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Characterization of the weak closure of the convex envelope of the transformation family

Han-Ping LI

Université Louis Pasteur, Strasbourg, France

Université de Wuhan, Wuhan, Chine

E-mail: lihp@irma.u-strasbg.fr

Résumé

Soit \mathbf{P} une mesure de probabilité tendue définie sur un espace mesurable localement compact (Ω, \mathcal{A}) , soit G un groupe localement compact opérant continûment sur (Ω, \mathcal{A}) . Pour tout élément g de G , l'image de \mathbf{P} par g $g\mathbf{P}$ est une mesure de probabilité sur le même l'espace. Si μ une probabilité sur G , alors $\mathbf{P}_\mu = \int g\mathbf{P}\mu(dg)$ est aussi une mesure de probabilité sur le même l'espace.

Il est facile de montrer que si une famille $\{\mu_\lambda, \lambda \in \Lambda\}$ converge faiblement (ou uniformément tendue), alors la famille $\{\mathbf{P}_{\mu_\lambda}, \lambda \in \Lambda\}$ converge faiblement (ou uniformément tendue, respectivement) ; par conséquent, \mathbf{P}_μ appartient à $\overline{\text{Conv}}(\{g\mathbf{P}, g \in G\})$, l'adhérence faible de l'enveloppe convexe de $\{g\mathbf{P}, g \in G\}$.

Nous donnons un exemple pour montrer qu'en général

$\{\mathbf{P}_\mu, \mu \text{ est une probabilité sur } G\} \neq \overline{\text{Conv}}(\{g\mathbf{P}, g \in G\})$.

Nous montrons que l'action par G sur (Ω, \mathcal{A}) doit être presque propre pour que tout $\{\mu_\lambda, \lambda \in \Lambda\}$ rendant $\{\mathbf{P}_{\mu_\lambda}, \lambda \in \Lambda\}$ uniformément tendue soit uniformément tendue.

Finalement, nous montrons que si $\{\mathbf{P}_{\mu_\lambda}, \lambda \in \Lambda\}$ est uniformément tendue, et si l'action par G sur (Ω, \mathcal{A}) est presque propre, alors $\{\mu_\lambda, \lambda \in \Lambda\}$ est uniformément tendue. Autrement dit, si l'action par un groupe métrisable G est presque propre sur (Ω, \mathcal{A}) , alors $\{\mathbf{P}_\mu, \mu \text{ est une probabilité sur } G\} = \overline{\text{Conv}}(\{g\mathbf{P}, g \in G\})$.

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In his paper of 1970 [5, Stepan] , J. Stepan showed that if \mathbf{P} is a tight probability measures on an Abelian normal topological group G , then the weak closure of the convex envelope of the translations family $\{g\mathbf{P}, g \in G\}$ is just the family defined by $\{\mathbf{P}_\mu = \int g\mathbf{P}\mu(dg), \mu$ being a probability measures on $G\}$.

In this study, we investigate the same problem in a much more general context. In stead of limiting to probability measures on a group, we consider here probability measures on a general measurable space (Ω, \mathcal{A}) , with a group G acting on the left of it. Under very general conditions, we establish the same relationship as above.

First of all, we shall state the basic assumptions used throughout the paper.

- (1) (Ω, \mathcal{A}) is a Hausdorff locally compact measurable space.
- (2) G is a Hausdorff locally compact group with a countable topological base (LCD space), endowed with its Borel field $\sigma(G)$.
- (3) The map $(g, \omega) \mapsto g \circ \omega \hat{=} g\omega$ is continuous.

Remark:

For $g \in G$, we denote indifferently $g\mathbf{P} = \mathbf{P}g^{-1}$ the distribution \mathbf{P} lifted by g : $\mathbf{P}g^{-1}(B) = \mathbf{P}(g^{-1}\{B\})$, for every $B \in \mathcal{A}$.

Denote by $\text{Cap}(Y, \mathcal{B})$, the class of all probability measures defined on (Y, \mathcal{B}) .

