = (-m_{01} - j + 1)v_{j-1} & (2) \end{cases}$. The integer N is such that $v_N = 0$ and $v_{N-1} \neq 0$. From (1), one deduces that $v_j = 0$ for all $j \geq N$ and from (2), that $-m_{01} - N + 1 = 0$ therefore $N = 1 - m_{01}$. And the result follows. The others equalities are proved the same way. □

Proposition 2.7.7. *The algebra $\hat{\mathfrak{g}}$ is an integrable \mathfrak{a} -module.*

Proof : The algebra $\hat{\mathfrak{g}}$ is a \mathfrak{c} -diagonalisable \mathfrak{a} -module and the weight-spaces are $V_\omega = \bigoplus_{\rho(\gamma)=\omega} \hat{\mathfrak{g}}^\gamma$ for $\omega \in \mathfrak{c}$. Let us prove that X_j, Y_j are locally nilpotent for any j . It is enough to prove they are locally nilpotent on the generators of $\hat{\mathfrak{g}}$ ([K] Lemma 3.4. and Lemma 3.5.). Let j be a fixed integer in $\{0, \dots, k\}$. Recall that $X_j = \sum_{\alpha \in \Pi^j} X_\alpha$ (resp. $Y_j = \sum_{\alpha \in \Pi^j} Y_\alpha$), where X_α, Y_α are associated to the simple root α . Let $h \in \mathfrak{h}$. One has

$$(adX_j)^2(h) = \sum_{\alpha \in \Pi^j} \left(\sum_{\beta \in \Pi^j} -\beta(h)[X_\alpha, X_\beta] \right) = - \sum_{\alpha, \beta \in \Pi^j} \beta(h)[X_\alpha, X_\beta]$$

As roots of Π^j are orthogonal, we have $(adX_j)^2(h) = 0$.

Let $i_0 = 0, \dots, n$. If $\alpha_{i_0} \in \Pi^j$ one has clearly $(adX_j)(X_{\alpha_{i_0}}) = 0$. If $\alpha_{i_0} \notin \Pi^j$, for any integer N one has $(adX_j)^N(X_{\alpha_{i_0}}) \in \sum \hat{\mathfrak{g}}^{\alpha_{i_0} + \beta}$ where β is the sum of N roots of Π^j all different from α_{i_0} . For N great enough, $\alpha_{i_0} + \beta$ is not a root of Δ hence $(adX_j)^N(X_{\alpha_{i_0}}) = 0$.

Let us prove now that $(adX_j)^3(Y_{\alpha_{i_0}}) = 0$.

If $\alpha_{i_0} \notin \Pi^j$ then $(adX_j)(Y_{\alpha_{i_0}}) = \sum_{\alpha \in \Pi^j} [X_\alpha, Y_{\alpha_{i_0}}] = 0$.

If $\alpha_{i_0} \in \Pi^j$ then $(adX_j)(Y_{\alpha_{i_0}}) = \sum_{\alpha \in \Pi^j} [X_\alpha, Y_{\alpha_{i_0}}] = \alpha_{i_0}^\vee$. One deduces that

$$(adX_j)^2(Y_{\alpha_{i_0}}) = \sum_{\alpha \in \Pi^j} [X_\alpha, \alpha_{i_0}^\vee] = -2X_{\alpha_{i_0}}$$

hence $(adX_j)^3(Y_{\alpha_{i_0}}) = 0$. □

Proposition 2.7.8. *The root system Σ is the set $\rho(\Delta)$.*

Proof : We know that $\Sigma \subset \rho(\Delta)$, let us prove the converse inclusion. As in Theorem 1.5.8. we show that for any $\alpha \in \Delta$, there exists an integer n and a root $\bar{\omega}$ in Σ such that $\rho(\alpha) = n\bar{\omega}$. If $\bar{\omega}$ is an imaginary root, then $\rho(\alpha) \in \Sigma$. If $\bar{\omega}$ is a real root then up to

conjugation by an element of the Weyl group of \mathfrak{a} , we can assume that $\bar{\omega}$ is a simple root. Assume that n is a positive integer. This means that α is the sum of n roots of Π whose restrictions to \mathfrak{c} are $\bar{\omega}$. Since such simple roots are orthogonal, this is impossible if $n \geq 2$. Hence $\rho(\alpha) = \bar{\omega}$ and $\rho(\Delta) \subset \Sigma$.

□

Theorem 2.7.9. *The root system Σ graduates the algebra $\hat{\mathfrak{g}}$ and \mathfrak{a} is the grading subalgebra.*

Proof : Let $\bar{\alpha} \in \Sigma \cup \{0\}$, set

$$V_{\bar{\alpha}} = \bigoplus_{\rho(\gamma)=\bar{\alpha}} \hat{\mathfrak{g}}^{\gamma}$$

One has clearly $V_0 = \mathfrak{h}$ and $V_{\bar{\alpha}} = \{x \in \hat{\mathfrak{g}}, [h, x] = \bar{\alpha}(h)x \ \forall h \in \mathfrak{c}\}$. Since $\rho(\Delta) = \Sigma$, we have $\hat{\mathfrak{g}} = \bigoplus_{\bar{\alpha} \in \Sigma \cup \{0\}} V_{\bar{\alpha}}$ because $\rho(\Delta) = \Sigma$.

□

Finally, the diagrams in the table B are exactly those of the *n.t.m.* Σ -gradations of $\hat{\mathfrak{g}}$ and the classification is complete.

TABLE A: Admissible subalgebras

At the left hand is the Dynkin diagram of the affine algebra $\hat{\mathfrak{g}}$ and at the right hand is the Dynkin diagram of the corresponding admissible (eventually C-admissible) \mathfrak{a} .

