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**Effective Polynomial Upper Bounds to Perigees and
Numbers of $(3x + d)$ -Cycles of a Given Oddlength.**

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Abstract. For any positive odd integer d not divisible by 3, the arithmetical function $T_d(m) = \begin{cases} \frac{3m+d}{2}, & \text{if } m \text{ is odd} \\ \frac{m}{2}, & \text{otherwise} \end{cases}$ generates on the set \mathbf{N} of natural numbers a dynamical system \mathcal{D}_d . The $3x + d$ hypothesis, generalizing the well-known $3x + 1$ conjecture, asserts that \mathcal{D}_d has a finite number of cycles and no divergent trajectories. We study here the cyclic structure of the system \mathcal{D}_d , and prove in particular an effective and sharp polynomial upper bound to the number of cycles in \mathcal{D}_d with a given number of odd members.

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1. Introduction.

Let d be a positive odd integer not divisible by 3, and let T_d be the function defined on the set of positive integers, as follows :

$$\forall m \in \mathbf{N} \quad T_d(m) = \begin{cases} \frac{3m+d}{2}, & \text{if } m \text{ is odd ,} \\ \frac{m}{2}, & \text{otherwise .} \end{cases} \quad (1 : 1)$$

Repeated iterations of the function T_d generate $(3x + d)$ - (or T_d -) trajectories

$$\forall d \in \mathbf{D} = \{1, 5, 7, 11, 13, \dots\} \quad \forall m \in \mathbf{N} , \quad \tau_d(m) = \{m, T_d(m), T_d^2(m), \dots\} , \quad (1 : 2)$$

By definition, a trajectory $\tau_d(m)$ is a cycle of the length ℓ , $\mathbf{C} = \mathbf{C}(m, d) = \tau_d(m)$, $\text{length}(\mathbf{C}) = \ell$, if $T_d^\ell(m) = m$ and, for any $j \in [1, \ell - 1]$, $T_d^j(m) \neq m$ (note that $\ell > 1$, since the mapping T_d (1:1) has no fixed points). The minimal member of a T_d -cycle \mathbf{C} is odd, and is called its *perigee*, $n_0 = \text{prg}(\mathbf{C})$. Thus, the number k of odd members of a T_d -cycle, called here its *oddlength*, is a positive integer, $k \geq 1$. The length and oddlength of a cycle are related by the inequality $\ell \geq \lceil k \cdot \log_2 3 \rceil$ [Belaga, Mignotte 1998] (see Theorem 3.2(1) below). Note also that no member of a T_d -trajectory (1:2), excluding possibly the first one, is divisible by 3, and thus, all odd members of a T_d -cycle belong to \mathbf{D} .

It has been conjectured that the dynamical system $\mathcal{D}_d = \{\mathbf{N}, T_d\}$ has no divergent T_d -trajectories (1:2), and that the number $\varsigma(d)$ of cyclic T_d -trajectories is finite [Lagarias 1990], [Belaga, Mignotte 1998]. In the particular case $d = 1$, the well-known $3x + 1$ conjecture [Lagarias 1985], [Wirsching 1998] is even more specific : any trajectory $\tau_1(m)$ enters ultimately the (only) $3x + 1$ cycle $\{1 \rightarrow 2 \rightarrow 1\}$.

The present paper is concerned with the cyclic part of the above $3x + d$ conjecture, and more generally, with quantitative (and when available, numerical) characteristics of the cyclic structure of systems \mathcal{D}_d . Let $\mathcal{C}(d)$ and $\mathcal{C}_k(d)$ be the sets of all T_d -cycles and, respectively, of all such cycles with k odd members, or, in our terminology, of the oddlength $k \geq 1$.

Technically, our main result is the following general upper bound to the perigee of a T_d -cycle of the length ℓ and oddlength k :

$$\forall d \in \mathbf{D} \quad \forall \mathbf{C} \in \mathcal{C}(d) , \quad \left\{ \begin{array}{l} \text{length}(\mathbf{C}) = \ell, \\ \text{oddlength}(\mathbf{C}) = k, \end{array} \right\} \implies n_0 = \text{prg}(\mathbf{C}) \leq \frac{d}{2^{\frac{\ell}{k}} - 3} . \quad (1 : 3)$$

The inequality (1:3) has three important implications. The **first** one is an upper bound to the ratio of the length of a T_d -cycle to its oddlength, which, together with the well-known lower bound (2:5), Theorem 2.1(1), confines this ratio to the interval :

$$\forall d \in \mathbf{D} \quad \forall \mathbf{C} \in \mathcal{C}(d) , \quad \log_2 3 \leq \rho_d(\mathbf{C}) = \frac{\text{length}(\mathbf{C})}{\text{oddlength}(\mathbf{C})} \leq \log_2(d + 3) . \quad (1 : 4)$$

