



**HAL**  
open science

# Multi-geodesic tessellations, fractional Dehn twists and uniformization of algebraic curves

Samuel Lelièvre, Robert Silhol

► **To cite this version:**

Samuel Lelièvre, Robert Silhol. Multi-geodesic tessellations, fractional Dehn twists and uniformization of algebraic curves. 2007. hal-00129643

**HAL Id: hal-00129643**

**<https://hal.science/hal-00129643v1>**

Preprint submitted on 13 Feb 2007

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# MULTI-GEODESIC TESSELLATIONS, FRACTIONAL DEHN TWISTS AND UNIFORMIZATION OF ALGEBRAIC CURVES

SAMUEL LE LIÈVRE, ROBERT SILHOL

Université Paris-Sud Orsay, Université Montpellier II  
samuel.lelievre@math.u-psud.fr, rs@math.univ-montp2.fr

## 0. Introduction.

To illustrate what this paper is about we first consider a classical example of a translation surface. Consider two copies of a regular Euclidean pentagon as on the left of figure 1. Identifying opposite parallel sides by translations one obtains a Riemann surface with a natural locally flat metric with one cone type point of total angle  $6\pi$ . Since this is a compact Riemann surface it is also an algebraic curve and in this case it is well known that this is the curve defined by  $y^2 = x^5 - 1$ . Since the surface is of genus 2 it is also a hyperbolic surface. The hyperbolic metric can easily be recovered by taking two copies of a regular hyperbolic pentagon (with interior angle  $\pi/5$ ) in the unit disk, attached as on the right of figure 1 and with opposite sides identified by appropriate hyperbolic transformations (this construction is adapted from the one given in [Ku-Nä]).

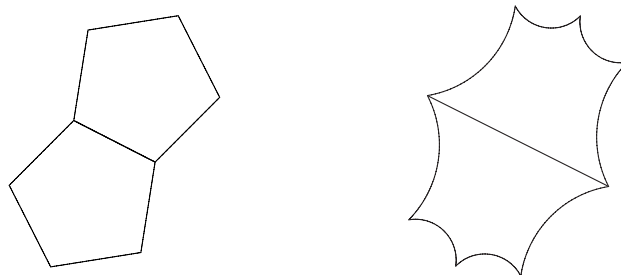


Fig. 1

The important point here is that in fact these two decompositions into two regular pentagons coincide exactly or otherwise said the pentagons are geodesic for both the locally flat metric defined above and the natural hyperbolic metric. In this case this follows easily from the fact that the edges of the pentagons are fixed points of anti-conformal reflections.

This situation is unfortunately far from being the generic one and in general two different metrics on a surface have no geodesic arcs in common. There are however infinitely many families for which we have a decomposition into polygons, geodesic

for two or more metrics. This is in particular the case for surfaces obtained by taking finitely many copies of an Euclidean rectangle and identifying edges of the rectangles pairwise, using translations or rotations of angle  $\pi$  (half-turns). These are the surfaces we will explore in this paper.

The existence of a multi-geodesic tessellation on such surfaces has interesting consequences. One of these is that it provides a mechanical way to reconstruct a Fuchsian group for the surface (see Propositions **3.2** and **3.3**).

Another consequence is that it allows for a description in terms of Fenchel-Nielsen coordinates of the Teichmüller disk generated by a surface tiled by squares (see Proposition **3.6**). Moreover such a description allows for an interpretation in terms of fractional Dehn twists of the natural  $\mathrm{PSL}_2(\mathbb{Z})$  action on the  $\mathrm{PSL}_2(\mathbb{R})$ -orbit of such surfaces, Corollary **3.7** and section **5**, where there are also examples of fractional Dehn twists that connect surfaces in different  $\mathrm{PSL}_2(\mathbb{R})$ -orbits, or even in different strata. These follow from the fact that actually not only is the tessellation multi-geodesic but the medians of the rectangles (vertical or horizontal) extend to simple closed curves that are geodesic for the different metrics.