TABLE A₁

The algebra $\hat{\mathfrak{g}}$ is of type **(Aff1)**

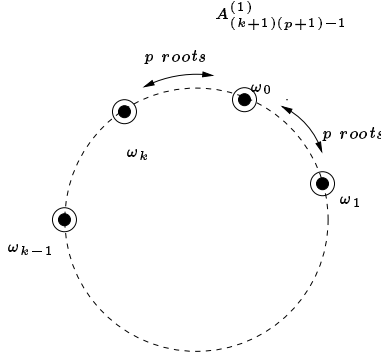
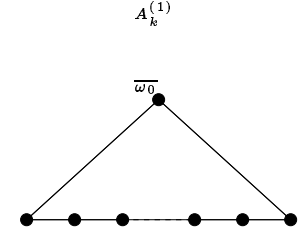
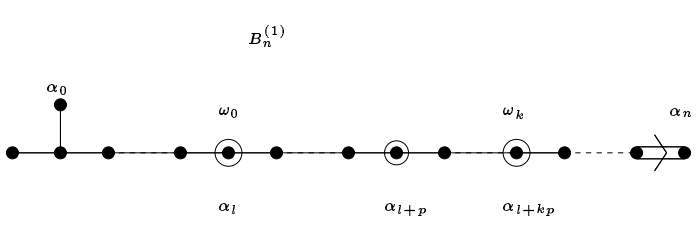
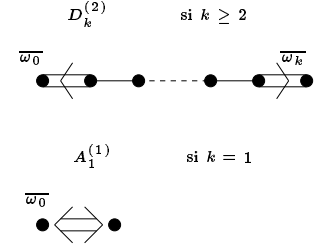
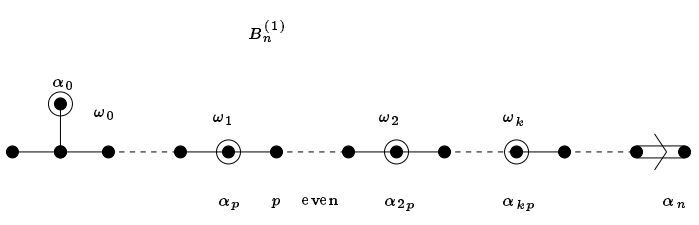
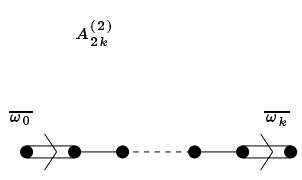
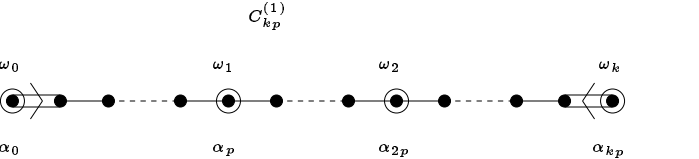
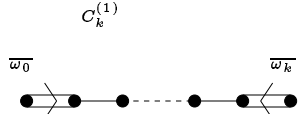
 <p style="text-align: center;">$A_{(k+1)(p+1)-1}^{(1)}$</p> <p style="text-align: center;">C-admissible</p>	 <p style="text-align: center;">$A_k^{(1)}$</p>
 <p style="text-align: center;">$B_n^{(1)}$</p> <p style="text-align: center;">C-admissible if $p = 1$</p>	 <p style="text-align: center;">$D_k^{(2)}$ si $k \geq 2$</p> <p style="text-align: center;">$A_1^{(1)}$ si $k = 1$</p>
 <p style="text-align: center;">$B_n^{(1)}$</p> <p style="text-align: center;">C-admissible</p>	 <p style="text-align: center;">$A_{2k}^{(2)}$</p>
 <p style="text-align: center;">$C_{kp}^{(1)}$</p>	 <p style="text-align: center;">$C_k^{(1)}$</p>

TABLE A₁ (following)

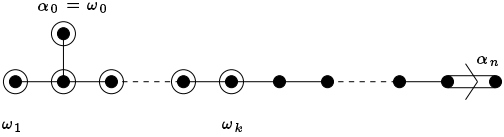
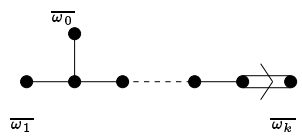
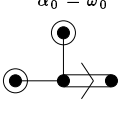
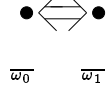
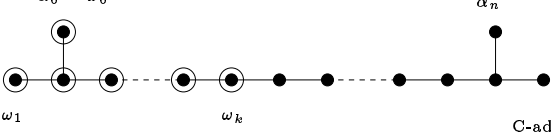
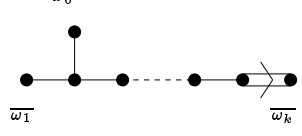
<p style="text-align: center;">$E_n^{(1)}$</p>  <p style="text-align: right;">C-admissible</p>	<p style="text-align: center;">$B_k^{(1)}$</p> 
<p style="text-align: center;">$B_3^{(1)}$</p>  <p style="text-align: right;">C-admissible</p>	<p style="text-align: center;">$A_1^{(1)}$</p> 
<p style="text-align: center;">$D_n^{(1)}$</p>  <p style="text-align: right;">C-admissible</p>	<p style="text-align: center;">$B_k^{(1)}$</p> 

TABLE A₁ (continued)