Second, the inequality (1:3) implies the following general and uniform upper bound to perigees of T_d -cycles of the oddlength $k \geq 1$:

$$\forall d \in \mathbf{D} \quad \forall k \in \mathbf{N} \quad \forall \mathbf{C} \in \mathcal{C}_k(d) \quad n_0 = \text{prg}(\mathbf{C}) \leq \mathbf{U}_{d,k} = \frac{d}{2^{\frac{\lceil k \cdot \log_2 3 \rceil}{k}} - 3} . \quad (1 : 5)$$

The bound (1:5) has an effective polynomial numerical equivalent (see the estimate (1:8) below). It is also sharp in the following natural sense (Theorem 3.2, (3:11(1))) : the average value of an odd member of a T_d -cycle of the oddlength $k \geq 1$ is bigger than $\mathbf{U}_{d,k}$. Thus, for example, the T_5 -cycle $\mathbf{C} = \{23 \rightarrow 37 \rightarrow 58 \rightarrow 29 \rightarrow 46\}$ has 3 odd members, $n_0 = \text{prg}(\mathbf{C}) = 23 < \mathbf{U}_{5,3} \approx 28.6038 < 29 < 37$.

Third, since no two T_d -cycles have a common member, any such cycle is fully determined by its perigee. Thus, the upper bound (1:5) not only implies that the set $\mathcal{C}_k(d)$ is finite, but supplies us with an effective general upper bound to the number $\varsigma_k(d) = \#\mathcal{C}_k(d)$ of T_d -cycles of the oddlength $k \geq 1$:

$$\forall d \in \mathbf{D} \quad \forall k \in \mathbf{N}, \quad \varsigma_k(d) \leq \frac{1}{3} \cdot \mathbf{U}_{d,k} = \frac{1}{3} \cdot \frac{d}{2^{\frac{[k \cdot \log_2 3]}{k}} - 3} \quad (1 : 6)$$

(the factor $\frac{1}{3}$ is due to the aforementioned inclusion $n_0 \in \mathbf{D}$).

Any numerical evaluation of the expression $\mathbf{U}_{d,k}$ depends on the state of our knowledge of effective *lower bounds* to diophantine approximations of linear combinations of logarithms $\log 2$ and $\log 3$. The best known at present lower bound belongs to [Rhin 1987], whose techniques lacks the generality of Baker's method (cf. [Baker, Wüstholz 1993] and the references there), but is more effective in the evaluation of linear forms of a few specific logarithms, including our case :

$$\forall r, s \in \mathbf{Z}, \quad \begin{cases} t = \max(|r|, |s|), \\ |r + s \log 2 + \log 3| > t^{-13.3}, \end{cases} \quad (1 : 7)$$

The inequality (1:7) implies the following effective polynomial upper bound to $\mathbf{U}_{d,k}$:

$$\forall d \in \mathbf{D} \quad \forall k \in \mathbf{N}, \quad \mathbf{U}_{d,k} \leq 102 \cdot d \cdot k^{14.3}. \quad (1 : 8)$$

Historical Remarks. The present author is not aware of any previous effective (and in any sense sharp) upper bound to the minimal odd member of a T_d -cycle. The following general exponential upper bound to the number $\varsigma_k(d)$ of T_d -cycles of the oddlength $k \geq 1$ has been actually proved in [Belaga, Mignotte 1998] (even if not explicitly articulated) and refined in [Belaga, Mignotte 2000] :

$$\forall d \in \mathbf{D} \quad \forall k \in \mathbf{N}, \quad \varsigma_k(d) < d \cdot \left(\frac{3}{2}\right)^k \cdot \frac{2^{\epsilon_k}}{2^{\epsilon_k} - 1}, \quad \epsilon_k = [k \cdot \log_2 3] - k \cdot \log_2 3. \quad (1 : 9)$$

The bound (1:9) has been derived from an identical upper bound to the *maximal odd member* of a cycle, the corresponding numerical upper bound being based on the estimate of [Baker, Wüstholz 1993] :

$$\forall d \in \mathbf{D} \quad \forall k \in \mathbf{N}, \quad \varsigma_k(d) < d \cdot k^C \cdot \left(\frac{3}{2}\right)^k, \quad (1 : 10)$$

with an effective but enormous constant C . Refining the proof of the inequality (1:9) and applying the estimate of [Rhin 1987] to (1:9), [Belaga, Mignotte 2000] have proved a better, but still exponential effective upper bound

$$\forall d \in \mathbf{D} \quad \forall k \in \mathbf{N}, \quad \varsigma_k(d) \leq 204 \cdot d \cdot \left(\frac{3}{2}\right)^k \cdot k^{13.3}. \quad (1 : 11)$$

