



HAL
open science

Effective Polynomial Upper Bounds to Perigees and Numbers of $(3x + d)$ -Cycles of a Given Oddlength..

Edward G. Belaga

► **To cite this version:**

Edward G. Belaga. Effective Polynomial Upper Bounds to Perigees and Numbers of $(3x + d)$ -Cycles of a Given Oddlength... 2000. <hal-00129652>

HAL Id: hal-00129652

<https://hal.science/hal-00129652v1>

Preprint submitted on 8 Feb 2007

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

**Effective Polynomial Upper Bounds to Perigees and
Numbers of $(3x + d)$ -Cycles of a Given Oddlength.**

Edward G. Belaga

*Université Louis Pasteur 7, rue René Descartes
F-67084 Strasbourg Cedex, FRANCE
e-mail : belaga@math.u-strasbg.fr*

Abstract. For any positive odd integer d not divisible by 3, the arithmetical function $T_d(m) = \begin{cases} \frac{3m+d}{2}, & \text{if } m \text{ is odd} \\ \frac{m}{2}, & \text{otherwise} \end{cases}$ generates on the set \mathbf{N} of natural numbers a dynamical system \mathcal{D}_d . The $3x + d$ hypothesis, generalizing the well-known $3x + 1$ conjecture, asserts that \mathcal{D}_d has a finite number of cycles and no divergent trajectories. We study here the cyclic structure of the system \mathcal{D}_d , and prove in particular an effective and sharp polynomial upper bound to the number of cycles in \mathcal{D}_d with a given number of odd members.

1991 Mathematical Subject Classification : Primary, 11K31, 11K38, 11K55; Secondary, 11B85.

Key words and phrases : $3x+1$ and $3x+d$ functions, $3x+1$ and $3x+d$ conjectures, divergent trajectory, cycle, perigee, length, oddlength, odd frame, Collatz signature of a cycle.

1. Introduction.

Let d be a positive odd integer not divisible by 3, and let T_d be the function defined on the set of positive integers, as follows :

$$\forall m \in \mathbf{N} \quad T_d(m) = \begin{cases} \frac{3m+d}{2}, & \text{if } m \text{ is odd ,} \\ \frac{m}{2}, & \text{otherwise .} \end{cases} \quad (1 : 1)$$

Repeated iterations of the function T_d generate $(3x + d)$ - (or T_d -) trajectories

$$\forall d \in \mathbf{D} = \{1, 5, 7, 11, 13, \dots\} \quad \forall m \in \mathbf{N} , \quad \tau_d(m) = \{m, T_d(m), T_d^2(m), \dots\} , \quad (1 : 2)$$

By definition, a trajectory $\tau_d(m)$ is a cycle of the length ℓ , $\mathbf{C} = \mathbf{C}(m, d) = \tau_d(m)$, $length(\mathbf{C}) = \ell$, if $T_d^\ell(m) = m$ and, for any $j \in [1, \ell - 1]$, $T_d^j(m) \neq m$ (note that $\ell > 1$, since the mapping T_d (1:1) has no fixed points). The minimal member of a T_d -cycle \mathbf{C} is odd, and is called its *perigee*, $n_0 = prg(\mathbf{C})$. Thus, the number k of odd members of a T_d -cycle, called here its *oddlength*, is a positive integer, $k \geq 1$. The length and oddlength of a cycle are related by the inequality $\ell \geq \lceil k \cdot \log_2 3 \rceil$ [Belaga, Mignotte 1998] (see Theorem 3.2(1) below). Note also that no member of a T_d -trajectory (1:2), excluding possibly the first one, is divisible by 3, and thus, all odd members of a T_d -cycle belong to \mathbf{D} .

It has been conjectured that the dynamical system $\mathcal{D}_d = \{\mathbf{N}, T_d\}$ has no divergent T_d -trajectories (1:2), and that the number $\varsigma(d)$ of cyclic T_d -trajectories is finite [Lagarias 1990], [Belaga, Mignotte 1998]. In the particular case $d = 1$, the well-known $3x + 1$ conjecture [Lagarias 1985], [Wirsching 1998] is even more specific : any trajectory $\tau_1(m)$ enters ultimately the (only) $3x + 1$ cycle $\{1 \rightarrow 2 \rightarrow 1\}$.

The present paper is concerned with the cyclic part of the above $3x + d$ conjecture, and more generally, with quantitative (and when available, numerical) characteristics of the cyclic structure of systems \mathcal{D}_d . Let $\mathcal{C}(d)$ and $\mathcal{C}_k(d)$ be the sets of all T_d -cycles and, respectively, of all such cycles with k odd members, or, in our terminology, of the oddlength $k \geq 1$.