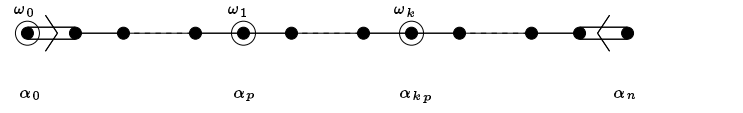
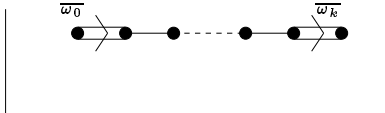
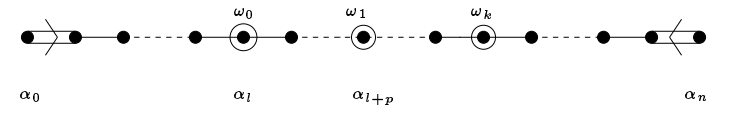
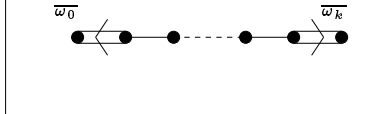
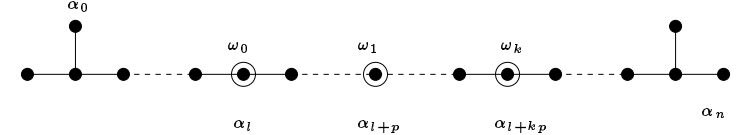
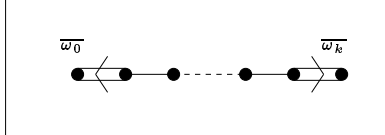
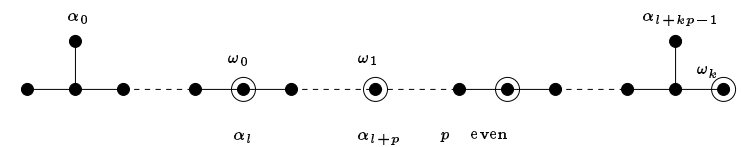
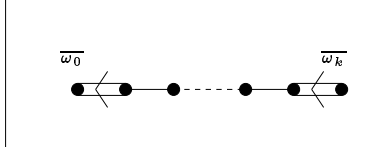
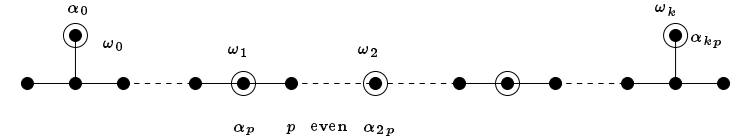
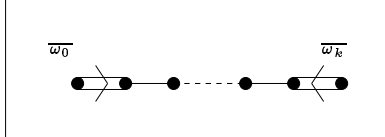

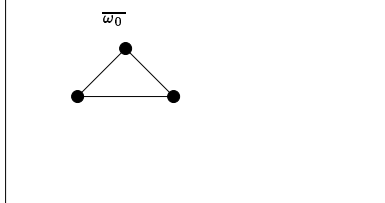
<p style="text-align: center;">$C_n^{(1)}$</p>  <p style="text-align: center;">$\alpha_0 \qquad \qquad \alpha_p \qquad \qquad \alpha_{kp} \qquad \qquad \alpha_n$</p>	<p style="text-align: center;">$A_{2k}^{(2)}$</p>  <p style="text-align: center;">$\overline{\omega_0} \qquad \qquad \overline{\omega_k}$</p>
<p style="text-align: center;">$C_n^{(1)}$</p>  <p style="text-align: center;">$\alpha_0 \qquad \qquad \alpha_l \qquad \qquad \alpha_{l+p} \qquad \qquad \alpha_k \qquad \qquad \alpha_n$</p>	<p style="text-align: center;">$D_k^{(2)}$</p>  <p style="text-align: center;">$\overline{\omega_0} \qquad \qquad \overline{\omega_k}$</p>
<p style="text-align: center;">$D_n^{(1)}$ C-admissible if $p = 1$</p>  <p style="text-align: center;">$\alpha_0 \qquad \qquad \alpha_l \qquad \qquad \alpha_{l+p} \qquad \qquad \alpha_{l+kp} \qquad \qquad \alpha_n$</p>	<p style="text-align: center;">$D_k^{(2)}$</p>  <p style="text-align: center;">$\overline{\omega_0} \qquad \qquad \overline{\omega_k}$</p>
<p style="text-align: center;">$D_n^{(1)}$</p>  <p style="text-align: center;">$\alpha_0 \qquad \qquad \alpha_l \qquad \qquad \alpha_{l+p} \qquad \qquad \alpha_{l+kp-1} \qquad \qquad \alpha_k$ $p \text{ even}$</p>	<p style="text-align: center;">$A_{2k}^{(2)}$</p>  <p style="text-align: center;">$\overline{\omega_0} \qquad \qquad \overline{\omega_k}$</p>
<p style="text-align: center;">$D_{kp}^{(1)}$ C-admissible</p>  <p style="text-align: center;">$\alpha_0 \qquad \qquad \alpha_p \qquad \qquad \alpha_{2p} \qquad \qquad \alpha_{kp}$ $p \text{ even}$</p>	<p style="text-align: center;">$C_k^{(1)}$</p>  <p style="text-align: center;">$\overline{\omega_0} \qquad \qquad \overline{\omega_k}$</p>
<p style="text-align: center;">$E_6^{(1)}$ C-admissible</p>  <p style="text-align: center;">$\omega_1 \qquad \qquad \omega_2$ ω_0 α_0</p>	<p style="text-align: center;">$A_2^{(1)}$</p>  <p style="text-align: center;">$\overline{\omega_0}$</p>

TABLE A₁ (continued)

