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**Cyclic Structure of Dynamical Systems**  
**Associated with  $3x + d$  Extensions of Collatz Problem.**

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**Abstract.** We study here, from both theoretical and experimental points of view, the cyclic structures, both general and primitive, of dynamical systems  $\mathcal{D}_d$  generated by iterations of the functions  $T_d$  acting, for all  $d \geq 1$  relatively prime to 6, on positive integers :

$$T_d : \mathbf{N} \longrightarrow \mathbf{N}; \quad T_d(n) = \begin{cases} \frac{n}{2} & , \text{ if } n \text{ is even;} \\ \frac{3n+d}{2} & , \text{ if } n \text{ is odd.} \end{cases}$$

In the case  $d = 1$ , the properties of the system  $\mathcal{D} = \mathcal{D}_1$  are the subject of the well-known  $3x + 1$  conjecture. For every one of 6667 systems  $\mathcal{D}_d$ ,  $1 \leq d \leq 19999$ , we calculate its (complete, as we argue) list of primitive cycles. We unite in a single conceptual framework of primitive memberships, and we experimentally confirm three primitive cycles conjectures of Jeff Lagarias. An in-deep analysis of the diophantine formulae for primitive cycles, together with new rich experimental data, suggest several new conjectures, theoretically studied and experimentally confirmed in the present paper. As a part of this program, we prove a new upper bound to the number of primitive cycles of a given oddlength.

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### **Chapter I. The $3x + d$ Extension of Collatz Problem.**

The present study is an experimental and theoretical sequel to our enquiry [Belaga, Mignotte 1998] into the cyclic structure of  $3x + d$  extensions of the  $3x + 1$  conjecture (cf. §1 below), the extensions remarkably simple in form, but very rich and powerful in substance, §§2-4, 7, 8. We describe and study this cyclic structure, §§3, 4, in terms of *primitive cycles*, §§4, 6, 7, introduced in [Lagarias 1990], the building blocks of the general cyclic structure of  $3x + d$  dynamical systems, §§2, 3, 10.

For 6667 such systems  $\mathcal{D}_d$ , with the *shift number*  $d$  relatively prime to 6,  $1 \leq d \leq 19999$ , we have found experimentally on the whole 42765 primitive cycles and have confirmed three “Primitive Cycles Existence” conjectures of Jeff Lagarias, §5, as well as few others theoretically predicted phenomena, §10. The new rich experimental data, Tables 1-6, Figures 1-9, delivered by an extremely effective searching engine, §12, and coupled with an in-depth analysis of the diophantine formulae for primitive cycles, §§7-10, have led to the discovery of new, mostly unexpected and sometimes rather spectacular phenomena, - the subjects of several new conjectures, §§5, 6, 10, 11, which are introduced, theoretically studied, and experimentally confirmed in the present paper.

Our theoretical contributions include the discovery of (a formal framework for) the disparated duality between shift numbers  $d$  and member numbers  $n$  of primitive cycles. In the framework of this duality, based on the *primitive membership concept*, finds its natural place and interpretation of experimentally discovered here *irreducible* and *strongly irreducible* memberships, §11.

We have also improved, both in form and substance, our upper bound [Belaga, Mignotte 1998] to the number of primitive cycles of the given oddlength, §10.

We have attempted to make the presentation of our results as full and accessible to the general mathematical public, as possible. In this regard, the present paper can be viewed as a tutorial-like, both theoretical and computational introduction into dynamical, recursive, and diophantine aspects of the  $3x + d$  conjecture.

#### **1. Preamble : the $3x + 1$ Conjecture, or Collatz Problem.**

We remind the reader the well-known and still open  $3x + 1$  *conjecture*, or *Collatz*

problem [Lagarias 1985]. The reminder is concentrated around, and gives the reader a foretast of, the central topics persued in this study. We also lay down in this introductory section the basic conceptional and notational groundwork of this study. (The numerations of conjectures, tables, and figures are thorough : Conjecture 1, §1, Conjecture 7, §2. Other declarative items, such as definitions, lemmas, etc., bear the number of a corresponding section and are enumerated in the order of their appearance : Note 5.1, Aside 5.2, Notation 5.3, Lemma 5.4.)

**Conjecture 1 : The  $3x + 1$  Conjecture.** For any positive integer  $m$ , define :

$$T : \mathbf{N} \longrightarrow \mathbf{N}; \quad T(m) = \begin{cases} \frac{m}{2} & , \text{ if } m \text{ is even;} \\ \frac{3m+1}{2} & , \text{ if } m \text{ is odd.} \end{cases} \quad (1 : 1)$$

Then, for any positive integer  $n$ , the  $T$ -trajectory  $\tau(n)$  of iterates of  $T$  starting at  $m$ ,

$$\tau(m) = \{T^0(m) = m, T(m), T^2(m) = T(T(m)), T^3(m), \dots\} \quad (1 : 2)$$

runs ultimately into the cycle

$$\mathbf{C}^o = \langle 1 \rightarrow 2 \rightarrow 1 \rangle. \quad (1 : 3)$$

The problem has many mathematical facets [Wirsching 1998]. But first and foremost it represents an intricate interplay between algorithmic (or recursive), dynamic, and diophantine insights. The embedding of Collatz problem into a wider  $3x + d$  context [Lagarias 1990], [Belaga, Mignotte 1998] both sheds new light on this interplay and reveals new and remarkably deep phenomena.

**(I) The  $3x + 1$  Dynamical System.** As a mapping defined on the set  $\mathbf{N}$  of positive integers, the function  $T$  (1:1) generates a dynamic system  $\mathcal{D}$  with the only known cycle  $\mathbf{C}^o$  (1:3). From the dynamical point of view, the  $3x + 1$  conjecture might fail for two reasons :

(i) Either there exists a divergent  $T$ -trajectory,

$$\lim_{k \rightarrow \infty} T^k(m) = \infty.$$

Thus, the *no-divergent-trajectories* conjecture :

**Conjecture 2 : (NDTC) $_{3x+1}$**   $T$  has no divergent trajectories.

(ii) Or the cycle  $\mathbf{C}^o$  is not unique. Hence, first, the *finite-number-of-cycles* conjecture, discarding the possibility of an infinite number of cycles :

**Conjecture 3 : (FCC) $_{3x+1}$**   $T$  has only a finite number of cycles.

And then its (probably, most difficult to prove) refinement, the *one-cycle* conjecture :

**Conjecture 4 : (UCC) $_{3x+1}$**   $T$  has one and only one (for short, *unique*) cycle,  $\mathbf{C}^o$  (1:3).

**(II) The  $3x + 1$  Algorithm.** The  $3x + 1$  conjecture can be naturally interpreted as a halting problem for the algorithm :

$$\mathbf{A}_{3x+1} : \begin{cases} m := n \\ \mathbf{while} \ m > 1 \ \mathbf{do} \\ \quad m := T(m) \\ \mathbf{endwhile} . \end{cases} \quad (1 : 4)$$

Hence, the *unsolvability* option, either in its strongest (and absolute) form of *algorithmical undecidability*, or in a weaker (and relative) sense of *unprovability* in an appropriate axiomatic setting [Belaga 1998] :

**Note 1.1 : On Undecidability and an Eventual  $3x + 1$  Paradox.** (1) It is true that the remarkable wealth of vast numerical checks and a host of plausible heuristic arguments [Lagarias 1985], [Wirsching 1998] leave little doubt in the veracity of the  $3x + 1$  conjecture.

(2) However, Conway’s example of a periodically linear function naturally generalizing (1:1) and simulating a universal Turing machine [Conway 1972] demonstrates how extraordinary close are we here to the unsolvability realm. (See also [Maslov 1967], [Matijasevič 1967] for similar earlier, albeit less accessible results.)

(3) The sets  $\mathbf{N}_{\text{cyclic}}$  and  $\mathbf{N}_{\text{ucyclic}}$  of integers  $m \in \mathbf{N}$  with, respectively, cyclic and ultimately cyclic trajectories  $\tau(m)$  are *effectively enumerable* ([Enderton 1977] and [Davis 1977] are good recursion theory and solvability reference sources). This can be shown by the standard universal argument of recursion theory, namely : if  $\mathbf{R}(n, q)$  is the value of  $m$  after the the  $q$ -th iteration of the loop **while** of the algorithm  $\mathbf{A}_{3x+1}$  (we assume  $\mathbf{R}(n, q) = m = 1$ , if the loop terminates on an earlier step  $1 \leq q' < q$ ), then any effective enumeration of the set  $\mathbf{N}^2$  of pairs of positive integers will permit us to effectively enumerate all  $n$  with at least one case of equality  $\mathbf{R}(n, q') = \mathbf{R}(n, q), 1 \leq q' < q$ .

(4) Moreover, the diophantine interpretation of the cyclic part of the  $3x + 1$  conjecture (see the subdivision (III) of this section and Chapter III below) gives rise to a much more transparent, pure diophantine enumeration procedures.

(5) However, nothing is known about recursive enumerability of the set  $\mathbf{N}_{\infty}$  of integers where divergent trajectories start, and therefore, about the recursive decidability of the no-divergent-trajectories conjecture (Conjecture 2).

(6) It might even happen that, on the one hand, a proof would be available of the recursive undecidability of Conjecture 2, whereas, on the other hand, one could be “absolutely certain” that, “in reality,” the set  $\mathbf{N}_{\infty}$  is empty : see [Belaga 1998] for the precise sense, implications and a plausible “ways out” of such a paradox.

**(III) Exponential Diophantine Machinery.** Iterating the arithmetic procedure (1:1), one gets explicit exponential diophantine formulae for iterations  $T^i$ . These cumbersome formulae are apparently of not much use in studying the no-divergent-trajectories conjecture, Conjecture 2. Not so with the cyclic conjectures, Conjectures 3, 4, with their beautiful arithmetical equivalents, Lemma 1.4 and Conjecture 5. (See more about this arithmetical connection below, §§7-10.)

The following definition is a considerably simplified and trimmed version of the below Definition 7.1.

**Definition 1.2.** For any pair  $(k, \ell) \in N \times N$  of positive integers, satisfying the inequality  $k \cdot \log_2 3 < \ell$ , we define the corresponding *Collatz number*  $B = B_{k, \ell}$  and its *Collatz corona*

$\mathcal{A} = \mathcal{A}_{k,\ell}$ , which is a finite set of positive integers, as follows.

(1) Collatz  $(k, \ell)$ -number :

$$B = B_{k,\ell} = 2^\ell - 3^k . \quad (1 : 5)$$

By definition, a Collatz number is an odd positive integer not divisible by 3.

(2) Collatz  $(k, \ell)$ -corona  $\mathcal{A} = \mathcal{A}_{k,\ell}$  of the Collatz number  $B = B_{k,\ell}$  :

(a) If  $k = 1$ , then

$$\mathcal{A}_{1,\ell} = \{1\} . \quad (1 : 6)$$

(b) If  $k > 1$ , then for any *aperiodic*  $(k - 1)$ -tuple  $\mathbf{P} = \{p_1, \dots, p_{k-1}\}$  of positive integers (for a definition of the term “aperiodic,” see Definition 7.1(2)) satisfying the inequality

$$p_1 + \dots + p_{k-1} < \ell , \quad (1 : 7)$$

the following number belongs to  $\mathcal{A}$  :

$$A = A(\mathbf{P}) = 3^{k-1} + 3^{k-2} \cdot 2^{p_1} + \dots + 3 \cdot 2^{p_1 + \dots + p_{k-2}} + 2^{p_1 + \dots + p_{k-1}} . \quad (1 : 8)$$

By definition, members of Collatz corona are odd positive integers not divisible by 3.

**Lemma 1.3.** (1) Collatz corona  $\mathcal{A}_{k,\ell}$  is a one-element set *iff*  $k = 1$  (1:6)

(2) Otherwise, Collatz corona  $\mathcal{A}$  of a Collatz number  $B$  is a finite set,  $\#\mathcal{A} > 1$  of mutually distinct positive integers not equal to  $B$ .

(3) Moreover, if  $\ell > 1 + k \cdot \log_2 3$ , then  $\mathcal{A}$  extends both below and above  $B$ .

See Lemma 7.2 for more general and detailed assertions, with explicitly calculated values of  $\#\mathcal{A}$ ,  $\inf \mathcal{A}$ , and  $\sup \mathcal{A}$ .

Here is a diophantine interpretation of the  $T$ -cyclicity condition (see Corollary 8.4(2) for a proof) :

**Lemma 1.4.** A  $T$ -trajectory  $\tau(m)$  (1:2) starting at a positive odd integer  $n$  is a cycle of the length  $\ell$  *iff* there exists such Collatz number  $B = B_{k,\ell}$  (1:5) that for some  $A \in \mathcal{A}_{k,\ell}$ ,

$$B \mid A , \quad n = \frac{A}{B} . \quad (1 : 9)$$

Note that any cycle has at least one odd member, so the oddness condition on  $n$  is not restrictive.

Thus, the following pure diophantine conjecture implies Conjecture 4 :

**Conjecture 5.** Excluding the case

$$B_{2,1} = 2^2 - 3^1 = 1 , \quad (1 : 10)$$

no Collatz number divides a number from its Collatz corona.

**Note 1.5. (1)** According to Lemma 1.4, the Collatz number  $B_{2,1}$  corresponds to the cycle (1:3), with the values  $n = A = 1$ .

**(2)** A Collatz number  $B_{k,\ell}$ ,  $k \geq 2$ , equal to 1 would be a divisor of any member of its Collatz corona. According to [Ribenoim 1994], already some mediaeval mathematicians knew that the number (1:10) is the only Collatz number equal to 1. (This is a particular, completely resolved case of still open Catalan’s conjecture [Ribenoim 1994]; see also below Conjecture 6.)

Conjecture 5 suggests the following diophantine hypothesis (see Conjecture 13, §2, for its  $d$  extension) :

**Conjecture 6.** For any pair of nonequal prime numbers  $r, q$ ,  $2 \leq r < q$  and any pair  $(k, \ell)$  of positive integers, with the exception of the following two cases,

$$r = 2, q = 3, k = 2, \ell = 1 \quad \text{and} \quad r = 2, q = 3, k = 3, \ell = 2, \quad (1 : 11)$$

there exist either no, or at most a finite number of  $(k - 1)$ -tuples of positive integers  $p_1, \dots, p_{k-1}$ ,  $p_1 + \dots + p_{k-1} < \ell$ , satisfying the congruence

$$q^{k-1} + r^{p_1} \cdot q^{k-2} + r^{p_1+p_2} \cdot q^{k-3} + \dots + r^{p_1+\dots+p_{k-2}} \cdot q + r^{p_1+\dots+p_{k-1}} \equiv 0 \pmod{|r^\ell - q^k|}. \quad (1 : 12)$$

Note that if  $|r^\ell - q^k| = 1$ , then the divisibility in (1:12) would be assured for any  $k$ -tuple  $p_1, \dots, p_k$ . This scenario (which generalizes the plot of Note 1.5(2)) is excluded by the above condition (1:11), according to the following special (and also completely resolved [Ribenoim 1994]) case of Catalan’s conjecture :

**Catalan’s Conjecture.** The only solutions in positive integers  $r, q, \ell, k$  of the equation  $|r^\ell - q^k| = 1$  are listed in (1:11).

**(IV) A Preliminary Sample of the Diophantine  $(3x + d)$  Extension.** To give the reader a foretast of Diophantine version of the coming  $(3x + d)$  extension of Collatz conjecture (before its precise definition in §2), let us call a positive odd integer  $d$  not divisible by 3 a key to a Collatz number  $B$ , if  $B$  divides some members of the set  $d \cdot \mathcal{A}$ , with  $\mathcal{A}$  being Collatz corona of  $B$  (since  $B$  and members of  $\mathcal{A}$  are odd and not divisible by 3, the key can be restricted to numbers of the same type).

Any Collatz number  $B = B_{k,\ell}$  has at least one key : for any  $A \in \mathcal{A}_{k,\ell}$ , take  $d = \frac{A}{\gcd(A,B)}$ . Now, the other way around, is any positive odd integer  $d$  not divisible by 3 a key to some Collatz number ?

The below  $(3x + d)$  conjectures imply (and our calculations strongly confirm) that any number  $d$  is a key for at least one and at most a finite number of Collatz numbers. And, as our calculations show, in many cases “relatively small”  $d$  are keys to “very big” Collatz numbers (a phenomenon conjectured in [Lagarias 1990]; see for details §10).

Thus, for  $d = 3299$ ,  $k = 1000$ ,  $\ell = 1992$  and some “very big”  $A \in \mathcal{A}_{1000,1992}$  (cf. also Example 2.4),

$$B = 2^{1922} - 3^{1000} > 10^{577} \quad \text{and} \quad B \quad \text{divides} \quad 3299 \cdot A \quad (1 : 13)$$

These new, surprising, and absolutely nontrivial insights into arithmetic of simple combinations of exponents of 2 and 3 represent a unique contribution of Collatz problem and its  $3x + d$  generalization to number theory.

## 2. The $3x + d$ Conjecture.

It was with the purpose of better understanding of the interplay of dynamic, algorithmic, and diophantine factors in the  $3x + 1$  case that its  $3x + d$  generalization, for all  $d \geq 1$  relatively prime to 6, has been proposed in [Lagarias 1990] and, independently, in [Belaga 1995] and [Belaga, Mignotte 1998]<sup>1</sup>. One can now say with hindsight that this quantum  $d$ -increment of the  $3x + 1$  conjecture turned out to be its most natural and elucidative extension :

(i) dynamically, only the  $3x + 1$  cycle uniqueness claim, Conjecture 5(UCC)<sub>3x+1</sub>, has to be withdrawn in the general  $3x + d$  context, with two other conjectures, no-divergent-trajectories (Conjectures 6(NDTC)<sub>3x+d</sub> and 7(URCC)<sub>3x+d</sub> below) and finite-number-of-cycles (Conjecture 8(NDTC)<sub>3x+d</sub>), upheld;

(ii) algorithmically, the respective change boils down to the replacement of the constant 1 in the algorithm (1:4) by an appropriate positive integer, see (2:5) below;

(iii) as it has been already mentioned above, §1(IV), the corresponding  $3x + d$  diophantine interpretation extends the problem of divisibility (1:6) of the number  $A$  by  $B$  (1:5) to the problem of divisibility of  $d \cdot A$  by  $B$ .

Define the  $3x + d$  transform, with the *shift number*  $d$  being a positive odd integer not divisible by 3 (or, in other words,  $d \geq 1$  relatively prime to 6), as follows :

$$T_d : \mathbf{N} \longrightarrow \mathbf{N} ; \quad T_d(m) = \begin{cases} \frac{m}{2} & , \text{ if } m \text{ is even;} \\ \frac{3m+d}{2} & , \text{ if } m \text{ is odd.} \end{cases} \quad (2 : 1)$$

Then, for any positive integer  $n \in \mathbf{N}$ , define a  $T_d$ -trajectory and  $T_d$ -cycle starting at  $m$ , as follows :

$$\left\{ \begin{array}{l} \tau_d(m) = \{m = T_d^0(m), m_1 = T_d(m), m_2 = T_d^2(m) = T_d(T_d(m)), \dots\} \\ \tau_d(m) \text{ is a } T_d\text{-cycle of the length } \ell \iff \begin{cases} \exists \ell \geq 1, m_\ell = T_d^\ell(m) = m \\ \forall r (1 \leq r < \ell), \\ m_r = T_d^r(m) \neq m. \end{cases} \end{array} \right. \quad (2 : 2)$$

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<sup>1</sup> We are grateful to the anonymous referee of [Belaga, Mignotte 1998], who brought our attention to the important paper [Lagarias 1990].

It is worth to mention here that, in [Belaga, Mignotte 1998], we also study the case  $d = -1$ . It has been dropped here, to avoid too much technical particularization of our notations and claims.

The last condition ( $T_d^r(m) \neq m$ , if  $r \not\equiv 0 \pmod{\ell}$ ) guarantees that a cycle of the length  $\ell$  is not a cycle of a smaller length.

**(I) The  $3x + d$  Dynamics.** We use the shorthand notations  $\mathbf{D}$ ,  $\{\xi, \eta\}$ , and  $\mathbf{D}_d$  for, respectively, the set of positive odd integers not divisible by 3, its enumeration functions, and its subset of integers relatively prime to  $d$  :

$$\left\{ \begin{array}{l} \mathbf{D} = \{n \in \mathbf{N} \mid \gcd(n, 6) = 1\} = 1, 5, 7, 11, 13, 17, 19 \dots ; \\ \mathbf{D}_d = \{n \in \mathbf{D} \mid \gcd(n, d) = 1\} ; \\ \left\{ \begin{array}{l} \forall t \in \mathbf{N}, \xi(t) = 6 \cdot \left\lfloor \frac{t}{2} \right\rfloor - (-1)^t \in \mathbf{D} ; \forall n \in \mathbf{D}, \eta(n) = \left\lceil \frac{n}{3} \right\rceil \in \mathbf{N} ; \\ \mathbf{N} \xrightarrow{\xi} \mathbf{D} \xrightarrow{\eta} \mathbf{N} \xrightarrow{\xi} \mathbf{D} ; \xi \circ \eta = \text{id}_{\mathbf{D}}, \eta \circ \xi = \text{id}_{\mathbf{N}} . \end{array} \right. \end{array} \right. \quad (2:3)$$

For any  $d \in \mathbf{D}$ , the dynamical system  $\mathcal{D}_d$  has a cycle closely resembling  $\mathbf{C}^o$  (1:3) :

$$\mathbf{C}_d^o = \{d \rightarrow 2d \rightarrow d\} \quad (2:4)$$

(cf. also Example 4.5 below). However, according to the first of the three conjectures of Lagarias, this cycle is never unique if  $d \neq 1$  (4:12). Moreover, in all 6667 dynamical systems, experimentally studied in the present paper, we have found at least two, and in many cases, much more cycles. Thus, for instance, for  $d = 6487$  and  $d = 14303$ , the systems  $\mathbf{D}_{6487}$  and  $\mathbf{D}_{14303}$  have at least 596 and 944 primitive cycles, respectively : cf. Examples 8.7(1), 10.5(2) and Table 5, §10.

This loss of the unique cyclic attractor, similar to  $\mathbf{C}^o$  (1:3) in the  $3x + 1$  case, does not apparently affect the general dynamical pattern. In the  $3x + d$  case, too, heuristic arguments and numerical checks carried in this study lead us to believe that, for all  $d \in \mathbf{D}$ , both no-divergent-trajectories and finite-number-of-cycles conjectures hold :

**Conjecture 7 : (NDTC) $_{3x+d}$**  For any  $d \in \mathbf{D}$ ,  $T_d$  has no divergent trajectories.

Or equivalently, but in the affirmative form :

**Conjecture 8 : (URCC) $_{3x+d}$**  For any  $d \in \mathbf{D}$ , a  $T_d$ -trajectory runs ultimately into a cycle.

The existence of at least one  $T_d$ -cycle being guaranteed (the cycle  $\mathbf{C}_d^o$  (2:4)), the total number is conjectured to be finite :

**Conjecture 9 : (FCC) $_{3x+d}$**  For any  $d \in \mathbf{D}$ ,  $T_d$  has only a finite number of cycles.

The total number of  $T_d$ -cycles will be the subject of our formula (4:12) and conjectures, old and new, §§5,6.

Taken together, the above conjectures imply that any  $T_d$ -trajectory collapses into a bounded vicinity of 1. We define below, §§3, 10, three numerical characteristics of systems  $\mathcal{D}_d$  implicit in this assumption. One of these characteristics, the *recurrence threshold*  $R_d$ , appears in the below  $3x + d$  extension (2:7) of the  $3x + 1$  halting problem (1:4).

An important dynamical insight into the  $3x + d$  conjecture represents the technique of primitive cycles [Lagarias 1990] and the related conjectures, old and new, §§4-6, 9. It is

noteworthy, however, that at least seven of the new conjectures, presented below, cannot be properly formulated and understood in the pure dynamical context : they are inspired by, and substantially rely on the diophantine interpretation and analysis of the conjecture Conjecture 9(FCC) $_{3x+d}$ , §§7, 8; see also the subsection (III) below.

**(II) The  $3x + d$  Conjecture as a Halting Problem.** Taken together, the above conjectures are equivalent to the following halting problem :

**Conjecture 10 : Algorithmic Version of Conjectures 7-9.** For any  $d \in \mathbf{D}$ , there exists such a positive integer  $R$ , that the following algorithm halts for all input (positive integer) values of  $n$  :

$$\mathbf{A}_{3x+d} : \begin{cases} m := n \\ \mathbf{while} \ m > R \ \mathbf{do} \\ \quad m := T_d(m) \\ \mathbf{endwhile} . \end{cases} \quad (2 : 5)$$

Note that in the special case  $d = R = 1$ , the procedure (2:5) becomes the  $3x + 1$  algorithm (1:4). Note also that the minimal value of  $R$ , for a given shift number  $d$ , is the recurrence threshold  $R_d$  mentioned in the subsection (II) above and defined below, §3.

The affirmative version of the no-divergent-trajectories conjecture can be naturally interpreted as the following separate halting problem. (For the proof, see §12, Lemma 12.2.) We do not know a similarly transparent interpretation for the finite-number-of-cycles conjecture.

**Conjecture 11 : Algorithmic Version of Conjecture 8.** For any  $d \in \mathbf{D}$ , the following algorithm always halts :

$$\mathbf{A}_{3x+d}^{\text{cyclic}} : \begin{cases} m := n := j \\ \mathbf{while} \ m \neq n \ \mathbf{do} \\ \quad m := T_d(m); \ n := T_d(n); \ n := T_d(n); \\ \mathbf{endwhile} . \end{cases} \quad (2 : 6)$$

For a proof that  $\mathbf{A}_{3x+d}^{\text{cyclic}}$  detects all  $T_d$ -cycles, see Proposition 12.2.

This equivalence intimates the idea of a simple algorithm for checking both Conjectures 7(NDTC) $_{3x+d}$  and 9(FCC) $_{3x+d}$  and, in the case they hold, for computing all  $T_d$ -cycles. Namely, starting with  $m = 1$ , and then going on and on, follow the  $T_d$ -trajectory  $\tau_d(m)$  (2:2), until it runs ultimately into a  $T_d$ -cycle. Using enhanced versions of this algorithm, §11, we will experimentally confirm (cf. Tables 1-6, Figures 1-9, and the searching algorithm, §12) several well-known and new  $3x + d$  related phenomena and conjectures, §§3-6, 8-11.