Comments and Future Prospects. (1) The upper bound (1:4) to the ratio $\rho_d(\mathbf{C})$ implies in the $3x + 1$ case that the length of a cycle with k odd members does not exceed

$2k$. Note that the only known at present $3x+1$ cycle $\{1 \rightarrow 2 \rightarrow 1\}$ has one odd member, is of the length two, and has ratio two. More generally, the series of T_d -cycles of the length $\ell \in \mathbf{N}$, $d = 2^\ell - 3$, $n_0 = 1$ and of the oddlength $k = 1$ demonstrates that the upper bound (1:4) is sharp.

(2) The bounds (1:5) and, especially, (1:6) can be apparently improved. In fact, the experimental discovery of 843 T_{14303} -cycles of the oddlength 17, with perigees varying from 385057 to $1391321 < \mathbf{U}_{14303,17} = 2099280$, suggests that the bound (1:5) is apparently sharp up to a one-digit constant, whereas the bound (1:6), $\varsigma_{17}(14303) = 843 < \frac{1}{3}\mathbf{U}_{14303,17} = 699760$, far from being sharp, is at least realistic : for some d, k , the dynamical system \mathcal{D}_d has “many” cycles of the oddlength k .

(3) At present, the bounds (1:3,4,6) look useless, or at least insufficient, for an eventual proof of the cyclic part of the $3x+d$ conjecture, i. e., of the finiteness of the number $\varsigma(d)$ of T_d -cycles. At its best, the straightforward application of these bounds yields the trivial infinite upper bound :

$$\forall d \in \mathbf{D} \quad \varsigma(d) = \sum_{k \geq 1} \varsigma_k(d) \leq 102 \cdot d \cdot \sum_{k \geq 1} k^{14.3} = \infty . \quad (1 : 12)$$

However, the obstacle of the infinite summation in (1:12) could be possibly circumvented by a refinement of the above scheme, to fit the purpose of yielding directly an *absolute* (i. e., not depending on k) upper bound to the number $\varsigma(d)$.

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2. Exponential Diophantine Formulae for $3x+d$ Cycles.

Let, as above, $\mathbf{C} = \mathbf{C}(m, d) = \tau_d(m)$ (1:2) be a T_d -cycle of the length ℓ , $length(\mathbf{C}) = \ell$. We remind the reader that, according to (1:1), the minimal member, or *perigee* of a T_d -cycle, $n_0 = prg(\mathbf{C})$, is odd, and that the total number $k \geq 1$ of odd members of a cycle is called its *oddlength*. Moreover, if n is an odd member of a cycle, then $n \in \mathbf{D}$ (1:2), since no number divisible by 3 can belong to a cycle.

Note that if $m' \neq m$ is a member of a T_d -cycle $\mathbf{C} = \tau_d(m)$, or in other words, if \mathbf{C} *meets* m' , one should view $\mathbf{C}' = \mathbf{C}(m', d)$ as just another name for the same cycle $\mathbf{C} = \mathbf{C}(m, d)$. Since a T_d -cycle is fully characterized by its minimal member, the following notation can be accepted as the canonical one :

$$\mathbf{C} = \mathbf{C}(n_0, d) = \mathbf{C}[n_0, d] = \tau_d(n_0), \quad n_0 = prg(\mathbf{C}) . \quad (2 : 1)$$

In this case, we also say that \mathbf{C} *starts at* n_0 .

For any positive integer $m \in \mathbf{N}$, let $odd(m)$ be the number obtained by factoring out of m the highest possible power of 2, say 2^j , and let $\nu_2(m) = j$. Thus $odd(m)$ is odd and $m = odd(m) \cdot 2^{\nu_2(m)}$. Define

$$S : \mathbf{D} \times \mathbf{D} \rightarrow \mathbf{D} ; \quad \forall n, d \in \mathbf{D} . \quad S_d(n) = \text{odd}(3n + d) . \quad (2 : 2)$$

The function S_d speeds up the action of T_d (1:1), skipping even members of T_d -trajectories. In particular, $m = 1$ becomes the fixed point of the function $S_1 = \text{odd}(3n + 1)$, $S_1(1) = 1$, corresponding to the (according to the $3x + 1$ conjecture, only) T -cycle $\mathbf{C}(1, 1) = \{1 \rightarrow 2 \rightarrow 1\}$.