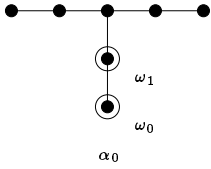
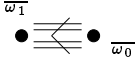
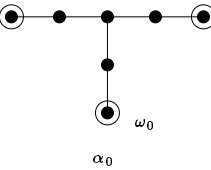
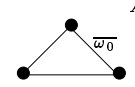
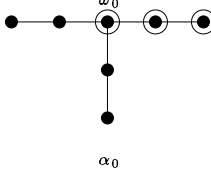
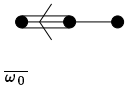
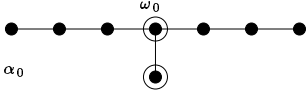
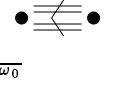
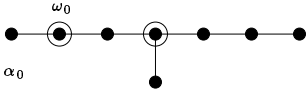
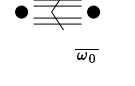
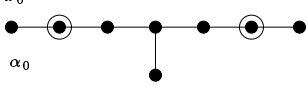
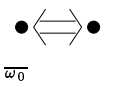
 <p style="text-align: center;">α_0</p>	$E_6^{(1)}$	 <p style="text-align: center;">$A_2^{(2)}$</p>
 <p style="text-align: center;">α_0</p>	$E_6^{(1)}$	 <p style="text-align: center;">$A_2^{(1)}$</p>
 <p style="text-align: center;">α_0</p>	$E_6^{(1)}$	 <p style="text-align: center;">$G_2^{(1)}$</p>
 <p style="text-align: center;">α_0</p>	$E_7^{(1)}$	 <p style="text-align: center;">$A_2^{(2)}$</p>
 <p style="text-align: center;">α_0</p>	$E_7^{(1)}$	 <p style="text-align: center;">$A_2^{(2)}$</p>
 <p style="text-align: center;">α_0</p>	$E_7^{(1)}$	 <p style="text-align: center;">$A_1^{(1)}$</p>

TABLE A₁ (continued)

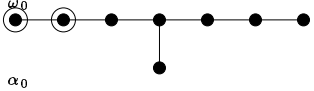

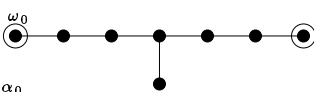

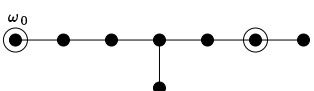

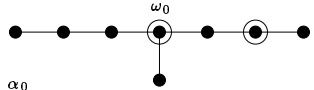

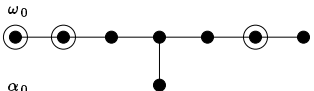
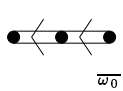
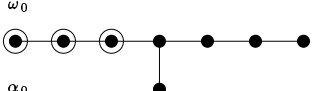
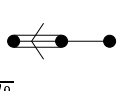
<p style="text-align: right;">$E_7^{(1)}$</p>  <p>α_0</p>	<p style="text-align: right;">$A_2^{(2)}$</p>  <p>$\overline{\omega_0}$</p>
<p style="text-align: right;">$E_7^{(1)}$</p>  <p>α_0</p>	<p style="text-align: right;">$A_2^{(1)}$</p>  <p>$\overline{\omega_0}$</p>
<p style="text-align: right;">$E_7^{(1)}$</p>  <p>α_0</p>	<p style="text-align: right;">$A_2^{(2)}$</p>  <p>$\overline{\omega_0}$</p>
<p style="text-align: right;">$E_7^{(1)}$</p>  <p>α_0</p>	<p style="text-align: right;">$A_2^{(2)}$</p>  <p>$\overline{\omega_0}$</p>
<p style="text-align: right;">$E_7^{(1)}$</p>  <p>α_0</p>	<p style="text-align: right;">$A_4^{(2)}$</p>  <p>$\overline{\omega_0}$</p>
<p style="text-align: right;">$E_7^{(1)}$</p>  <p>α_0</p>	<p style="text-align: right;">$G_2^{(1)}$</p>  <p>$\overline{\omega_0}$</p>

TABLE A₁ (continued)

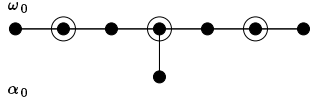
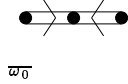
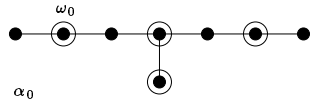
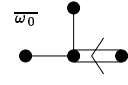
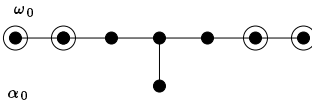
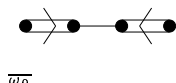
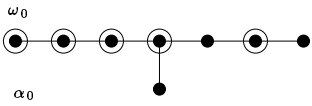
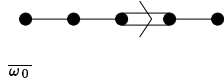
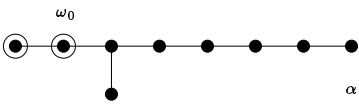
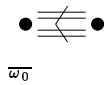
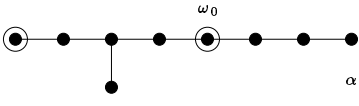
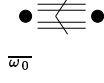
<p style="text-align: right;">$E_7^{(1)}$</p> 	<p style="text-align: right;">$C_2^{(1)}$</p> 
<p style="text-align: right;">$E_7^{(1)}$</p> 	<p style="text-align: right;">$A_5^{(2)}$</p> 
<p style="text-align: right;">$E_7^{(1)}$</p>  <p style="text-align: center;">C-admissible</p>	<p style="text-align: right;">$C_3^{(1)}$</p> 
<p style="text-align: right;">$E_7^{(1)}$</p>  <p style="text-align: center;">C-admissible</p>	<p style="text-align: right;">$F_4^{(1)}$</p> 
<p style="text-align: right;">$E_8^{(1)}$</p>  <p style="text-align: center;">C-admissible</p>	<p style="text-align: right;">$A_2^{(2)}$</p> 
<p style="text-align: right;">$E_8^{(1)}$</p> 	<p style="text-align: right;">$A_2^{(2)}$</p> 