**(III) The Exponential Diophantine Paraphrase of the Cyclic Part of the  $3x + d$  Conjecture.** The diophantine formalism turns out to be strictly complementary to the algorithmic one : it becomes effective where the algorithmic approach fails, and vice versa. Namely, as in the  $3x + 1$  case, explicit arithmetic formulae for  $T_d$  iterations are apparently of no help in studying the no-divergent-trajectories conjecture, in either

of its two forms (Conjectures 7, 8). Not so with the cyclic part of the  $3x + d$  conjecture (Conjecture 9) :

**Lemma 2.3.** [Belaga, Mignotte 1998] For any  $d \in \mathbf{D}$ , there exists a  $T_d$ -cycle meeting an odd positive integer  $n$  iff, there exists such a Collatz number  $B = B_{k,\ell}$  (1:5) that for some  $A \in \mathcal{A}_{k,\ell}$ ,

$$d \cdot A \equiv 0 \pmod{B}, \quad n = \frac{d \cdot A}{B}. \quad (2 : 7)$$

(A similar result has been proved in [Lagarias 1990].) Cf. these formulae with their  $3x + 1$  version (1:9), Lemma 1.4.

Hence, the following exponential diophantine version of the above finite-number-of-cycles conjecture (cf. §1(IV) and see for details §§7, 8) :

**Conjecture 12 :** (DFCC) $_{3x+1}$  For any  $d \in \mathbf{D}$ , the total number of Collatz numbers satisfying (2:7) is finite.

Our computations show that even for relatively small values of the shift number  $d \in \mathbf{D}$ , some  $T_d$ -cycles are very long, with the corresponding Collatz number  $B$  and its counterpart  $A$  from Collatz corona of  $B$  rather big.

**Example 2.4.** The biggest found in this study Collatz number corresponds to a  $T_{16819}$ -cycle of the length 4686 (see below Table 5, §10) :

$$B = B_{2292,4686} = 2^{4686} - 3^{2292} > 10^{1409} \quad \text{and} \quad B \text{ divides } 16819 \cdot A' \quad (2 : 8)$$

This should be expected [Lagarias 1990], if each system  $\mathcal{D}_d$  has at least one primitive  $T_d$ -cycle. Now, the experimental data show the pervasiveness of this phenomenon on a scale far surpassing the available theoretical expectations; cf. Table5, §9.

Finally, the  $3x+d$ -extention (2:7) of the diophantine version (1:5-9) of Collatz problem suggests the following  $3x + d$ -extension of our Conjecture 6 (1:12) :

**Conjecture 13.** For any pair of nonequal prime numbers  $r, q, 2 \leq r < q$  and any positive integer  $d$  relatively prime to both  $p$  and  $q$ ,  $\gcd(d, p \cdot q) = 1$ , the exist either no, or at most a finite number of  $k$ -tuples of positive integers  $p_1, \dots, p_k, 1 \leq k < \infty$ , satisfying the following congruence :

$$\begin{aligned} d \cdot (q^{k-1} + r^{p_1} \cdot q^{k-2} r^{p_1+p_2} \cdot q^{k-3} + \dots + r^{p_1+\dots+p_{k-2}} \cdot q + r^{p_1+\dots+p_{k-1}}) \\ \equiv 0 \pmod{|r^\ell - q^k|}. \end{aligned} \quad (2 : 9)$$

### 3. More of $3x + d$ Dynamics : Recurrence, Collapse, and Attraction.

Taken together, Conjecture 7(NDTC) $_{3x+d}$  and Conjecture 9(FCC) $_{3x+d}$  are obviously equivalent to each one of the following three conjectures (the second and third conjectures,

united in a single, less elaborated and slightly different hypothesis, can be found in [Belaga, Mignotte 1998] under the name of *Termination Set Conjecture*) :

**Conjecture 14 : Recurrence Threshold Conjecture.** For any  $d \in \mathbf{D}$ , there exists such a positive integer  $R$  that any  $T_d$ -trajectory (2:2) enters the interval  $1 \leq m \leq R$  infinitely many times. The minimal of such numbers  $R$  is denoted by  $R_d$  and called the  $T_d$ -recurrence threshold.

**Conjecture 15 : Collapse Threshold Conjecture.** For any  $d \in \mathbf{D}$ , there exists such a positive integer  $L$  that any  $T_d$ -trajectory  $\tau_d(m)$  runs ultimately into the interval  $1 \leq m \leq L$ , and there it remains. The minimal of such numbers  $L$  is denoted by  $L_d$  and called the  $T_d$ -collapse threshold.

**Conjecture 16 : Termination Set, or Attractor Conjecture.** For any  $d \in \mathbf{D}$ , there exists such a finite set  $\mathcal{U}$  of positive integers, that any  $T_d$ -trajectory  $\tau_d(m)$  runs ultimately into  $\mathcal{U}$ , and there it remains. The intersection of all such sets is denoted by  $\mathcal{U}_d$  and called the  $T_d$ -termination set, or  $T_d$ -attractor.

Cf. also the below 4.7(3) (4:14).

These equivalent re-formulations have at least three advantages. First, they spell explicitly out the existence of a “collapsing” phenomenon observed in all studied until now systems  $\mathcal{D}_d$ , - the possible “key” to understanding of other  $T_d$ -phenomena.

Second, they define three important characteristics of the cyclic structure of the system  $\mathcal{D}_d$ . Namely, assuming Conjectures 7(NDTC) $_{3x+d}$  and 9(FCC) $_{3x+d}$  hold,

- (i)  $T_d$ -recurrence threshold  $R_d$  is the maximal of minimal members of  $T_d$ -cycles;
- (ii)  $T_d$ -collapse threshold  $L_d$  is the maximal of maximal members of  $T_d$ -cycles;
- (iii)  $T_d$ -attractor  $\mathcal{U}_d$  is the union of members of all  $T_d$ -cycles.

These characteristics can be computed from their respective primitive  $T_d$ -analogues, §4 (4:11-14).

Finally, algorithmic re-formulations of the above conjectures makes manifest in a most transparent way the “halting” nature of  $3x+d$  related problems. In particular, Conjecture 12 is the precise equivalent of the halting problem (2:5), §2.

## Chapter II. Primitive Cyclic Structure.

Further progress in studying the fine cyclic structure of systems  $\mathcal{D}_d$  becomes possible thanks to the technique of primitive cycles [Lagarias 1990]. Intuitively, a primitive cycle is a  $T_d$ -cycle which is not a  $q$ -multiple of a  $T_r$ -cycle, with  $q \cdot r = d, r < d$ .

One of the main purposes of the present paper is to theoretically and experimentally explore the primitive cyclic structure of dynamical systems  $\mathcal{D}_d$ . In particular, for 6667 dynamical systems  $\mathcal{D}_d$ , in the interval  $d \in \mathbf{D}, d \leq 19999$ , we calculate (as we argue, §13, all) their primitive cycles.

The analysis of this experimental data permits us to confirm with different degrees of certainty (which should, and will be made exact below, Note 5.2, §13) the aforementioned no-divergent-trajectories conjecture and, *via* the conjectures of Jeff Lagarias [Lagarias

1990], the finite-number-of-cycles conjecture, §11. Our dual interpretation of Lagarias conjectures is new, §§5,7. So are the insights which lead inextricably to our shift number and member number conjectures, §§10, 11. All our new conjectures, §§6,8-11, are solidly confirmed by the experimental data, cf. Tables 1-6, Figures 1-9, and the searching algorithm, §12.

#### 4. Primitive $3x + d$ Cycles.

We start with a few trivial technical observations.

**Notation 4.1.** We remind that the shift number  $d \geq 1$  is relatively prime to 6,  $d \in \mathbf{D}$  (2:3). For any positive integer  $m \in \mathbf{N}$ , define the number  $odd(m)$  obtained by factoring out of  $m$  the highest possible power of 2; thus  $odd(m)$  is odd and  $m = odd(m) \cdot 2^j$ , for some  $j$ . Hence the notations :

$$\forall m \in \mathbf{N}, \quad \nu_2(m) = \max \{j \geq 0 \mid m \cdot 2^{-j} \in \mathbf{N}\}, \quad m = odd(m) \cdot 2^{\nu_2(m)}. \quad (4:1)$$

**Lemma-Definition 4.2. (1)** Any  $T_d$ -trajectory passes by an odd number, and whatever might be  $m \in \mathbf{N}$ , all other members  $T_d(m), T_d^2(m), \dots$  of a  $T_d$ -trajectory  $\tau_d(m)$  (2:2) are not divisible by 3 :

$$\forall m, r \in \mathbf{N} \quad \forall d \in \mathbf{D}, \quad \gcd(3, T_d^r(m)) = 1. \quad (4:2)$$

Thus, all odd members of the trajectory  $\tau_d(m)$  different from  $m$  (and, in the case of a  $T_d$ -cycle, all its odd members),  $n, n_1, \dots$ , are relatively prime to 6,  $n, n_1, \dots \in \mathbf{D}$ .

(2). In particular, the minimal member  $n$  of a  $T_d$ -cycle is odd and  $n \in \mathbf{D}$ . And without the loss of generality, we can assume that any  $T_d$ -trajectory starts at an integer relatively prime to 6. Then, the full sequence of odd members, in the order of their appearance in a  $T_d$ -trajectory  $\tau_d(n)$  starting at  $m = n_0 \in \mathbf{D}$  (cf. (2:2), (4:1)), is denoted by  $Odd(n, d)$ , and the full sequence of the corresponding exponents of 2, by  $Even(n, d)$  :

$$\forall n, d \in \mathbf{D} \quad \left\{ \begin{array}{l} \text{(1)} \quad \tau_d(n) = \{m_0 = n, m_1, m_2, \dots\}; \\ \text{(2)} \quad Even(n, d) = \{e_1, e_2, \dots\} \subset \mathbf{N}, \\ \quad \quad \quad \forall j \geq 1, \quad e_j = \nu_2(3n_{j-1} + d); \\ \text{(3)} \quad Odd(n, d) = \{n_0 = n, n_1, n_2, \dots\} \subset \mathbf{D}, \\ \quad \quad \quad \forall j \geq 1, \quad n_j = odd(3n_{j-1} + d) = m_{r_j}, \quad r_j = \sum_{1 \leq i \leq j} e_i. \end{array} \right. \quad (4:3)$$

(3) In the case of a  $T_d$ -cycle  $\mathbf{C}$  of the length  $\ell$  (2:2), the periodic sequence (4:3(3)) is called the *oddcycle* associated with  $\mathbf{C}$  and denoted by  $Odd(\mathbf{C})$ , with the period  $k < \ell$  called the *oddlength* of the cycle, with the minimal member  $n = n_0 \in \mathbf{D}$  listed as the first member of the cycle  $\mathbf{C}$  (cf. (4:4(2) below), and with the respective list  $Even(\mathbf{C})$  of

exponents defined as above (4:3(2)). The sets of members and of odd members of a cycle are denoted by  $Set(\mathbf{C})$  and  $Oddset(\mathbf{C})$ , respectively :

$$\left\{ \begin{array}{l} \text{(1)} \quad \mathbf{C} = \langle m_0, \dots, m_{\ell-1} \rangle; \quad m_0 = \min_{0 \leq j \leq \ell-1} \{m_j\}; \\ \quad \quad \quad T_d(m_{\ell-1}) = m_0, \quad T_d(m_i) = m_{i+1} \neq m_0, \quad 0 \leq i \leq \ell-2; \\ \text{(2)} \quad Odd(\mathbf{C}) = \langle n_0, \dots, n_{k-1} \rangle; \\ \quad \quad \quad n_0 = m_0 = \min\{m_0, \dots, m_{\ell-1}\} \neq n_j, \quad 1 \leq j \leq k-1, \\ \quad \quad \quad n_j = odd(3n_{j-1} + d), \quad 1 \leq j \leq k-1; \quad n_0 = odd(3n_{\ell-1} + d); \\ \text{(3)} \quad Even(\mathbf{C}) = \langle e_1, \dots, e_k \rangle; \\ \quad \quad \quad e_j = \nu_2(3n_{j-1} + d), \quad 1 \leq j \leq k; \quad e_1 + \dots + e_k = \ell; \\ \text{(4)} \quad Set(\mathbf{C}) = \{m_i \mid 0 \leq i \leq \ell-1\}; \quad Oddset(\mathbf{C}) = \{n_j \mid 0 \leq j \leq k-1\}. \end{array} \right. \quad (4:4)$$

Thus, e. g., a  $T_{2^r-3}$ -cycle  $\mathbf{C}_{2^r-3}^o$ ,  $r > 1$ , of the length  $r+1$ , starting at (the odd number) 1, has no other odd members (4:4),  $oddlength(\mathbf{C}_{2^r-3}^o)1$  :

$$\mathbf{C}_{2^r-3}^o = \langle 1, 2^{r-1}, 2^{r-2}, \dots, 2 \rangle; \quad Odd(\mathbf{C}_{2^r-3}^o) = \langle 1 \rangle; \quad Even(\mathbf{C}_{2^r-3}^o) = \langle r \rangle.$$

(4) For any odd  $q > 1$ , the  $q$ -multiple of a  $T_d$ -trajectory  $\tau_d(m)$  is identical to the  $T_{q \cdot d}$ -trajectory starting at  $q \cdot m$  (2:2) :

$$\tau_{q \cdot d}(q \cdot m) = q \cdot \tau_d(m) = \{q \cdot m, q \cdot T_d(m), q \cdot T_d^2(m), q \cdot T_d^3(m), \dots\}. \quad (4:5)$$

In particular, if  $\mathbf{C} = \tau_d(m)$  is a  $T_d$ -cycle of the length  $\ell$  and  $oddlength$   $k$ , then  $q \cdot \mathbf{C} = \tau_{q \cdot d}(q \cdot m)$  is a  $T_{q \cdot d}$ -cycle of the same length and  $oddlength$ .

(5) And vice versa, if, for some odd  $q > 1$ ,  $\gcd(d, m) = q \geq 1$ , then the  $T_d$ -trajectory  $\tau_d(m)$  is the  $q$ -multiple of a  $T_{\frac{d}{q}}$ -trajectory :

$$\gcd(d, m) = q > 1 \iff \tau_d(m) = q \cdot \tau_{\frac{d}{q}}\left(\frac{m}{q}\right). \quad (4:6)$$

*Proof* : Straightforward verification.

*End of the Proof.*

**Definition 4.3.** (1) A  $T_d$ -cycle  $\mathbf{C} = \tau_d(m)$  is called primitive, if

$$\gcd(d, m) = \gcd(d, T_d^r(m)) = 1, \quad \text{for all } r \geq 1.$$

Or, in other words, if for any divisor  $q > 1$  of  $d$ ,  $\mathbf{C}$  is not a  $q$ -multiple of a  $T_{\frac{d}{q}}$ -cycle.

(2) For any  $d \in \mathbf{D}$ , let  $\mathcal{C}(d)$ ,  $\mathcal{P}(d)$ , and  $\mathcal{P}\langle n \rangle$  be, respectively, the sets of all  $T_d$ -cycles, all primitive  $T_d$ -cycles, and all primitive cycles meeting an integer  $n$  (we postpone the detailed definition of sets  $\mathcal{P}\langle n \rangle$  till §6) :

$$\forall d \in \mathbf{D}, \quad \mathcal{P}(d) \subset \mathcal{C}(d); \quad \bigcup_{d \in \mathbf{D}} \mathcal{P}(d) = \mathcal{P} \subset \mathcal{C} = \bigcup_{d \in \mathbf{D}} \mathcal{C}(d). \quad (4:7)$$

A priori, all three sets might be finite (in particular, empty) or infinite, with  $\mathcal{P}(d) \neq \mathcal{C}(d)$  for all  $d > 1$  : cf. Example 4.5(2). And so might be their cardinalities :

$$\forall d \in \mathbf{D} , \quad \varpi(d) = \#\mathcal{P}(d) < \varsigma(d) = \#\mathcal{C}(d). \quad (4 : 8)$$

**Corollary 4.4.** (1) An odd member  $n$  of a primitive cycle belongs to  $\mathbf{D}_d$  (2:3). Conversely, a non-divergent trajectory  $\tau_d(m)$  starting at  $n \in \mathbf{D}_d$  either is a primitive cycle, or ultimately enters such a cycle.

(2) If a cycle  $\mathbf{C} = \tau_d(m)$  is not primitive,  $\gcd(d, m) = q > 1$ , then the cycle  $\frac{1}{q} \cdot \mathbf{C} = \tau_{\frac{d}{q}}\left(\frac{m}{q}\right)$  (4:6) is primitive.

**Example 4.5.** (1) The  $T$ -cycle  $\mathbf{C}^o$  (1:3) is primitive.

(2) According to (2:4), for any  $d > 1, d \in \mathbf{D}$ , the  $T_d$ -cycle  $\mathbf{C}_d^o$  is a  $d$ -multiple of the primitive  $T$ -cycle  $\mathbf{C}^o$  (1:3).

(3) Keith Matthews asserts in [Matthews 1999] that, in our terminology, the system  $\mathcal{D}_{371}$  has four cycles, with the minimal members 25, 265, 371, 721 and lengths 222, 4, 2, 29, respectively, or for short,

$$[25(222)]_{371}, [265(4)]_{371}, [371(2)]_{371}, [721(29)]_{371}.$$

Note that  $371 = 7 \cdot 53$  and that only the  $T_{371}$ -cycle  $[25(222)]_{371}$  is primitive. The other three cycles are 53-, 371-, and 7-multiples of, respectively, the primitive  $T_7$ -,  $T_1$ -, and  $T_{53}$ -cycles

$$[5(4)]_7, [1(2)]_1, [103(29)]_{53}.$$

Our calculations confirm Matthews assertion. We have found that each one of the systems  $\mathcal{D}_7$ ,  $\mathcal{D}_{53}$ , and  $\mathcal{D}_{371}$  has only one primitive cycle, presented above.

The following claims about basic structural properties of the set  $\mathcal{C}(d)$  of all  $T_d$ -cycles (Definition 4.3) formalize intuitive insights gained with the above examples. These propositions do not depend on any of assumptions or conjectures concerning the dynamical systems  $\mathcal{D}_d$ . In particular, the below formulae are valid when the set  $\mathcal{C}(d)$  is infinite.

**Lemma 4.6.** For any  $d \in \mathbf{D}$ , let  $\mathcal{C}(d)$  and  $\mathcal{P}(d)$  be, respectively, the sets of all  $T_d$ -cycles and all primitive  $T_d$ -cycles,  $\mathcal{P}(d) \subset \mathcal{C}(d)$  (Definition 4.3(2)). For any positive integer  $q \in \mathbf{D}$  and any  $T_d$ -cycle  $\mathbf{C}$  (4:1) define the  $T_{q \cdot d}$ -cycle  $q \cdot \mathbf{C}$  as the collection of  $q$ -multiples of members of  $\mathbf{C}$  (4:2). Define further

$$q \cdot \mathcal{P}(d) = \{q \cdot \mathbf{C} \mid \forall \mathbf{C} \in \mathcal{P}(d)\} \subset \mathcal{C}(q \cdot d). \quad (4 : 9)$$

Then the following equality holds :

$$\mathcal{C}(d) = \bigcup_{\substack{d=q \cdot r \\ q, r \in \mathbf{D}}} q \cdot \mathcal{P}(r). \quad (4 : 10)$$

*Proof* : Straightforward verification.

*End of Proof.*

**Corollary 4.7. (1)** Let  $\varpi(d)$  and  $\varsigma(d)$  be the total numbers of primitive and, respectively, all  $T_d$ -cycles (4:8). If for some  $d \in \mathbf{D}$ , the set  $\mathcal{C}(d)$  is infinite, then so is the set  $\mathcal{P}(r)$  for at least one divisor  $r$  of  $d$  ( $r|d, 1 \leq r \leq d$ ) :

$$\forall d \in \mathbf{D}, \varsigma(d) = \infty \iff \exists r | d (1 \leq r \leq d), \varpi(r) = \infty. \quad (4:11)$$

**(2)** Moreover, the equality (4:9) implies the following formula (which, according to (4:11), becomes the trivial identity  $\infty = \infty$ , if the set  $\mathcal{C}(d)$  would be infinite) :

$$\varsigma(d) = \sum_{\substack{d=q \cdot r \\ q, r \in \mathbf{D}}} \varpi(r) = 1 + \sum_{\substack{d=q \cdot r, r \geq 2 \\ q, r \in \mathbf{D}}} \varpi(r). \quad (4:12)$$

**(3)** Let

$$n_o(\mathbf{C}) = \min(m, m_1, \dots, m_{\ell-1}), \quad m^o(\mathbf{C}) = \max(m, m_1, \dots, m_{\ell-1}) \quad (4:13)$$

be the minimal, respectively, maximal member of a  $T_d$ -cycle  $\mathbf{C}$  of the length  $\ell$ . Clearly,  $n_o$  is odd and  $\in \mathbf{D}$ , and  $m^o$  is even. If, for a given  $d \in \mathbf{D}$ , the combination  $(\text{NDTC})_{3x+d} + (\text{FCC})_{3x+d}$  of conjectures holds, §2, then the  $T_d$ -recurrence and  $T_d$ -collapse thresholds  $R_d, L_d$ , and the  $T_d$ -attractor  $\mathcal{U}_d$ , §3, are given by the formulae :

$$\left\{ \begin{array}{l} (1) \quad R_d^o = \max \{ n_o(\mathbf{C}) \mid \mathbf{C} \in \mathcal{P}(d) \}; \\ (2) \quad R_d = \max \{ q \cdot R_r^o \mid d = q \cdot r \ \& \ q, r \in \mathbf{D} \}; \\ (3) \quad L_d^o = \max \{ m^o(\mathbf{C}) \mid \mathbf{C} \in \mathcal{P}(d) \}; \\ (4) \quad L_d = \max \{ q \cdot L_r^o \mid d = q \cdot r \ \& \ q, r \in \mathbf{D} \}; \\ (5) \quad \mathcal{U}_d = \{ m \in \mathbf{C} \mid \mathbf{C} \in \mathcal{P}(d) \}. \end{array} \right. \quad (4:14)$$

**Note 4.8. (1)** For all  $d \in \mathbf{D}$ ,  $1 \leq d \leq 19999$ , denote by

$$\tilde{\mathcal{P}}_d, \tilde{\mathcal{C}}_d, \tilde{R}_d^o, \tilde{R}_d, \tilde{L}_d^o, \tilde{L}_d, \tilde{\mathcal{U}}_d, \tilde{\varpi}(d), \tilde{\varsigma}(d), \quad (4:15)$$

respectively, the sets of primitive and general cycles found experimentally in this study, as well as the corresponding  $\tilde{T}_d$ -characteristics calculated according to the formulae (4:11-14), with  $\mathcal{P}(d)$  everywhere replaced by  $\tilde{\mathcal{P}}_d$ .

**(2)** This *tilde*-device will be used throughout the paper, to distinguish between theoretical (or absolute) values of parameters and their experimental estimates.

**(3)** It is clear that all *tilde*-parameters represent experimental lower bounds to their respective original counterparts, as, e. g.,  $\tilde{R}_d, \tilde{L}_d, \tilde{\mathcal{U}}_d, \tilde{\varpi}(d)$ , and  $\tilde{\varsigma}_d$  :

$$\tilde{R}_d \leq R_d, \tilde{L}_d \leq L_d, \tilde{\mathcal{A}}_d \subseteq \mathcal{A}_d, \tilde{\varpi}(d) \leq \varpi(d), \tilde{\varsigma}(d) \leq \varsigma(d). \quad (4:16)$$

**Conjectural Claim 4.9.** In the interval  $d \in \mathbf{D}$ ,  $1 \leq d \leq 19999$  of values  $d$  experimentally studied in the present paper, the inequalities and inclusion of (4:16) are, in fact, equalities.

We justify this claim below, §12, where we will provide the evidence that, in this study, we have actually found all primitive  $T_d$ -cycles (and, thus, all general  $T_d$ -cycles, too), for all  $d$  from the above interval.

**Example 4.10. (1)** We have found only two primitive  $T_{1715}$ -cycles, with the minimal and maximal members 941, 773, and 28,876, 55,876, respectively. On the other hand, the calculated  $T_{1715}$ -recurrence and -collapse thresholds are, respectively,

$$\tilde{R}_{1715} = 886,685 = 5 \cdot 177337, \quad \tilde{L}_{1715} = 795268030 = 5 \cdot 159053606.$$

Both these numbers are coming from a non-primitive  $T_{1715}$ -cycle (remark that  $1715 = 5 \cdot 343$ ), the 5-multiple of a primitive  $T_{343}$ -cycle. The minimal and maximal members of this  $T_{343}$ -cycle are also the  $\tilde{T}_{343}$ -recurrence and -collapse thresholds,

$$\tilde{R}_{343} = \tilde{R}_{343}^o = 177337, \quad \tilde{L}_{343} = \tilde{L}_{343}^o = 159053606.$$

## 5. Three Basic Primitive Cycles Conjectures.

But do primitive cycles exist in all systems  $\mathcal{D}_d$ ? And what are the chances of an integer  $n \in \mathbf{D}$  (cf. Lemma 4.2(1), Corollary 4.4(1)) to be met by such a cycle, for some  $d \in \mathbf{D}$ ,  $\gcd(d, n) = 1$ ? These are the subjects of three original conjectures of Lagarias [Lagarias 1990].

The two first conjectures of Lagarias are straightforward restrictions to primitive cycles of, respectively, the no-divergent-trajectories and finite-number-of-cycles conjectures, §2. Conversely, the second conjecture implies, according to Corollary 4.7(1,2), (4:12), the finite-number-of-cycles conjecture.

The no-divergent-trajectories conjecture (in its “ultimately-cyclic” form, Conjecture 8, §2), together with Corollary 4.4 (1) (a non-divergent trajectory  $\tau_d(m)$  starting at  $n \in \mathbf{D}_d$  (2:3) either is a primitive cycle, or ultimately enters such a cycle) immediately imply the first conjecture of Lagarias :

**Conjecture 17 : Existence of a Primitive  $T_d$ -cycle.** For any  $d \in \mathbf{D}$ , there exists at least one primitive  $T_d$ -cycle (cf. (4:8)) :

$$\forall d \in \mathbf{D}, \quad \varpi(d) \geq 1. \tag{5 : 1}$$

The second conjecture of Lagarias is a formal restriction of the finite-number-of-cycles conjecture (Conjecture 9, §2) to primitive cycles. In fact, according to (4:12), Corollary 4.7(1.2), both conjectures are equivalent :

**Conjecture 18 : Finiteness of the Set of Primitive  $T_d$ -cycles.** For any  $d \in \mathbf{D}$ , the number  $\varpi(d)$  (4:8) of primitive  $T_d$ -cycles is finite :

$$\forall d \in \mathbf{D}, \quad \varpi(d) < \infty. \tag{5 : 2}$$

The third conjecture of Lagarias has no immediately apparent  $3x + 1$  analogue. We interpret it here and in §§6, 8, as a double conjecture, dual to the first and second conjectures, with the emerging *disparate duality* between  $d$ - and  $n$ -related phenomena being an important new and all-pervading intuition behind the present study.