Technically, our main result is the following general upper bound to the perigee of a T_d -cycle of the length ℓ and oddlength k :

$$\forall d \in \mathbf{D} \quad \forall \mathbf{C} \in \mathcal{C}(d) , \quad \left\{ \begin{array}{l} length(\mathbf{C}) = \ell, \\ oddlength(\mathbf{C}) = k, \end{array} \right\} \implies n_0 = prg(\mathbf{C}) \leq \frac{d}{2^{\frac{\ell}{k}} - 3} . \quad (1 : 3)$$

The inequality (1:3) has three important implications. The **first** one is an upper bound to the ratio of the length of a T_d -cycle to its oddlength, which, together with the well-known lower bound (2:5), Theorem 2.1(1), confines this ratio to the interval :

$$\forall d \in \mathbf{D} \quad \forall \mathbf{C} \in \mathcal{C}(d) , \quad \log_2 3 \leq \rho_d(\mathbf{C}) = \frac{length(\mathbf{C})}{oddlength(\mathbf{C})} \leq \log_2(d + 3) . \quad (1 : 4)$$

Second, the inequality (1:3) implies the following general and uniform upper bound to perigees of T_d -cycles of the oddlength $k \geq 1$:

$$\forall d \in \mathbf{D} \quad \forall k \in \mathbf{N} \quad \forall \mathbf{C} \in \mathcal{C}_k(d) \quad n_0 = prg(\mathbf{C}) \leq \mathbf{U}_{d,k} = \frac{d}{2^{\frac{\lceil k \cdot \log_2 3 \rceil}{k}} - 3} . \quad (1 : 5)$$

The bound (1:5) has an effective polynomial numerical equivalent (see the estimate (1:8) below). It is also sharp in the following natural sense (Theorem 3.2, (3:11(1))) : the average value of an odd member of a T_d -cycle of the oddlength $k \geq 1$ is bigger than $\mathbf{U}_{d,k}$. Thus, for example, the T_5 -cycle $\mathbf{C} = \{23 \rightarrow 37 \rightarrow 58 \rightarrow 29 \rightarrow 46\}$ has 3 odd members, $n_0 = prg(\mathbf{C}) = 23 < \mathbf{U}_{5,3} \approx 28.6038 < 29 < 37$.

Third, since no two T_d -cycles have a common member, any such cycle is fully determined by its perigee. Thus, the upper bound (1:5) not only implies that the set $\mathcal{C}_k(d)$ is finite, but supplies us with an effective general upper bound to the number $\varsigma_k(d) = \#\mathcal{C}_k(d)$ of T_d -cycles of the oddlength $k \geq 1$:

$$\forall d \in \mathbf{D} \quad \forall k \in \mathbf{N}, \quad \varsigma_k(d) \leq \frac{1}{3} \cdot \mathbf{U}_{d,k} = \frac{1}{3} \cdot \frac{d}{2^{\frac{[k \cdot \log_2 3]}{k}} - 3} \quad (1 : 6)$$

(the factor $\frac{1}{3}$ is due to the aforementioned inclusion $n_0 \in \mathbf{D}$).

Any numerical evaluation of the expression $\mathbf{U}_{d,k}$ depends on the state of our knowledge of effective *lower bounds* to diophantine approximations of linear combinations of logarithms $\log 2$ and $\log 3$. The best known at present lower bound belongs to [Rhin 1987], whose techniques lacks the generality of Baker's method (cf. [Baker, Wüstholz 1993] and the references there), but is more effective in the evaluation of linear forms of a few specific logarithms, including our case :

$$\forall r, s \in \mathbf{Z}, \quad \begin{cases} t = \max(|r|, |s|), \\ |r + s \log 2 + \log 3| > t^{-13.3}, \end{cases} \quad (1 : 7)$$

The inequality (1:7) implies the following effective polynomial upper bound to $\mathbf{U}_{d,k}$:

$$\forall d \in \mathbf{D} \quad \forall k \in \mathbf{N}, \quad \mathbf{U}_{d,k} \leq 102 \cdot d \cdot k^{14.3}. \quad (1 : 8)$$

Historical Remarks. The present author is not aware of any previous effective (and in any sense sharp) upper bound to the minimal odd member of a T_d -cycle. The following general exponential upper bound to the number $\varsigma_k(d)$ of T_d -cycles of the oddlength $k \geq 1$ has been actually proved in [Belaga, Mignotte 1998] (even if not explicitly articulated) and refined in [Belaga, Mignotte 2000] :

$$\forall d \in \mathbf{D} \quad \forall k \in \mathbf{N}, \quad \varsigma_k(d) < d \cdot \left(\frac{3}{2}\right)^k \cdot \frac{2^{\epsilon_k}}{2^{\epsilon_k} - 1}, \quad \epsilon_k = [k \cdot \log_2 3] - k \cdot \log_2 3. \quad (1 : 9)$$