We associate with any T_d -cycle $\mathbf{C} = \mathbf{C}[n_0, d]$ its odd frame, $\mathbf{F} = \text{Odd}(\mathbf{C})$, the list of odd members of the cycle, in the order of their appearance in $\tau_d(n_0)$ (1:2), as the T_d -iterations of n_0 proceed. By definition, the frame is a S_d -cycle (2:2) starting at n_0 , and its length is called the *oddlength* of the cycle \mathbf{C} :

$$\left\{ \begin{array}{l} \forall \mathbf{C} = \mathbf{C}[n_0, d] , \quad n_0, d \in \mathbf{D} , \\ \ell = \text{length}(\mathbf{C}) = \min \{i \in \mathbf{N} \mid m_i = T_d^i(n_0) = n_0\} ; \\ k = \text{oddlength}(\mathbf{C}) = \min \{j \in \mathbf{N} \mid n_j = S_d^j(n_0) = n_0\} ; \\ \mathbf{F} = \text{Odd}(\mathbf{C}) = \langle n_0, n_1, \dots, n_{k-1} \rangle \in \mathbf{D}^k . \end{array} \right. \quad (2 : 3)$$

The even members of the T_d -cycle $\mathbf{C} = \mathbf{C}[n_0, d]$ can be recovered from its frame with the help of the cycle *Collatz signature* $\mathbf{P} = \theta(\mathbf{C})$, the vector of exponents of 2 factoring out from the values of the function T_d (1:1) at odd members of \mathbf{C} , as follows :

$$\left\{ \begin{array}{l} \forall \mathbf{C} = \mathbf{C}[n_0, d] , \quad n_0, d \in \mathbf{D} , \\ \mathbf{F} = \text{Odd}(\mathbf{C}) = \langle n_0, n_1, \dots, n_{k-1} \rangle ; \\ p_1 = \nu_2(T_d(n_0)), \quad p_2 = \nu_2(T_d(n_1)), \quad \dots, \quad p_k = \nu_2(T_d(n_{k-1})) ; \\ \mathbf{P} = \theta(\mathbf{C}) = \langle p_1, p_2, \dots, p_k \rangle \in \mathbf{N}^k ; \\ \ell = \text{length}(\mathbf{C}) = |\mathbf{P}| = p_1 + p_2 + \dots + p_k ; \\ \forall j \in [1, k-1] , \quad \left\{ \begin{array}{l} m_{p_1 + \dots + p_j} = n_j ; \\ p_j > 1 \implies \forall i \in [1, p_j - 1] , \quad m_{p_1 + \dots + p_j - i} = 2^i \cdot n_j . \end{array} \right. \end{array} \right. \quad (2 : 4)$$

Moreover, the Collatz signature $\mathbf{P} = \theta(\mathbf{C})$ of a cycle $\mathbf{C} = \mathbf{C}[n_0, d]$, $n_0, d \in \mathbf{D}$, completely characterizes it :

Theorem 2.1. [Belaga, Mignotte 1998] **(1)** The Collatz signature $\mathbf{P} = \theta(\mathbf{C})$ satisfies the inequality :

$$\ell = |\mathbf{P}| = p_1 + \dots + p_k \geq \lceil k \cdot \log_2 3 \rceil . \quad (2 : 5)$$

(2) Define the exponential diophantine function $A = a_k : \mathbf{N}^k \longrightarrow \mathbf{N}$, as follows :

$$\left\{ \begin{array}{l} \forall k \geq 1 \quad \forall \mathbf{P} = \langle p_1, \dots, p_k \rangle \in \mathbf{N}^k , \\ A = a_k(\mathbf{P}) = \left\{ \begin{array}{ll} 1, & \text{if } k = 1 ; \\ 3^{k-1} + 2^{p_1} \cdot 3^{k-2} + \dots + 2^{p_1 + \dots + p_{k-2}} \cdot 3 + 2^{p_1 + \dots + p_{k-1}}, & \text{otherwise .} \end{array} \right. \end{array} \right. \quad (2 : 6)$$

Let $\sigma = \sigma_k$ be the circular (counterclockwise) permutation on k -tuples :

$$\forall \mathbf{P} = \langle p_1, \dots, p_k \rangle \in \mathbf{N}^k . \quad \sigma(\mathbf{P}) = \sigma_k(\mathbf{P}) = \langle p_2, \dots, p_k, p_1 \rangle . \quad (2 : 7)$$