Let \mathbf{P} be a probability measure defined on (Ω, \mathcal{A}) . The group G generates in a natural way the following well known families:

- (1) **Parametric transformation family:**

$$\mathcal{M}_1(\mathbf{P}) \hat{=} \{g\mathbf{P}, , g \in G\}.$$

The parametric transformation model can be mixed by a priori law from the Bayesian point of view, that is, this model in his turn generates also the following family:

- (2) **Semi-parametric transformation family:**

$$\mathcal{M}_2(\mathbf{P}) \hat{=} \{\mathbf{P}_\mu : \mathbf{P}_\mu(B) = \int g\mathbf{P}(B)\mu(dg), \forall B \in \mathcal{A}, \mu \in \text{Cap}(G, \sigma(G))\}$$

If we use Conv to denote the convex envelope, then we shall first show that

$$\text{Conv}(\mathcal{M}_1(\mathbf{P})) \subset \mathcal{M}_2(\mathbf{P}) \subset \overline{\text{Conv}(\mathcal{M}_1(\mathbf{P}))}.$$

where the closure is taken w.r.t. the weak topology on $\text{Cap}(\Omega, \mathcal{A})$, and then establish an equality by means of the weak convergence.

Theorem 1 *The following relation is true:*

$$\mathcal{M}_2(\mathbf{P}) \subset \overline{\text{Conv}}(\mathcal{M}_1(\mathbf{P}))$$

where the closure is taken w.r.t. the weak topology on $\text{Cap}(\Omega, \mathcal{A})$.

Proof:

1) If $\mathbf{Q} \in \mathcal{M}_2(\mathbf{P})$,
then there is a $\mu \in \text{Cap}(G, \sigma(G))$ such that

$$\mathbf{Q}(B) = \int g\mathbf{P}(B)\mu(dg), \quad \forall B \in \mathcal{A}.$$

Let $E \subset G$ be a countable dense set of G . Then by the theorem 6.3 , p. 44 of [4, Parthasarathy]., there is a sequence of measures whose supports are finite subsets of E , and which converges weakly to μ , that is

$$\mu_n = \sum_{i=1}^{m_n} \alpha_i^{(n)} \delta_{g_i^{(n)}} \text{ converges weakly to } \mu.$$

It is clear that $\mathbf{Q}_n(B) = \int g\mathbf{P}(B)\mu_n(dg) \in \text{Conv}(\mathcal{M}_p(\mathbf{P}))$, and that the weak convergence of μ_n to μ implies the weak convergence of $\mu_n \times \mathbf{P}$ to $\mu \times \mathbf{P}$ which implies finally the weak convergence of \mathbf{Q}_n to \mathbf{Q} . Consequently, $P = \text{Lim}P_n \in \overline{\text{Conv}}(\mathcal{M}_1(\mathbf{P}))$.

Lemma If $\{\mu_\lambda, \lambda \in \Lambda\}$ is uniformly tight, then $\{\mathbf{P}_{\mu_\lambda}, \lambda \in \Lambda\}$ is also uniformly tight.

Proof:

We show that the uniform tightness of $\{\mu \in \Lambda\}$ implies also the uniform tightness of $\{\mathbf{P}_\mu, \mu \in \Lambda\}$. Take $H \subset G, K \subset \Omega$ two compact sets and denote $HK = \{gx \in \Omega, g \in G, x \in \Omega\}$. Since HK is the image of a continuous mapping of a compact set $H \times K$, it is also compact.

Note that for $g \in H$, $g^{-1}HK \supseteq K$, therefore

$$\mathbf{P}_\mu(HK) \geq \int_H \mathbf{P}(g^{-1}HK)\mu(dg) \geq \mathbf{P}(K)\mu(H).$$

Without the further condition, the reverse of Theorem 1 is not true. Here is a counter example:

Let $\mathbf{P} = \mathcal{N}(0,1)$ be a standard normal distribution defined on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$. For $\sigma \in \mathbf{R}_+ = (0, \infty)$, define $g_\sigma : x \rightarrow \sigma x$. Let $G = \{\sigma \in \mathbf{R}_+\}$ be the multiplication group. So $g_\sigma \mathbf{P} = \mathcal{N}(0, \sigma^2)$ Therefore $\left(g_{\frac{1}{n}} \mathbf{P}\right)_{n \rightarrow \infty}$ converges weakly to $\mathcal{N}(0, 0) = \delta_0 \in \overline{\text{Conv}}(\mathcal{M}_1(\mathbf{P}))$ on $(\mathbf{R}, \mathcal{B}(\mathbf{R}))$. But $\delta_0 \notin \mathcal{M}_2(\mathbf{P})$ for $0 \notin G$.