The bound (1:9) has been derived from an identical upper bound to the *maximal odd member* of a cycle, the corresponding numerical upper bound being based on the estimate of [Baker, Wüstholz 1993] :

$$\forall d \in \mathbf{D} \quad \forall k \in \mathbf{N}, \quad \varsigma_k(d) < d \cdot k^C \cdot \left(\frac{3}{2}\right)^k, \quad (1 : 10)$$

with an effective but enormous constant C . Refining the proof of the inequality (1:9) and applying the estimate of [Rhin 1987] to (1:9), [Belaga, Mignotte 2000] have proved a better, but still exponential effective upper bound

$$\forall d \in \mathbf{D} \quad \forall k \in \mathbf{N}, \quad \varsigma_k(d) \leq 204 \cdot d \cdot \left(\frac{3}{2}\right)^k \cdot k^{13.3}. \quad (1 : 11)$$

Comments and Future Prospects. (1) The upper bound (1:4) to the ratio $\rho_d(\mathbf{C})$ implies in the $3x + 1$ case that the length of a cycle with k odd members does not exceed

$2k$. Note that the only known at present $3x+1$ cycle $\{1 \rightarrow 2 \rightarrow 1\}$ has one odd member, is of the length two, and has ratio two. More generally, the series of T_d -cycles of the length $\ell \in \mathbf{N}$, $d = 2^\ell - 3$, $n_0 = 1$ and of the oddlength $k = 1$ demonstrates that the upper bound (1:4) is sharp.

(2) The bounds (1:5) and, especially, (1:6) can be apparently improved. In fact, the experimental discovery of 843 T_{14303} -cycles of the oddlength 17, with perigees varying from 385057 to $1391321 < \mathbf{U}_{14303,17} = 2099280$, suggests that the bound (1:5) is apparently sharp up to a one-digit constant, whereas the bound (1:6), $\varsigma_{17}(14303) = 843 < \frac{1}{3}\mathbf{U}_{14303,17} = 699760$, far from being sharp, is at least realistic : for some d, k , the dynamical system \mathcal{D}_d has “many” cycles of the oddlength k .

(3) At present, the bounds (1:3,4,6) look useless, or at least insufficient, for an eventual proof of the cyclic part of the $3x+d$ conjecture, i. e., of the finiteness of the number $\varsigma(d)$ of T_d -cycles. At its best, the straightforward application of these bounds yields the trivial infinite upper bound :

$$\forall d \in \mathbf{D} \quad \varsigma(d) = \sum_{k \geq 1} \varsigma_k(d) \leq 102 \cdot d \cdot \sum_{k \geq 1} k^{14.3} = \infty . \quad (1 : 12)$$

However, the obstacle of the infinite summation in (1:12) could be possibly circumvented by a refinement of the above scheme, to fit the purpose of yielding directly an *absolute* (i. e., not depending on k) upper bound to the number $\varsigma(d)$.

Acknowledgments. The author is grateful to Professor Maurice Mignotte for the permission to use here the aforementioned experimental results of our common project [Belaga, Mignotte 2000].

2. Exponential Diophantine Formulae for $3x+d$ Cycles.

Let, as above, $\mathbf{C} = \mathbf{C}(m, d) = \tau_d(m)$ (1:2) be a T_d -cycle of the length ℓ , $length(\mathbf{C}) = \ell$. We remind the reader that, according to (1:1), the minimal member, or *perigee* of a T_d -cycle, $n_0 = prg(\mathbf{C})$, is odd, and that the total number $k \geq 1$ of odd members of a cycle is called its *oddlength*. Moreover, if n is an odd member of a cycle, then $n \in \mathbf{D}$ (1:2), since no number divisible by 3 can belong to a cycle.

Note that if $m' \neq m$ is a member of a T_d -cycle $\mathbf{C} = \tau_d(m)$, or in other words, if \mathbf{C} *meets* m' , one should view $\mathbf{C}' = \mathbf{C}(m', d)$ as just another name for the same cycle $\mathbf{C} = \mathbf{C}(m, d)$. Since a T_d -cycle is fully characterized by its minimal member, the following notation can be accepted as the canonical one :

$$\mathbf{C} = \mathbf{C}(n_0, d) = \mathbf{C}[n_0, d] = \tau_d(n_0), \quad n_0 = prg(\mathbf{C}) . \quad (2 : 1)$$

In this case, we also say that \mathbf{C} *starts at* n_0 .