Finally we note that in many cases one can use the tiling by rectangles to recover an equation for the corresponding algebraic curve (see sections **4** and **5**). Hence the tiling by rectangles provides a bridge between the algebraic equation and the hyperbolic structure deduced from the multi-geodesic tessellation. In other words this solves the uniformization problem for such curves. In fact it also gives a scheme to do uniformization for infinitely many families of curves (see section **4** for some examples). Finally in section **6** and the Appendix we discuss some number theoretic aspects.

The authors would like to thank Hugo Akrouit and Peter Buser for many useful discussions. The authors would also like to thank Le centre Bernoulli in Lausanne for its hospitality while working on this paper.

## 1. Multi-geodesics in genus 1 and genus 0.

Let  $G_1$  be discrete subgroup of  $\mathrm{Aut}(\mathbb{D}) \cong \mathrm{PSU}(1, 1)$  of genus 1 and generated by two hyperbolic elements  $A$  and  $B$ . The commutator  $[A, B]$  of  $A$  and  $B$  may be either elliptic of order  $n$  or parabolic and hence the signature of the group is  $(1; n)$  or  $(1; \infty)$ .

The quotient  $S_1 = \mathbb{D}/G_1$  is a hyperbolic genus 1 surface with a cone point of total angle  $2\pi/n$ . The surface has also a distinguished homology basis  $(\alpha, \beta)$  given by the images of the (oriented) axes of the transformations  $A$  and  $B$ . Replacing  $B$  by  $B^{-1}$  if necessary we may assume that this is a canonical basis. In this situation there is a unique  $\tau$  such that we have a conformal equivalence from  $S_1$  to  $\mathbb{C}/\Lambda$ , where  $\Lambda$  is the lattice generated by 2 and  $2\tau$ , and the image of  $[-1, 1]$  (resp.  $[-\tau, \tau]$ ) under the canonical projection  $\pi_2 : \mathbb{C} \rightarrow \mathbb{C}/\Lambda = S_1$  is in the same homology class as  $\alpha$  (resp.  $\beta$ ). The conditions define the equivalence up to a translation in  $\mathbb{C}/\Lambda$ . We make it unique by requiring that the intersection of the axes maps to  $\pi_2(0)$  and write  $S_1 = \mathbb{C}/\Lambda$ .

Call  $\pi_1$  the covering map  $\mathbb{D} \rightarrow S_1$ . Since  $\pi_2 : \mathbb{C} \rightarrow S_1$  is the universal cover of  $S_1$  and  $\mathbb{D}$  is simply connected,  $\pi_1$  lifts to a map  $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ , and this lifting is unique if we impose  $\varphi(0) = 0$ .

By construction the surface  $S_1$  comes equipped with two metrics, the natural flat metric and the metric induced by the Poincaré metric on  $\mathbb{D}$ . In the general situation

the geodesic arcs for these two metrics bear no relations. A typical situation is represented in figure 2 where on the left is a fundamental domain in the unit disk and on the right are the images under  $\varphi$  of the hyperbolic geodesics shown on the left. Although the difference is slight none of the arcs shown on the right are straight line segments.