If now $\mathbf{P} = \theta(\mathbf{C})$ is the Collatz signature of a cycle $\mathbf{C} = \mathbf{C}[n_0, d]$, $n_0, d \in \mathbf{D}$ of the length ℓ , oddlength $k \geq 1$, and with the frame $\mathbf{F} = \langle n_0, n_1, \dots, n_{k-1} \rangle$, then

$$\left. \begin{array}{l} \text{(1)} \quad B = b_k(\mathbf{P}) = B_{k,\ell} = 2^\ell - 3^k > 0 ; \quad \text{(cf. (2 : 5))} \\ \text{(2)} \quad n_0 = d \cdot \frac{A}{B} , \quad A = a_k(\mathbf{P}) ; \quad \text{(cf. (2 : 6))} \\ \text{(3)} \quad \forall j \in [1, k-1] , \quad n_j = d \cdot \frac{a_k(\sigma^j(\mathbf{P}))}{B} . \quad \text{(cf. (2 : 7))} \end{array} \right\} \quad (2 : 8)$$

3. Upper Bound to the Number of $3x + d$ Cycles of a Given Oddlength.

According to the formulae (2:8(2,3)), the odd members of a T_d -cycle of the oddlength k satisfy the inequality

$$\forall j \in [1, k-1] , \quad n_j \leq \mathbf{W}_{d,k} = d \cdot \sup_{\substack{\mathbf{P} \in \mathbf{N}^k \\ |\mathbf{P}| \geq k \log_2 3}} \frac{a_k(\mathbf{P})}{2^{|\mathbf{P}|} - 3^k} = d \cdot \sup_{\ell \geq k \log_2 3} \frac{\max_{\substack{\mathbf{P} \in \mathbf{D}^k \\ |\mathbf{P}| = \ell}} a_k(\mathbf{P})}{2^\ell - 3^k} . \quad (3 : 1)$$

Simple calculations show that (cf. (1:9) above)

$$\mathbf{W}_{d,k} = d \cdot \left(\frac{3}{2}\right)^k \cdot \frac{2^{\epsilon_k}}{2^{\epsilon_k} - 1} , \quad \epsilon_k = \lceil k \cdot \log_2 3 \rceil - k \cdot \log_2 3 . \quad (3 : 2)$$

An analogue of this general upper bound and its more explicit version (1:11), have been, respectively, proved in [Belaga, Mignotte 1998] and refined in [Belaga, Mignotte 2000].

We will be able to improve these bounds thanks, first, to a more careful analysis of the formulae (2:8), and then, to a remarkable inequality (3:5) proved below, Theorem 3.1. Namely, instead of evaluating from above *all members* of a T_d -cycle of the oddlength k , we evaluate here its *minimal member* $n_0 = \text{prg}(\mathbf{C})$. Since the different T_d -cycles have different perigees $n_0 \in \mathbf{D}$, an upper bound $n_0 \leq \mathbf{V}'_{d,k}$ would imply the bound $\varsigma_k(d) \leq \frac{1}{3} \cdot \mathbf{V}'_{d,k}$ to the number of T_d -cycles of the oddlength k .

More formally, if $\mathbf{P} = \theta(\mathbf{C})$ (2:4) is the Collatz signature of the cycle $\mathbf{C} = \mathbf{C}[n_0, d]$, $n_0, d \in \mathbf{D}$, of the length $\ell = |\mathbf{P}|$ and oddlength $k \geq 1$, then, according to (2:8),

$$n_0 \leq \min\{n_0, n_1, \dots, n_{k-1}\} = d \cdot \frac{\min_{j \in [0, k-1]} \left\{ a_k(\sigma^j(\mathbf{P})) \right\}}{2^\ell - 3^k} . \quad (3 : 3)$$

Define for any k -tuple of positive integers \mathbf{P} its *average* $\bar{\mathbf{P}}$, the arithmetical mean of all its counterclockwise permutations. This k -tuple of positive (generally speaking, rational) numbers depends only on the dimension k and *length* $\ell = |\mathbf{P}|$ of \mathbf{P} :

$$\forall k \geq 1 \quad \forall \mathbf{P} = \{p_1, \dots, p_k\} \in \mathbf{N}^k, \quad \ell = |\mathbf{P}|, \quad \bar{\mathbf{P}} = \frac{1}{k} \sum_{j \in [0, k-1]} \sigma^j(\mathbf{P}) = \left\{ \frac{\ell}{k}, \dots, \frac{\ell}{k} \right\}. \quad (3:4)$$

Extending the definition of the function a_k (2:6) to k -tuples of positive reals, we will prove below (Theorem 3.2) the inequality

$$\forall k \geq 1 \quad \forall \mathbf{P} \in \mathbf{N}^k, \quad \tilde{a}_k(\mathbf{P}) = \min_{j \in [0, k-1]} \left\{ a_k(\sigma^j(\mathbf{P})) \right\} \leq a_k(\bar{\mathbf{P}}) = \frac{2^\ell - 3^k}{2^{\frac{\ell}{k}} - 3}. \quad (3:5)$$

The inequalities (3:3) and (3:5) imply the general upper bound (1:5), depending only on d and k , to the minimal member $n_0 = \text{prg}(\mathbf{C})$ of any T_d -cycle of the oddlength k :

$$\forall n, d \in \mathbf{D}, \quad \mathbf{C} = \mathbf{C}[n, d] \implies n \leq \mathbf{U}_{d, k} = \frac{d}{2^{\frac{\ell}{k}} - 3}, \quad (3:6)$$

and, finally, the upper bound (1:6).