**Conjecture 19 : Any Suitable Integer Is a Member of at Least One Primitive  $T_d$ -cycle.** For any  $n \in \mathbf{D}$ , there exists at least one  $d \in \mathbf{D}$  with a primitive  $T_d$ -cycle meeting  $n$ . Or, in other words, the the set of odd members of all primitive  $T_d$ -cycles covers the set  $\mathbf{D}$  of odd positive integers not divisible by 3 (cf. the notations (4:4)) :

$$\bigcup_{d \in \mathbf{D}} \bigcup_{\mathbf{C} \in \mathcal{P}(d)} \text{Set}(\mathbf{C}) = \mathbf{D} . \quad (5 : 3)$$

**Conjecture 20 : Any Suitable Integer Is, in Fact, a Member of an Infinite Number of Primitive  $T_d$ -cycles.**

Of course, Conjecture 20 tautologically implies Conjecture 19. Still, it is worth to state explicitly both conjectures, and for the following reasons :

**Note 5.1. (1)** The apparently weaker Conjecture 19 might be more amenable to a proof.

**(2)** On the other hand, as it has been proved in [Lagarias 1990], an integer  $n \in \mathbf{D}$  belonging to a primitive cycle of the oddlength  $k$ , belongs to an infinite series of primitive cycles of the same oddlength and with an almost identical diophantine structure (see for details §11, Theorem 11.3(3)). Thus, Conjecture 19 formally implies the apparently stronger Conjecture 20.

**(3)** However, as our experimental data (which stronly confirm Conjecture 19 and provide a reasonable, in fact, the best possible in the circumstrances, Note 5.2, confirmation of Conjecture 20 : cf. Figure 2 below) show, the proliferation of primitive cycles with the same values of length and oddlength does not account for the actual abundance of recurring memberships of a given number  $n \in \mathbf{D}$  in primitive cycles, §11.

**Note 5.2 : Cogency of Experimental Confirmation. (1)** The experimental search for primitive  $T_d$ -cycles, for 6667 systems  $\mathcal{D}_d$ , within the range  $1 \leq d \leq 19999$ , confirm Conjectures 17-20, albeit, inevitably, with different degrees of certainty.

**(2)** In fact, to fully experimentally confirm, in the chosen range of input parameters, the claims of Conjectures 17, 19 (the existence of at least one primitive cycle with a fixed shift number or member), it is necessary and sufficient to find such a primitive cycle, with  $d$  or  $n$  from the chosen range : cf. Figures 3,4 and, respectively, 2.

**(3)** Another matter, however, is to experimentally confirm a conjecture which claims that some set of primitive cycles is not just nonempty (Conjectures 17, 19), but is finite (i. e., *not infinite*, - Conjecture 18) or even infinite (Conjecture 20). Here the confirmation depends on the quality of available evidence that the search was exhaustive. We postpone this discussion to §12.

We start with some preliminary notations.

**Notation 5.3.** For any pair of odd integers  $Q, R \in \mathbf{D}$ , define an interval  $\mathbf{I}_{Q,R} \subset \mathbf{D}$ , as follows :

$$Q, R \in \mathbf{D} \implies \mathbf{I}_{Q,R} = [Q, R] \cap \mathbf{D} . \quad (5 : 4)$$

By definition,  $\mathbf{I}_{Q,R} = \emptyset$  if  $Q > R$ .

**Lemma 5.4.** According to (2:3), the number  $\iota(Q, R)$  of integers in the interval  $\mathbf{I}_{Q,R}$  (5:4) is equal to

$$\iota(Q, R) = \#\mathbf{I}_{Q,R} = \begin{cases} \lfloor \frac{R}{3} \rfloor - \lfloor \frac{Q}{3} \rfloor + 1, & \text{if } Q \leq R ; \\ 0, & \text{otherwise .} \end{cases} \quad (5 : 5)$$

In these notations, the range of our experimental study becomes:

$$d \in \mathbf{I}_{1,19999} \quad , \quad \iota(1, 19999) = 6667 , \quad (5 : 6)$$

*Proof* : Straightforward verification.

*End of the Proof.*

**Experimental Evidence 5.5. (1) Conjecture 17.** We have found at least one primitive cycle, for any  $d$  from the chosen range. With the above notation  $\tilde{\varpi}(d)$  (4:15,16), we have :

$$\forall d \in \mathbf{I}_{1,19999} \quad , \quad \varpi(d) \geq \tilde{\varpi}(d) \geq 1 . \quad (5 : 7)$$

Table 1 lists the first 500 values of the function  $\tilde{\varpi}(d)$ ,  $d \in \mathbf{I}_{1,1499}$ .

**(2) Conjecture 18.** As it has been mentioned above, Note 5.2, the experimental evidence for this conjecture can be only circumstantial. The experimentally found sets  $\tilde{\mathcal{P}}_d$  of primitive cycles and their cardinalities  $\tilde{\varpi}(d)$  (4:15) cannot be but finite. We will argue below, §12, why the sets and numbers, experimentally found in this study, are not just experimental samples, providing us with the lower bounds (4:16), but, in fact, are the actual sets and numbers of primitive cycles in the given range of shift numbers  $d$ . What follows, is a digest of our experimental findings.

**(a)** The total number of found here primitive  $T_d$ -cycles is 42765, with the maximal value  $\tilde{\varpi}(d) = 944$  attained at  $d = 14303$  :

$$\sum_{d \in \mathbf{I}_{1,19999}} \tilde{\varpi}(d) = 42765 ; \quad \max_{d \in \mathbf{I}_{1,19999}} \tilde{\varpi}(d) = 944 = \tilde{\varpi}(14303). \quad (5 : 8)$$

**(b)** More than a half of 6667 dynamical systems  $\mathcal{D}_d$  explored in this study have less than 3 primitive cycles. To describe the distribution of these systems according to the values  $\tilde{\varpi}(d)$ , let us define  $\delta_{D,E}(r)$  as the total number of systems  $\mathcal{D}_d$ , with  $d \in \mathbf{I}_{D,E}$  and with exactly  $r$  primitive cycles; then the corresponding function  $\tilde{\delta}$  will be its experimentally found lower bound to :

$$\delta_{D,E}(r) = \#\left\{ d \in \mathbf{I}_{D,E} \mid \varpi(d) = r \right\} . \quad (5 : 9)$$

The below table (Table 3) gives the values of the experimental function  $\tilde{\delta}_{1,19999}$  in the interval  $1 \leq P \leq 160$ . We have found only eleven systems  $\mathcal{D}_d$  with the numbers of primitive cycles bigger than 160, see Table 2.

**Table 1.** The numbers  $\tilde{\omega}(d)$  of the primitive  $T_d$ -cycles traced in this study, for 500 values of  $d$  from 1 to 1499. For a given  $d \in \mathbf{I}_{1,1499}$ , find  $d' \in \mathbf{I}_{1,59}, d' \equiv d \pmod{60}$  and  $q = \frac{d-d'}{60}$ . The value  $\tilde{\omega}(d)$  can be found on the intersection of the  $q$ -th row and  $d'$ -th column.

1	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49	53	55	59
1	5	1	2	9	2	1	3	2	4	1	2	3	1	1	7	1	1	3	7
2	1	1	7	3	1	4	3	1	1	3	3	2	7	2	1	1	1	2	5
2	4	2	3	2	5	1	3	3	2	2	1	1	4	2	3	2	2	7	1
3	1	2	3	4	1	2	2	1	4	1	3	2	1	2	1	8	19	3	4
2	2	6	2	3	3	7	3	3	18	1	1	1	7	3	6	2	1	17	3
2	3	7	2	4	2	1	1	2	4	2	3	8	2	3	2	2	5	20	11
10	4	1	1	2	2	3	2	3	3	4	5	1	1	1	4	3	3	1	2
2	4	1	23	4	3	10	5	2	1	1	4	4	1	1	3	1	1	2	4
13	15	2	2	1	7	52	7	5	2	5	2	12	1	3	3	3	1	3	2
1	2	3	10	3	4	1	1	2	1	2	2	10	1	1	3	2	4	3	3
12	2	5	5	3	3	1	2	2	3	1	2	1	1	6	3	3	1	1	2
2	2	10	6	3	2	4	1	2	1	1	2	11	1	2	2	1	2	4	6
3	4	5	2	3	1	1	3	13	2	1	8	2	1	2	2	3	2	1	3
31	6	2	2	2	2	2	9	2	1	5	4	4	1	2	7	2	4	6	4
1	1	1	4	1	1	2	3	7	2	1	1	1	6	2	4	8	23	2	2
3	2	3	4	2	1	4	2	8	3	2	2	7	1	24	1	2	1	2	6
2	9	4	1	3	1	1	3	4	3	6	1	13	2	1	5	2	3	6	1
4	1	6	2	4	2	1	3	7	2	4	2	3	1	2	6	2	2	3	7
6	4	3	2	7	10	1	3	6	1	1	1	1	1	4	5	4	2	1	1
6	2	3	2	7	3	1	1	1	4	1	3	6	2	2	2	2	1	3	2
33	5	4	2	1	2	2	6	2	1	2	10	1	6	1	3	2	1	1	5
10	9	17	3	2	3	2	3	1	1	1	9	5	10	3	1	3	1	3	53
3	3	10	1	5	3	2	3	15	3	9	2	7	2	5	3	1	1	4	2
1	1	3	5	17	2	22	12	1	5	2	3	7	2	3	1	1	2	2	3
3	2	1	2	1	3	2	4	3	2	1	1	1	2	2	5	4	4	9	1

**Table 2.** Eleven systems  $\mathcal{D}_d$  with the numbers of primitive cycles bigger than 160.

d	7463	18359	7727	15655	10289	9823	17021	14197	13085	6487	14303
$\tilde{\omega}(d)$	162	164	198	207	214	241	258	329	335	534	944

These data strongly suggest the following new conjectures :

**Conjecture 21.** (1) The number of primitive cycles attains all positive integer values :

$$\varpi(d) : \mathbf{D} \longrightarrow \mathbf{N} \text{ is an onto mapping} \iff \forall r \in \mathcal{N} \exists d \in \mathcal{D}, \varpi(d) = r. \quad (5 : 10)$$

(2) For any  $D \in \mathbf{D}$  and any sufficiently big  $E \in \mathbf{D}, E \gg D$ , the distribution function  $\delta_{D,E}(r)$  (5:9) attains its maximal values for  $r = 1$  and/or 2, and exponentially descends with  $r \geq 2$  growing.

**Table 3.**

The number  $\tilde{\delta}_{1,19999}(P)$  of systems  $\mathcal{D}_d$  with exactly  $P, 1 \leq P \leq 160$ , experimentally found primitive  $T_d$ -cycles. On the intersection of  $i$ -th row ( $1 \leq i \leq 16$ ) and  $j$ -th column ( $1 \leq j \leq 10$ ) is the number of systems with  $P = j + 10 \cdot (i - 1)$  primitive cycles.

	<b>j = 1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>	<b>10</b>
<b>i=1</b>	1481	1507	1005	605	392	259	216	172	121	121
<b>2</b>	87	72	45	53	48	42	35	19	30	24
<b>3</b>	25	20	17	19	13	16	11	8	13	11
<b>4</b>	5	7	7	8	3	3	8	6	2	4
<b>5</b>	2	5	3	3	3	2	4	0	1	3
<b>6</b>	3	3	3	4	4	2	4	0	2	2
<b>7</b>	2	1	4	2	4	1	4	3	2	2
<b>8</b>	0	0	1	1	0	3	0	1	2	1
<b>9</b>	2	0	2	3	1	1	0	0	2	0
<b>10</b>	2	0	0	0	0	0	2	0	0	1
<b>11</b>	0	0	0	0	0	1	1	0	0	0
<b>12</b>	1	0	0	1	0	0	1	0	0	1
<b>13</b>	0	0	0	0	0	0	0	0	0	0
<b>14</b>	0	0	0	0	0	0	0	2	1	0
<b>15</b>	0	2	0	0	0	0	0	0	0	0
<b>16</b>	0	0	1	0	0	2	0	0	1	0

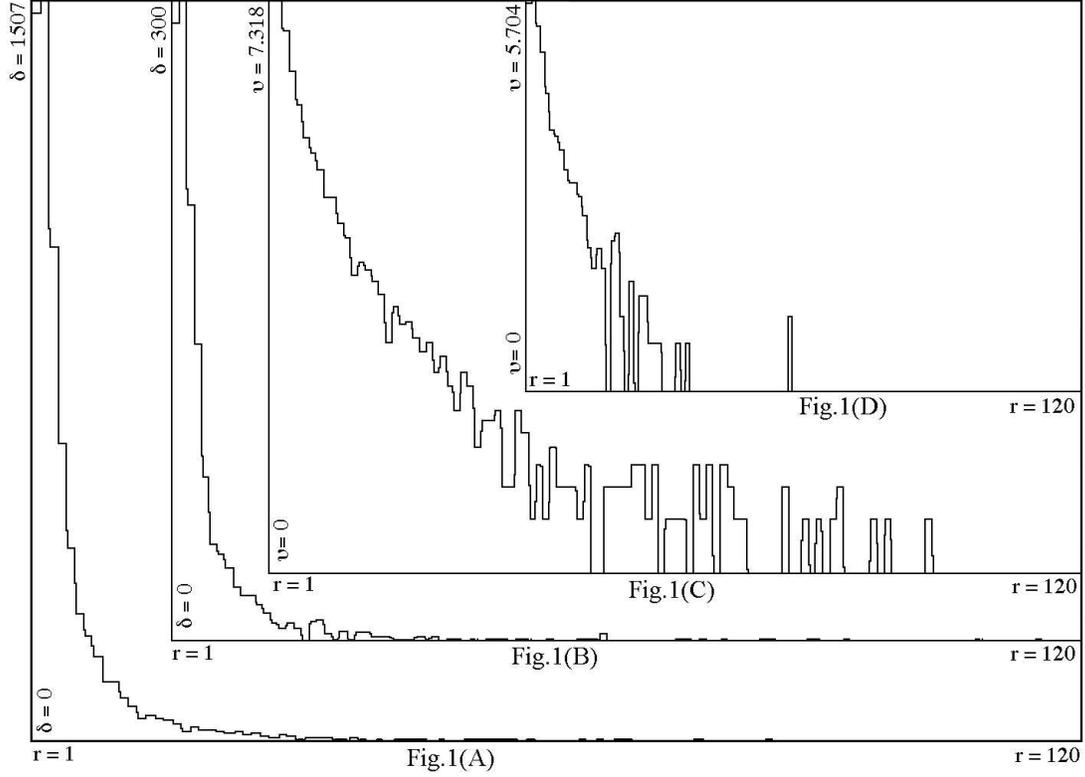
**Experimental Evidence 5.6 : Conjecture 21.** Figure 1 displays two distributions  $\delta$  of numbers  $\tilde{\omega}(d)$  of primitive  $T_d$ -cycles, as well as the logarithms  $v = \log(\delta)$  of these distributions.

The first distribution (cf. Figure 1(A) and, for its logarithm, Figure 1(C)) describes all 6667 systems experimentally studied in this paper,  $1 \leq d \leq 19999$ . The second one deals with the portion of 1333 systems corresponding to the (arbitrary chosen) interval  $12001 \leq d \leq 15997$  (Figure 1(B,D)).

Both distributions count only systems  $\mathcal{D}_d$  with the numbers of cycles ranging from 1 to 120. (We remind the reader that the maximal number  $\tilde{\omega}(d)$  of cycles found in this study is 944 (5:8), and, according to Table 3, only twenty out of 6667 systems have more than 120 primitive cycles).

$$\forall r (1 \leq r \leq 120) , \begin{cases} \delta = \delta_{1,19999}(r), v = \ln(\delta_{1,19999}(r)) ; \\ \delta = \delta_{8003,11999}, v = \ln(\delta_{8003,11999}(r)) . \end{cases} \quad (5 : 11)$$

**Experimental Evidence 5.7 : Conjecture 19. (1)** The conjecture has been strongly confirmed for all  $n \in \mathcal{D}, 1 \leq n \leq 59999$  (altogether, 20000 values) in the band  $d \in \mathbf{I}_{1,19999}$ . We have found that any  $n$  from the chosen range is a member of at least 18 primitive  $T_d$ -cycles. Or, using a version of the notations (4:8) :



**Figure 1.** The distributions of numbers of primitive cycles (see Experimental Evidence 5.6 for explanations). Note that systems with 1 to 2 cycles predominate.

$$\forall n \in \mathcal{D} \quad \forall D, E \in \mathbf{I}_{1,19999}, \quad \tilde{\omega}\langle n \rangle \Big|_D^E = \#\left\{d \in \mathbf{I}_{D,E} \mid \exists \mathbf{C} \in \tilde{\mathcal{P}}(d) \mid n \in \mathbf{C}\right\}, \quad (5 : 12)$$

it has been found that

$$\forall n \in \mathbf{I}_{1,59999}, \quad 18 \leq \tilde{\omega}\langle n \rangle \Big|_1^{19999} \leq 452. \quad (5 : 13)$$

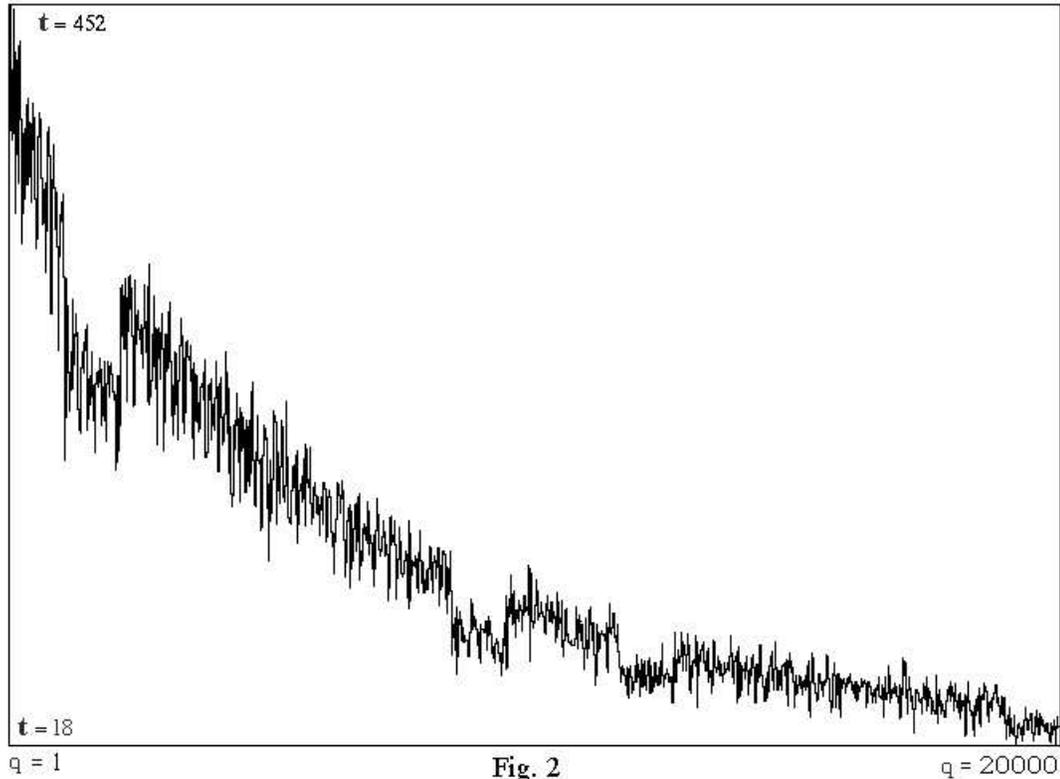
(2) To present this statistic handily in a graphic form, Figure 2, we number the 20000 values of  $n \in \mathbf{I}_{1,59999}$  by the parameter  $q \in [1, 20000]$ , according to (2:3), and denote by  $\mathbf{t}(q)$  the number of primitive  $T_d$ -cycles meeting  $n = \xi(q)$ ,  $d \in \mathbf{I}_{1,19999}$ :

$$\left\{ \begin{array}{l} \mathbf{t}(q) = \tilde{\omega}\langle \xi(q) \rangle \Big|_1^{19999} = \tilde{\omega}\left\langle 6 \cdot \left\lfloor \frac{q}{2} \right\rfloor - (-1)^q \right\rangle \Big|_1^{19999}; \\ 1 \leq q \leq 20000 \iff n = \xi(q) \in \mathbf{I}_{1,59999}. \end{array} \right. \quad (5 : 14)$$

(3) We give here also a more detailed sample of the experimental data, the first twenty values of the function  $\mathbf{t}(q) = \tilde{\omega}\langle n \rangle \Big|_1^{19999}$ ,  $1 \leq n \leq 59$ , Table 4.

**Table 4.** The first twenty values of the function  $\mathbf{t}(q) = \tilde{\omega}\langle n(q) \rangle|_1^{19999}$ ,  $1 \leq q \leq 20$ . Thus, for example, among 42765 primitive cycles discovered here, 452 cycles start at the first value of  $n$ ,  $n = 1$ , and 440 meet the 18–th value,  $n = 53$ .

$q$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$n$	1	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49	53	55	59
$\mathbf{t}(q)$	452,354,359,414,396,410,423,399,324,409,426,265,407,415,418,422,395,440,300,406																			



**Figure 2.** The graph of the function  $\mathbf{t}(q) = \tilde{\omega}\langle n(q) \rangle|_1^{19999}$ ,  $1 \leq q \leq 20000$ . Notice three distinct and mysterious “depressions”.

## 6. Shift Number – Member Number Disparate Duality.

In this section, we lay down a groundwork for the study of (disparate) duality between related *shift number* and *member number* properties. The impetus for this approach comes from the comparison between two sets of conjectures of §5 (Conjectures 18,19 versus Conjecture 20). The true meaning of this duality will become apparent, and the adequate

diophantine formalism for its study will emerge in the next section.

The three conjectures of Lagarias can be viewed as a dual conjecture about projections of the *set of memberships*, as follows :

**Definition 6.1. (1)** If a primitive  $T_d$ -cycle  $\mathbf{C}$  meets a number  $n \in \mathbf{D}$ , then  $\gcd(n, d) = 1$  (Corollary 4.4(1)) and the pair  $\langle \overline{n, d} \rangle$  is called a *primitive membership pair*, or *primitive membership*, or simply *membership*. We denote by  $\mathcal{P}$  the set of all primitive cycles and by  $\mathcal{M}$  the set of all memberships.

**(2)** Since, for a given membership  $\langle \overline{n, d} \rangle$ , the corresponding primitive  $T_d$ -cycle meeting  $n$  is unique,  $\mathbf{C} = \tau_d(n)$  (2:2), one can extend to memberships the cyclic notations of *length* and *oddlength*, Lemma-Definition 4.2 (4:4). A membership of the length  $\ell$  and oddlength  $k$  will be sometimes referred to as a  $(k, \ell)$ -membership, with  $\mathcal{M}_\ell^{(k)}$  being the set of  $(k, \ell)$ -memberships :

$$\left\{ \begin{array}{l} \Delta = \{(n, d) \in \mathbf{D}^2 \mid \gcd(n, d) = 1\} ; \\ \mathcal{M} = \{(n, d) \in \Delta \mid \langle \overline{n, d} \rangle \text{ is a membership} \iff \mathbf{C} = \tau_d(n) \in \mathcal{P}(d)\} ; \\ \zeta : \mathcal{M} \longrightarrow \mathcal{P} ; \quad \forall \langle \overline{n, d} \rangle \in \mathcal{M} , \quad \zeta \langle \overline{n, d} \rangle = \mathbf{C} \in \mathcal{P}(d) ; \\ \text{length} \langle \overline{n, d} \rangle = \text{length}(\mathbf{C}) = \ell ; \quad \text{oddlength} \langle \overline{n, d} \rangle = \text{oddlength}(\mathbf{C}) = k ; \\ \mathcal{M}_\ell^{(k)} = \{\langle \overline{n, d} \rangle \in \mathcal{M} \mid \text{oddlength} \langle \overline{n, d} \rangle = k , \text{length} \langle \overline{n, d} \rangle = \ell\} ; \\ \mu_\ell^{(k)} = \#\mathcal{M}_\ell^{(k)} . \end{array} \right. \quad (6 : 1)$$

A diophantine interpretation of sets  $\mathcal{M}_\ell^{(k)}$ , §§7-9, (8:6), will result in a simple formula for their cardinalities (9:2).

**(3)** For the same reason, the set  $\mathcal{M}\langle n \rangle$  of all memberships with a fixed first component (or, *member number*)  $n$  can be identified with both the set  $\mathcal{P}\langle n \rangle$  of all primitive cycles meeting  $n$  and the (sub)set of all shift numbers  $d \in \mathbf{D}_n$  (2:3), with a  $T_d$ -cycle (always exactly one) meeting  $n$  :

$$\forall n \in \mathbf{D} , \quad \mathcal{M}\langle n \rangle = \{\mathbf{C} \in \mathcal{P}\langle n \rangle \mid n \in \text{Set}(\mathbf{C})\} \cong \mathcal{P}\langle n \rangle = \{d \in \mathbf{D}_n \mid \mathbf{C} = \tau_d(n) \in \mathcal{P}(d)\} . \quad (6 : 2)$$

**(4)** On the other hand, the set  $\mathcal{M}(d)$  of all memberships with a fixed second component (or, *shift number*)  $d$  can be identified with the set of members of all  $T_d$ -cycles (cf. the notations (4:4)) :

$$\forall d \in \mathbf{D} , \quad \mathcal{M}(d) = \bigcup_{\mathbf{C} \in \mathcal{P}(d)} \text{Set}(\mathbf{C}) . \quad (6 : 3)$$

**(5)** Through the mapping  $\zeta$  (6:1) and natural projections of the set  $\mathcal{M}$  of memberships,

$$\mathbf{D} \xleftarrow{\nabla_{\text{memb}}} \mathcal{M} \xrightarrow{\nabla_{\text{shift}}} \mathbf{D} , \quad \forall \langle \overline{n, d} \rangle \in \mathcal{M} , \quad \nabla_{\text{memb}} \langle \overline{n, d} \rangle = n , \quad \nabla_{\text{shift}} \langle \overline{n, d} \rangle = d , \quad (6 : 4)$$

the four above sets are formally related, as follows :

$$\left\{ \begin{array}{l} \forall n \in \mathbf{D} , \mathcal{M}\langle n \rangle = \nabla_{\text{memb}}^{-1}(n) \subset \mathcal{M} ; \quad \zeta(\mathcal{M}\langle n \rangle) = \mathcal{P}\langle n \rangle \\ \zeta : \mathcal{M}\langle n \rangle \xrightarrow{\cong} \mathcal{P}\langle n \rangle ; \\ \forall d \in \mathbf{D} , \mathcal{M}(d) = \nabla_{\text{shift}}^{-1}(d) \subset \mathcal{M} ; \quad \zeta(\mathcal{M}(d)) = \mathcal{P}(d) . \end{array} \right. \quad (6 : 5)$$

**Lemma 6.2.** (1) If  $\ell < k \cdot \log_2 3$ , then the set  $\mathcal{M}_\ell^{(k)}$  is empty.

(2) For any  $d \in \mathbf{D}$ , the sets  $\mathcal{M}(d), \mathcal{P}(d)$  are simultaneously either empty, finite, or infinite, with their cardinalities  $\mu(d)$  and  $\varpi(d)$  (4:8) related by the following formula (cf. the notations (4:4)) :

$$\left\{ \begin{array}{l} \forall d \in \mathbf{D} , \quad \mathcal{M}(d) = \zeta^{-1}(\mathcal{P}(d)) = \bigcup_{\mathbf{C} \in \mathcal{P}(d)} \text{Oddset}(\mathbf{C}) ; \\ \mu(d) = \#\mathcal{M}(d) = \sum_{\mathbf{C} \in \mathcal{P}(d)} \text{oddlength}(\mathbf{C}) ; \\ \varpi(d) \leq \mu(d) \leq \varpi(d) \cdot \max_{\mathbf{C} \in \mathcal{P}(d)} \text{oddlength}(\mathbf{C}) . \end{array} \right. \quad (6 : 6)$$

*Proof :* (1) See below Theorem 8.3(1). (2) Straightforward verification. *End of the Proof.*

Assuming  $\varpi(d)$  and  $\mu(d)$  are finite, the second numerical characteristics of dynamical systems  $\mathcal{D}_d$ , as our calculations show, is less volatile and, on the average, steadily growing one : cf. Figures 3,4.