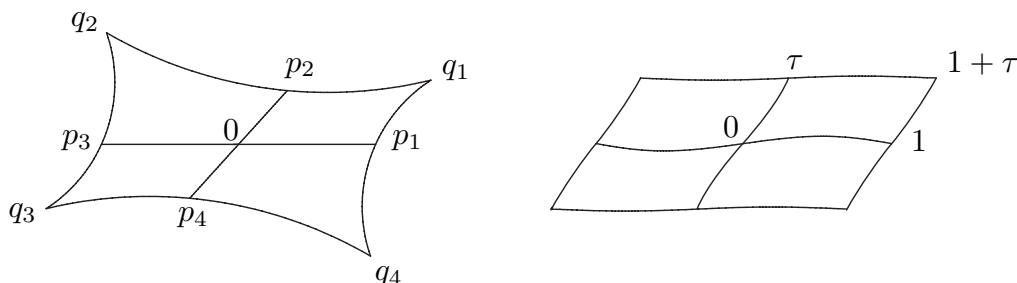


Fig. 2

We have nevertheless the following elementary statement

**1.1 Lemma.** *Let  $G_1, S_1, \tau$  and  $\varphi : \mathbb{D} \rightarrow \mathbb{C}$  be as above. Let  $q_1 = \varphi^{-1}(1 + \tau), \dots, q_4 = \varphi^{-1}(1 - \tau)$  be the pull-back of the vertices of the fundamental parallelogram for  $S_1$ . Then  $p_1 = \varphi^{-1}(1), p_2 = \varphi^{-1}(\tau), p_3 = \varphi^{-1}(-1)$  and  $p_4 = \varphi^{-1}(-\tau)$  are the hyperbolic midpoints of  $(q_4q_1), (q_1q_2), (q_2q_3)$  and  $(q_3q_4)$  respectively. Moreover  $\varphi^{-1}(0)$  is the hyperbolic midpoint of  $(p_1p_3)$  and  $(p_2p_4)$ .*

PROOF. By construction the pull-back via  $\varphi$  of  $z \mapsto -z$  is the order two elliptic transformation centered at  $\varphi^{-1}(0)$  and the  $q_i$  and  $p_i$  are preimages of the fixed points of the induced transformation in  $S_1$ .

If the axes of  $A$  and  $B$  are orthogonal we have a much stronger statement. In this case we may always assume that up to conjugation the axis of  $A$  is the real axis and the axis of  $B$  is the pure imaginary axis. The reflections with respect to the real axis and the pure imaginary axis induce anti-holomorphic involutions on  $S_1$  or in other words define real structures on  $S_1$ . See left of figure 3 where a fundamental domain for  $G_1$  is represented. In order to avoid the denomination ‘‘hyperbolic rectangle’’ we will call such a domain an *equiquadrangle*.

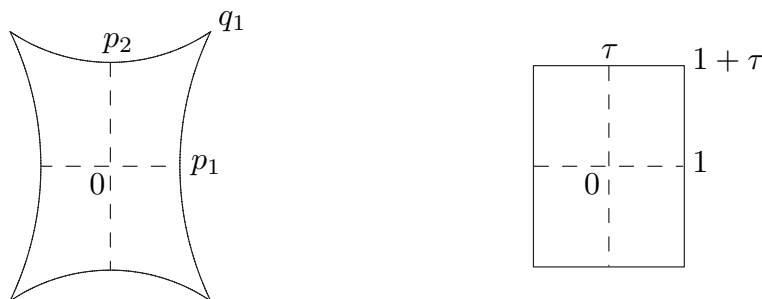


Fig. 3

These real structures obviously have two real components, one is the image of the real (resp. pure imaginary) axis, the other is the image of two identified opposite sides of the fundamental domain. Looking at the lattice  $\Lambda = \langle 2, 2\tau \rangle$  introduced above, this means that  $\tau$  is pure imaginary, or in other words that the natural

fundamental domain for  $\Lambda$  is a rectangle. By the uniqueness of the map  $\varphi$ , this map commutes with complex conjugation on both sides, i.e.  $\overline{\varphi(\bar{z})} = \varphi(z)$  and hence maps  $\mathbb{D} \cap \mathbb{R}$  to  $\mathbb{R}$  and  $\mathbb{D} \cap i\mathbb{R}$  to  $i\mathbb{R}$ .

**1.2 Lemma.** *Let  $G_1$  be a discrete subgroup of  $\text{Aut}(\mathbb{D})$  of signature  $(1; n)$  or  $(1; \infty)$  and generated by two hyperbolic elements  $A$  and  $B$  with respective axes the real axis and the pure imaginary axis. Let  $A_1$  (resp.  $B_1$ ) be the unique hyperbolic element such that  $A_1^2 = A$  (resp.  $B_1^2 = B$ ).*

*Let  $c_n$  be the images of the pure imaginary axis under  $A_1^n$  and  $d_n$  the image of the real axis under  $B_1^n$ . Let  $S_1 = \mathbb{D}/G_1$  and  $\pi_1$  the natural projection  $\mathbb{D} \rightarrow S_1$ . Then for all  $n$ ,  $\pi_1(c_n)$  and  $\pi_1(d_n)$  are geodesic arcs in  $S_1$  for both the hyperbolic metric induced by that of  $\mathbb{D}$  and for the natural conformal flat metric on the torus  $S_1$ .*