TABLE A₁ (continued)

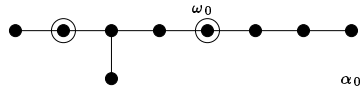

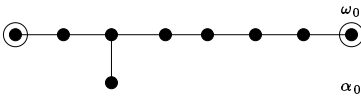

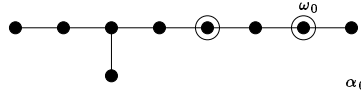

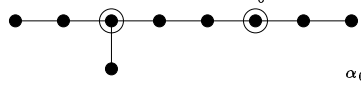

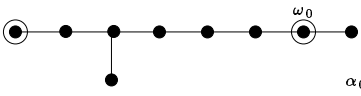

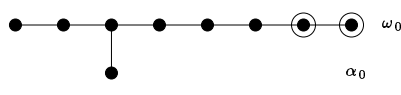

<p style="text-align: right;">$E_8^{(1)}$</p>  <p style="text-align: right;">α_0</p>	<p style="text-align: right;">$A_1^{(1)}$</p> 
<p style="text-align: right;">$E_8^{(1)}$</p>  <p style="text-align: right;">α_0</p>	<p style="text-align: right;">$A_2^{(2)}$</p> 
<p style="text-align: right;">$E_8^{(1)}$</p>  <p style="text-align: right;">α_0</p>	<p style="text-align: right;">$A_2^{(2)}$</p> 
<p style="text-align: right;">$E_8^{(1)}$</p>  <p style="text-align: right;">α_0</p>	<p style="text-align: right;">$A_2^{(2)}$</p> 
<p style="text-align: right;">$E_8^{(1)}$</p>  <p style="text-align: right;">α_0</p>	<p style="text-align: right;">$A_1^{(1)}$</p> 
<p style="text-align: right;">$E_8^{(1)}$</p>  <p style="text-align: right;">α_0</p>	<p style="text-align: right;">$A_2^{(2)}$</p> 

TABLE A₁ (continued)

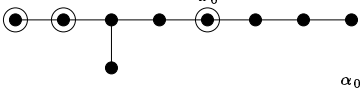
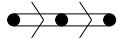
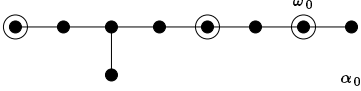
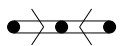
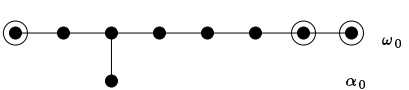

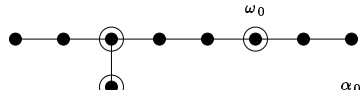

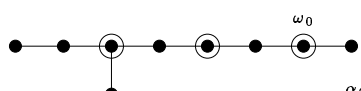

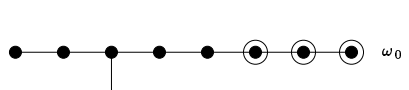

 <p style="text-align: right;">$E_8^{(1)}$ α_0</p>	C-admissible	 <p style="text-align: right;">$A_4^{(2)}$ $\overline{\omega_0}$</p>
 <p style="text-align: right;">$E_8^{(1)}$ α_0</p>		 <p style="text-align: right;">$C_2^{(1)}$ $\overline{\omega_0}$</p>
 <p style="text-align: right;">$E_8^{(1)}$ α_0</p>		 <p style="text-align: right;">$A_4^{(2)}$ $\overline{\omega_0}$</p>
 <p style="text-align: right;">$E_8^{(1)}$ α_0</p>	C-admissible	 <p style="text-align: right;">$D_4^{(3)}$ $\overline{\omega_0}$</p>
 <p style="text-align: right;">$E_8^{(1)}$ α_0</p>		 <p style="text-align: right;">$G_2^{(1)}$ $\overline{\omega_0}$</p>
 <p style="text-align: right;">$E_8^{(1)}$ α_0</p>	C-admissible	 <p style="text-align: right;">$G_2^{(1)}$ $\overline{\omega_0}$</p>

TABLE A₁ (continued)

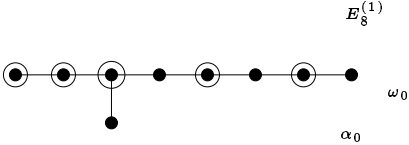
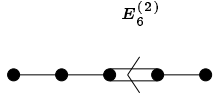
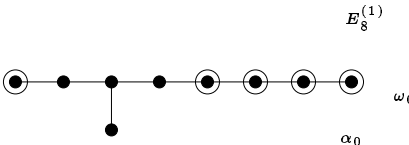
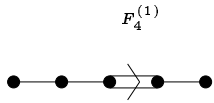
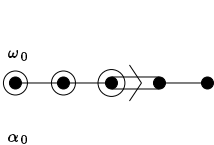
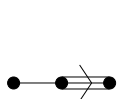
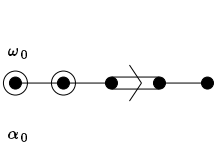

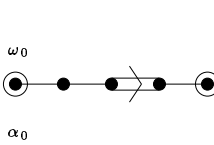
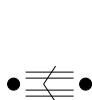
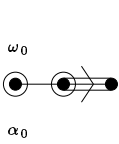