Definition 3.1. (1) Let Λ be the set of pairs of positive integers (k, ℓ) satisfying the inequality implied by (2:5),

$$\Lambda = \left\{ (k, \ell) \in \mathbf{N}^2 \mid \lambda(k, \ell) = \ell - \lceil k \cdot \log_2 3 \rceil \geq 0 \right\}; \quad (3:7)$$

Extend the definition of the function $A = a_k$ (2:6) to k -tuples of positive reals from the $(k-1)$ -dimensional tetrahedron $\mathbf{T}_{k, \ell}$,

$$\forall (k, \ell) \in \Lambda, \quad \mathbf{T}_{k, \ell} = \left\{ \mathbf{X} \in \mathbf{R}^k \mid |\mathbf{X}| = x_1 + \dots + x_k = \ell \wedge \forall j \in [1, k], x_j \geq 1 \right\}, \quad (3:8)$$

with k vertices $\mathbf{V}_1, \dots, \mathbf{S}_k$,

$$\mathbf{V}_1 = \{\ell - k + 1, 1, \dots, 1\}, \mathbf{V}_2 = \{1, \ell - k + 1, \dots, 1\}, \dots, \mathbf{V}_k = \{1, 1, \dots, \ell - k + 1\} \quad (3:9)$$

(2) The permutation σ (2:7) induces on $\mathbf{T}_{k, \ell}$ the rotation σ , with the center \mathbf{O} of the tetrahedron being the only fixed point :

$$\left\{ \begin{array}{l} \forall \mathbf{X} = \langle x_1, x_2, \dots, x_{k-1}, x_k \rangle \in \mathbf{T}_{k, \ell}, \quad \sigma(\mathbf{X}) = \langle x_2, x_3, \dots, x_k, x_1 \rangle; \\ \bar{\mathbf{X}} = \frac{1}{k} \cdot \sum_{j \in [1, k]} \sigma^j(\mathbf{X}) = \left\langle \frac{\ell}{k}, \dots, \frac{\ell}{k} \right\rangle = \mathbf{O}, \quad \sigma(\mathbf{O}) = \mathbf{O}; \\ a_k(\mathbf{O}) = \sum_{j=1}^k 3^{k-j} \cdot 2^{\frac{\ell}{k}} = \frac{2^\ell - 3^k}{2^{\frac{\ell}{k}} - 3}. \end{array} \right. \quad (3:10)$$

Theorem 3.2. For any k -tuple \mathbf{X} from $\mathbf{T}_{k,\ell}$, the inequalities hold :

$$\forall \mathbf{X} \in \mathbf{T}_{k,\ell} \left\{ \begin{array}{l} \text{(1)} \quad \bar{a}_k(\mathbf{X}) = \frac{1}{k} \cdot \sum_{j=0}^{k-1} a_k(\sigma^j(\mathbf{X})) \geq a_k(\mathbf{O}) = \frac{2^\ell - 3^k}{2^{\frac{\ell}{k}} - 3} ; \\ \text{(2)} \quad \tilde{a}_k(\mathbf{X}) = \min_{j \in [0, k-1]} \{ a_k(\sigma^j(\mathbf{X})) \} \leq a_k(\mathbf{O}) = \frac{2^\ell - 3^k}{2^{\frac{\ell}{k}} - 3} , \end{array} \right. \quad (3 : 11)$$

with the equalities holding only in the case $\mathbf{X} = \mathbf{O}$.

4.Proof of Theorem 3.2.

Note that, according to (2:6), if $k = 1$, then $\ell \geq 2$ and

$$\forall \mathbf{X} \in \mathbf{T}_{1,\ell} , \quad \bar{\mathbf{X}} = \mathbf{X} = \mathbf{O} , \quad \bar{a}_k(\mathbf{X}) = \tilde{a}_k(\mathbf{X}) = \frac{2^\ell - 3}{2^\ell - 3} = 1 . \quad (4 : 1)$$

Thus, we can assume henceforth that $k \geq 2$.