Let us introduce some conditions to avoid this. Let A, B be subset of Ω , denote

$$((A, B)) = \{g \in G | gA \cap B \neq \emptyset\}.$$

Consider the following condition for G and Ω

(*) $((K, K))$ is a compact set in G whenever K is compact in Ω .

Remark:

1. If we define the mapping $\theta : (g, x) \mapsto (x, gx)$, then $((A, B)) = \text{proj}_1(\theta^{-1}(A \times B))$. Since all spaces are locally compact and separated, $((K, K))$ is always closed.

2. The condition (*) is equivalent to

(*) $((K_1, K_2))$ is a compact set in G whenever K_1, K_2 are compact in Ω .

since $((K_1, K_2)) \subset ((K_1 \cup K_2, K_1 \cup K_2))$.

3. If $(\Omega, \mathcal{A}) = (G, \sigma(G))$, then the condition (*) is trivially verified.

4. A mapping f is said proper if $f \times I_d$ is a continuous and closed mapping. (TG I section 10.1 [2, Bourbaki].). In the same reference, it is showed that a mapping f in a locally compact space is proper if and only if the inverse image of any compact set is compact. (TG I section 10.3 Proposition 7 [2, Bourbaki].).

The group G is said to act properly on Ω if the function θ : defined by $\theta : (g, x) \mapsto (x, gx)$ from $G \times \Omega$ to $\Omega \times \Omega$ is proper.

5. If G acts properly on Ω if and only if the condition (*) is verified, according to the Theorem 1 of (TG III section 4.5 [2, Bourbaki].).

6. If Ω is compact and G acts properly on it, then G is necessarily compact. We introduce the following weaker condition.

Definition

We say that G acts almost properly on Ω if for any $\{\mu_\lambda, \lambda \in \Lambda\}$, for any compact set $K \subset \Omega$ and for any $\epsilon > 0$, there is a compact subset $H \subset ((K, K))$ such that

$$\mu_\lambda(H) > \mu_\lambda((K, K)) - \epsilon \quad \text{for all } \lambda \in \Lambda.$$

Theorem 2 *If $\{\mu, \mu \in \Lambda\}$ is uniformly tight, then the group G acts almost properly on Ω .*

Proof:

By the above lemma, the uniform tightness of $\{\mu \in \Lambda\}$ implies also the uniform tightness of $\{\mathbf{P}_\mu, \mu \in \Lambda\}$. The two families $\{\mathbf{P}_\mu, \mu \in \Lambda\}$ and $\{\mu, \mu \in \Lambda\}$ being uniformly tight, that is, for all $\epsilon > 0$, there exists a compact set $K_\epsilon \subset \Omega$ a compact set $H_\epsilon \subset G$ such that $\mathbf{P}_\mu(K_\epsilon) > 1 - \epsilon$, $\mu(H_\epsilon) > 1 - \epsilon$ for all $\mu \in \Lambda$, and $\mathbf{P}(K_\epsilon) > 1 - \epsilon$.

$$\begin{aligned} 1 - \epsilon &< \mathbf{P}_\mu(K_\epsilon) = \int g \mathbf{P}(K_\epsilon) \mu(dg) \\ &= \int \int I_{\{gx \in K_\epsilon\}} \mathbf{P}(dx) \mu(dg) \\ &= \int_{K_\epsilon} \mu(\left(\left(\{x\}, K_\epsilon\right)\right)) \mathbf{P}(dx) + \int_{K_\epsilon} \mu(\left(\left(\{x\}, K_\epsilon\right)\right)) \mathbf{P}(dx) \\ &\leq \epsilon + \int_{K_\epsilon} \mu(\left(\left(\{x\}, K_\epsilon\right)\right)) \mathbf{P}(dx) \end{aligned}$$