 <p style="text-align: right;">$E_8^{(1)}$ ω_0 α_0</p>	C-admissible	 <p style="text-align: center;">$E_6^{(2)}$ $\overline{\omega_0}$</p>
 <p style="text-align: right;">$E_8^{(1)}$ ω_0 α_0</p>	C-admissible	 <p style="text-align: center;">$F_4^{(1)}$ $\overline{\omega_0}$</p>
 <p style="text-align: right;">$F_4^{(1)}$ ω_0 α_0</p>	C-admissible	 <p style="text-align: center;">$G_2^{(1)}$ $\overline{\omega_0}$</p>
 <p style="text-align: right;">$F_4^{(1)}$ ω_0 α_0</p>		 <p style="text-align: center;">$A_2^{(2)}$ $\overline{\omega_0}$</p>
 <p style="text-align: right;">$F_4^{(1)}$ ω_0 α_0</p>		 <p style="text-align: center;">$A_2^{(2)}$ $\overline{\omega_0}$</p>
 <p style="text-align: right;">$G_2^{(1)}$ ω_0 α_0</p>		 <p style="text-align: center;">$A_2^{(2)}$ $\overline{\omega_0}$</p>

TABLE A₂

The algebra $\hat{\mathfrak{g}}$ is of type (Aff2)

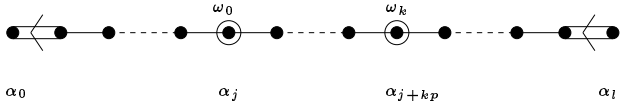
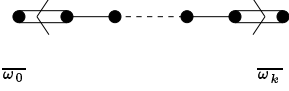
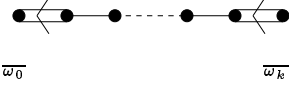
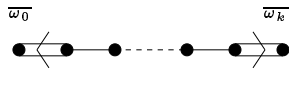
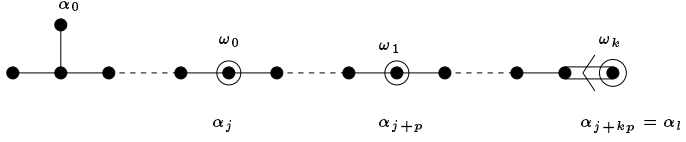
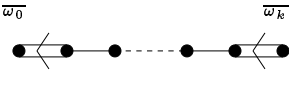
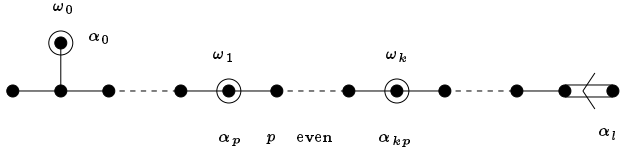

$A_{2l}^{(2)}$  <p>α_0 α_j α_{j+kp} α_l</p>	$D_{k+1}^{(2)}$  <p>$\overline{\omega}_0$ $\overline{\omega}_k$</p>
$A_{2l}^{(2)}$  <p>α_0 α_j α_{j+p} $\alpha_{j+kp} = \alpha_l$</p>	$A_{2k}^{(2)}$  <p>$\overline{\omega}_0$ $\overline{\omega}_k$</p>
$A_{2l-1}^{(2)}$  <p>α_0 α_j α_{j+kp} α_l</p>	$D_{k+1}^{(2)}$  <p>$\overline{\omega}_0$ $\overline{\omega}_k$</p>
$A_{2l-1}^{(2)}$  <p>α_0 α_j α_{j+p} $\alpha_{j+kp} = \alpha_l$</p>	$A_{2k}^{(2)}$  <p>$\overline{\omega}_0$ $\overline{\omega}_k$</p>
$A_{2l-1}^{(2)}$  <p>α_0 α_p p even α_{kp} α_l</p>	$A_{2k}^{(2)}$  <p>$\overline{\omega}_k$ $\overline{\omega}_0$</p>

TABLE A₂ (continued)

<p style="text-align: center;">$A_{2kp-1}^{(2)}$</p> <p style="text-align: center;">C-admissible</p>	<p style="text-align: center;">$C_k^{(1)}$</p>
<p style="text-align: center;">$D_{l+1}^{(2)}$</p> <p style="text-align: center;">C-admissible if $p = 1$</p>	<p style="text-align: center;">$D_{k+1}^{(2)}$</p>
<p style="text-align: center;">$E_6^{(2)}$</p>	<p style="text-align: center;">$A_2^{(2)}$</p>
<p style="text-align: center;">$E_6^{(2)}$</p>	<p style="text-align: center;">$A_2^{(2)}$</p>
<p style="text-align: center;">$E_6^{(2)}$</p>	<p style="text-align: center;">$A_2^{(2)}$</p>
<p style="text-align: center;">$E_6^{(2)}$</p> <p style="text-align: center;">C-admissible</p>	<p style="text-align: center;">$A_2^{(2)}$</p>
<p style="text-align: center;">$E_6^{(2)}$</p> <p style="text-align: center;">C-admissible</p>	<p style="text-align: center;">$A_4^{(2)}$</p>

Table B

The Dynkin diagram of $\hat{\mathfrak{g}}$ takes place in the left column (with arrows joining roots which have the same restriction) while the right column contains the Dynkin diagram of the grading subalgebra \mathfrak{a} .