For any positive integer $m \in \mathbf{N}$, let $odd(m)$ be the number obtained by factoring out of m the highest possible power of 2, say 2^j , and let $\nu_2(m) = j$. Thus $odd(m)$ is odd and $m = odd(m) \cdot 2^{\nu_2(m)}$. Define

$$S : \mathbf{D} \times \mathbf{D} \rightarrow \mathbf{D} ; \quad \forall n, d \in \mathbf{D} . \quad S_d(n) = \text{odd}(3n + d) . \quad (2 : 2)$$

The function S_d speeds up the action of T_d (1:1), skipping even members of T_d -trajectories. In particular, $m = 1$ becomes the fixed point of the function $S_1 = \text{odd}(3n + 1)$, $S_1(1) = 1$, corresponding to the (according to the $3x + 1$ conjecture, only) T -cycle $\mathbf{C}(1, 1) = \{1 \rightarrow 2 \rightarrow 1\}$.

We associate with any T_d -cycle $\mathbf{C} = \mathbf{C}[n_0, d]$ its odd frame, $\mathbf{F} = \text{Odd}(\mathbf{C})$, the list of odd members of the cycle, in the order of their appearance in $\tau_d(n_0)$ (1:2), as the T_d -iterations of n_0 proceed. By definition, the frame is a S_d -cycle (2:2) starting at n_0 , and its length is called the *oddlength* of the cycle \mathbf{C} :

$$\left\{ \begin{array}{l} \forall \mathbf{C} = \mathbf{C}[n_0, d] , \quad n_0, d \in \mathbf{D} , \\ \ell = \text{length}(\mathbf{C}) = \min \{i \in \mathbf{N} \mid m_i = T_d^i(n_0) = n_0\} ; \\ k = \text{oddlength}(\mathbf{C}) = \min \{j \in \mathbf{N} \mid n_j = S_d^j(n_0) = n_0\} ; \\ \mathbf{F} = \text{Odd}(\mathbf{C}) = \langle n_0, n_1, \dots, n_{k-1} \rangle \in \mathbf{D}^k . \end{array} \right. \quad (2 : 3)$$

The even members of the T_d -cycle $\mathbf{C} = \mathbf{C}[n_0, d]$ can be recovered from its frame with the help of the cycle *Collatz signature* $\mathbf{P} = \theta(\mathbf{C})$, the vector of exponents of 2 factoring out from the values of the function T_d (1:1) at odd members of \mathbf{C} , as follows :

$$\left\{ \begin{array}{l} \forall \mathbf{C} = \mathbf{C}[n_0, d] , \quad n_0, d \in \mathbf{D} , \\ \mathbf{F} = \text{Odd}(\mathbf{C}) = \langle n_0, n_1, \dots, n_{k-1} \rangle ; \\ p_1 = \nu_2(T_d(n_0)), \quad p_2 = \nu_2(T_d(n_1)), \quad \dots, \quad p_k = \nu_2(T_d(n_{k-1})) ; \\ \mathbf{P} = \theta(\mathbf{C}) = \langle p_1, p_2, \dots, p_k \rangle \in \mathbf{N}^k ; \\ \ell = \text{length}(\mathbf{C}) = |\mathbf{P}| = p_1 + p_2 + \dots + p_k ; \\ \forall j \in [1, k-1] , \quad \left\{ \begin{array}{l} m_{p_1 + \dots + p_j} = n_j ; \\ p_j > 1 \implies \forall i \in [1, p_j - 1] , \quad m_{p_1 + \dots + p_j - i} = 2^i \cdot n_j . \end{array} \right. \end{array} \right. \quad (2 : 4)$$

Moreover, the Collatz signature $\mathbf{P} = \theta(\mathbf{C})$ of a cycle $\mathbf{C} = \mathbf{C}[n_0, d]$, $n_0, d \in \mathbf{D}$, completely characterizes it :

Theorem 2.1. [Belaga, Mignotte 1998] **(1)** The Collatz signature $\mathbf{P} = \theta(\mathbf{C})$ satisfies the inequality :

$$\ell = |\mathbf{P}| = p_1 + \dots + p_k \geq \lceil k \cdot \log_2 3 \rceil . \quad (2 : 5)$$

(2) Define the exponential diophantine function $A = a_k : \mathbf{N}^k \longrightarrow \mathbf{N}$, as follows :

$$\left\{ \begin{array}{l} \forall k \geq 1 \quad \forall \mathbf{P} = \langle p_1, \dots, p_k \rangle \in \mathbf{N}^k , \\ A = a_k(\mathbf{P}) = \left\{ \begin{array}{ll} 1, & \text{if } k = 1 ; \\ 3^{k-1} + 2^{p_1} \cdot 3^{k-2} + \dots + 2^{p_1 + \dots + p_{k-2}} \cdot 3 + 2^{p_1 + \dots + p_{k-1}}, & \text{otherwise} . \end{array} \right. \end{array} \right. \quad (2 : 6)$$