Now, the three basic conjectures of §5 can be summed up, as follows :

**Conjecture 22.** (1) All sets  $\mathcal{M}\langle n \rangle$  are infinite. (2) On the other hand, all sets  $\mathcal{M}(d)$  are non-empty and finite :

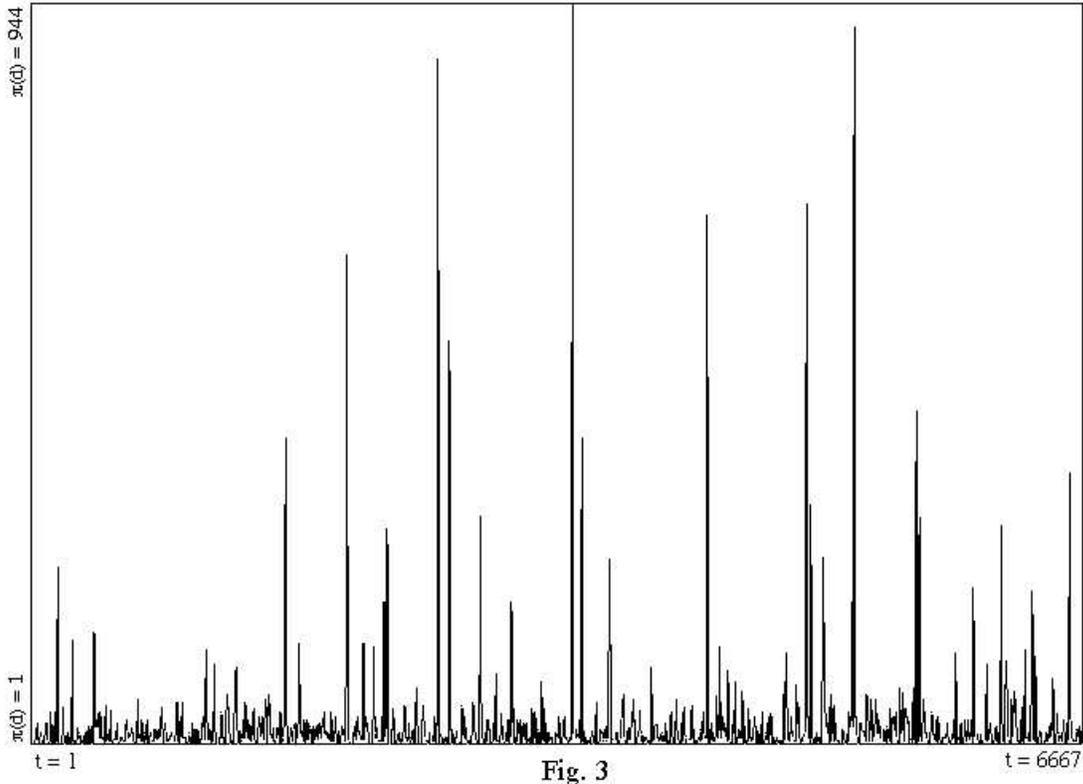
$$\left\{ \begin{array}{l} \text{(1)} \quad \forall n \in \mathbf{D} , \quad 1 < \mu\langle n \rangle = \#\mathcal{M}\langle n \rangle = \#\nabla_{\text{memb}}^{-1}(n) = \infty ; \\ \text{(2)} \quad \forall d \in \mathbf{D} , \quad 1 \leq \mu(d) = \#\mathcal{M}(d) = \#\nabla_{\text{shift}}^{-1}(d) < \infty . \end{array} \right. \quad (6 : 7)$$

**Corollary 6.3.** If, for a given  $d \in \mathbf{D}$ , the system  $\mathcal{D}_d$  has no divergent trajectories and if, according to the conjecture (6:7),  $\mu(d) = \#\mathcal{M}(d) < \infty$ , then  $\mathcal{M}(d)$  is the  $T_d$ -attractor of  $\mathcal{D}_d$  (cf. Conjecture 16, §3, and formula (4:14)) :

$$\mathcal{U}_d = \mathcal{M}(d) . \quad (6 : 8)$$

Diophantine insights of the next chapter will shed new light on this disparate duality between the member-number- and shift-number-projections.

### Chapter III. Cyclic Diophantine Formalism.

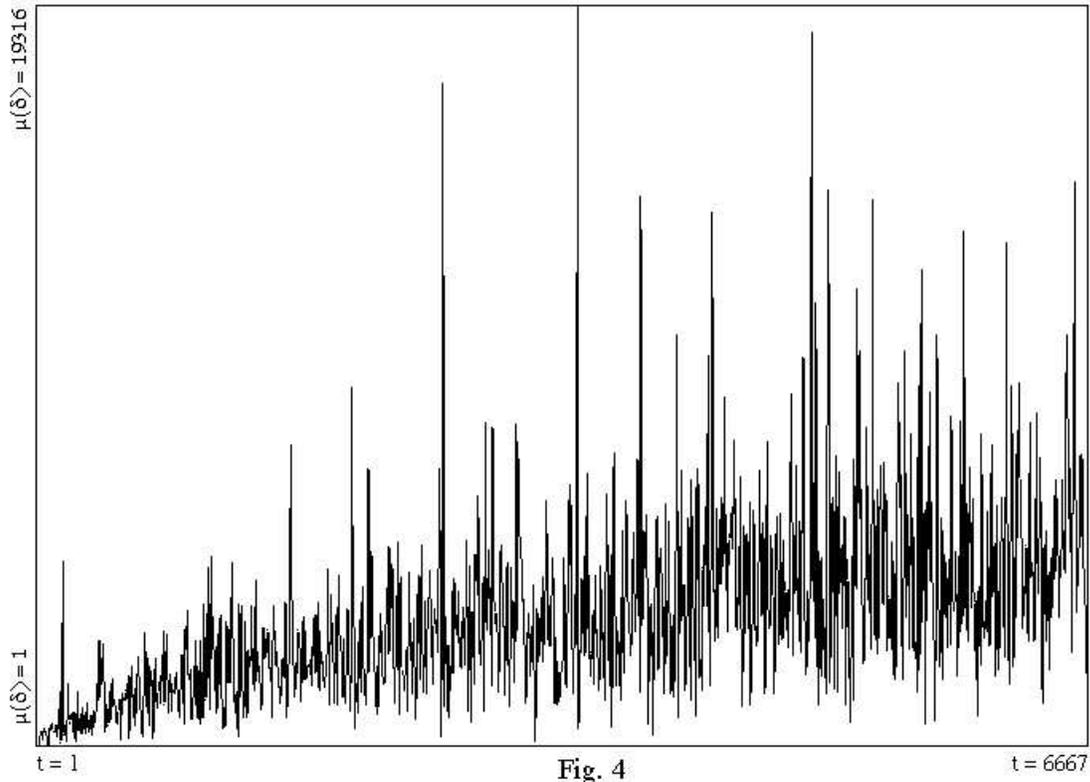


**Figure 3.** The graph of the function  $\varpi(\xi(t))$ ,  $\varpi(d)$  being the number of primitive cycles, with  $d = \xi(t)$  ranging from 1 to 19999 over  $\mathcal{D}(2:3)$  when  $1 \leq t \leq 6667$ .

We introduce here and in the next section some well-known, but still very poorly understood diophantine formulae for primitive  $T_d$ -cycles [Lagarias1990], [Belaga, Mignotte 1998], and we extend them to primitive memberships (Definition 5.4). These formulae are an indispensable tool in our analysis, §§9,10, of the above old and new conjectures, §§5, 6, as well as of the related experimental data. This analysis will result, in particular, in both the refinement of some known  $3x + d$  conjectures and the introduction of a few new ones, §§9,10.

Note that all our previous theoretical and experimental constructions, facts, claims, and observations, §§2-6, are restricted to, or derived from the study of (numerical results of) iterations of the algorithms (2:1,5,6).

The strength of our diophantine machinery is best illustrated by the fact that it permits to *effectively construct* (which means here, to produce a simple, transparent, and straightforward procedure for computation of) all primitive memberships and cycles, – and not just *effectively enumerate* primitive cycles, Note 1.1(3,4), – which means in recursion theory, to prove by “*recursive trial and error*” that there exists a recursive, or more strongly, primitively recursive one-to-one mapping from the set of natural numbers onto the set of



**Figure 4.** The graph of the function  $\mu(d)$  majorizing the function  $\varpi(d)$  (6:6).

primitive cycles.

To be sure, the specific “recursive trial and error” procedure developed in this study turns out to be very effective in localizing (as we believe, all) primitive  $T_d$ -cycles in the rather extensive range  $1 \leq d \leq 19999, d \in \mathbf{D}$ , cf. §§13,14. On the other hand, many important questions concerning primitive cycles and memberships remain out of reach even for the diophantine formalism.

It is true that it is only using this formalism that one is able not only to directly construct primitive cycles and memberships, but also to effectively evaluate the cardinalities of sets of such cycles or memberships with given lengths and/or oddlengths, §9.

Yet, no such formulae are known either for the number  $\mu(d)$  (6:6) of primitive memberships with a given shift number  $d$ , or for the numbers  $\mu_\ell^{(k)}(d)$ ,  $\mu^{(k)}(d)$ ,  $\mu_\ell(d)$  and  $\mu_\ell^{(k)}(n)$ ,  $\mu_\ell(n)$  of such memberships with either second, or first component fixed and of the given length and/or oddlength, cf. §9 (according to Conjecture 22 (6:7), the cardinalities  $\mu(n)$  and  $\mu^{(k)}(n)$  are presumed to be infinite). Moreover, one cannot as yet discard the option

[Belaga 1998] that the very existence of such a formula is an *unsolvable problem*, - in the spirit of the negative solution [Jones, Matijasevic 1991] of Hilbert's Tenth Problem.

The actuality of the unsolvability option is best illustrated by the fact that many  $(3x + d)$ -related question actually depend on intricate properties of exponential diophantine polynomials involving the numbers 2 and 3. Take, for example, the erratic behaviour, depending on the obscure rate of diophantine approximations of the logarithm  $\log_2 3$ , of *marginal Collatz numbers*, Definition 7.1(5). As our calculations show (Figure 6, §10), such numbers are responsible for a disproportionately big chunk of primitive cycles with relatively small shift numbers  $d$ . This is why their unpredictable diophantine oscillations (Lemma 7.2(4), (7:13)) represent one of the most explicitly mysterious facets (and, probably, the most formidable single technical challenge in our search for a solution) of the  $3x + 1$  and  $3x + d$  problems.

## 7. Collatz Configurations, Numbers, and Coronas.

We describe in this and the next sections a simple diophantine procedure which, from a given  $k$ -tuple  $\mathbf{P}$  of positive integers (satisfying some elementary supplementary conditions and called here *Collatz configuration*), constructs a primitive membership of the oddlength  $k$  and length  $\ell = |\mathbf{P}|$ , with the norm  $|\dots|$  (or length) of a  $k$ -tuple defined as the sum of its components. And we show that this construction yields a natural one-to-one correspondence between the sets of primitive  $(k, \ell)$ -memberships (Definition 6.1, above) and Collatz  $(k, \ell)$ -configurations (Definition 7.1, below).

Most of the formulae of these two sections are well known (cf., e. g., Corollary 8.5), but their extensions to primitive memberships (Definition 7.1, Lemma 7.2, Theorems 8.1,3, Lemmas 8.4, 8.9) are new and carry some important theoretical and methodological implications.

In this section, we give precise definitions and detailed treatment of Collatz numbers and coronas, mentioned already in §1. Our notations follow those of [Belaga, Mignotte 1998], with some modifications and extensions.

**Definition 7.1.** (1) Let  $\Lambda \subset \mathbf{N}^2$  be the set of all pairs of positive integers  $(k, \ell)$ , satisfying the following equivalent inequalities :

$$\lambda_{k,\ell} = \ell - \lceil k \cdot \log_2 3 \rceil \geq 0 \iff 2^\ell - 3^k > 0 , \quad (7 : 1)$$

with the number  $\lambda_{k,\ell}$  called the  $(k, \ell)$ -margin.

(2) Let  $\sigma_k$  be the circular (counterclockwise) permutation on  $k$ -tuples of objects from a given domain. In this paper, we deal only with  $k$ -tuples of positive integers omit the subscript  $k$  when its value is implied by the context) :

$$\forall \mathbf{P} = \langle p_1, \dots, p_k \rangle \in \mathbf{N}^k , \quad \sigma(\mathbf{P}) = \sigma_k(\mathbf{P}) = \langle p_2, \dots, p_k, p_1 \rangle . \quad (7 : 2)$$

For any  $k \geq 1$ , a  $k$ -tuple  $\mathbf{P}_k$  of positive integers is called *aperiodic*, if all  $k$  outcomes of its circular permutations are different :

$$\sigma^j(\mathbf{P}) \neq \mathbf{P} = \sigma^k(\mathbf{P}) , \quad 1 \leq j < k . \quad (7 : 3)$$

(3) An aperiodic  $k$ -tuple  $\mathbf{P}$ , satisfying the condition (cf. (7:1))

$$\ell = |\mathbf{P}| = p_1 + \dots + p_k \geq \lceil k \cdot \log_2 3 \rceil , \quad (7 : 4)$$

is called a *Collatz*  $(k, \ell)$ -*configuration*, or *(Collatz)  $k$ -configuration of the length  $\ell$* . We denote by  $\Pi_\ell^{(k)}$  the set of all Collatz  $(k, \ell)$ -configurations, and by  $\Pi$  their (disjoint) union :

$$\forall (k, \ell) \in \Lambda , \quad \Pi_\ell^{(k)} \subset \mathbf{N}^k , \quad \pi_\ell^{(k)} = \#\Pi_\ell^{(k)} ; \quad \Pi = \bigsqcup_{(k, \ell) \in \Lambda} \Pi_\ell^{(k)} . \quad (7 : 5)$$

(4) For any Collatz configuration  $\mathbf{P}$  (7:1-5), define the integers  $A, B, F, G, H \in \mathbf{D}$ , as follows :

$$a, b, f, g, h : \quad \Pi \longrightarrow \mathbf{D} ;$$

$$\forall (k, \ell) \in \Lambda \quad \forall \mathbf{P} = \langle p_1, \dots, p_k \rangle \in \Pi_\ell^{(k)} ,$$

$$\left\{ \begin{array}{l} A = a(\mathbf{P}) , \quad B = B_{k, \ell} = b(\mathbf{P}) , \quad F = f(\mathbf{P}) , \quad G = g(\mathbf{P}) , \quad H = h(\mathbf{P}) ; \\ A = \begin{cases} 1, & \text{if } k = 1 ; \\ 3^{k-1} + 2^{p_1} \cdot 3^{k-2} + \dots + 2^{p_1 + \dots + p_{k-2}} \cdot 3 + 2^{p_1 + \dots + p_{k-1}}, & \text{otherwise ;} \end{cases} \\ B = 2^\ell - 3^k ; \quad H = \gcd(A, B) ; \quad F = \frac{A}{H} ; \quad G = \frac{B}{H} ; \quad \gcd(F, G) = 1 . \end{array} \right. \quad (7 : 6)$$

(5) According to (7:1), the integer  $B = B_{k, \ell}$  is positive; it is called here *Collatz*  $(k, \ell)$ -*number*. A Collatz number  $B_{k, \ell}$  is called *marginal*, if  $\ell = \lceil k \cdot \log_2 3 \rceil$  or, equivalently, if the  $(k, \ell)$ -margin is zero,  $\lambda_{k, \ell} = 0$  (7:1).

(6) The set of values  $A$  for all Collatz  $(k, \ell)$ -configurations,

$$\mathcal{A}_{k, \ell} = \text{Image}(a|_{\Pi_\ell^{(k)}}) = a(\Pi_\ell^{(k)}) \subset \mathbf{D} , \quad (7 : 7)$$

is called *Collatz*  $(k, \ell)$ -*corona*, or *Collatz*  $B_{k, \ell}$ -*corona*, or simply *corona*.

**Lemma 7.2.** (1) The total number  $\chi_{k, \ell}$  of Collatz  $(k, \ell)$ -configurations can be calculated according to the following formula :

$$\forall (k, \ell) \in \Lambda \quad \left\{ \begin{array}{l} \chi_{k, \ell} = \#\Pi_\ell^{(k)} = \begin{cases} \binom{\ell-1}{k-1} , & \text{if } \gcd(k, \ell) = 1 , \\ \sum_{r|\gcd(k, \ell)} \mu(r) \cdot \binom{\frac{\ell}{r}-1}{\frac{k}{r}-1} , & \text{otherwise ,} \end{cases} \\ \chi_{k, \ell} \equiv 0 \pmod{k} , \end{array} \right. \quad (7 : 8)$$

where  $\mu$  is the Möbius function [Nathanson 2000] :

$$\mu(m) = \begin{cases} 1 , & \text{if } m = 1 , \\ (-1)^q , & \text{if } m \text{ is the product of } q \text{ distinct primes ,} \\ 0 , & \text{if } m \text{ is divisible by a square of a prime .} \end{cases}$$

The low part of the formula (7:8), defined for pairs  $(k, \ell)$  with the property  $\gcd(k, \ell) > 1$ , is universal and covers – but also obscures – the special upper case  $\gcd(k, \ell) = 1$ . (See [Lagarias 1990] for an analogue of the formula (7:8).)

(2) The below lower and upper bounds to members  $A$  of Collatz corona are sharp :

$$\forall(k, \ell) \in \Lambda \left\{ \begin{array}{l} \underline{A}_{k, \ell} = 3^k - 2^k, \quad \overline{A}_{k, \ell} = 2^{\ell-k+1} \cdot (3^{k-1} - 2^{k-1}) + 3^{k-1}; \\ \forall A \in \mathcal{A}_{k, \ell} \left\{ \begin{array}{l} 3^{k-1} < \underline{A}_{k, \ell} \leq A \leq \overline{A}_{k, \ell} < 2^{\ell-k+1} \cdot 3^{k-1}. \end{array} \right. \end{array} \right. \quad (7:9)$$

(3) The function  $a$  does not actually depend on the  $k$ -th component  $p_k$  of a Collatz configuration (7:2-4), and its values for two  $k$ -configurations of different lengths can be equal. More precisely,

$$\forall(k, \ell) \in \Lambda \quad \forall j \geq 1, \quad (k, \ell + j) \in \Lambda \quad \& \quad \mathcal{A}_{k, \ell} \subset \mathcal{A}_{k, \ell + j}. \quad (7:10)$$

(4) With the exception of the trivial case (corresponding to the primitive  $T_1$ -cycle (1:3) : cf. Corollary 8.5)

$$k = 1, \quad \ell = 2, \quad \mathcal{A}_{1,2} = \{1\}, \quad B = 1,$$

a Collatz number never belongs to the respective Collatz corona and, if  $k \geq 4$ , is located below the upper bound (7:9) of the corona,

$$\forall(k, \ell) \in \Lambda, \quad B = B_{k, \ell} < 2^\ell \quad (< \overline{A}_{k, \ell}, \text{ if } k \geq 4). \quad (7:11)$$

Non-marginal Collatz numbers (Definition 7.1(5)) are located above the lower bound (7:9) to Collatz corona (cf. (7:1)) :

$$\forall(k, \ell) \in \Lambda, \quad \lambda_{k, \ell} = \ell - \lceil k \cdot \log_2 3 \rceil \geq 1 \implies B_{k, \ell} > 3^k \cdot 2^{\lambda_{k, \ell} - 1} > \underline{A}_{k, \ell}. \quad (7:12)$$

Some marginal Collatz numbers “descend” far below this limit, with the following (apparently, the best known and absolutely) nontrivial lower bound (10:25) [Rhin 1987] :

$$\forall(k, \ell) \in \Lambda, \quad \ell = \lceil k \cdot \log_2 3 \rceil \implies \left\{ \begin{array}{l} B_{k, \ell} = 3^k \cdot (2^\ell \cdot 3^{-k} - 1) = 3^k \cdot (2^{\epsilon_k} - 1), \\ 0 < \epsilon_k = \lceil k \cdot \log_2 3 \rceil - k \cdot \log_2 3 < 1, \\ 3^{k-1} \cdot k^{-13.3} < B_{k, \ell} < 20 \cdot \epsilon_k^{-1} \cdot \underline{A}_{k, \ell} \end{array} \right. \quad (7:13)$$

(5) The mappings

$$\left\{ \begin{array}{l} \forall(k, \ell) \in \Lambda, \quad a_\ell^{(k)} : \Pi_\ell^{(k)} \rightarrow \mathbf{D}; \\ b : \Lambda \rightarrow \mathbf{D}; \\ \beta = (a, b) : \Pi \rightarrow \mathbf{D}^2; \quad \gamma = (f, g) : \Pi \rightarrow \mathbf{D}^2, \end{array} \right. \quad (7:14)$$

are injective. The injectivity of  $a_\ell^{(k)}$  implies, according to (7:8),

$$\forall (k, \ell) \in \Lambda, \# \mathcal{A}_{k, \ell} = \# a(\Pi_\ell^{(k)}) = \# \Pi_\ell^{(k)} = \chi_{k, \ell}. \quad (7:15)$$

**(6)** The Collatz number  $B = B_{k, \ell}$  (7:6) depends only on the dimension  $k$  and length  $\ell$  of its Collatz configuration. In particular, the mapping  $b$  is invariant under the action of the circular permutation  $\sigma$  (7:2),

$$\forall (k, \ell) \in \Lambda \quad \forall \mathbf{P} \in \Pi_\ell^{(k)}, b(\mathbf{P}) = b(\sigma(\mathbf{P})). \quad (7:16)$$

**(7)** As to the mapping  $a$ , its injectivity implies that its actions on all  $k$  iterations of the permutation  $\sigma$  of any Collatz  $k$ -configuration are different :

$$\forall (k, \ell) \in \Lambda \quad \forall \mathbf{P} \in \Pi_\ell^{(k)}, a(\sigma^i(\mathbf{P})) \neq a(\sigma^j(\mathbf{P})), \quad 0 \leq i < j \leq k-1. \quad (7:17)$$

This property justifies the following shorthand notation (cf. Definition 7.1(2)) :

$$\left\{ \begin{array}{l} \forall (k, \ell) \in \Lambda \quad \forall A \in \mathcal{A}_{k, \ell} \quad \exists! \mathbf{P} = a^{-1}(A) \in \Pi_\ell^{(k)}, \quad A = a(\mathbf{P}); \\ \text{define :} \quad \sigma(A) = a(\sigma(\mathbf{P})). \end{array} \right. \quad (7:18)$$

**(8)** The numbers  $A \in \mathcal{A}_{k, \ell}$ ,  $\sigma(A)$ , and  $B = B_{k, \ell}$  are related by the formulae :

$$\left. \begin{array}{l} \forall (k, \ell) \in \Lambda \\ \forall \mathbf{P} = \langle p_1, \dots, p_k \rangle \in \Pi_\ell^{(k)} \end{array} \right\} \left\{ \begin{array}{l} A = a(\mathbf{P}) \\ B = b(\mathbf{P}) \end{array} \right\} \implies \left\{ \begin{array}{l} 3A + B = 2^{p_1} \cdot \sigma(A); \\ \sigma(A) = \text{odd}(3A + B). \end{array} \right. \quad (7:19)$$

**(9)** The last identity implies that, according to (7:6),

$$\forall (k, \ell) \in \Lambda \quad \forall A \in \mathcal{A}_{k, \ell}, H = \gcd(A, B) = \gcd(A, \sigma(A), \sigma^2(A), \dots, \sigma^{k-1}(A), B). \quad (7:20)$$

**(10)** In particular, according to (4:1), (7:16-18,19), the mapping  $h$  is invariant under the action of  $\sigma$ , albeit it substantially depends on  $\mathbf{P} \in \Pi_{k, \ell}$ , and not just, as  $b$ , only on its dimension parameters  $k, \ell$ . And so is the mapping  $g$  :

$$\forall \mathbf{P} \in \Pi \quad \left\{ \begin{array}{l} \sigma(H) = h(\sigma(\mathbf{P})) = \gcd(a(\sigma(\mathbf{P})), b(\sigma(\mathbf{P}))) = \gcd(\sigma(A), B) = H; \\ \sigma(G) = g(\sigma(\mathbf{P})) = \frac{\sigma(B)}{\sigma(H)} = \frac{B}{H} = G. \end{array} \right. \quad (7:21)$$

**(10)** Respectively, the action of  $\sigma$  on  $f$  is described by the formula similar to that for  $a$  (7:19) :

$$\forall \mathbf{P} \in \Pi, \sigma(F) = f(\sigma(\mathbf{P})) = \frac{\sigma(A)}{\sigma(H)} = \text{odd}\left(\frac{3A}{H} + \frac{B}{H}\right) = \text{odd}(3F + G), \quad (7:22)$$

with all  $k$  numbers  $\sigma^j(F) = f(\sigma^j(\mathbf{P}))$  being different :

$$\forall(k, \ell) \in \Lambda \quad \forall \mathbf{P} \in \Pi_\ell^{(k)}, \quad \sigma^i(F) \neq \sigma^j(F), \quad 0 \leq i < j \leq k-1. \quad (7:23)$$

*Proof* : **(1)** (A binary version of the formula (7:8) can be found in [Lagarias 1990]) We omit first the aperiodicity condition and prove the upper part of the formula (7:8) : the number of different representations of a positive integer  $\ell$  as a sum of  $k < \ell$  positive integers,  $p_1 + \dots + p_k$ , with the fixed order of summands  $p_j$ , is equal to  $\binom{\ell-1}{k-1}$ . A pure combinatorial proof is to decompose a  $\ell$ -tuple of ones in  $k$  consecutive (say, from left to right) non-empty segments of ones, by choosing the ends of these segments, with the end of the  $k$ -th (rightmost) segment being the rightmost one of the  $\ell$ -tuple. This leaves us with the choice of  $k-1$  remaining ends out of  $\ell-1$  free ones.