PROOF: The real structure on  $S_1$  induced by complex conjugation in  $\mathbb{D}$  has two real connected components  $\pi_1(\mathbb{R} \cap \mathbb{D})$  and  $\pi_1(d_1)$ . Since these are the fixed points of an anti-conformal reflection they are geodesic for any metric compatible with the conformal structure and for which the reflection is anti-conformal. The same is true for  $\pi_1(i\mathbb{R} \cap \mathbb{D})$  and  $\pi_1(c_1)$ . Since the  $c_n$  and  $d_n$  are just the images of  $c_0, c_1, d_0$  and  $d_1$  under  $G_1$  we are done.

**1.3 Corollary.** *Let  $S_1$  be as in 1.2 and let  $\varphi : \mathbb{D} \rightarrow \mathbb{C}$  be the lifting of the projection map  $\pi_1 : \mathbb{D} \rightarrow S_1$  normalized as above. Let  $G$  be the group generated by  $A_1$  and  $B_1$  — see 1.2. Then the images of  $\mathbb{D} \cap \mathbb{R}$  and  $\mathbb{D} \cap i\mathbb{R}$  under  $G$  are mapped by  $\varphi$  onto vertical lines through the integers and horizontal lines through  $n\tau$ ,  $n \in \mathbb{Z}$ .*

Another case where we have multi-geodesic arcs is the case of genus 1 surfaces with a half-Dehn twist (see figure 4).

**1.4 Lemma.** *Let  $A, B, A_1, B_1$  and  $G_1$  be as in Lemma 1.2. Let  $G'_1$  be the group generated by  $A$  and  $B' = A_1 B$ . Then  $G'_1$  has same signature as  $G_1$  (i.e.  $(1; n)$  or  $(1; \infty)$ ). Let  $S'_1 = \mathbb{D}/G'_1$  and let  $\pi'_1 : \mathbb{D} \rightarrow S'_1$  be the natural projection.*

*Then  $\pi'_1(i\mathbb{R} \cap \mathbb{D})$ ,  $\pi'_1(A_1(i\mathbb{R} \cap \mathbb{D}))$  and  $\pi(B_1^{-1}(\mathbb{R} \cap \mathbb{D}))$  are geodesic arcs in  $S'_1$  for both the hyperbolic metric induced by  $\mathbb{D}$  and the natural flat metric on the torus.*

PROOF. Since  $A$  and  $A_1$  commute, the commutator  $[A, B']$  is conjugate to the commutator  $[A, B]$ . This proves the first assertion.

For the rest we note that  $S'_1$  is obtained from  $S_1$  by applying a half-Dehn twist along the image of the axis of  $A$ . Since this is a real component of  $S_1$ ,  $S'_1$  also has real structures, but with only one real component this time (see for example [Bu-Se]). The real structures compatible with the hyperbolic metric must keep fixed the elliptic point (or the cusp, depending on the signature). To describe these, let  $\sigma$  be complex conjugation and let  $\sigma_1 = B_1^{-1} \cdot \sigma \cdot B_1$  and  $\sigma_2 = -\sigma$ . We have  $B_1^{-1} \cdot \sigma = \sigma \cdot B_1$ , hence we have  $\sigma_1 = B^{-1} \cdot \sigma = \sigma \cdot B$ . We also have  $\sigma \cdot A = A \cdot \sigma$  and similarly  $\sigma \cdot A_1 = A_1 \cdot \sigma$ . With these relations it is easy to prove that

- (i)  $\sigma_1 \cdot A \cdot \sigma_1 = B'^{-1} \cdot A \cdot B'$ ;
- (ii)  $\sigma_1 \cdot B' \cdot \sigma_1 = B'^{-1} \cdot A$ .