(1) The inequality (3:10(1)) is implied by the standard inequality $\frac{1}{k}(a + b + \dots) \geq \sqrt[k]{a \cdot b \cdot \dots}$, as follows :

$$\begin{aligned} \forall k \geq 2 \quad \forall (k, \ell) \in \Lambda \quad \forall \mathbf{X} \in \mathbf{T}_{k,\ell} , \quad & \frac{1}{k} \sum_{0 \leq j \leq k-1} a_k(\sigma^j(\mathbf{X})) = \\ & 3^{k-1} + \sum_{1 \leq j \leq k-1} \frac{3^{k-j-1}}{k} \sum_{0 \leq r \leq k-1} 2^{\sigma^r(x_1 + \dots + x_j)} \geq \\ & 3^{k-1} + \sum_{1 \leq j \leq k-1} 3^{k-j-1} \cdot 2^{\frac{1}{k} \sum_{0 \leq r \leq k-1} \sigma^r(x_1 + \dots + x_j)} = \\ & a_k(\mathbf{O}) = \frac{2^\ell - 3^k}{2^{\frac{\ell}{k}} - 3} . \quad \text{cf. (3 : 10)} \end{aligned}$$

(2) If $\mathbf{X} = \mathbf{O} \in \mathbf{T}_{k,\ell}$, then (3:10(2)) becomes a trivial identity. Otherwise, $\mathbf{O} \neq \mathbf{X} \in \mathbf{T}_{k,\ell}$ ($k \geq 2$, $(k, \ell) \in \Lambda$), and between the k k -tuples $\sigma^j(\mathbf{X})$, $0 \leq j \leq k-1$, there exists at least two different ones :

$$\forall k \geq 2 \quad \forall (k, \ell) \in \Lambda \quad \forall \mathbf{X} \in \mathbf{T}_{k,\ell} \setminus \{\mathbf{O}\} \quad \left\{ \begin{array}{l} \forall j \in [0, k-1] , \quad \sigma^j(\mathbf{X}) \neq \bar{\mathbf{X}} = \mathbf{O} ; \\ \exists j \in [1, k-1] , \quad \mathbf{X} \neq \sigma^j(\mathbf{X}) . \end{array} \right. \quad (4 : 2)$$

Now the proof proceeds *ad absurdum* : the assumption $a_k(\sigma^j(\mathbf{X})) > a_k(\mathbf{O})$ for all $j \in [0, k-1]$ would imply that $a_k(\bar{\mathbf{X}}) > a_k(\mathbf{O})$ as well, – a contradiction, since $\bar{\mathbf{X}} = \mathbf{O}$ (3:10), (4:2).

The equation $a_k(\mathbf{X}) = a_k(\mathbf{O})$ induces a break up of the $(k - 1)$ -dimensional tetrahedron $\mathbf{T}_{k,\ell}$ (3:8) into three disjoint subsets : the closed $(k - 2)$ -dimensional submanifold $\mathbf{T}^0 = \mathbf{T}_{k,\ell}^0$ defined by this equation, and two $(k - 1)$ -dimensional and open in $\mathbf{T}_{k,\ell}$ submanifolds $\mathbf{T}^+ = \mathbf{T}_{k,\ell}^+$, and $\mathbf{T}^- = \mathbf{T}_{k,\ell}^-$, defined by the inequalities $a_k(\mathbf{X}) > a_k(\mathbf{O})$ and $a_k(\mathbf{X}) < a_k(\mathbf{O})$, respectively :

$$\forall k \geq 2 \quad \forall (k, \ell) \in \Lambda, \quad \left\{ \begin{array}{l} \mathbf{T}^0 = \mathbf{T}_{k,\ell}^0 = \left\{ \mathbf{X} \in \mathbf{T}_{k,\ell} \mid a_k(\mathbf{X}) = a_k(\mathbf{O}) = \frac{2^\ell - 3}{2^{\frac{\ell}{k}} - 3} \right\}; \\ \mathbf{T}^+ = \mathbf{T}_{k,\ell}^+ = \left\{ \mathbf{X} \in \mathbf{T}_{k,\ell} \mid a_k(\mathbf{X}) > a_k(\mathbf{O}) \right\}; \\ \mathbf{T}^- = \mathbf{T}_{k,\ell}^- = \left\{ \mathbf{X} \in \mathbf{T}_{k,\ell} \mid a_k(\mathbf{X}) < a_k(\mathbf{O}) \right\}. \end{array} \right. \quad (4 : 3)$$

We shall prove below the following properties of these three submanifolds :

(A) \mathbf{T}^0 is a smooth (in fact, analytical) submanifold.