Let $\sigma = \sigma_k$ be the circular (counterclockwise) permutation on k -tuples :

$$\forall \mathbf{P} = \langle p_1, \dots, p_k \rangle \in \mathbf{N}^k . \quad \sigma(\mathbf{P}) = \sigma_k(\mathbf{P}) = \langle p_2, \dots, p_k, p_1 \rangle. \quad (2 : 7)$$

If now $\mathbf{P} = \theta(\mathbf{C})$ is the Collatz signature of a cycle $\mathbf{C} = \mathbf{C}[n_0, d]$, $n_0, d \in \mathbf{D}$ of the length ℓ , oddlength $k \geq 1$, and with the frame $\mathbf{F} = \langle n_0, n_1, \dots, n_{k-1} \rangle$, then

$$\left\{ \begin{array}{ll} \text{(1)} & B = b_k(\mathbf{P}) = B_{k,\ell} = 2^\ell - 3^k > 0 ; \quad (\text{cf. (2 : 5)}) \\ \text{(2)} & n_0 = d \cdot \frac{A}{B}, \quad A = a_k(\mathbf{P}) ; \quad (\text{cf. (2 : 6)}) \\ \text{(3)} & \forall j \in [1, k-1], \quad n_j = d \cdot \frac{a_k(\sigma^j(\mathbf{P}))}{B}. \quad (\text{cf. (2 : 7)}) \end{array} \right\} \quad (2 : 8)$$

3. Upper Bound to the Number of $3x + d$ Cycles of a Given Oddlength.

According to the formulae (2:8(2,3)), the odd members of a T_d -cycle of the oddlength k satisfy the inequality

$$\forall j \in [1, k-1], \quad n_j \leq \mathbf{W}_{d,k} = d \cdot \sup_{\substack{\mathbf{P} \in \mathbf{N}^k \\ |\mathbf{P}| \geq k \log_2 3}} \frac{a_k(\mathbf{P})}{2^{|\mathbf{P}|} - 3^k} = d \cdot \sup_{\ell \geq k \log_2 3} \frac{\max_{\substack{\mathbf{P} \in \mathbf{D}^k \\ |\mathbf{P}| = \ell}} a_k(\mathbf{P})}{2^\ell - 3^k}. \quad (3 : 1)$$

Simple calculations show that (cf. (1:9) above)

$$\mathbf{W}_{d,k} = d \cdot \left(\frac{3}{2}\right)^k \cdot \frac{2^{\epsilon_k}}{2^{\epsilon_k} - 1}, \quad \epsilon_k = \lceil k \cdot \log_2 3 \rceil - k \cdot \log_2 3. \quad (3 : 2)$$

An analogue of this general upper bound and its more explicit version (1:11), have been, respectively, proved in [Belaga, Mignotte 1998] and refined in [Belaga, Mignotte 2000].

We will be able to improve these bounds thanks, first, to a more careful analysis of the formulae (2:8), and then, to a remarkable inequality (3:5) proved below, Theorem 3.1. Namely, instead of evaluating from above *all members* of a T_d -cycle of the oddlength k , we evaluate here its *minimal member* $n_0 = \text{prg}(\mathbf{C})$. Since the different T_d -cycles have different perigees $n_0 \in \mathbf{D}$, an upper bound $n_0 \leq \mathbf{V}'_{d,k}$ would imply the bound $\varsigma_k(d) \leq \frac{1}{3} \cdot \mathbf{V}'_{d,k}$ to the number of T_d -cycles of the oddlength k .

More formally, if $\mathbf{P} = \theta(\mathbf{C})$ (2:4) is the Collatz signature of the cycle $\mathbf{C} = \mathbf{C}[n_0, d]$, $n_0, d \in \mathbf{D}$, of the length $\ell = |\mathbf{P}|$ and oddlength $k \geq 1$, then, according to (2:8),

$$n_0 \leq \min\{n_0, n_1, \dots, n_{k-1}\} = d \cdot \frac{\min_{j \in [0, k-1]} \left\{ a_k(\sigma^j(\mathbf{P})) \right\}}{2^\ell - 3^k}. \quad (3 : 3)$$

Define for any k -tuple of positive integers \mathbf{P} its *average* $\bar{\mathbf{P}}$, the arithmetical mean of all its counterclockwise permutations. This k -tuple of positive (generally speaking, rational) numbers depends only on the dimension k and *length* $\ell = |\mathbf{P}|$ of \mathbf{P} :

$$\forall k \geq 1 \quad \forall \mathbf{P} = \{p_1, \dots, p_k\} \in \mathbf{N}^k, \quad \ell = |\mathbf{P}|, \quad \bar{\mathbf{P}} = \frac{1}{k} \sum_{j \in [0, k-1]} \sigma^j(\mathbf{P}) = \left\{ \frac{\ell}{k}, \dots, \frac{\ell}{k} \right\}. \quad (3:4)$$