$G_2^{(1)}$	$A_2^{(2)}$
	$\rho(\alpha_0)$
$F_4^{(1)}$	$D_4^{(3)}$
	$\rho(\alpha_0)$
$D_{p+1}^{(2)}$	$A_{2p+2}^{(2)}$
	$\rho(\alpha_0)$
$E_7^{(1)}$	$E_6^{(2)}$
	$\rho(\alpha_0)$
$E_6^{(1)}$	$F_4^{(1)}$
	$\rho(\alpha_0)$
$E_6^{(1)}$	$D_4^{(3)}$
	$\rho(\alpha_0)$
$B_3^{(1)}$	$G_2^{(1)}$
	$\rho(\delta) = \delta_{\mathfrak{a}}$

TABLE B (continued)

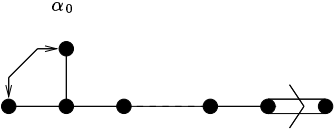
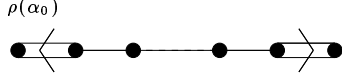
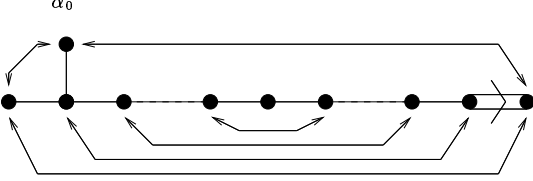
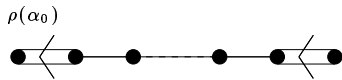
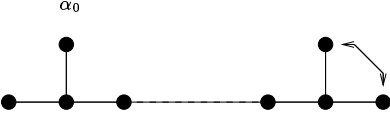
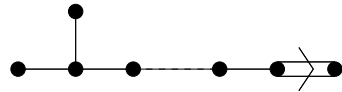
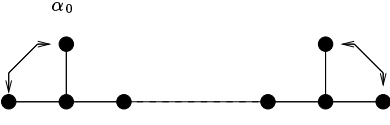
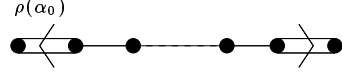
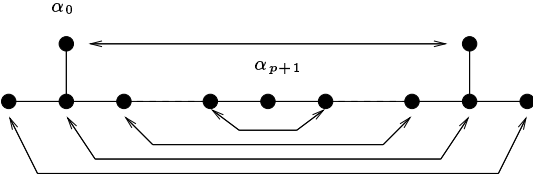
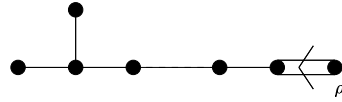
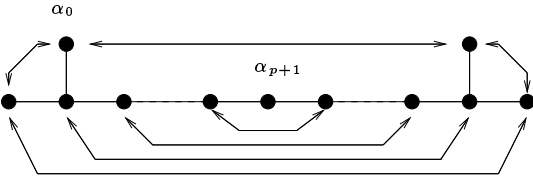
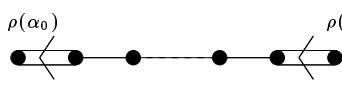
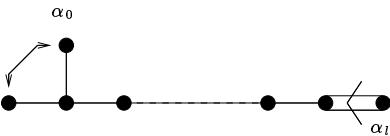
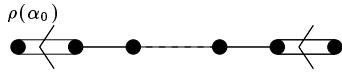
<p style="text-align: center;">$B_l^{(1)}$</p> 	<p style="text-align: center;">$D_l^{(2)}$</p>  <p style="text-align: right;">$\rho(\delta) = 2\delta_a$</p>
<p style="text-align: center;">$B_{2p+1}^{(1)}$</p> 	<p style="text-align: center;">$A_{2p}^{(2)}$</p>  <p style="text-align: right;">$\rho(\delta) = 2\delta_a$</p>
<p style="text-align: center;">$D_l^{(1)}$</p> 	<p style="text-align: center;">$B_{l-1}^{(1)}$</p>  <p style="text-align: right;">$\rho(\delta) = \delta_a$</p>
<p style="text-align: center;">$D_l^{(1)}$</p> 	<p style="text-align: center;">$D_{l-1}^{(2)}$</p>  <p style="text-align: right;">$\rho(\delta) = 2\delta_a$</p>
<p style="text-align: center;">$D_{2p+2}^{(1)}$</p> 	<p style="text-align: center;">$A_{2p+1}^{(2)}$</p>  <p style="text-align: right;">$\rho(\delta) = 2\delta_a$</p>
<p style="text-align: center;">$D_{2p+2}^{(1)}$</p> 	<p style="text-align: center;">$A_{2p+2}^{(2)}$</p>  <p style="text-align: right;">$\rho(\delta) = 2\delta_a$</p>
<p style="text-align: center;">$A_{2l-1}^{(2)}$</p> 	<p style="text-align: center;">$A_{2l-2}^{(2)}$</p>  <p style="text-align: right;">$\rho(\delta) = \delta_a$</p>

TABLE B (continued)