Now, if  $\gcd(k, \ell) = 1$ , all such representations are aperiodic. Otherwise,  $\gcd(k, \ell) = q > 1$ , with a periodic representation appearing, for any divisor  $r \mid q$ , as the concatenation of  $r$  identical aperiodic representations of  $\frac{\ell}{r}$  by  $\frac{k}{r}$  summands. The rest is the usual techniques of *Möbius inclusion-exclusion formula*.

**(2-10)** Straightforward verification.

*End of the Proof.*

## 8. Exponential Diophantine Criteria for Memberships and Cycles.

The recursive formulae (7:21,22) for the action of  $\sigma$  on the pair of mappings  $f, g$  mirrors the iterative formulae (4:4(2,3)) for calculation of an odd member of a primitive cycle from the previous one. This mirroring is at the heart of the following pair of dual constructions.

The first construction builds, from any Collatz configuration, a primitive membership, Theorem 8.1. The second one, inversely, reconstructs from any primitive membership its unique Collatz configuration prototype, Theorem 8.3.

**Theorem 8.1.** **(1)** For any Collatz configuration  $\mathbf{P}$ , the pair  $\gamma(\mathbf{P})$  of positive integers (7:14) is a primitive membership :

$$\gamma : \Pi \longrightarrow \mathcal{M}; \quad \forall(k, \ell) \in \Lambda \quad \left\{ \begin{array}{l} \forall \mathbf{P} \in \Pi_\ell^{(k)}, \quad \gamma_{k,\ell} : \Pi_\ell^{(k)} \longrightarrow \mathcal{M}_\ell^{(k)}; \\ \gamma_{k,\ell}(\mathbf{P}) = \langle f(\mathbf{P}), g(\mathbf{P}) \rangle = \langle \overline{F}, \overline{G} \rangle \in \mathcal{M}_\ell^{(k)}. \end{array} \right. \quad (8:1)$$

**(2)** The corresponding primitive  $T_G$ -cycle with the odd member number  $F, \mathbf{C} = \zeta \langle \overline{F}, \overline{G} \rangle$  (6:1) is constructed, as follows. According to (7:23), for any  $k$ -configuration  $\mathbf{P}$ , all components of the  $k$ -tuple

$$\Phi(\mathbf{P}) = \left\{ F = f(\mathbf{P}), \sigma(F) = f(\sigma(\mathbf{P})), \dots, \sigma^{k-1}(F) = f(\sigma^{k-1}(\mathbf{P})) \right\} \in \mathbf{D}^k \quad (8:2)$$

are different. Suppose  $n_0 = \sigma^j(F)$ , for some  $j, 0 \leq j \leq k-1$ , is the minimal member of the  $k$ -tuple  $\Phi(\mathbf{P})$  (8:2). Then the  $k$ -tuple  $\sigma^j(\Phi(\mathbf{P}))$  is the odd-part  $Odd(\mathbf{C})$  (4:4(2)) of a primitive  $T_G$ -cycle  $\mathbf{C}$ .

According to Lemma 7.2(5), the mapping  $\gamma$  is injective : to different Collatz configurations correspond different primitive memberships. It is also surjective, and thus, represents a natural one-to-one correspondence between Collatz configurations and memberships. We shall prove it by displaying a procedure  $\theta$ , inverse to  $\gamma$ , which assigns to a membership its unique Collatz configuration counterpart.

The procedure  $\theta$  will be initially defined on the set  $\tilde{\mathcal{M}}$  of lowermost primitive memberships (Definition 8.2, Theorem 8.3(1-4)), and then extended to all memberships (Theorem 8.3(5)) :

**Definition 8.2.** (1) A primitive membership  $\langle \overline{n, d} \rangle$  is called *lowermost* if its left component  $n$  is the minimal (odd) member of the corresponding primitive cycle. Thus, in accordance with the notations (4:4(2)),

$$\langle \overline{n, d} \rangle \text{ is lowermost} \iff \mathbf{C} = \zeta \langle \overline{n, d} \rangle = \left\{ n = m_0, m_1, \dots, m_{\ell-1} \right\}. \quad (8:3)$$

(2) Respectively, Collatz  $(k, \ell)$ -configuration  $\mathbf{P}$  is called *lowermost* if the primitive membership  $\gamma(\mathbf{P})$  (8:1) is lowermost. We use tilde sign for notations for sets of lowermost primitive memberships and Collatz configurations :  $\tilde{\mathcal{M}}, \tilde{\Pi}^{(k)}$ , etc.

(Some basic properties of lowermost memberships and configuration will be the subject of the below Lemma 8.7, and their applications to Conjecture 20, §5, can be found in §9.)

**Theorem 8.3.** (1) Let  $\mathbf{C}$  be a primitive  $T_d$ -cycle (4:4) of the length  $\ell$  and oddlength  $k$ , and let  $\mathcal{P}_\ell^k$  be the set of all such cycles. Then  $(k, \ell) \in \Lambda$ .

(2) Let  $\mathcal{P}_\ell^k$  be the set of all primitive cycles of the length  $\ell$  and oddlength  $k$ .

Let  $\mathbf{C}$  be a primitive  $T_d$ -cycle, and let  $n_0$  be its minimal member. Then, tautologically, the pair  $(n_0, d) = \omega(\mathbf{C})$  is the lowermost primitive membership uniquely characterising  $\mathbf{C}$ , and this one-to-one correspondence is inverse to  $\zeta$ , Definition 6.12 (6:1) :

$$\mathcal{P} \begin{array}{c} \xrightarrow{\omega} \\ \xleftarrow{\zeta} \end{array} \tilde{\mathcal{M}} : \quad \forall (k, \ell) \in \Lambda, \quad \mathcal{P}_\ell^k \begin{array}{c} \xrightarrow{\omega} \\ \xleftarrow{\zeta} \end{array} \tilde{\mathcal{M}}_\ell^{(k)}. \quad (8:4)$$

(3) For any  $\mathbf{C} \in \mathcal{P}_\ell^k$ , let the  $k$ -tuple  $Even(\mathbf{C})$  be defined as above, (4:4(3)). Then,  $Even(\mathbf{C})$  is Collatz  $(k, \ell)$ -configuration.

(4) Moreover,  $Even(\mathbf{C})$  is a lowermost configuration, such that  $\gamma(Even(\mathbf{C})) = \omega(\mathbf{C})$ . Accordingly, the construction  $\theta$  can be defined for the subsets  $\tilde{\mathcal{M}}, \tilde{\Pi}$  of lowermost memberships and configurations, as follows :

$$\begin{array}{ccc} \mathcal{P} & \begin{array}{c} \xrightarrow{\omega} \\ \xleftarrow{\zeta} \end{array} & \tilde{\mathcal{M}} \\ & & \\ Even & \searrow \nearrow \gamma & \\ & \tilde{\Pi} & \end{array} \quad \theta|_{\tilde{\mathcal{M}}} = Even \circ \zeta : \quad \tilde{\mathcal{M}} \xrightarrow{\omega} \mathcal{P} \xrightarrow{Even} \tilde{\Pi}. \quad (8:5)$$

(5) Let now, for some  $(k, \ell) \in \Lambda$ ,  $\langle \overline{n, d} \rangle \in \mathcal{M}_\ell^{(k)}$  be a primitive (not necessary, low-ermost) membership, and let  $\mathbf{C} = \zeta \langle \overline{n, d} \rangle$  be the corresponding primitive  $T_d$ -cycle (6:1) meeting  $n$ , of the length  $\ell$  and oddlength  $k$ . Being an odd member of  $\mathbf{C}$ , the integer  $n$  can be identified with some  $n_r \in \text{Odd}(\mathbf{C}), 0 \leq r \leq k-1$  (4:4(2)). The  $k$ -tuple  $\text{Even}(\mathbf{C})$  being Collatz configuration, so is its  $\sigma^r$ -permutation. Define  $\theta \langle \overline{n, d} \rangle = \sigma^r(\text{Even}(\mathbf{C}))$  and verify that  $\theta$  is inverse to  $\gamma$  :

$$\Pi \xrightleftharpoons[\theta]{\gamma} \mathcal{M} : \left\{ \begin{array}{l} \forall \langle \overline{n, d} \rangle \in \tilde{\mathcal{M}} \subset \mathcal{M}, \theta \langle \overline{n, d} \rangle = \text{Even}(\zeta \langle \overline{n, d} \rangle); \quad \text{cf. (8:5)} \\ \forall \langle \overline{n, d} \rangle \in \mathcal{M}, \left\{ \begin{array}{l} \theta \langle \overline{n, d} \rangle = \sigma^r(\text{Even}(\zeta \langle \overline{n, d} \rangle)), \\ n = n_r \in \text{Odd}(\zeta \langle \overline{n, d} \rangle); \\ \gamma(\theta \langle \overline{n, d} \rangle) = \langle \overline{n, d} \rangle; \end{array} \right. \\ \forall \mathbf{P} \in \Pi, \theta(\gamma(\mathbf{P})) = \mathbf{P}; \\ \forall (k, \ell) \in \Lambda, \Pi_\ell^{(k)} \cong \mathcal{M}_\ell^{(k)} : \Pi_\ell^{(k)} \xrightleftharpoons[\theta_\ell^{(k)}]{\gamma_\ell^{(k)}} \mathcal{M}_\ell^{(k)}; \\ \mu_\ell^{(k)} = \#\mathcal{M}_\ell^{(k)} = \pi_\ell^{(k)} = \#\Pi_\ell^{(k)}. \end{array} \right. \quad (8:6)$$

Or at greater length,

$$\left. \begin{array}{l} \forall (k, \ell) \in \Lambda \\ \forall \langle \overline{n, d} \rangle \in \mathcal{M}_\ell^{(k)} \end{array} \right\} \left\{ \begin{array}{l} \mathbf{C} = \zeta \langle \overline{n, d} \rangle, k = \text{oddlength}(\mathbf{C}), \ell = \text{length}(\mathbf{C}), \\ n = n_r \in \text{Odd}(\mathbf{C}), 0 \leq r \leq k-1; \\ \theta \langle \overline{n, d} \rangle = \mathbf{P} = \langle p_1, \dots, p_k \rangle = \sigma^r(\text{Even}(\mathbf{C})) = \\ \sigma^r \langle e_1, \dots, e_k \rangle \in \mathbf{N}^k; \\ \forall j (1 \leq j \leq k), p_j = \begin{cases} e_{j+r}, & \text{if } 1 \leq j \leq k-r \\ e_{j+r-k}, & \text{otherwise} \end{cases}; \\ |\mathbf{P}| = p_1 + \dots + p_k = e_1 + \dots + e_k = \ell. \end{array} \right. \quad (8:7)$$

*Proof* : Straightforward verification based on the assertions of Lemma 7.2. *End of the Proof.*

The constructions of Theorems 8.1, 8.3 imply the following well-known diophantine criteria [Lagarias1990], [Belaga, Mignotte 1998] for the existence of primitive cycles with given parameters  $n$  and/or  $d$  :

**Corollary 8.4.** (1) Given a shift number  $d$  and an integer  $n$  ( $d, n \in \mathbf{D}$ ), there exists a (not necessary primitive) cycle  $T_d$ -cycle of the length  $\ell$  and oddlength  $k$  meeting  $n$  iff there exists a member  $A$  of Collatz  $B_{k, \ell}$ -corona,  $A \in \mathcal{A}_{k, \ell}$ , such that

$$d \cdot A = n \cdot B_{k, \ell}. \quad (8:8)$$

(2) Given a shift number  $d \in \mathbf{D}$ , there exists a primitive  $T_d$ -cycle of the length  $\ell$  and oddlength  $k$ , *iff* there exists a member  $A$  of Collatz  $B_{k,\ell}$ -corona, such that

$$d = \frac{B_{k,\ell}}{\gcd(A, B_{k,\ell})} . \quad (8:9)$$

Using (8:9), one can “translate” Conjectures 17, 18 into their diophantine equivalent, Conjecture 23 below.

(3) The formula (8:9) implies the following congruences :

$$d = \frac{B_{k,\ell}}{\gcd(A, B_{k,\ell})} \implies \begin{cases} (1) & B_{k,\ell} \equiv 0 \pmod{d} ; \\ (2) & d \cdot A \equiv 0 \pmod{B_{k,\ell}} . \end{cases} \quad (8:10)$$

The first congruence is a necessary condition for a  $T_d$ -cycle to be of the length  $\ell$  and oddlength  $k$ . The necessary condition provided by the second congruence is also sufficient, if  $d = 1$  or if  $d$  is prime. In particular, if  $d = 1$ , then  $B_{k,\ell} \mid A$ . Which implies the equivalence between the cyclic part of the  $3x + 1$  conjecture, Conjecture 4, §1, and its diophantine version, Conjecture 5, §1.

(4) According to (8:10), below Conjecture 24 is formally stronger than to the finite-number-of-primitive-cycles conjecture (Conjectures 18, 22(2) (6:7(2 right))), but, in fact, is equivalent to it.

(5) Similarly, given an integer  $n \in \mathbf{D}$ , there exists a primitive cycle of the length  $\ell$  and oddlength  $k$  meeting  $n$  *iff* there exists a member  $A$  of Collatz  $(k, \ell)$ -corona, such that

$$n = \frac{A}{\gcd(A, B_{k,\ell})} . \quad (8:11)$$

Thus, Conjectures 19, 20 are equivalent to the below diophantine Conjecture 25.

(6) The formula (8:9) implies the following congruence :

$$n = \frac{A}{\gcd(A, B_{k,\ell})} \implies \begin{cases} (1) & A \equiv 0 \pmod{n} ; \\ (2) & n \cdot B_{k,\ell} \equiv 0 \pmod{A} . \end{cases} \quad (8:12)$$

In particular, if  $n = 1$  or  $n \geq 5$  is prime and  $(k, \ell) \in \Lambda$ , there exists a primitive  $(k, \ell)$ -membership with the member number  $n$ , *iff* the congruence (8:12(2)) holds for some  $A \in \mathcal{A}_{k,\ell}$ .

(7) For composite  $n \in \mathbf{D}$ , the member-number analogue, based on the condition (8:12), of below Conjecture 24 does not imply the infinite-number-of-member-numbers Conjecture 20. In fact, an assumption that the congruence in (8:12) has an infinite number of solutions formally implies only that (i)  $n$  is a member of an infinite number of cycles, not necessary primitive, and (ii) that some divisor  $q$  of  $n$ ,  $1 < q < n$ , belongs to an infinite number of primitive cycles.

(8) Finally, the combination of the above claims (8:9,11) yields the following criterion. A pair of integers  $(n, d) \in \mathbf{D}^2$  is a primitive membership of the length  $\ell$  and oddlength  $k$  *iff*  $\gcd(n, d) = 1$  and there exists a member  $A$  of Collatz  $(k, \ell)$ -corona, such that

$$n = \frac{A}{\gcd(A, B_{k,\ell})} , \quad d = \frac{B_{k,\ell}}{\gcd(A, B_{k,\ell})} . \quad (8 : 13)$$

**Conjecture 23.** For any  $d \in \mathbf{D}$ , there exists (1) at least one and (2) at most finite number of pairs  $(k, \ell) \in \Lambda$ , such that the equality (8:9) holds for at least one  $A \in \mathcal{A}_{k,\ell}$ . Moreover (cf. the notations (6:6)),

$$\forall d \in \mathbf{D} \quad \forall (k, \ell) \in \Lambda , \quad \mu_\ell^{(k)}(d) = \#\{\mathbf{P} \in \Pi_\ell^{(k)} \mid B_{k,\ell} = d \cdot \gcd(a(\mathbf{P}), B_{k,\ell})\} . \quad (8 : 14)$$

**Conjecture 24.** For any  $d \in \mathbf{D}$ , there exists only finite number of pairs  $(k, \ell) \in \Lambda$ , such that the congruence (8:10) holds for some  $A \in \mathcal{A}_{k,\ell}$ .

**Conjecture 25.** For any  $n \in \mathbf{D}$ , there exists an infinite number of pairs  $(k, \ell) \in \Lambda$ , such that the congruence (8:11) holds for some  $A \in \mathcal{A}_{k,\ell}$ . Moreover (6:6),

$$\forall n \in \mathbf{D} \quad \forall (k, \ell) \in \Lambda , \quad \mu_\ell^{(k)}(n) = \#\{\mathbf{P} \in \Pi_\ell^{(k)} \mid a(\mathbf{P}) = n \cdot \gcd(a(\mathbf{P}), B_{k,\ell})\} . \quad (8 : 15)$$

*Proof of Corollary 8.4 :* Only the claim (4) needs a hint of a proof (by induction). According to the above claim (3), Conjectures 18, 23 are equivalent in the case of a prime shift number  $d = 5, 7, 11, \dots$ . Let now  $d \in \mathbf{D}, d \geq 5$  be composite, and suppose there exists an infinite set of triples  $\{k, \ell, A_{k,\ell}\}$ , with  $(k, \ell) \in \Lambda, A_{k,\ell} \in \mathcal{A}_{k,\ell}$ , satisfying the congruence (8:10). Then, for any such triplet, the pair

$$n_{k,\ell} = \frac{A_{k,\ell}}{\gcd(A_{k,\ell}, B_{k,\ell})} , \quad d_{k,\ell} = \frac{B_{k,\ell}}{\gcd(A_{k,\ell}, B_{k,\ell})} ,$$

defined according to (8:13), is a primitive membership, with  $d_{k,\ell} < d$  being a divisor of  $d$ . The number of divisors of  $d$  being finite, at least one of those divisors  $< d$  should be the shift number for an infinite number of primitive memberships (cf. also Corollary 4.7(10), (4:10-12)). Contradicts the inductive assumption clause. *End of the Proof.*

**Example 8.5.** (1) The shift number  $d = 6487 = 13 \cdot 499$  has the representation  $6487 = 2^{16} - 3^{10}$ . Thus, any combination of parameters  $p_1, \dots, p_{10}$ , such that

$$p_1 + \dots + p_{10} = 16 > 10 \cdot \log_2 3 = 15.8496 \quad , \quad \gcd(a(p_1, \dots, p_{10}), 6487) = 1 ,$$

is a solution of the equation

$$d = g(p_1, \dots, p_{10}) = \frac{B_{10,16}}{\gcd(a(p_1, \dots, p_{10}), B_{10,16})} = 6487 ,$$

and thus, corresponds to a primitive  $T_{6487}$ -cycle of the length 16 and oddlength 10. It is feasible to check even by hand that there are 456 such solutions. We have found also 78 solutions (among 534 primitive  $T_{6487}$ -cycles altogether), corresponding to the below congruences,

$$2^{32} - 3^{20} \equiv 2^{48} - 3^{30} \equiv 2^{64} - 3^{40} \equiv 2^{96} - 3^{60} \equiv 0 \pmod{6487} .$$

But “why” are eliminated an infinite number of other possible candidates for the congruence  $B_{k,\ell} \equiv 0 \pmod{d}$  (8:10(1)) ? **(2)** In the case  $d = 827$ , the smallest congruence of the type (8:10(1)) is :

$$15\ 182\ 893 = 2^{24} - 3^{13} \equiv 0 \pmod{827},$$

with experimentally found 4 solutions. Two more solutions correspond to the pair  $(\ell, k) = (48, 26)$ , with one solution corresponding to  $(72, 39)$ . But, again, “why” not others ? Say,  $2^{96} - 3^{52}$ , or maybe  $2^{82} - 3^{10}$ , with its prime factorization :

$$\begin{aligned} 2^{82} - 3^{10} &= 4\ 835\ 703\ 278\ 458\ 516\ 698\ 765\ 655 = \\ &5 \cdot 827 \cdot 3\ 557 \cdot 6\ 793 \cdot 91\ 009 \cdot 531\ 807\ 317. \end{aligned}$$

**(3)** The answer, suggested by the last example, might be that the prime factorization “signature” of numbers  $2^\ell - 3^k$  cannot be “captured” in the congruence (8:10(2)), beyond a certain finite limit, by numbers of the type  $A \in \mathcal{A}_{k,\ell}$ . In other words, relatively big prime factors are, apparently, common to all Collatz numbers above certain limits, and such factors are beyond the prime factorization “power” of members of respective Collatz coronas.

**(4)** On the other hand, as Conjectures 20, §5, and 28, 29, §11, suggest, there exist no similar obstacles in solving the congruence (8:12(2)), probably because there exist enough Collatz numbers with relatively big divisors whose prime factorizations comprise only relatively small prime factors. These insights show the apparent importance of surprising sporadic properties of numbers  $2^\ell - 3^k$  in an eventual solution of  $3x + 1$  and  $3x + d$  problems.

**Notations 8.6.** The mutually inverse isomorphisms  $\gamma, \theta$  (8:6) between the corresponding (sub)sets of Collatz configurations and primitive memberships permit us to extend to Collatz configurations the notations (6:5,6) for the sets of Collatz configurations yielding either  $n$ , or  $d$ , as, respectively, the corresponding member- or shift-number :

$$\left\{ \begin{array}{l}
(1) \quad \forall n \in \mathbf{D} , \Pi\langle n \rangle = \theta(\mathcal{M}\langle n \rangle) ; \pi\langle n \rangle = \#\Pi\langle n \rangle ; \\
(2) \quad \quad \quad \forall (k, \ell) \in \Lambda , \Pi_\ell^{(k)}\langle n \rangle = \theta_\ell^{(k)}(\mathcal{M}_\ell^{(k)}\langle n \rangle) ; \\
(3) \quad \quad \quad \pi_\ell^{(k)}\langle n \rangle = \#\Pi_\ell^{(k)}\langle n \rangle = \mu_\ell^{(k)}\langle n \rangle = \#\mathcal{M}_\ell^{(k)}\langle n \rangle ; \\
(4) \quad \quad \quad \forall \geq 1 , \Pi^{(k)}\langle n \rangle = \theta^{(k)}(\mathcal{M}^{(k)}\langle n \rangle) ; \\
(5) \quad \quad \quad \pi^{(k)}\langle n \rangle = \#\Pi^{(k)}\langle n \rangle = \mu^{(k)}\langle n \rangle = \#\mathcal{M}^{(k)}\langle n \rangle ; \\
(6) \quad \forall d \in \mathbf{D} , \Pi(d) = \theta(\mathcal{M}(d)) ; \pi(d) = \#\Pi(d) ; \\
(7) \quad \quad \quad \forall (k, \ell) \in \Lambda , \Pi_\ell^{(k)}(d) = \theta_\ell^{(k)}(\mathcal{M}_\ell^{(k)}(d)) ; \\
(8) \quad \quad \quad \pi_\ell^{(k)}(d) = \#\Pi_\ell^{(k)}(d) = \mu_\ell^{(k)}(d) = \#\mathcal{M}_\ell^{(k)}(d) . \\
(9) \quad \quad \quad \forall k \geq 1 , \Pi^{(k)}(d) = \theta^k(\mathcal{M}^{(k)}(d)) ; \\
(10) \quad \quad \quad \pi^{(k)}(d) = \#\Pi^{(k)}(d) = \mu^{(k)}(d) = \#\mathcal{M}^{(k)}(d) .
\end{array} \right. \quad (8 : 16)$$

#### Chapter IV. The Primitive Cyclic Structure of Systems $\mathcal{D}_d$ .

The diophantine formalism of Chapter III, with its fundamental isomorphism between sets of Collatz configurations and primitive memberships,  $\Pi \xrightarrow[\theta]{\gamma} \mathcal{M}$  (8:6), can be used in two ways to extract new information about properties of (sets of) primitive memberships and/or cycles.

Namely, the formalism can be applied either directly, by constructing (sets of) primitive memberships/cycles as  $\gamma$ -images of some well-defined (sets of) Collatz configurations. Or inversely, one can study properties of  $\omega$ -images of well-defined (sets of) primitive memberships/cycles.

The direct approach will be the main subject of the next two sections. The inverse procedure will be instrumental in the study of (sets of) primitive memberships/cycles with a given, first, shift number, §10, and then, member number  $n$ , §11.

#### 9. Memberships and Cycles with Fixed Shift Number and/or Member Number.

A fine example of the effectiveness of the direct approach mentioned above represent elementary solutions of the following problems (see also [Lagarias 1990]) :

**Problem 9.1.** (1) For a given pair  $(k, \ell) \in \Lambda$ , find the number of all primitive cycles of the length  $\ell$  and oddlength  $k$ .

(2) Find the number of all primitive cycles of the length  $\ell \geq 2$ .