Extending the definition of the function a_k (2:6) to k -tuples of positive reals, we will prove below (Theorem 3.2) the inequality

$$\forall k \geq 1 \quad \forall \mathbf{P} \in \mathbf{N}^k, \quad \tilde{a}_k(\mathbf{P}) = \min_{j \in [0, k-1]} \left\{ a_k(\sigma^j(\mathbf{P})) \right\} \leq a_k(\bar{\mathbf{P}}) = \frac{2^\ell - 3^k}{2^{\frac{\ell}{k}} - 3}. \quad (3:5)$$

The inequalities (3:3) and (3:5) imply the general upper bound (1:5), depending only on d and k , to the minimal member $n_0 = \text{prg}(\mathbf{C})$ of any T_d -cycle of the oddlength k :

$$\forall n, d \in \mathbf{D}, \quad \mathbf{C} = \mathbf{C}[n, d] \implies n \leq \mathbf{U}_{d, k} = \frac{d}{2^{\frac{\ell}{k}} - 3}, \quad (3:6)$$

and, finally, the upper bound (1:6).

Definition 3.1. (1) Let Λ be the set of pairs of positive integers (k, ℓ) satisfying the inequality implied by (2:5),

$$\Lambda = \left\{ (k, \ell) \in \mathbf{N}^2 \mid \lambda(k, \ell) = \ell - \lceil k \cdot \log_2 3 \rceil \geq 0 \right\}; \quad (3:7)$$

Extend the definition of the function $A = a_k$ (2:6) to k -tuples of positive reals from the $(k-1)$ -dimensional tetrahedron $\mathbf{T}_{k, \ell}$,

$$\forall (k, \ell) \in \Lambda, \quad \mathbf{T}_{k, \ell} = \left\{ \mathbf{X} \in \mathbf{R}^k \mid |\mathbf{X}| = x_1 + \dots + x_k = \ell \wedge \forall j \in [1, k], x_j \geq 1 \right\}, \quad (3:8)$$

with k vertices $\mathbf{V}_1, \dots, \mathbf{S}_k$,

$$\mathbf{V}_1 = \{\ell - k + 1, 1, \dots, 1\}, \quad \mathbf{V}_2 = \{1, \ell - k + 1, \dots, 1\}, \dots, \quad \mathbf{V}_k = \{1, 1, \dots, \ell - k + 1\} \quad (3:9)$$

(2) The permutation σ (2:7) induces on $\mathbf{T}_{k, \ell}$ the rotation σ , with the center \mathbf{O} of the tetrahedron being the only fixed point :

$$\left\{ \begin{array}{l} \forall \mathbf{X} = \langle x_1, x_2, \dots, x_{k-1}, x_k \rangle \in \mathbf{T}_{k, \ell}, \quad \sigma(\mathbf{X}) = \langle x_2, x_3, \dots, x_k, x_1 \rangle; \\ \bar{\mathbf{X}} = \frac{1}{k} \cdot \sum_{j \in [1, k]} \sigma^j(\mathbf{X}) = \left\langle \frac{\ell}{k}, \dots, \frac{\ell}{k} \right\rangle = \mathbf{O}, \quad \sigma(\mathbf{O}) = \mathbf{O}; \\ a_k(\mathbf{O}) = \sum_{j=1}^k 3^{k-j} \cdot 2^{\frac{\ell}{k}} = \frac{2^\ell - 3^k}{2^{\frac{\ell}{k}} - 3}. \end{array} \right. \quad (3:10)$$

Theorem 3.2. For any k -tuple \mathbf{X} from $\mathbf{T}_{k,\ell}$, the inequalities hold :

$$\forall \mathbf{X} \in \mathbf{T}_{k,\ell} \left\{ \begin{array}{l} \text{(1)} \quad \bar{a}_k(\mathbf{X}) = \frac{1}{k} \cdot \sum_{j=0}^{k-1} a_k(\sigma^j(\mathbf{X})) \geq a_k(\mathbf{O}) = \frac{2^\ell - 3^k}{2^{\frac{\ell}{k}} - 3} ; \\ \text{(2)} \quad \tilde{a}_k(\mathbf{X}) = \min_{j \in [0, k-1]} \left\{ a_k(\sigma^j(\mathbf{X})) \right\} \leq a_k(\mathbf{O}) = \frac{2^\ell - 3^k}{2^{\frac{\ell}{k}} - 3} , \end{array} \right. \quad (3 : 11)$$

with the equalities holding only in the case $\mathbf{X} = \mathbf{O}$.

4.Proof of Theorem 3.2.

Note that, according to (2:6), if $k = 1$, then $\ell \geq 2$ and

$$\forall \mathbf{X} \in \mathbf{T}_{1,\ell} , \quad \bar{\mathbf{X}} = \mathbf{X} = \mathbf{O} , \quad \bar{a}_k(\mathbf{X}) = \tilde{a}_k(\mathbf{X}) = \frac{2^\ell - 3}{2^\ell - 3} = 1 . \quad (4 : 1)$$

Thus, we can assume henceforth that $k \geq 2$.