(B) The submanifolds \mathbf{T}^0 , \mathbf{T}^+ , \mathbf{T}^- are connected and simply connected sets.

(C) The closed set $\mathbf{T}^{0+} = \mathbf{T}^0 \cup \mathbf{T}^+$ is strictly convex : the convex closure $\mathcal{P}(S)$ of a finite set S of k -tuples from \mathbf{T}^{0+} belongs to \mathbf{T}^+ , excluding, if necessary, the tuples from S belonging to \mathbf{T}^0 .

The last property immediately implies the validity of the above argument *ad absurdum*.

To prove (A-C), one needs to look at the first and second partial derivatives of the function $a_k(\mathbf{X})$:

$$\left\{ \begin{array}{l} \forall k \geq 2 \quad \forall (k, \ell) \in \Lambda \quad \forall \mathbf{X} = \langle x_1, \dots, x_k \rangle \in \mathbf{T}_{k,\ell}, \\ \mathbf{A}_k(\mathbf{X}) = \left\{ \frac{\partial a_k}{\partial x_1}, \dots, \frac{\partial a_k}{\partial x_{k-1}}, \frac{\partial a_k}{\partial x_k} \right\}(\mathbf{X}) = \\ \ln 2 \cdot \left\{ 2^{x_1} a_{k-1}(x_2, \dots, x_k), \dots, 2^{x_1 + \dots + x_{k-1}} a_1(x_k), 0 \right\}; \\ \frac{\partial \mathbf{A}_k(\mathbf{X})}{\partial x_i} = \left(\frac{\partial^2 a_k}{\partial x_i \partial x_j} \right)_{i,j \in [1,k]}(\mathbf{X}); \\ \forall i, j \in [1, k], \quad r = \max(i, j), \\ \frac{1}{(\ln 2)^2} \cdot \left(\frac{\partial^2 a_k}{\partial x_i \partial x_j} \right)(\mathbf{X}) = 2^{x_1 + \dots + x_r} a_{k-r}(x_{r+1}, \dots, x_k). \end{array} \right. \quad (4 : 4)$$

The properties (A,B) of the submanifolds \mathbf{T}^0 , \mathbf{T}^+ , \mathbf{T}^- are immediately implied by the character of the first derivative $\mathbf{A}_k(\mathbf{X})$. To prove the property (C), consider the second differential of the function a_k , the quadratic form

$$\left\{ \begin{array}{l} d^2 a_k(\mathbf{X}) = \sum_{i,j \in [1,k]} \frac{\partial^2 a_k}{\partial x_i \partial x_j}(\mathbf{X}) dx_i dx_j = \\ (\ln 2)^2 \sum_{r \in [1,k]} 2^{x_1 + \dots + x_r} a_{k-r}(x_{r+1}, \dots, x_k) \left(\sum_{i,j \in [1,r]} dx_i dx_j \right) = \\ (\ln 2)^2 \sum_{r \in [1,k]} 2^{x_1 + \dots + x_r} a_{k-r}(x_{r+1}, \dots, x_k) (dx_1 + \dots + dx_r)^2 > 0 . \end{array} \right. \quad (4 : 5)$$

End of the Proof.

References

[Baker, Wüstholz 1993]

Alan Baker, Gisbert Wüstholz [1993] : Logarithmic forms and group varieties, J. Reine Angew. Math. 442, 19–62.

[Belaga, Mignotte 1998]

Edward G. Belaga, Maurice Mignotte [1998] : Embedding the $3x+1$ Conjecture in a $3x+d$ Context, Experimental Mathematics 7:2, 145-151.

[Belaga, Mignotte 2000]

Edward G. Belaga, Maurice Mignotte [2000] : Cyclic Structure of Dynamical Systems Associated with $3x+d$ Extensions of Collatz Problem, Preprint 2000/018, Univ. Louis Pasteur, Strasbourg, <http://www-irma.u-strasbg.fr/irma/publications/2000/00018.ps.gz>.

[Lagarias 1985]

Jeffrey C. Lagarias [1985] : The $3x+1$ Problem and Its Generalizations, Amer. Math. Monthly 92, 3-23.

[Lagarias 1990]

Jeffrey C. Lagarias [1990] : The Set of Rational Cycles for the $3x+1$ Problem, Acta Arith. 56, 33-53.

[Rhin 1987]

Georges Rhin [1987] : Approximants de Padé et mesures effectives d'irrationalité, Séminaire de Théorie des Nombres, Paris 1985–86, 155–164, Progr. Math., 71, Birkhäuser Boston, Boston, MA, 1987

Günther J. Wirsching [1998] : The Dynamical System Generated by the $3n+1$ Function, LNM 1681, Springer, Berlin.