According to the below notations (9:1), we need actually to calculate the parameters  $\varpi_\ell^{(k)}$  and  $\varpi^\ell$  :

**Notations 9.2.** Let  $Z, Z^{(k)}, Z_\ell, Z_\ell^{(k)}$  and be the generic notations for the sets of primitive memberships  $\mathcal{M}$ , Collatz configurations  $\Pi$ , or primitive cycles  $\mathcal{P}$ , respectively, of the length  $\ell$  and/or oddlength  $k$ , and let  $z, z^{(k)}, z_\ell, z_\ell^{(k)}$  and  $z(x), z^{(k)}(x), z_\ell(x), z_\ell^{(k)}(x)$  be the corresponding notations for the cardinalities of such sets,  $\mu\langle n \rangle, \mu\langle d \rangle, \pi\langle n \rangle, \pi\langle d \rangle, \varpi\langle d \rangle$  (defined in (6:6), (8:16), and (4:8), respectively). These sets and cardinalities are related by the following obvious equalities :

$$\left\{ \begin{array}{l} Z_\ell = \bigcup_{k=1}^{\lfloor \ell \cdot \log_3 2 \rfloor} Z_\ell^{(k)}, \quad Z^{(k)} = \bigcup_{\ell=\lceil k \cdot \log_2 3 \rceil}^{\infty} Z_\ell^{(k)}, \quad Z = \bigcup_{k \geq 1} Z^{(k)} = \bigcup_{\ell \geq 2} Z_\ell; \\ z_\ell = \sum_{k=1}^{\lfloor \ell \cdot \log_3 2 \rfloor} z_\ell^{(k)}, \quad z^{(k)} = \sum_{\ell=\lceil k \cdot \log_2 3 \rceil}^{\infty} z_\ell^{(k)}, \quad z = \sum_{\substack{k \geq 1 \\ \ell \geq \lceil k \cdot \log_2 3 \rceil}} z_\ell^{(k)} = \sum_{k \geq 1} z^{(k)} = \sum_{\ell \geq 2} z_\ell; \\ z_\ell(x) = \sum_{k=1}^{\lfloor \ell \cdot \log_3 2 \rfloor} z_\ell^{(k)}(x), \quad z^{(k)}(x) = \dots, \quad z(x) = \dots \end{array} \right. \quad (9:1)$$

In the above notations, the following lemma solves Problem 9.1 (a different solution can be found in [Lagarias 1990]) :

**Lemma 9.3. (1)** According to Theorems 8.1, 8.3, Lemma 7.2(1), and the formulae (8:6) the number of primitive memberships of the length  $\ell$  and oddlength  $k$  (6:1) is equal to  $\chi_{k,\ell}$  (7:8) :

$$\forall (k, \ell) \in \Lambda, \quad \mu_\ell^{(k)} = \pi_\ell^{(k)} = \chi_{k,\ell} \equiv 0 \pmod{k}. \quad (9:2)$$

Respectively, the number  $\varpi_\ell^{(k)}$  of primitive cycles of the length  $\ell$  and oddlength  $k$  is equal to :

$$\forall (k, \ell) \in \Lambda, \quad \varpi_\ell^{(k)} = \#\mathcal{P}_\ell^{(k)} = \frac{\chi_{k,\ell}}{k}. \quad (9:3)$$

**(2)** The number  $\varpi_\ell$  of primitive cycles of a fixed length  $\ell \geq 2$  is equal to (cf. the summation formulae (9:1)) :

$$\forall \ell \in \mathbf{N}, \ell \geq 2, \quad \varpi_\ell = \sum_{k=1}^{\lfloor \ell \cdot \log_3 2 \rfloor} \frac{\chi_{k,\ell}}{k}. \quad (9:4)$$

**Example 9.4 : Case  $\ell = 14, k = 3 < 8 = \lfloor \ell \cdot \log_3 2 \rfloor$ .** According to (7:8),  $\gcd(3, 14) = 1$  implies :

$$\varpi_{14}^3 = \frac{\chi_{3,14}}{3} = \frac{1}{3} \binom{2}{13} = 26.$$

We have found experimentally all these 26 primitive cycles, namely : two  $T_{1487}$ -cycles, starting at  $n = 11, 13$ , respectively, and 24  $T_{16357}$ -cycles, starting at, respectively,

$$n = 19, 23, 29, 31, 37, 47, 49, 53, 65, 79, 85, 89, 97, \\ 149, 161, 169, 185, 277, 233, 271, 289, 313, 361, 569 .$$

This example is representative of one of the series of (successful) controlling checks of the exhaustiveness of our search procedure. In fact, according to the below limits (9:5) on values of shift numbers of primitive cycles with given length and oddlength, the possible values of  $d$  for a cycle  $\mathbf{C} \in \mathcal{P}_{14}^3$  belong to the range of this study,  $d \leq B_{3,14} = 16357 < 19999$ , and a really thorough search should locate all 26 candidates.

As to the “full” cardinalities  $\mu\langle n \rangle, \mu\langle d \rangle$  and  $\varpi\langle d \rangle$  (the subjects of Conjectures 17-22, §§5, 6), or even their “oddlength” versions,  $\mu^{(k)}\langle n \rangle, \mu^{(k)}\langle d \rangle, \varpi^{(k)}\langle d \rangle$ , there do not apparently exist as elementary or, at least, as transparent formulae for their  $(k, \ell)$ – and  $\ell$ –refinements, as the above (9:2-4). Still, the below basic diophantine reductions of Conjecture 22, §6, provide us with new insights, otherwise unavailable, into the properties of the parameters in question : see Conjecture 23, §10, and Conjecture 25, §11.

The following, only partially understood phenomena, illustrate the underlying arithmetical difficulty : the absence of almost any predictable correlation between the pair of parameters  $(k, \ell)$  and the values of components of the corresponding primitive membership  $\langle \overline{n}, \overline{d} \rangle$ . Another salient feature of the emerging picture is the pervasiveness of the disparate  $(n, d)$ –duality, §5 : not only are the corresponding  $(n, d)$ –properties complementary, the levels of difficulty of their existing and/or prospective proofs are complementary, too.

**Problem 9.5** The bounds (7:9) and (7:11-13) to values of, respectively,  $A \in \mathcal{A}_{k,\ell}$  and  $B = B_{k,\ell}$  imply the following, *a priori* exponentially wide bounds to the components  $n = f(\mathbf{P}), d = g(\mathbf{P})$  (7:6) of a primitive membership of given length and oddlength :

$$\forall(k, \ell) \in \Lambda \quad \forall \mathbf{P} \in \Pi_{\ell}^{(k)}, \quad \begin{cases} 1 \leq n = f(\mathbf{P}) \leq \overline{A}_{k,\ell} < 2^{\ell-k+1} \cdot 3^{k-1} ; \\ 1 \leq d = g(\mathbf{P}) \leq B_{k,\ell} < 2^{\ell} . \end{cases} \quad (9 : 5)$$

Are these bounds sharp or, at least, “relatively” sharp?

Note that if some of those bounds are (relatively) sharp, and remain so with  $\ell$  growing, it would imply, in particular, that to relatively “small” pairs  $(k, \ell)$  correspond sometimes exponentially big values of  $n$  and/or  $d$ , whereas for some relatively “big”  $(k, \ell)$ , the values of  $n$  and/or  $d$  are (absolutely or relatively) small. See §§10-11 for the study of these and related phenomena for, respectively, shift numbers  $d$  and member numbers  $n$ .

## 10. Shift Number Conjectures and Claims.

The shift number conjectures 17, 18, §5, or equivalently, 22(2), §6, 23, §8, assert that, for any  $d \in \mathbf{D}$  the number  $\varpi\langle d \rangle$  (5:1,2) (or equivalently,  $\mu\langle d \rangle$  (6:7(2))) of primitive  $T_d$ –cycles (respectively, memberships) is a positive integer (cf. the above generic notations (9:1)) :

$$\forall d \in \mathbf{D} \left\{ \begin{array}{l} 1 \leq \varpi(d) = \left\{ \begin{array}{l} \sum_{\ell \geq 2} \sum_{1 \leq k < \ell \cdot \log_3 2} \varpi_\ell^{(k)}(d) = \sum_{\ell \geq 2} \varpi_\ell(d) \\ \sum_{k \geq 1} \sum_{\ell > k \cdot \log_2 3} \varpi_\ell^{(k)}(d) = \sum_{k \geq 1} \varpi^{(k)}(d) \end{array} \right\} < \infty ; \\ 1 \leq \mu(d) = \left\{ \begin{array}{l} \sum_{\ell \geq 2} \sum_{1 \leq k < \ell \cdot \log_3 2} \mu_\ell^{(k)}(d) = \sum_{\ell \geq 2} \mu_\ell(d) \\ \sum_{k \geq 1} \sum_{\ell > k \cdot \log_2 3} \mu_\ell^{(k)}(d) = \sum_{k \geq 1} \mu^{(k)}(d) \end{array} \right\} < \infty . \end{array} \right. \quad (10 : 1)$$

We are concerned in this section with the finite-number-of-primitive-cycles conjecture (Conjectures 18, §5, 23, §8,) or, in other words, with the finiteness of the sums in (10:1). The double summations of (10:1) suggest two similar strategies : in each case, first prove that every summand is finite (which either have been already done above, for the  $\ell$ -summands, or will be done in this sections, for their  $k$ -counterparts), and then prove that the number of non-zero summands is finite (the most difficult, and still open parts of the problem).

In the first case, the summands are  $\varpi_\ell(d)$  or  $\mu_\ell(d)$  (i. e. the numbers of  $T_d$ -cycles or memberships of the given length  $\ell$ ), and the task to prove the finiteness of summands (which are themselves sums of a finite number of finite summands) is quite easy. In the first, and longest, part of this section, we study the dependence of the numbers of primitive cycles and memberships with a given shift number on their lengths and margins (Corollary 10.1, Lemmas 10.2, 10.5, 10.7, Conjectures 26, 27).

In the second case, the summands are, by definition, sums of infinite number of finite summands  $\varpi_\ell^{(k)}(d)$  or  $\mu_\ell^{(k)}(d)$ , and their finiteness cannot be taken for granted. We conclude this section with an improved (both in substance and in methodology) version of the upper bound [Belaga, Mignotte 1998] to the numbers of primitive cycles and memberships with given shift number and of the given oddlength.

**Corollary 10.1.** According to (9:1-4),

$$\forall (k, \ell) \in \Lambda \left\{ \begin{array}{l} \mathcal{M}_\ell^{(k)} = \bigcup_{d \in \mathbf{D}} \mathcal{M}_\ell^{(k)}(d) \implies \left\{ \begin{array}{l} \mu_\ell^{(k)}(d) \leq \chi_{k, \ell} < \infty ; \\ \mu_\ell(d) \leq \sum_{k=1}^{\lfloor \ell \cdot \log_3 2 \rfloor} \chi_{k, \ell} < \infty ; \end{array} \right. \\ \mathcal{P}_\ell^k = \bigcup_{d \in \mathbf{D}} \mathcal{P}_\ell^k(d) \implies \left\{ \begin{array}{l} \varpi_\ell^{(k)}(d) \leq \frac{\chi_{k, \ell}}{k} < \infty ; \\ \varpi_\ell(d) \leq \sum_{k=1}^{\lfloor \ell \cdot \log_3 2 \rfloor} \frac{\chi_{k, \ell}}{k} < \infty . \end{array} \right. \end{array} \right. \quad (10 : 2)$$

In other words, the number of primitive memberships (cycles) with a given shift number and of a given length is finite.

This fact implies (Lemma 10.2 below) that the following conjecture is equivalent to the finite-number-of-primitive- $T_d$ -cycles conjecture (Conjectures 18, 22(2) (6:7(2 right))) :

**Conjecture 26.** For any  $d \in \mathbf{D}$ , there exists an upper bound  $L(d) \geq 2$  to the lengths of primitive  $T_d$ -memberships and cycles :

$$\forall d \in \mathbf{D} \quad \exists L = L(d) \geq 2 \quad \forall \ell > L, \quad \mu_\ell(d) = \varpi_\ell(d) = 0. \quad (10 : 3)$$

**Lemma 10.2.** Conjecture 26 is equivalent to Conjectures 18, 22(2) (6:7(2 right)), 24.

*Proof :* (1) If  $\varpi(d) < \mu(d) < \infty$  (Conjectures 18, 22(2) (6:7(2 right))), then only finite number of summands in the following sums can be positive, – which is equivalent to (10:3) :

$$\varpi(d) = \sum_{\ell \geq 2} \varpi_\ell(d) < \mu(d) = \sum_{\ell \geq 2} \mu_\ell(d) < \infty. \quad (10 : 4)$$

(2) According to Corollary 10.1, all summands in two sums of (10:4) are finite. If, starting with some  $\ell = L(d)+1$  (Conjecture 26), they are equal to zero, then the parameters  $\varpi(d), \mu(d)$  are finite. *End of the Proof.*

**Problem 10.3.** (1) How small can be the minimal length of a primitive  $T_d$ -membership or cycle?

(2) How big can be its maximal length ?

**Definition 10.4.** (1) Assuming Conjectures 17 and 18, §5, i. e. (cf. (6:5,6(2))),

$$1 \leq \varpi(d) < \mu(d) < \infty ,$$

define, for any  $d \in \mathbf{D}$ , its *upper* and *lower length limits*,  $\bar{\ell}(d)$  and  $\underline{\ell}(d)$ , as follows :

$$\forall d \in \mathbf{D} , \quad \bar{\ell}(d) = \max\{\ell \geq 2 \mid \mu_\ell(d) \neq 0\} , \quad \underline{\ell}(d) = \min\{\ell \geq 2 \mid \mu_\ell(d) \neq 0\} . \quad (10 : 5)$$

(2) Wild fluctuations of the functions  $\bar{\ell}(d)$  and  $\underline{\ell}(d)$  (Figure 5) are cancelled, and their occasional, but steady grows is preserved, by the following *upper* and *lower length limit threshold functions* :

$$\forall d \in \mathbf{D} , \quad \begin{cases} \bar{\mathcal{L}}(d) = \max\{\bar{\ell}(d') \mid d' \in \mathbf{I}_{1,d}\} ; \\ \underline{\mathcal{L}}(d) = \max\{\underline{\ell}(d') \mid d' \in \mathbf{I}_{1,d}\} . \end{cases} \quad (10 : 6)$$

(For the definition of the lower length limit  $\underline{\ell}(d)$  (10:5) to be sound, one needs Conjecture 17 : there exists at least one primitive  $T_d$ -membership. For the soundness of the definition of the upper length limit  $\bar{\ell}(d)$ , one needs both conjectures : cf. Conjecture 26, with  $L = L(d) = \bar{\ell}(d)$  chosen as the upper bound in (10:3).)

The below solution of Problem 10.3(1) gives a trivial but rather realistic universal lower bound to lengths of  $T_d$ -memberships (Lemma 10.5(1)), and demonstrates that in

an infinite number of cases this universal lower bound can be considerably strengthened (Lemma 10.5(2) [Lagarias 1990]).

Neither upper, nor nontrivial lower bounds to the upper length limit  $\bar{\ell}(d)$  are known (according to Conjecture 26 and Lemma 10.2, any universal upper bound would prove the finite-number-of-primitive-cycles conjecture, Conjecture 18). However, since, trivially,  $\bar{\ell}(d) \geq \underline{\ell}(d)$ , the below lower bounds (10:8) for an infinite series of shift numbers provide the only available now nontrivial lower bounds to the upper length limit for at least some shift numbers.

**Lemma 10.5.** (1) The following lower bound to the lower length limit is universal :

$$\forall d \in \mathbf{D} , \quad \underline{\ell}(d) \geq \log_2 d . \quad (10 : 7)$$

(2) For an infinitely growing series of shift numbers,  $d_1, d_2, \dots, d_j, \dots$ , the lower length limit has the following lower bound :

$$\exists f : \mathbf{N} \longrightarrow \mathbf{N} , \quad i < j \iff f(i) < f(j) \quad \& \quad \forall j \in \mathbf{N} , \quad \underline{\ell}(d_{f(j)}) \geq \frac{5}{4} \cdot (d_{f(j)})^{\frac{1}{3}} . \quad (10 : 8)$$

*Proof :* (1) According to (8:9), :

$$\forall \ell \geq 2 \quad \forall (k, \ell) \in \Lambda \quad \forall \mathbf{P} \in \Pi_\ell^{(k)} , \quad d = \frac{B_{k, \ell}}{\gcd(a(\mathbf{P}), B_{k, \ell})} \leq B_{k, \ell} < 2^\ell , \quad (10 : 9)$$

which implies the bound (10:7). As our calculations show, in many cases, the bound (10:7) is very close to be sharp.

(2) See [Lagarias 1990].

*End of the Proof.*

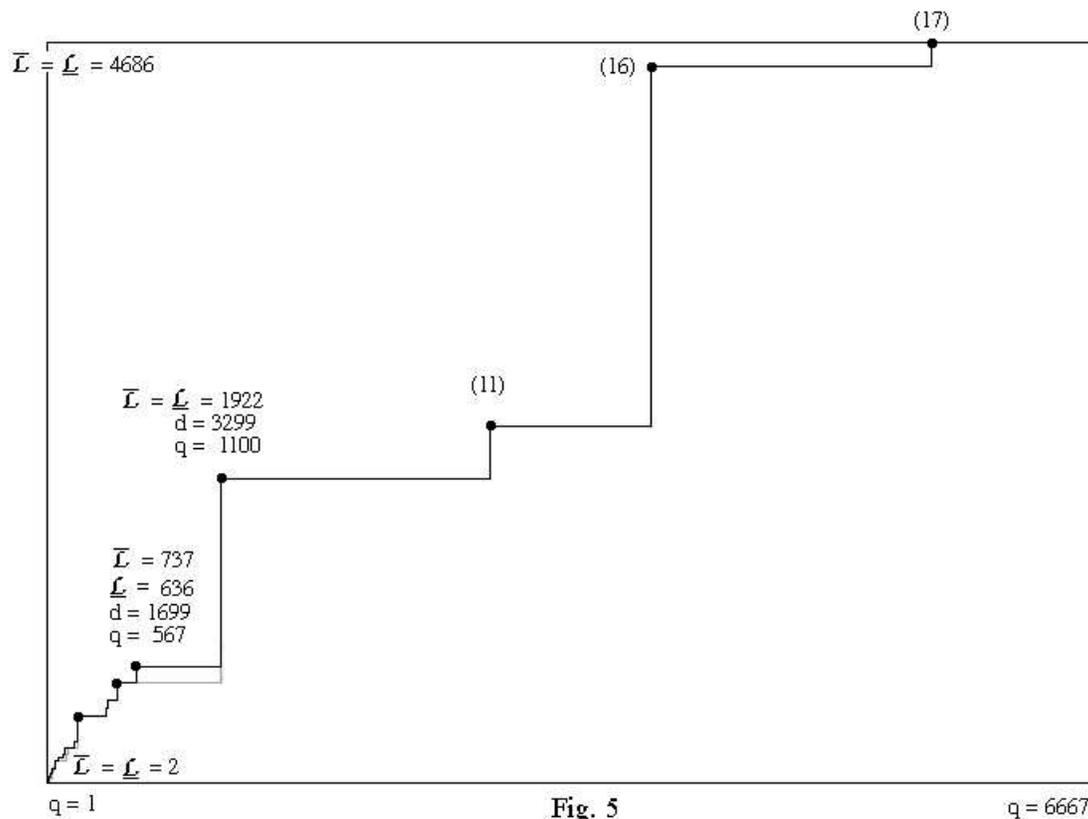
**Comments and Experimental Evidence 10.6.** (1) The smaller is the cancellation factor  $\gcd((a(\mathbf{P}), B_{k, \ell})) \geq 1$  in (10:9), the closer is the corresponding lower length limit  $\underline{\ell}(d)$  to the lower bound (10:7). Thus, for example, the primitive membership  $\langle \overline{1, 16381} \rangle$  has the length  $\ell = 14 > \log_2 16381 = 13.9997$ .

(2) However, as Lagarias has discovered, in some cases, this factor is necessary very huge : this is the phenomenon behind the lower bounds (10:8). Cf. (1:13) and Example 2.4.

(3) The problem of evaluating the precise (or sharp) rate of growth of the functions  $\bar{\mathcal{L}}(d), \underline{\mathcal{L}}(d)$  remains open. However, the experimental data (cf. (10:10) and Table 5) suggest that this rate is, most probably, much higher than that suggested by (10:8), and as high as  $d \log d$  : cf. (10:10), Table 5, Conjecture 27, and Figure 5 below.

(4) Moreover, excluding probably an infinite, but “very scarce” subset of shift numbers, the functions  $\bar{\mathcal{L}}(d)$  and  $\underline{\mathcal{L}}(d)$  are identical : an overwhelming majority of experimentally studied here systems  $\mathcal{D}_d$  with at least one “very long” primitive cycle have only one cycle,  $\varpi(d) = 1$ ; see (10:10), subsection (7), and Table 5 below.

(5) There are only 61 pairs  $(k, \ell) \in \Lambda$ , with the corresponding Collatz numbers  $B_{k, \ell}$  belonging to the range  $\mathbf{I}_{1, 19999}$ . And there are only 106 values of  $d$ , out of the total number



**Figure 5.** The graphs of the functions (10:6)  $\bar{\mathcal{L}}(d)$  (black) and  $\underline{\mathcal{L}}(d)$  (gray) superimposed. Points (11), (16), (17) have the parameters  $\ell, d$  as given in the corresponding columns of Table 6.

of 6667 explored here, with the corresponding primitive  $T_d$ -cycles of such  $(k, \ell)$  lengths and oddlengths.

(6) The last observation illustrates why the great majority of primitive cycles, experimentally recovered in this study, should come from Collatz configurations of much bigger values of lengths, oddlengths, and Collatz numbers. Among them there are 11 primitive cycles of lengths varying from 2004 to 2604, one cycle of the length 3918 and one of the length 4531, Table 5. In particular, we have found (cf. Figure 6) :

**Table 5.** 17 biggest values of lengths  $\ell = \bar{\ell}(d)$  of primitive  $T_d$ -cycles,  $d \in \mathbf{I}_{1,19999}$  (5:6), with shift numbers  $d$ , oddlengths  $k$ , margins  $\lambda_{k,\ell}$  (7:12), numbers  $\varpi(d)$  of primitive  $T_d$ -cycles, decimal exponents  $\log_{10} B$  of their Collatz numbers ( e. g.,  $B_{982,1942} \approx 10^{448}$  ), and the minimal odd member  $n_0$ .

	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	
$\bar{\ell}(d)$	1942	1954	1956	1966	2004	2014	2088	2151	
$k$	982	978	972	947	1024	1010	1044	1047	
$\lambda_{k,\ell}$	385	403	415	465	380	413	433	491	
$d$	14123	13187	13291	14627	16409	15683	15661	16411	
$\varpi(d)$	3	1	1	1	1	1	1	1	
$\log_{10} B$	448	587	587	590	602	605	627	646	
$n_0$	37	17	5	19	119	11	103	53	
	<b>9</b>	<b>10</b>	<b>11</b>	<b>12</b>	<b>13</b>	<b>14</b>	<b>15</b>	<b>16</b>	<b>17</b>
$\bar{\ell}(d)$	2155	2242	2254	2270	2588	2604	3918	4531	4686
$k$	1092	1146	1136	1166	1304	1356	1954	2253	2292
$\lambda_{k,\ell}$	424	425	453	421	521	454	820	960	1053
$d$	14303	10531	8431	13495	15563	16381	13829	11491	16819
$\varpi(d)$	944	1	1	1	1	2	1	1	1
$\log_{10} B$	647	673	677	682	778	782	1178	1362	1410
$n_0$	101	25	23	143	83	23	13	5	7

$$\left\{ \begin{array}{l} \bar{\ell}(107) = \underline{\ell}(107) = 106 , \\ \bar{\ell}(139) = \underline{\ell}(139) = 136, \\ \bar{\ell}(143) = 140 , \underline{\ell}(143) = 16, \\ \bar{\ell}(197) = \underline{\ell}(197) = 141 , \dots , \\ \bar{\ell}(8431) = \underline{\ell}(8431) = 2254 , \\ \bar{\ell}(11491) = \underline{\ell}(11491) = 4531 . \\ \bar{\ell}(16819) = \underline{\ell}(16819) = 4686 . \end{array} \right. \quad (10 : 10)$$

(7) Only three out of 16 systems  $\mathcal{D}_d$ , presented in Table 5, with at least one “long” primitive cycle have more than one primitive cycle,  $\varpi(d) > 1$ . (Curiously enough, one of those systems,  $\mathcal{D}_{14303}$ , has the maximal number of primitive cycles with a given shift number discovered in our experimental study,  $\varpi(14303) = 944$  : cf. Figure 3.) The other 13 systems fit into the pattern of the lower bound (10:8) : cf. the above subsection (4).

(8) All primitive cycles of Table 5 have rather big margins (7:1,12). The general statistics belies this particular pattern, apparently, typical for cycles with “big” lengths and “small” shift numbers. The graph of the distribution  $w(\lambda)$  (9:10) of values of the margin  $\lambda = \lambda_{k,\ell}$  (Figure 6) shows that marginal Collatz numbers (Lemma 7.2(4) (7:12)) and, more generally, Collatz numbers with small margins, account for a disproportionately big number of primitive cycles with shift numbers  $d$  from a fixed interval,  $d \in \mathbf{D}_{1,D}$ . This should not come as a surprise to us : for a given  $\ell \geq 2$ , to smaller margins correspond smaller Collatz numbers and, finally, primitive cycles with smaller shift and member number which are readily captured by our search procedure, §12.

(9) Figure 6 represents the graph of the function  $w$ , defined as follows (cf. (7:12)) :

$$\forall \lambda \geq 0 \quad \left\{ \begin{array}{l} \forall d_1, d_2 \in \mathbf{D} \quad \left\{ \begin{array}{l} W(\lambda; d_1, d_2) = \\ \{ \mathbf{C} \in \mathcal{P}(d) \mid d_1 \leq d \leq d_2 \ \& \ \lambda_{k,\ell} = \lambda \} , \\ w(\lambda; d_1, d_2) = \#W(\lambda; d_1, d_2) ; \end{array} \right. \\ w(\lambda) = w(\lambda; 1, 19999) . \end{array} \right. \quad (10 : 11)$$

(10) Similar phenomena characterize the distribution of values of “big” shift numbers with cycles of a “small” length, as well (cf. Problem 9.5, (9:5)). Thus, the above Example 9.4 shows the attainability of the upper bound to  $d$ , with  $d$  equal to the corresponding Collatz number  $B_{3,14}$ . This question is more amenable to a theoretical analysis. In particular, the  $3x + 1$  conjecture implies that if  $d = B_{k,\ell}$  is prime, then there exist  $\frac{1}{k}\chi_{k,\ell}$  primitive  $T_d$ -cycles (Lemma 10.7 below). We have confirmed this implication for all 29 prime (out of the total number of 61) Collatz numbers  $B < 20000$ . This is another example of successful controlling checks of the thoroughness of our search procedure.

**Lemma 10.7.** (1) According to (7:6), a primitive  $T_d$ -cycle,  $d = B_{k,\ell}$ , exists *iff* at least one member  $A$  of Collatz  $(k, \ell)$ -corona is relatively prime to  $B_{k,\ell}$  :

$$\forall (k, \ell) \in \Lambda , \quad \mathcal{P}(B_{k,\ell}) \neq \emptyset \iff 1 \in \{ \gcd(A, B_{k,\ell}) \mid A \in \mathcal{A}_{k,\ell} \} . \quad (10 : 12)$$

(2) The diophantine equivalent (Conjecture 5, §1, Corollary 8.4(2)) of the  $3x + 1$  conjecture implies that, if  $d = B_{k,\ell}$  is prime, then there exist  $\frac{1}{k}\chi_{k,\ell}$  (7:8), (9:2-4) primitive  $T_d$ -cycles.