(1) The inequality (3:10(1)) is implied by the standard inequality $\frac{1}{k}(a + b + \dots) \geq \sqrt[k]{a \cdot b \cdot \dots}$, as follows :

$$\begin{aligned} \forall k \geq 2 \quad \forall (k, \ell) \in \Lambda \quad \forall \mathbf{X} \in \mathbf{T}_{k,\ell} , \quad & \frac{1}{k} \sum_{0 \leq j \leq k-1} a_k(\sigma^j(\mathbf{X})) = \\ & 3^{k-1} + \sum_{1 \leq j \leq k-1} \frac{3^{k-j-1}}{k} \sum_{0 \leq r \leq k-1} 2^{\sigma^r(x_1 + \dots + x_j)} \geq \\ & 3^{k-1} + \sum_{1 \leq j \leq k-1} 3^{k-j-1} \cdot 2^{\frac{1}{k} \sum_{0 \leq r \leq k-1} \sigma^r(x_1 + \dots + x_j)} = \\ & a_k(\mathbf{O}) = \frac{2^\ell - 3^k}{2^{\frac{\ell}{k}} - 3} . \quad \text{cf. (3 : 10)} \end{aligned}$$

(2) If $\mathbf{X} = \mathbf{O} \in \mathbf{T}_{k,\ell}$, then (3:10(2)) becomes a trivial identity. Otherwise, $\mathbf{O} \neq \mathbf{X} \in \mathbf{T}_{k,\ell}$ ($k \geq 2$, $(k, \ell) \in \Lambda$), and between the k k -tuples $\sigma^j(\mathbf{X})$, $0 \leq j \leq k-1$, there exists at least two different ones :

$$\forall k \geq 2 \quad \forall (k, \ell) \in \Lambda \quad \forall \mathbf{X} \in \mathbf{T}_{k,\ell} \setminus \{\mathbf{O}\} \quad \left\{ \begin{array}{l} \forall j \in [0, k-1] , \quad \sigma^j(\mathbf{X}) \neq \bar{\mathbf{X}} = \mathbf{O} ; \\ \exists j \in [1, k-1] , \quad \mathbf{X} \neq \sigma^j(\mathbf{X}) . \end{array} \right. \quad (4 : 2)$$

Now the proof proceeds *ad absurdum* : the assumption $a_k(\sigma^j(\mathbf{X})) > a_k(\mathbf{O})$ for all $j \in [0, k-1]$ would imply that $a_k(\bar{\mathbf{X}}) > a_k(\mathbf{O})$ as well, – a contradiction, since $\bar{\mathbf{X}} = \mathbf{O}$ (3:10), (4:2).

The equation $a_k(\mathbf{X}) = a_k(\mathbf{O})$ induces a break up of the $(k - 1)$ -dimensional tetrahedron $\mathbf{T}_{k,\ell}$ (3:8) into three disjoint subsets : the closed $(k - 2)$ -dimensional submanifold $\mathbf{T}^0 = \mathbf{T}_{k,\ell}^0$ defined by this equation, and two $(k - 1)$ -dimensional and open in $\mathbf{T}_{k,\ell}$ submanifolds $\mathbf{T}^+ = \mathbf{T}_{k,\ell}^+$, and $\mathbf{T}^- = \mathbf{T}_{k,\ell}^-$, defined by the inequalities $a_k(\mathbf{X}) > a_k(\mathbf{O})$ and $a_k(\mathbf{X}) < a_k(\mathbf{O})$, respectively :

$$\forall k \geq 2 \quad \forall (k, \ell) \in \Lambda, \quad \left\{ \begin{array}{l} \mathbf{T}^0 = \mathbf{T}_{k,\ell}^0 = \left\{ \mathbf{X} \in \mathbf{T}_{k,\ell} \mid a_k(\mathbf{X}) = a_k(\mathbf{O}) = \frac{2^\ell - 3}{2^{\frac{\ell}{k}} - 3} \right\}; \\ \mathbf{T}^+ = \mathbf{T}_{k,\ell}^+ = \left\{ \mathbf{X} \in \mathbf{T}_{k,\ell} \mid a_k(\mathbf{X}) > a_k(\mathbf{O}) \right\}; \\ \mathbf{T}^- = \mathbf{T}_{k,\ell}^- = \left\{ \mathbf{X} \in \mathbf{T}_{k,\ell} \mid a_k(\mathbf{X}) < a_k(\mathbf{O}) \right\}. \end{array} \right. \quad (4 : 3)$$

We shall prove below the following properties of these three submanifolds :

(A) \mathbf{T}^0 is a smooth (in fact, analytical) submanifold.