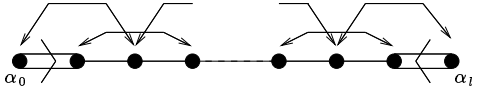
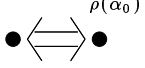
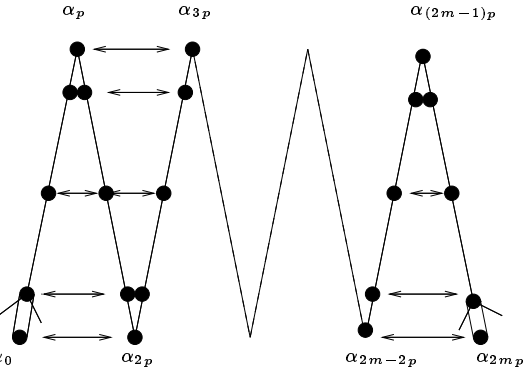
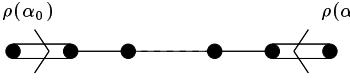
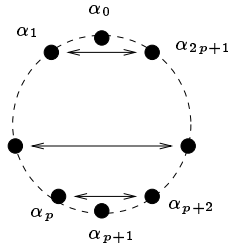
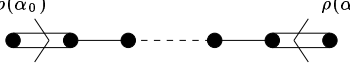
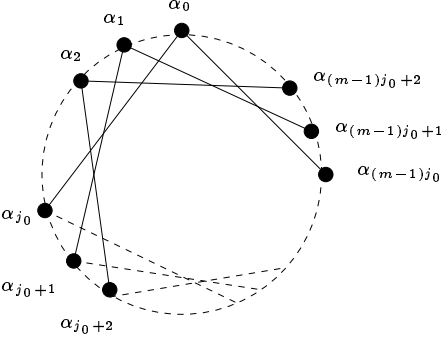
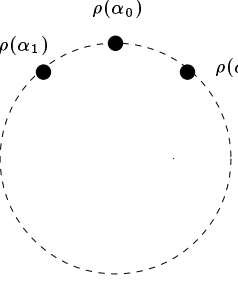
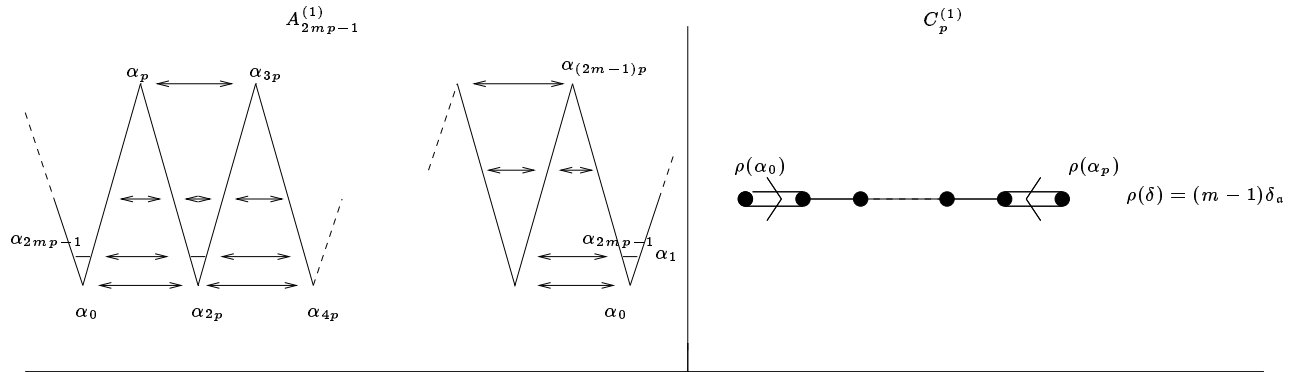
<p style="text-align: center;">$C_l^{(1)}$</p> 	<p style="text-align: center;">$A_1^{(1)}$</p>  <p style="text-align: right;">$\rho(\delta) = (l-1)\delta_a$</p>
<p style="text-align: center;">$C_{2m}^{(1)}$</p> 	<p style="text-align: center;">$C_p^{(1)}$</p>  <p style="text-align: right;">$\rho(\delta) = 2m\delta_a$</p>
<p style="text-align: center;">$A_{2p+1}^{(1)}$</p> 	<p style="text-align: center;">$C_{p+1}^{(1)}$</p>  <p style="text-align: right;">$\rho(\delta) = 2\delta_a$</p>
<p style="text-align: center;">$A_{mj_0-1}^{(1)}$</p> 	<p style="text-align: center;">$A_{j_0-1}^{(1)}$</p>  <p style="text-align: right;">$\rho(\delta) = (m-1)\delta_a$</p>

TABLE B (continued)



References

- [ABG] B. ALLISON - G. BENKART - Y. GAO. *Central extensions of Lie algebras graded by finite root systems*, Math. Ann. 316, 499-527 (2000), Springer-Verlag
- [BM] S. BERMAN - R. MOODY. *Lie algebras graded by finite root systems*, Inv. Math. 108, p. 323-347, 1992.
- [BZ] G. BENKART - E. ZELMANOV. *Lie algebras graded by finite root systems and intersection matrix algebras*, Invent. Math. 126, p. 1-45, 1996.
- [Bou6] N. BOURBAKI. *Groupes et algèbres de Lie*, ch 4, 5, 6 Masson Paris 1981.
- [K] V.G. KAC. *Infinite dimensional Lie algebras* Cambridge Univ. Press. Third edition.
- [KP] V.G. KAC - D. H. PETERSON. *Infinite flag varieties and conjugacy theorems* Proc. Natl. Acad. Sci. USA 80, 1983 p. 1778-1782.
- [Ne] E. NEHER. *Lie algebras graded by J -graded root systems and Jordan pairs covered by grids*, Amer. J. Math., Volume 118, Number 2, April 1996 p.439. The Johns Hopkins Univ. Press.
- [N] J. NERVI. *Algèbres de Lie simples graduées par un système de racines et sous-algèbres C -admissibles*, J. Alg., 223, 2000 p. 307-343.
- [Ru1] H. RUBENTHALER. *Espaces préhomogènes de type parabolique*, Thèse, Université de Strasbourg 1982.
- [Ru2] H. RUBENTHALER. *Construction de certaines sous-algèbres remarquables dans les algèbres de Lie semi-simples*, J. Alg., 81, 1983, p. 268-278.
- [Ru3] H. RUBENTHALER. *Les paires duales dans les algèbres de Lie réductives*, Astérisque n° 219, 1994.
- [SK] M. SATO - T. KIMURA. *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J., 65, 1977, p. 1-155.
- [Se] J.P. SERRE *Algèbres de Lie semi-simples complexes* W.A. Benjamin, INC, 1966.
- [Vi] E.B. VINBERG. *On the classification of the nilpotent elements of graded Lie algebras*, Soviet Math. Dokl. 16, 1975, p. 1517-1520.