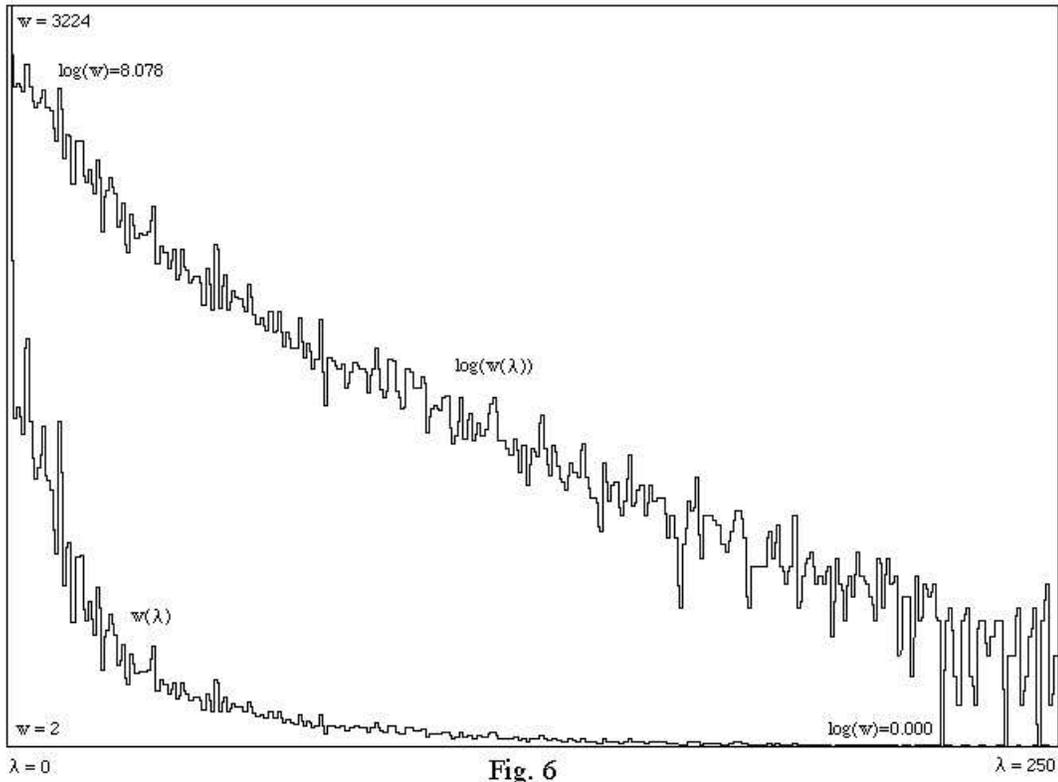
*Proof* : Selfevident.

*End of the Proof.*

The following conjectures formalize the above experimental insights (cf. Figure 5) :

**Conjecture 27.** (1) Define inductively the *upper* and *lower length limit threshold sets* (cf. Definition 10.4)  $\overline{D}, \underline{D}$ , as follows :

$$\left\{ \begin{array}{l} \overline{D} = \{ \overline{d}_1 < \overline{d}_2 < \dots \} \subset \mathbf{D}, \quad \overline{d}_1 = 1, \quad \overline{d}_{k+1} = \min \{ d \in \mathbf{D} \mid \overline{\mathcal{L}}(d) > \overline{\mathcal{L}}(\overline{d}_k) \} ; \\ \underline{D} = \{ \underline{d}_1 < \underline{d}_2 < \dots \} \subset \mathbf{D}, \quad \underline{d}_1 = 1, \quad \underline{d}_{k+1} = \min \{ d \in \mathbf{D} \mid \underline{\mathcal{L}}(d) > \underline{\mathcal{L}}(\underline{d}_k) \} . \end{array} \right. \quad (10 : 13)$$



**Figure 6.** The graph of the functions  $w(\lambda)$  (10:11) and  $\log(w)$ . Thus, there are 3224 primitive  $(k, \ell)$ -cycles with the zero margin  $\lambda = \ell - \lceil k \cdot \log_2 3 \rceil = 0$  (7:1), out of the total number of 42765 primitive  $T_d$ -cycles calculated here.

The sets  $\overline{D}, \underline{D} \subset \mathbf{D}$  (which, according to Lemma 10.5, are infinite) have density 0 (note that the set  $\mathbf{D}$  is of the density  $\frac{1}{3}$ ).

(2) The set  $\overline{D} = \overline{D} \cap \underline{D}$  is infinite and has relative density 1 in both sets  $\overline{D}$  and  $\underline{D}$ .

(3) The set  $\overline{D}' \subset \overline{D}$  of shift numbers  $d$  with a single primitive cycle,  $\varpi(d) = 1$ , is infinite and has relative density 1 in  $\overline{D}$ .

(4) The upper and lower length limit threshold functions  $\overline{\mathcal{L}}$  and  $\underline{\mathcal{L}}$  satisfy the following inequalities, with some “relatively small” constants  $c_1, c_2, c_3, c_4$  :

$$\forall d \in \overline{D}, c_1 \cdot \frac{d}{\log d} < \overline{\mathcal{L}}(d) < c_2 \cdot d; \quad \forall d \in \underline{D}, c_3 \cdot \frac{d}{\log d} < \underline{\mathcal{L}}(d) < c_4 \cdot d. \quad (10:14)$$

From the length-related aspects of the  $3x+d$  conjecture, we turn now to the oddlength-related ones. The finiteness of the number  $\mu^{(k)}(d)$  of primitive  $T_d$ -memberships of the

odddlength  $k$ , with an effective (but, in practice, very huge) upper bound, has been proved in [Belaga, Mignotte 1998]. The proof used a general inequality of [Baker, Wüstholz 1993] on diophantine approximations of linear  $\mathbf{Z}$ -combinations of logarithms of natural numbers.

We improve here our finiteness result, both methodologically and quantitatively. More precisely, the finiteness of the number  $\mu^{(k)}(d)$  is proved below in a quite elementary way, without any allusion to diophantine approximation theory or any other nontrivial fact outside our elementary diophantine formalism, §§7, 8.

On the other hand, we are able to improve our previous upper bound to  $\mu^{(k)}(d)$ , thanks to a more ancient, less general, but much more effective result from [Rhin 1987], on diophantine approximation of linear combinations of the logarithms  $\log 2$  and  $\log 3$  (cf. the inequality (10:25) below).

Note, finally, that the crucial common step in the below proofs of the upper bounds (10:15,16) is based on the general diophantine cyclicity criterion (8:8) (or (10:17) below). Thus, the below bounds hold also for the numbers  $\zeta(d)$  of all  $T_d$ -cycles, not necessary primitive ones : cf. (4:11,12).

**Theorem 10.8.** (1) For any  $d \in \mathbf{D}$  and any  $k \geq 1$ , an elementary proof yields the following exponential upper bound to the number of primitive  $T_d$ -memberships of the oddlength  $k$  (or, equivalently, of Collatz configurations of the oddlength  $k$  yielding  $n$  as the member number of the corresponding primitive membership (8:16(10))) :

$$\forall d \in \mathbf{D} \quad \forall k \geq 1, \quad \mu^{(k)}(d) < d \cdot (4.5)^k. \quad (10 : 15)$$

(2) A non-trivial and, as a matter of fact, the best known at present improvement of the above upper bound, based on a deep result [Rhin 1987] from diophantine approximation theory, still remains, quite disappointingly, exponential :

$$\forall d \in \mathbf{D} \quad \forall k \geq 1, \quad \mu^{(k)}(d) < 3d \cdot (1.5)^k \cdot k^{13.3}. \quad (10 : 16)$$

*Proof:* (1) As in [Belaga, Mignotte 1998], we count the number of respective memberships, using our combined formula (8:8) for shift and member numbers of primitive memberships, with the shift number  $d$  and the oddlength  $k$  fixed :

$$n = d \cdot \frac{A}{B_{k,\ell}}. \quad (10 : 17)$$

Define now the number (which, *a priori*, might be infinite, too)

$$\overline{N}(d, k) = \sup \left\{ d \cdot \frac{A}{B_{k,\ell}} \mid \forall \ell > k \cdot \log_2 3 \quad \& \quad \forall A \in \mathcal{A}_{k,\ell} \right\}. \quad (10 : 18)$$

It is clear that  $n \leq \overline{N}(d, k)$  and that the same upper bound holds for the number of different primitive  $T_d$ -membership of the oddlength  $k$  :

$$\mu^{(k)}(d) \leq \overline{N}(d, k). \quad (10 : 19)$$

(2) To evaluate from above  $\overline{N}(d, k)$ , we slightly modify the definition (10:18), evaluating separately  $d \cdot \frac{A}{B_{k,\ell}}$ , first, for a marginal Collatz number  $B_{k, \lceil k \cdot \log_2 3 \rceil}$ , and then for all other Collatz numbers  $B_{k,\ell}$ ,  $\ell > \lceil k \cdot \log_2 3 \rceil$  :

$$\left\{ \begin{array}{l} \overline{N}(d, k) = \max \left( \overline{N}_0(d, k), \overline{N}_1(d, k) \right) ; \\ \overline{N}_0(d, k) = \max_{\substack{\ell = \lceil k \cdot \log_2 3 \rceil \\ A \in \mathcal{A}_{k,\ell}}} \left\{ d \frac{A}{B_{k,\ell}} \right\} ; \\ \overline{N}_+(d, k) = \sup_{\substack{\ell > \lceil k \cdot \log_2 3 \rceil \\ A \in \mathcal{A}_{k,\ell}}} \left\{ d \frac{A}{B_{k,\ell}} \right\} . \end{array} \right. \quad (10 : 20)$$

Note that, since “sup” of the second line of (10:20) is defined on the finit set  $\mathcal{A}_{k, \lceil k \cdot \log_2 3 \rceil}$ , it is replaced here by “max”.

(3) According to (7:7,9),

$$\overline{N}_0(d, k) = d \cdot \frac{\overline{A}_{k, \lceil k \cdot \log_2 3 \rceil}}{B_{k, \lceil k \cdot \log_2 3 \rceil}} ; \quad \overline{A}_{k,\ell} = 2^{\ell-k+1} \cdot (3^{k-1} - 2^{k-1}) + 3^{k-1} < 2^{\ell-k+1} \cdot 3^{k-1} . \quad (10 : 21)$$

The trivial lower bound  $B_{k,\ell} \geq 1$  (in other words,  $2^\ell \neq 3^k$ , for any  $(k, \ell) \in \mathbf{N}$ ) gives us :

$$\overline{N}_0(d, k) = d \cdot \frac{\overline{A}_{k, \lceil k \cdot \log_2 3 \rceil}}{B_{k, \lceil k \cdot \log_2 3 \rceil}} < d \cdot \overline{A}_{k, \lceil k \cdot \log_2 3 \rceil} < d \cdot (4.5)^k . \quad (10 : 22)$$

On the other hand, for any  $\ell > \lceil k \cdot \log_2 3 \rceil$ ,

$$\left\{ \begin{array}{l} B_{k,\ell} = 2^\ell \cdot \left( 1 - \frac{3^k}{2^\ell} \right) > 2^{\ell-1} ; \\ \overline{N}_+(d, k) < d \cdot \frac{2^{\ell-k+1} \cdot 3^{k-1}}{B_{k,\ell}} < \frac{4}{3} d \cdot (1.5)^k . \end{array} \right. \quad (10 : 23)$$

Hence,

$$\overline{N}(d, k) < d \cdot (4.5)^k . \quad (10 : 24)$$

(4) The trivial lower bound  $B \geq 1$  for marginal Collatz numbers can be improved. According to [Rhin 1987], whose techniques lacks the generality of Baker’s method (cf. [Baker, Wüstholz 1993] and the references there), but is more effective in the evaluation of linear forms of several specific logarithms, including the case  $p + q \log 2 + r \log 3$  (cf. (7:13) for the definition of  $\epsilon_k$ ) :

$$\left\{ \begin{array}{l} \forall p, q, r \in \mathbf{Z} , |p + q \log 2 + r \log 3| > \left( \max(|p|, |q|, |r|) \right)^{-13.3} \\ \Downarrow \\ \forall k \geq 1 , 2^{\lceil k \cdot \log_2 3 \rceil} - 3^k = 3^k \cdot (2^{\epsilon_k} - 1) > 3^{k-1} \cdot k^{-13.3} . \end{array} \right. \quad (10 : 25)$$

Together with (10:20,22), this inequality implies the bound (10:16). *End of the Proof.*

## 11. Member Number Conjectures and Claims.

We study here some theoretical implications of new experimental data concerning distributions of member numbers in primitive cycles. This distributions are the subject of Conjectures 19, 20, §5 [Lagarias 1990]. We remind the reader that Conjecture 19 claims that any odd positive integer  $n$  not divisible by 3,  $n \in \mathbf{D}$ , is met by at least one primitive cycle, and Conjectures 20 strengthens this claim, asserting that  $n$  is met by an infinite number of such cycles: cf. Experimental Evidence 5.7, Table 4, and Figure 2, §5.

Here, our main technical tool will be the formula (7:6), (8:11) for the member number  $n$  of the primitive membership  $\langle n, d \rangle$  corresponding to Collatz  $(k, \ell)$ -configuration  $\mathbf{P}$  :

$$\left. \begin{array}{l} \forall (k, \ell) \in \Delta \\ \forall \mathbf{P} \in \Pi_{k, \ell} \end{array} \right\} \gamma(\mathbf{P}) = \langle n, d \rangle, \quad n = f(\mathbf{P}) = \frac{A}{\gcd(A, B_{k, \ell})} = \frac{a(\mathbf{P})}{\gcd(a(\mathbf{P}), B_{k, \ell})}. \quad (11 : 1)$$

As we have already observed above (Lemma 7.2 (7:10)), the function  $a(\mathbf{P})$  (7:6) does not actually depend on the rightmost component  $p_k$  of a configuration  $\mathbf{P}$ . This means that the expression (11:1) for  $n$  carries  $p_k$  only as a summand of the exponent  $\ell = p_1 + \dots + p_k$  of the Collatz number  $B_{k, \ell} = 2^\ell - 3^k$ . Thus, one can vary  $p_k$  in such a way that a value  $n$  once attained will be recapitulated infinitely many times.

This is the idea behind the proof ([Lagarias 1990], Lemma 11.1 below) that Conjectures 19, 20 are, in fact, equivalent : if  $n$  is met by at least one primitive cycle, then it is met by an infinite number of primitive cycles of the same oddlength and with an identical diophantine structure. This makes the rather audacious infinity claim of Conjecture 20 as plausible as that of Conjecture 19, dual to Conjecture 17 for shift numbers.

However, as our calculations show (cf. Experimental Evidence 5.7, Table 4, and Figure 2, §5, above and Table 6, Figures 7-9 below), the replicating mechanism of Lemma 11.1 accounts only for a tiny fraction of primitive memberships with a given member number  $n$ .

The below analysis (Definition 11.2, Lemma 11.3) of Lagarias' construction and of our experimental data leads to the notions of *irreducible* and *strongly irreducible* Collatz configurations, and, as the result, to two new infinity conjectures. Conjecture 28 strengthens Conjecture 20 by asserting the existence of infinitely many irreducible configurations yielding  $n$ . Conjecture 29 further strenghtens Conjecture 20, by claiming the same property for strongly irreducible configurations yielding  $n$ .

In what follows, we assume that a positive odd integer not divisible by 3 is fixed,  $n \in \mathbf{D}$ , together with the set  $\Pi(n)$  (8:18) of Collatz configurations yielding  $n$  as the member number of the corresponding primitive membership (cf. (7:6), (8:1)) :

$$\forall n \in \mathbf{D} \quad \forall \mathbf{P} \in \Pi(n) \quad f(\mathbf{P}) = n. \quad (11 : 2)$$

The below re-statements of the original claim from [Lagarias 1990] is adapted to our aims and notations.

**Lemma 11.1 (1)** For any odd positive integer  $r > 1$ , there exists such a minimal integer exponent  $m = \mathbf{m}(r) \geq 2$ , a divisor of Euler function  $\phi(r)$ , that  $2^r \equiv 1 \pmod{r}$  and, for any integer  $t \geq 1$ , if  $2^t \equiv 1 \pmod{r}$ , then  $t \equiv 0 \pmod{r}$  :

$$\forall r \in \mathbf{N} \quad \exists m = \mathbf{m}(r) \in \mathbf{N} \quad \left\{ \begin{array}{l} \phi(r) \equiv 0 \pmod{m} ; \\ 2^m \equiv 1 \pmod{r} ; \\ \forall t \in \mathbf{N} \quad \left\{ \begin{array}{l} 2^t \equiv 1 \pmod{r} \\ \updownarrow \\ t \equiv 0 \pmod{m} \end{array} \right\} \end{array} \right\} . \quad (11 : 3)$$

We remind that Euler function  $\phi(r)$  is defined as the number of different divisors of  $r$ .

**(2)** For any Collatz configuration  $\mathbf{P}$  corresponding to the primitive membership  $\gamma(\mathbf{P}) = \langle n, d \rangle$  (8:6,16),

$$\mathbf{P} = \langle p_1, \dots, p_{k-1}, p_k \rangle, \quad p_1 + \dots + p_k = \ell, \quad \mathbf{P} \in \Pi_\ell^{(k)}, \quad n = f(\mathbf{P}), \quad d = g(\mathbf{P}), \quad (11 : 4)$$

the below Collatz configuration  $\mathbf{P}'$ , corresponds to the primitive membership  $\gamma(\mathbf{P}') = \langle n, d' \rangle$  with the same member number, as follows :

$$\left\{ \begin{array}{l} (1) \mathbf{P}' = \langle p_1, \dots, p_{k-1}, p_k + t \rangle, \quad t = q \cdot \mathbf{m}(a(\mathbf{P})), \quad q \geq 1 ; \\ (2) \ell' = \ell + t ; \\ (3) a(\mathbf{P}') = a(\mathbf{P}), \quad (\text{cf. Lemma 7.2 (7 : 10)}) ; \\ (4) B_{k,\ell'} = 2^{\ell+t} - 3^k = 2^\ell \cdot (1 + r \cdot a(\mathbf{P})) - 3^k > B_{k,\ell}, \quad r \in \mathbf{N} \quad (\text{cf. (11 : 3)}) ; \\ (5) \gcd(a(\mathbf{P}'), B_{k,\ell'}) = \gcd(a(\mathbf{P}), B_{k,\ell}) ; \\ (6) n' = f(\mathbf{P}') = \frac{a(\mathbf{P}')}{\gcd(a(\mathbf{P}'), B_{k,\ell'})} = \frac{a(\mathbf{P})}{\gcd(a(\mathbf{P}), B_{k,\ell})} = f(\mathbf{P}) = n ; \\ (7) d' = g(\mathbf{P}') = \frac{B_{k,\ell'}}{\gcd(a(\mathbf{P}'), B_{k,\ell'})} > \frac{B_{k,\ell}}{\gcd(a(\mathbf{P}), B_{k,\ell})} = g(\mathbf{P}) = d . \end{array} \right\} \quad (11 : 5)$$

**(3)** Moreover, for sufficiently large  $q$ , Collatz configurations (11:5) are *lowermost*, Definition 8.2, i. e.,  $n$  is the minimal odd member of the corresponding primitive cycle.

*Proof :* (1) This is an elementary number theoretical fact : see, e. g., [Nathanson 2000].

(2) The statement (11:5) claims, rather implicitly, that *if (11:5) is a Collatz configuration, then it yields  $n$  as the member number.*

The last claim can be instantly verified, according to (11:5).

Now, for a  $k$ -tuple of positive integers,  $\mathbf{P}' \in \mathbf{N}$  (11:5), to be a Collatz configuration, two conditions need to be met : the inequality (7:4),  $|\mathbf{P}'| > k \cdot \log_2 3$ , and the aperiodicity.  $\mathbf{P}$  being a Collatz configuration,  $|\mathbf{P}| > k \cdot \log_2 3$ , we have :

$$|\mathbf{P}'| = |\mathbf{P}| + q \cdot \mathbf{m}(\mathbf{a}(\mathbf{P})) > k \cdot \log_2 3 .$$

As to the aperiodicity of  $\mathbf{P}'$ , it is obviously assured for all but, possibly, one value of  $q$ ; for example, the 6-tuple  $\langle 1, 1, 4, 1, 1, 2 \rangle$  is marginal Collatz (6, 10)-configuration, and all 6-tuples  $\langle 1, 1, 4, 1, 1, 2 + t \rangle$ ,  $t \geq 1$ , but the periodic one,  $\langle 1, 1, 4, 1, 1, 4 \rangle$ , are Collatz configurations, too.

In other words, there exists at most one  $q \geq 1$ , such that the corresponding  $k$ -tuple  $\mathbf{P}'$  is periodic and equal to the exponent vector  $Even(\mathbf{C})$  (4:4) of a *nonprimitive* cycle  $\mathbf{C}$  meeting  $n$ . This exceptional case has been, in fact, implicitly excluded from the claim (2), which concerns only *Collatz* configurations of the form (11:5).

(3) If  $k = 1$ , the claim is trivially true : the only odd member of a primitive cycle is always the lowermost (Lemma 4.2(2)). Consider now the following decomposition formulae for the functions  $a(\mathbf{P}) = a_k(\mathbf{P})$ ,  $a_k(\sigma^j(\mathbf{P}))$ ,  $k \geq 2, j \in [1, k - 1]$  (7:4,6) :

$$\left. \begin{array}{l} \forall k \geq 2 \\ \forall \mathbf{P} \in \Pi^{(k)} \\ \forall j \in [1, k - 1] \\ \forall t \geq 1 \end{array} \right\} \left\{ \begin{array}{l} \text{(1) } \mathbf{P} = \langle p_1, \dots, p_k \rangle, p_1 + \dots + p_k = \ell ; \\ \text{(2) } a_k(\sigma^j(\mathbf{P})) = 3^j \cdot a_{k-j}(p_{j+1}, \dots, p_k) + \\ \qquad \qquad \qquad 2^{p_{j+1} + \dots + p_k} \cdot a_j(p_1, \dots, p_j) . \\ \text{(3) } \mathbf{P}' = \langle p_1, \dots, p_k + t \rangle, \ell' = \ell + t ; \\ \text{(4) } a_k(\sigma^j(\mathbf{P}')) = 3^j \cdot a_{k-j}(p_{j+1}, \dots, p_k) + \\ \qquad \qquad \qquad 2^t \cdot 2^{p_{j+1} + \dots + p_k} \cdot a_j(p_1, \dots, p_j) = \\ \qquad \qquad \qquad 2^t \cdot U_{j,k}(\mathbf{P}) + V_{j,k}(\mathbf{P}) . \end{array} \right. \quad (11 : 6)$$

Applying now these formulae to Collatz configurations  $\mathbf{P}' = \langle p_1, \dots, p_k + t \rangle$ ,  $\ell' = \ell + t$ , and  $\sigma^j(\mathbf{P}')$ , we find that, for all  $t \geq 1$ ,  $a_k(\mathbf{P}') = a_k(\mathbf{P}) = A$  (the function  $a_k$  does not depend on the coordinate  $p_k$  (7:4)), but, according to (11:6(4)), for  $t$  sufficiently big,

$$\forall j \in [1, k - 1], a_k(\sigma^j(\mathbf{P}')) = 2^t \cdot U_{j,k} + V_{j,k} > A .$$

*End of the Proof.*

**Experimental Evidence 11.2.** (1) We have found altogether  $\tilde{\mu}(d)|_1^{19999} = 3584942$  primitive memberships  $\langle n, d \rangle$  in the band  $1 \leq d \leq 19999$  (cf. 6:6),

$$\tilde{\mu}(d)|_1^{19999} = \sum_{d \in \mathbf{I}_{1,19999}} \tilde{\mu}(d) = \sum_{d \in \mathbf{I}_{1,19999}} \sum_{\mathbf{C} \in \tilde{\mathcal{P}}(d)} \text{oddlength}(\mathbf{C}) .$$

(As we argue in §12, this is the exhaustive list of primitive memberships in the above interval of shift number values.) Figure 2 renders the graph of the distribution, according to the value of the member number  $n$ ,

$$\mathbf{t}(q) = \tilde{\omega}(n|_{n=\xi(q)})|_1^{19999}$$

**Table 6.** The first twenty values of the functions  $\mathbf{t}(q), \mathbf{i}(q), \mathbf{s}(q), \mathbf{r}(q) = \mathbf{t}(q) - \mathbf{i}(q), \frac{\mathbf{i}(q)}{\mathbf{s}(q)}$ . Cf the notations (11:15).

$q$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$n$	1	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49	53	55	59
$\mathbf{t}(q)$	452,354,359,414,396,410,423,399,324,409,426,265,407,415,418,422,395,440,300,406																			
$\mathbf{r}(q)$	018,011,008,012,005,006,021,012,007,001,009,006,016,001,004,008,014,001,002,001																			
$\mathbf{i}(q)$	434,343,351,402,391,404,402,387,317,399,417,259,391,405,414,414,381,430,298,405																			
$\mathbf{s}(q)$	223,203,194,217,231,221,227,206,192,225,210,177,218,227,232,221,220,241,201,220																			
$\frac{\mathbf{i}}{\mathbf{s}}(q)$	1.9, 1.7, 1.8, 1.8, 1.7, 1.8, 1.9, 1.9, 1.6, 1.7, 2.0, 1.5, 1.8, 1.8, 1.8, 1.9, 1.7, 1.8, 1.5, 1.8																			

of primitive memberships found in this study (according to Note 4.8(2), the *tilde*-mark indicates that we deal with the experimental value of the corresponding theoretically defined parameter), for 20000 values  $n$  of their member number component, in the rectangle  $\mathcal{R}_{1,19999}^{1,59999}$  :

$$\forall N, N', D, D' \in \mathbf{D}, \mathcal{R}_{D,D'}^{N,N'} = \{(n, d) \in \Delta \cap \{\mathbf{I}_{N,N'} \times \mathbf{I}_{D,D'}\}\}. \quad (11 : 7)$$

(Cf. also Experimental Evidence 5.7 and Table 4, §5.) We remind the definition (2:3) of the function  $n = \xi(q) = \left(6 \cdot \left\lfloor \frac{q}{2} \right\rfloor - (-1)^q\right)$  and the definition (6:1) of the set  $\Delta : \Delta = \{(n, d) \in \mathbf{D}^2 \mid \gcd(n, d) = 1\}$ . See also the definition (5:4) of an interval  $\mathbf{I}_{Q,R} \subset \mathbf{D}$ .

(2) In a smaller rectangle,  $\mathcal{R}_{1,19999}^{1,8249}$ , for 2750 values of the member number  $n$ , we have calculated also the distribution of *reducible* memberships, i. e., memberships which are (11:c,d)-type replicas of other memberships (see for details Definition 11.3). The below sample (the third line of Table 6, which is an extension of the above Table 4, §5) of experimentally found values of the number  $\mathbf{r}(q)$  of reducible memberships demonstrates that they account for only a tiny fraction of memberships. Thus, for example :

(i) out of 452 of experimentally discovered in the rectangle  $\mathcal{R}_{1,19999}^{1,8249}$  memberships of the form  $\langle 1, d \rangle$ , only 18, or less than 4%, are reducible;

(ii) and out of the total number 823954 of calculated in this rectangle primitive memberships, there are only 9383 reducible memberships, or less than 1.2%.