So $\int_{K_\epsilon} \mu(\left(\left(\{x\}, K_\epsilon\right)\right)) \mathbf{P}(dx) \geq 1 - 2\epsilon$, for all $\mu \in \Lambda$. This implies the existence of a point $x_\mu \in \Omega$ such that

$$\mu(\left(\left(\{x_\mu\}, K_\epsilon\right)\right)) \geq 1 - 3\epsilon.$$

Since $\left(\left(\{x_\mu\}, K_\epsilon\right)\right) \subset \left(\left(K_\epsilon, K_\epsilon\right)\right)$, $\mu(\left(\left(K_\epsilon, K_\epsilon\right)\right)) \geq 1 - 3\epsilon$. Therefore

$$\mu(H_1) \geq 1 - 4\epsilon.$$

for all $\mu \in \Lambda$. Since $H_1 = H_\epsilon \cap \left(\left(K_\epsilon, K_\epsilon\right)\right)$ is a closed subset of $\left(\left(K_\epsilon, K_\epsilon\right)\right)$ which is compact, H_1 is also compact. It follows that G acts almost properly on Ω .

Theorem 3 *(Weak closure characterization)*

Suppose the group G is metrisable and it acts almost properly on Ω , then

$$\mathcal{M}_2(\mathbf{P}) = \overline{\text{Conv}}(\mathcal{M}_1(\mathbf{P}))$$

Proof:

Suppose now that the group G acts almost properly on Ω .

If $\mathbf{Q} \in \overline{\text{Conv}}(\mathcal{M}_1(\mathbf{P}))$, then there is a sequence

$$\mathbf{Q}_n = \int g \mathbf{P} \mu_n(dg) = \sum_{i=1}^{m_n} \alpha_i^{(n)} g_i^{(n)} \mathbf{P} \in \text{Conv}(\mathcal{M}_1(\mathbf{P}))$$

such that \mathbf{Q}_n converges weakly to \mathbf{Q} . It is well known that \mathbf{Q}_n must be uniformly tight, that is, for all $\epsilon > 0$, there exists a compact set K_ϵ such that $\mathbf{Q}_n(K_\epsilon) > 1 - \epsilon$ for all $n \in N$. We can of course choose K_ϵ so that $\mathbf{P}(K_\epsilon) > 1 - \epsilon$.

$$\begin{aligned} 1 - \epsilon &< \mathbf{Q}_n(K_\epsilon) = \int g \mathbf{P}(K_\epsilon) \mu_n(dg) \\ &= \int \int I_{\{gx \in K_\epsilon\}} \mathbf{P}(dx) \mu_n(dg) \\ &= \int_{K_\epsilon} \mu_n(\{(\{x\}, K_\epsilon)\}) \mathbf{P}(dx) + \int_{K_\epsilon^c} \mu_n(\{(\{x\}, K_\epsilon)\}) \mathbf{P}(dx) \\ &\leq \epsilon + \int_{K_\epsilon} \mu_n(\{(\{x\}, K_\epsilon)\}) \mathbf{P}(dx) \end{aligned}$$

So $\int_{K_\epsilon} \mu_n(\{(\{x\}, K_\epsilon)\}) \mathbf{P}(dx) \geq 1 - 2\epsilon$. This implies the existence of a point $x_n \in \Omega$ such that

$$\mu_n(\{(\{x_n\}, K_\epsilon)\}) \geq 1 - 3\epsilon.$$

Since

$$\{(\{x_n\}, K_\epsilon)\} \subset \{(\{K_\epsilon\}, K_\epsilon)\}, \quad \mu_n(\{(\{K_\epsilon\}, K_\epsilon)\}) \geq 1 - 3\epsilon.$$

for all n . Since is almost proper, there is a compact subset $H \subset \{(\{K_\epsilon\}, K_\epsilon)\}$ such that $\mu_n(H) \geq 1 - 4\epsilon$. it follows that $\{\mu_n\}$ is uniformly tight. Since G is metrisable, then the uniform tightness implies the relative compactity. Denote μ the limit point of $\{\mu_n\}$. We have finally

$$\mathbf{Q}(B) = \int g \mathbf{P}(B) \mu(dg)$$

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Han-Ping LI
Département de Mathématique
Université Louis Pasteur
7 rue René-Descartes
67084 Strasbourg France