(B) The submanifolds \mathbf{T}^0 , \mathbf{T}^+ , \mathbf{T}^- are connected and simply connected sets.

(C) The closed set $\mathbf{T}^{0+} = \mathbf{T}^0 \cup \mathbf{T}^+$ is strictly convex : the convex closure $\mathcal{P}(S)$ of a finite set S of k -tuples from \mathbf{T}^{0+} belongs to \mathbf{T}^+ , excluding, if necessary, the tuples from S belonging to \mathbf{T}^0 .

The last property immediately implies the validity of the above argument *ad absurdum*.

To prove (A-C), one needs to look at the first and second partial derivatives of the function $a_k(\mathbf{X})$:

$$\left\{ \begin{array}{l} \forall k \geq 2 \quad \forall (k, \ell) \in \Lambda \quad \forall \mathbf{X} = \langle x_1, \dots, x_k \rangle \in \mathbf{T}_{k,\ell}, \\ \mathbf{A}_k(\mathbf{X}) = \left\{ \frac{\partial a_k}{\partial x_1}, \dots, \frac{\partial a_k}{\partial x_{k-1}}, \frac{\partial a_k}{\partial x_k} \right\}(\mathbf{X}) = \\ \ln 2 \cdot \left\{ 2^{x_1} a_{k-1}(x_2, \dots, x_k), \dots, 2^{x_1 + \dots + x_{k-1}} a_1(x_k), 0 \right\}; \\ \frac{\partial \mathbf{A}_k(\mathbf{X})}{\partial x_i} = \left(\frac{\partial^2 a_k}{\partial x_i \partial x_j} \right)_{i,j \in [1,k]}(\mathbf{X}); \\ \forall i, j \in [1, k], \quad r = \max(i, j), \\ \frac{1}{(\ln 2)^2} \cdot \left(\frac{\partial^2 a_k}{\partial x_i \partial x_j} \right)(\mathbf{X}) = 2^{x_1 + \dots + x_r} a_{k-r}(x_{r+1}, \dots, x_k). \end{array} \right. \quad (4 : 4)$$

The properties (A,B) of the submanifolds \mathbf{T}^0 , \mathbf{T}^+ , \mathbf{T}^- are immediately implied by the character of the first derivative $\mathbf{A}_k(\mathbf{X})$. To prove the property (C), consider the second differential of the function a_k , the quadratic form

$$\left\{ \begin{array}{l} d^2 a_k(\mathbf{X}) = \sum_{i,j \in [1,k]} \frac{\partial^2 a_k}{\partial x_i \partial x_j}(\mathbf{X}) dx_i dx_j = \\ (\ln 2)^2 \sum_{r \in [1,k]} 2^{x_1 + \dots + x_r} a_{k-r}(x_{r+1}, \dots, x_k) \left(\sum_{i,j \in [1,r]} dx_i dx_j \right) = \\ (\ln 2)^2 \sum_{r \in [1,k]} 2^{x_1 + \dots + x_r} a_{k-r}(x_{r+1}, \dots, x_k) (dx_1 + \dots + dx_r)^2 > 0 . \end{array} \right. \quad (4 : 5)$$

End of the Proof.

References

[Baker, Wüstholz 1993]

Alan Baker, Gisbert Wüstholz [1993] : Logarithmic forms and group varieties, J. Reine Angew. Math. 442, 19–62.

[Belaga, Mignotte 1998]

Edward G. Belaga, Maurice Mignotte [1998] : Embedding the $3x+1$ Conjecture in a $3x+d$ Context, Experimental Mathematics 7:2, 145-151.

[Belaga, Mignotte 2000]

Edward G. Belaga, Maurice Mignotte [2000] : Cyclic Structure of Dynamical Systems Associated with $3x+d$ Extensions of Collatz Problem, Preprint 2000/018, Univ. Louis Pasteur, Strasbourg, <http://www-irma.u-strasbg.fr/irma/publications/2000/00018.ps.gz>.

[Lagarias 1985]

Jeffrey C. Lagarias [1985] : The $3x+1$ Problem and Its Generalizations, Amer. Math. Monthly 92, 3-23.

[Lagarias 1990]

Jeffrey C. Lagarias [1990] : The Set of Rational Cycles for the $3x+1$ Problem, Acta Arith. 56, 33-53.

[Rhin 1987]

Georges Rhin [1987] : Approximants de Padé et mesures effectives d'irrationalité, Séminaire de Théorie des Nombres, Paris 1985–86, 155–164, Progr. Math., 71, Birkhäuser Boston, Boston, MA, 1987

Günther J. Wirsching [1998] : The Dynamical System Generated by the $3n+1$ Function, LNM 1681, Springer, Berlin.