(3) Moreover, according to Table 7, for 173 values of  $n \in \mathbf{I}_{1,8249}$ , all memberships are reducible (i. e.,  $\mathbf{r} = 0$ ), in 428 cases there exists only one reducible membership, and only in 463 cases the number of reducible memberships is 6 or more. These experimental data are the motivation behind the below definitions of, and constructions associated with *irreducible* and *strongly irreducible* memberships. (Cf. Definition 11.3, Figures 7-9 below.)

**Definition 11.3.** (1) To simplify and make uniform our notations, we define the empty 0-tuple  $\langle \rangle$  of positive integers,  $\mathbf{N}^0 = \{\langle \rangle\}$  and extend to it the definition (7:4) of the norm  $|\dots|$  :

**Table 7.** The number  $N(R)$  of values of  $n \in \mathbf{N}_{1,19999}$  with  $R$  reducible memberships,  $0 \leq R \leq 21$ , in the rectangle  $\mathcal{R}_{1,19999}^{1,8249}$ .

$R$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$N(R)$	173	428	568	521	359	238	167	103	60	52	29	16	16	4	7	3	3	0	1	1	0	1

$$\forall k \geq 0 \quad \forall \mathbf{P} = (p_1, \dots, p_k) \in \mathbf{N}^k, \quad |\mathbf{P}| = \begin{cases} 0, & \text{if } k = 0, \\ p_1 + \dots + p_k, & \text{otherwise.} \end{cases} \quad (11 : 8)$$

(2) The trivial decomposition

$$\mathbf{N}^k = \mathbf{N}^{k-1} \times \mathbf{N}, \quad k \geq 1,$$

can be interpreted both as a breaking a given  $k$ -tuple into its  $(k-1)$ -component *prequel* and 1-component *sequel*, and as an assembling this  $k$ -tuple from given prequel and sequel. Hence, the two following dual notations :

$$\begin{aligned} (1) \quad & \begin{cases} \forall k \geq 1 \quad \forall \mathbf{P} = (p_1, \dots, p_k) \in \mathbf{N}^k, \\ \mathbf{P} \Big|_k = \langle p_1, \dots, p_{k-1} \rangle \in \mathbf{N}^{k-1}, \quad \mathbf{P} \Big|_k = p_k. \end{cases} \\ (2) \quad & \begin{cases} \forall j \geq 0 \quad \forall \mathbf{P} = (p_1, \dots, p_k) \in \mathbf{N}^j \quad \forall p \in \mathbf{N}, \\ \mathbf{P}' = \langle \mathbf{P}, p \rangle \in \mathbf{N}^{j+1}. \end{cases} \end{aligned} \quad (11 : 9)$$

(3) For any  $k \in \mathbf{N}$  and  $n \in \mathbf{D}$ , let  $\Pi^{(k)}\langle n \rangle$  be the set of all Collatz configurations  $\mathbf{P}$  of the oddlength  $k$  yielding  $n = f(\mathbf{P})$ . A Collatz configuration  $\mathbf{P} \in \Pi^{(k)}\langle n \rangle$  is called *irreducible* if, for any  $p \in \mathbf{N}$ ,  $p < p_k = \mathbf{P} \Big|_k$ , the  $k$ -tuple  $\mathbf{P}' = \langle \mathbf{P} \Big|_k, p \rangle$  is either not Collatz configuration,  $\mathbf{P}' \notin \Pi$ , or it yields a member number  $n'$  different from  $n$ ,  $\mathbf{P}' \notin \Pi^{(k)}\langle n \rangle$ .

(4) A membership  $\langle n, d \rangle$  is called *irreducible*, if the corresponding Collatz configuration  $\mathbf{P} = \theta\langle n, d \rangle$  (8:6) is irreducible. We denote by  $\mathcal{I}(n) \subset \mathcal{M}\langle n \rangle$  (8:16(1)) the set of all irreducible primitive memberships with the member number equal to  $n$ .

(5) Let  $\prec$  denotes the alphabetical order on the set  $\mathbf{N}^k$  of  $k$ -tuples of positive integers (in this order, for example,  $\langle 1, 1, 3, 1, 1, 3 \rangle \prec \langle 1, 1, 3, 1, 1, 5 \rangle$ ). A Collatz configuration  $\mathbf{P} \in \Pi^{(k)}\langle n \rangle$  is called *strongly  $(n, k)$ -irreducible* (or, for short, if no ambiguity is in sight, *strongly irreducible*), if it is  $\prec$ -minimal in  $\Pi^{(k)}\langle n \rangle$ .

(6) A membership  $\langle n, d \rangle$  is called *strongly irreducible*, if the corresponding Collatz configuration  $\mathbf{P} = \theta\langle n, d \rangle$  is strongly irreducible. For any  $n \in \mathbf{D}$ , we denote by  $\mathcal{S}(n) \subset \mathcal{M}\langle n \rangle$  the set of all strongly irreducible primitive memberships with the member number equal to  $n$ . By definition, any two elements of  $\mathcal{S}(n)$  have different oddlengths (cf. the notations (8:16(1,4)) :

$$\forall k \geq 1 \quad \forall n \in \mathbf{D} \quad \begin{cases} \mathcal{S}^k(n) = \mathcal{S}(n) \cap \mathcal{M}^{(k)}(n) , \\ s(n, k) = \#\mathcal{S}^k(n) = \begin{cases} 0 , & \text{if } \mathcal{M}^{(k)}(n) = \emptyset , \\ 1 , & \text{otherwise .} \end{cases} \end{cases} \quad (11 : 10)$$

**Note 11.5.** (1) The above definitions (5, 6) of the  $\prec$  –order and of strong irreducibility imply :

$$\forall k \geq 1 \quad \forall \mathbf{P} \in \mathbf{N}^k \quad \forall p \in \mathbf{N} , \quad p < \mathbf{P}|_k , \quad \mathbf{P}' = \langle \mathbf{P}|_k, p \rangle \prec \mathbf{P} . \quad (11 : 11)$$

Thus, a strongly irreducible Collatz configuration is irreducible, and the following inclusions hold :

$$\forall n \in \mathbf{D} , \quad \Pi(n) \neq \emptyset \implies \mathcal{S}(n) \subseteq \mathcal{I}(n) \subset \Pi(n) . \quad (11 : 12)$$

(2) The choice of the alphabetical order in the above definition of strongly irreducible configuration is convenient and, probably, optimal for our goals, but not compulsory. What we need, in fact, is the characteristic function (11:10), and to define it effectively, any choice function choosing an element from *every nonempty* set  $\Pi^{(k)}(n)$  of Collatz configurations of the oddlength  $k$  yielding  $n$  would do..

As we have already mentioned above (Experimental Evidence 5.7, Table 4, Figure 2), our calculations strongly confirm the existence conjecture for member numbers (Conjecture 19, §5). They present also a convincing circumstantial evidence in the favour of infinity Conjecture 20, §5, which, as our experimental data suggest (Experimental Evidence 11.2), cannot be explained away by the replicating mechanism of Lemma 11.1. Hence, two new infinity conjectures :

**Conjecture 28.** For any  $n \in \mathbf{D}$ , there exists an infinite number of irreducible memberships  $\langle n, d \rangle$ , or in other words (cf. the inclusion (11:12)),

$$\forall n \in \mathbf{D} , \quad \#\mathcal{I}(n) = \#\Pi(n) = \infty . \quad (11 : 13)$$

**Conjecture 29.** For any  $n \in \mathbf{D}$ , there exists an infinite number of strongly irreducible memberships  $\langle n, d \rangle$ , or in other words (cf. the inclusion (11:12)),

$$\forall n \in \mathbf{D} , \quad \#\mathcal{S}(n) = \#\mathcal{I}(n) = \#\Pi(n) = \infty . \quad (11 : 14)$$

**Experimental Evidence 11.6.** (1) We do not know the values of the parameters  $\#\mathcal{S}(n)$ ,  $\#\mathcal{I}(n)$ , and  $\#\Pi(n)$  (this last one, according to Conjecture 20, is infinite), but we are able to calculate and compare the experimental lower bounds to these functions in the rectangle  $\mathcal{R}_{1,19999}^{1,8249}$  (11:7), for 2750 values of the member number  $n$ .

(2) Namely, define, similarly to the notations (2:3), (5:14),

$$\forall q \in [1, 2750] \quad \begin{cases} n = \xi(q) \left( 6 \cdot \left\lfloor \frac{q}{2} \right\rfloor - (-1)^a \right) \in \mathbf{I}_{1,8249} , \\ \mathbf{t}(q) = \#\mathcal{M}\langle n \rangle \Big|_{d \in \mathbf{I}_{1,19999}}^{n=\xi(q)} = \#\Pi\langle n \rangle \Big|_{d \in \mathbf{I}_{1,19999}}^{n=\xi(q)} , \\ \mathbf{i}(q) = \#\mathcal{I}\langle n \rangle \Big|_{d \in \mathbf{I}_{1,19999}}^{n=\xi(q)} , \\ \mathbf{s}(q) = \#\mathcal{S}\langle n \rangle \Big|_{d \in \mathbf{I}_{1,19999}}^{n=\xi(q)} . \end{cases} \quad (11 : 15)$$

Thus,  $\mathbf{t}(q)$ ,  $q \in [1, 2750]$  is the number of primitive  $T_d$ -cycles  $d \in \mathbf{I}_{1,19999}$  meeting  $n = \xi(q) \in \mathbf{I}_{1,19999}$ .

(3) Table 6 presents these functions, together with the difference  $\mathbf{r}(q) = \mathbf{t}(q) - \mathbf{i}(q) =$  and the ratio  $\frac{\mathbf{i}(q)}{\mathbf{s}(q)}$ , in the interval  $1 \leq q \leq 20$ .

(4) As we have already mentioned above (Experimental Evidence 5.7(2)), we were able to calculate the function  $\mathbf{t}(q)$  in a bigger interval ( $q \in [1, 20000]$ , Figure 2) than other functions. Figures 7-9 present, respectively, the graphs of the functions  $\mathbf{i}(q)$ ,  $\mathbf{s}(q)$ , and a superposition of the graphs  $\mathbf{t}(q)$ ,  $\mathbf{i}(q)$ , and  $\mathbf{s}(q)$  on the interval  $q \in [1, 2750]$ . The graphic scenario of Figure 9 chosen here needs a short comment included in its caption.

(5) As Table 6 and Figure 9 show, the ratio  $\frac{\mathbf{i}(q)}{\mathbf{s}(q)}$  stays around 2, and with growing  $q \in [1, 2750]$ , actually slightly descends. Thus, for example,

$$q = 2437, n = 7309 \quad \begin{cases} \mathbf{t}(2437) = \min_{q \in [1, 2750]} \mathbf{t}(q) = 133 ; \\ \mathbf{i}(2437) = \min_{q \in [1, 2750]} \mathbf{i}(q) = 133 ; \\ \mathbf{s}(2437) = \min_{q \in [1, 2750]} \mathbf{S}(q) = 118 ; \\ \frac{\mathbf{i}}{\mathbf{s}}(2437) = \min_{q \in [1, 2750]} \frac{\mathbf{i}}{\mathbf{s}}(q) = 1.1 . \end{cases}$$

This means, that for any relevant  $n$  and  $k$ , there exists on the average 0 to 2 irreducible memberships  $\langle n, d \rangle$ ,  $d \in \mathbf{I}_{1,19999}$ , of the oddlength  $k$ . This observation suggest below Conjecture 30.

We start by refining the notations (11:10,12) for irreducible and strongly irreducible Collatz configurations and primitive memberships (Definition 11.4) :

**Notations 11.7.** For any  $k \geq 1$  and  $n \in \mathbf{D}$ , define (cf. (8:16(4)) and (10:10)) :

$$\forall k \geq 1 \quad \forall n \in \mathbf{D} \quad \begin{cases} \Pi^{(k)} = \bigcup_{n \in \mathbf{D}} \Pi^{(k)}\langle n \rangle , \quad \mathcal{M}^{(k)} = \bigcup_{n \in \mathbf{D}} \mathcal{M}^{(k)}\langle n \rangle , \\ \mathcal{I}^k(n) = \mathcal{I}(n) \cap \mathcal{M}^{(k)}\langle n \rangle , \quad \mathbf{i}(n, k) = \#\mathcal{I}^k(n) , \\ \mathcal{S}^k(n) = \mathcal{S}(n) \cap \mathcal{M}^{(k)}\langle n \rangle , \quad \mathbf{s}(n, k) = \#\mathcal{S}^k(n) , \\ \mathcal{I}^k = \bigcup_{n \in \mathbf{D}} \mathcal{I}^k(n) , \quad \mathcal{S}^k = \bigcup_{n \in \mathbf{D}} \mathcal{S}^k(n) . \end{cases} \quad (11 : 16)$$



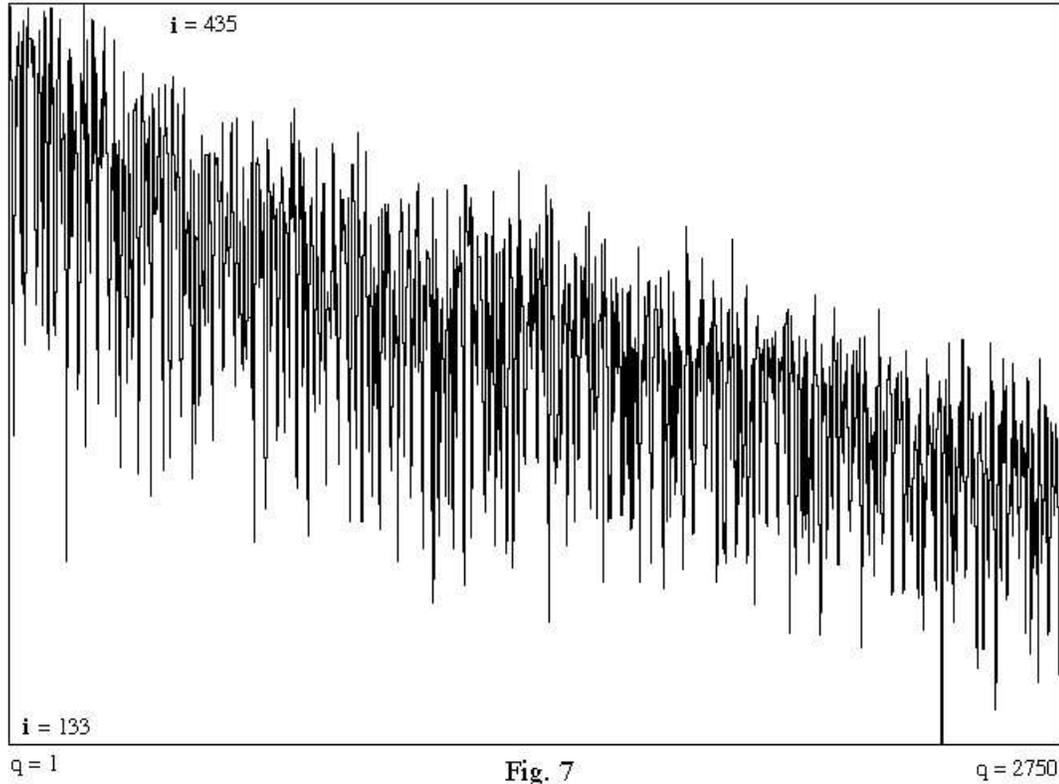


Fig. 7

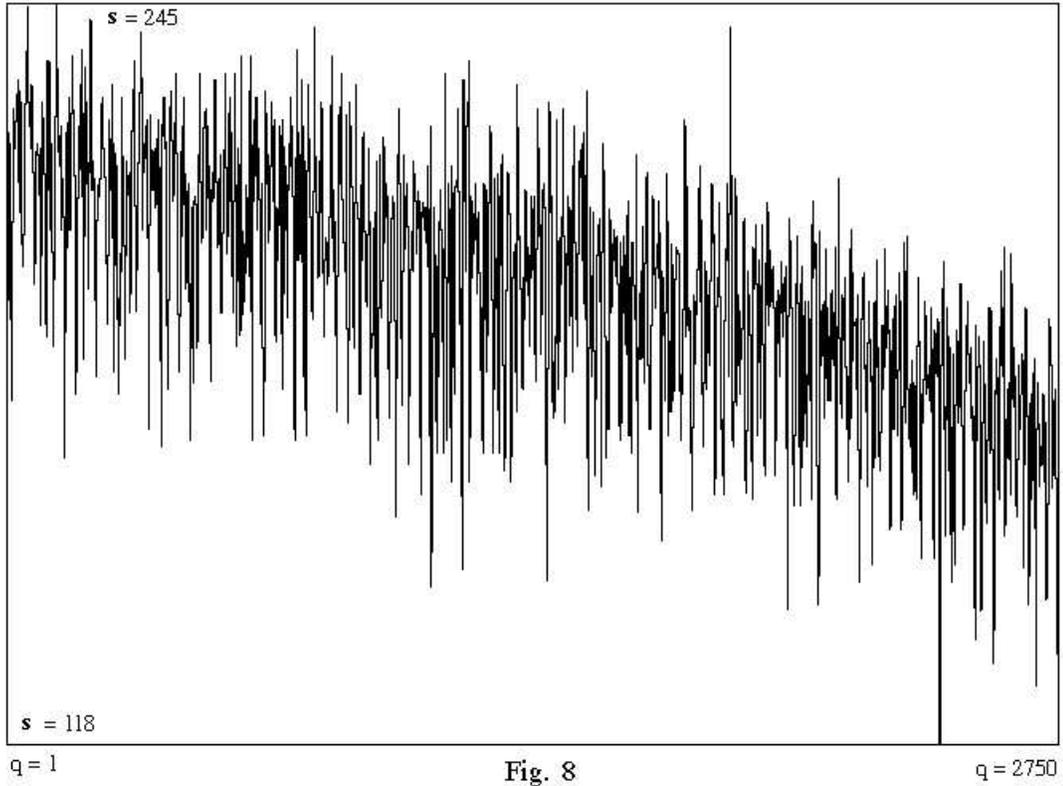
**Figure 7.** The graph of the function  $\mathbf{i}(q)$ ,  $1 \leq q \leq 2750$ ,  $\mathbf{i}(2437) = 133 \leq \mathbf{i}(q) \leq 435 = \mathbf{i}(196)$ . Thus, for example,  $n = \xi(1) = 1$  and  $n = \xi(196) = 1187$  (2:3), Experimental Evidence 11.2(1), are the member numbers of, respectively, 434 and 435 irreducible primitive memberships. These two values are maximal ones for the numbers of irreducible memberships in the rectangle  $\mathcal{R}_{1,1999}^{1,8249}$  (11:7).

This classical cycle detection device, remarkable for its simplicity, has been invented by Robert W. Floyd. (Never published by the author. The standard reference is [Knuth 1969], pp. 4-7, Exercise 7. See also [Cohen 1993], §8.5.2, for an update on cycles detection methods.) :

**Proposition 12.2.** Let  $f : \mathbf{N} \rightarrow \mathbf{N}$  be a function. A  $f$ -trajectory starting at a positive integer  $m$ ,

$$\tau(m) = \{f^0(m) = m, f(m), f^2(m) = f(f(m)), f^3(m), \dots\} \quad (12 : 2)$$

is ultimately cyclic *iff*, for some  $j \geq 1$ ,



**Figure 8.** The graph of the function  $s(q)$ ,  $1 \leq q \leq 2750$ ,  $s(2437) = 133 \leq s(q) \leq 245 = s(36)$ .

$$f^j(m) = f^{2^j}(m) , \quad (12 : 3)$$

*Proof :* (1) The *if* (or sufficiency) condition (12:3) is obvious.

(2) The *only if* (or necessary) condition. Suppose  $\tau(m)$  runs at the point  $s = f^z(m)$ ,  $z \geq 1$ , into a cycle of the length  $w$ , so that, for any  $x \geq 0$ ,  $f^{z+x}(n) = f^{z+x+w}(m)$ .

Assuming  $t = z + x$ , the condition (12:2) will be satisfied if, for some positive integers  $x, y \geq 1$ , the equality holds :

$$2t = 2(z + x) = z + x + y \cdot w = t + y \cdot w.$$

The choice  $y = \lceil \frac{z}{w} \rceil$  and  $x = y \cdot w - z$  would do.

*End of Proof.*

The above Working Assumption has been verified in more than 75,000,000 cases within the intervals (cf. the notations (5:4-6))

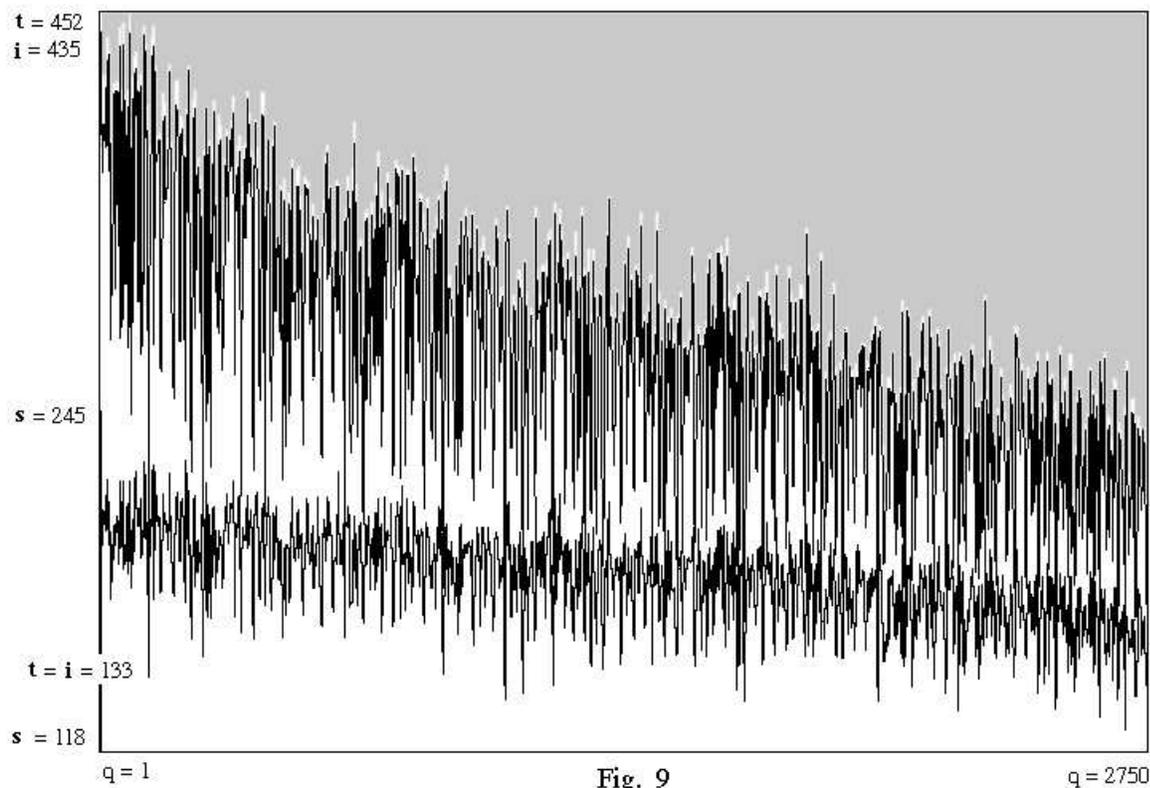


Fig. 9

**Figure 9.** (1) The graphic scenario of this superposition of the graphs of the functions  $\mathbf{t}(q)$ ,  $\mathbf{i}(q)$ ,  $\mathbf{s}(q)$  calls for an explanation. Since on the interval  $q \in [1, 2750]$  the difference between the functions  $\mathbf{t}(q)$  and  $\mathbf{i}(q)$  is almost imperceptible, we have chosen to present  $\mathbf{t}(q)$  as a white graph on the gray upper background. (2) The comparison of the graphs  $\mathbf{i}(q)$ ,  $\mathbf{s}(q)$  makes almost palpable both the remarkable stability and a slow growth of the ratio  $\frac{\mathbf{i}(q)}{\mathbf{s}(q)}$ .

$$d \in \mathbf{I}_{1,19999} , \quad n \in \mathbf{I}_{1,600 \cdot d - 1} . \quad (12 : 4)$$

and separately (when we were running a huge controlling check), in more than  $24 \cdot 10^9$  cases within the intervals

$$d \in \mathbf{I}_{1,4999} , \quad n \in \mathbf{I}_{1,3000 \cdot d - 1} . \quad (12 : 5)$$

To speed the search, trajectories have been traced in the increasing order of the starting number  $m \in \mathbf{D}$ . Then, if the trajectory  $\tau_d(m)$  fell at some point behind the initial

point  $m$ ,  $n = T_d^j(m) < m$ , the search along  $\tau_d(m)$  was abandoned, since its continuation, the trajectory  $\tau_d(n)$  starting at  $n$ , has been already treated before. Every one of the thus traced trajectories either run ultimately into a primitive cycle, or descended under  $m$ , and thus, has been already traced to a cycle earlier.

We have carried out some additional controlling checks, too :

(1) In 82 cases, for all  $d \in \mathbf{I}_{2755,2999}$ , an independent search has been carried out, with the bigger initial interval  $1 \leq n \leq 5000 \cdot d$ . No new primitive cycles have been discovered.

(2) In one particularly interesting case,  $d = 343$ , mentioned below, the chosen initial controlling interval was  $1 \leq n \leq 60,000 \cdot 343 > 2 \cdot 10^7$ . The calculations have confirmed the existence of only three primitive cycles discovered earlier.

(3) Cf. also the above examples of exhaustiveness of our experimental search : Note 5.2, Example 9.4, Comments and Experimental Evidence 10.6. (10).

Note also that the minimal members of all primitive cycles, discovered thus far, fall under the upper limit  $600 \cdot d$  in (12:4), with only two cases coming relatively close to this limit :

(1)  $d = 343$ ; one of primitive  $T_{343}$ -cycles (out of three discovered) has the minimal member (cf. (10:3))  $n_o = 177,337$ ;  $517 \cdot 343 < n_o < 518 \cdot 343 < 600 \cdot 343$  (and the maximal member  $m^o = 159,053,606$ ).

(2)  $d = 551$ ; one of primitive  $T_{551}$ -cycles (out of ten) has the minimal member  $n_o = 212,665$ ;  $385 \cdot 551 < n_o < 386 \cdot 551 < 600 \cdot 551$  (with  $m^o = 8,332,648$ ).

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