In exactly the same way we can also prove

- (iii)  $\sigma_2 \cdot A \cdot \sigma_2 = A^{-1}$ ;
- (iv)  $\sigma_2 \cdot B' \cdot \sigma_2 = A^{-1} \cdot B'$ .

To end the proof we only need to note that the fixed part of  $\sigma_1$  is  $B_1^{-1}(\mathbb{R} \cap \mathbb{D})$  and the fixed part of  $\sigma_2$  is  $i\mathbb{R} \cap \mathbb{D}$  which has same image as  $A_1(i\mathbb{R} \cap \mathbb{D})$ .

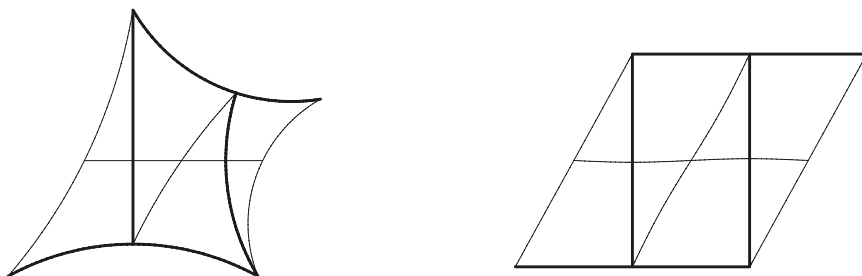


Fig. 4

In general no other geodesic gets mapped in  $\mathbb{C}$  onto a straight line. There are however some noteworthy exceptions. Of particular interest are the quadrangles corresponding to  $\tau = i$ ,  $\tau = \frac{1}{2} + \frac{i}{2}$  and  $\tau = (1 + i\sqrt{3})/2$ . In these cases the existence of additional real structures implies that the arcs marked in figure 5 are also geodesic for the hyperbolic metric.

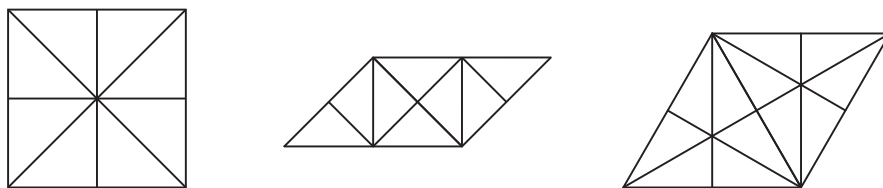


Fig. 5

In the sequel we will need to use a genus 0 variant of **1.2**. Let  $A$  be as above a hyperbolic element with axis the real line. Let  $e_1$  be the elliptic element of order 2 with center 0 and let  $e_2$  be a second elliptic element of order 2 with center in a point  $\tau$  on the pure imaginary axis. This data being subject to the condition that the group  $G_0 = \langle A, e_1, e_2 \rangle$  is a discrete subgroup of  $\text{Aut}(\mathbb{D})$  with signature of the form  $(0; 2, 2, 2, n)$  or  $(0; 2, 2, 2, \infty)$ . By construction the quotient  $\mathbb{D}/G_0$  is of genus 0 and is naturally equipped with a singular hyperbolic metric with 4 cone points, 3 with total angle  $\pi$  and one with angle  $2\pi/n$  or 3 cone points of angle  $\pi$  and a cusp.



Fig. 6

The fundamental domain described on the left of figure 6 is conformally equivalent to a rectangle with vertices  $\pm 1$  and  $\pm 1 + \tau$ . Using this rectangle we can also

equip  $S_0$  with a singular flat metric (see right of figure 6). For the same reasons as those indicated for  $S_1$  the sides of the fundamental domain coincide with the sides of the rectangle, and in particular are geodesic for both metrics. For further use we write this formally.

**1.5 Lemma.** *Let  $G_0$  be a discrete subgroup of  $\text{Aut}(\mathbb{D})$  of signature  $(0; 2, 2, 2, n)$  or  $(0; 2, 2, 2, \infty)$  and generated by an elliptic element  $e_1$  of order 2 centered at 0 and two hyperbolic elements  $A$  and  $B$  with respective axes the real axis and the pure imaginary axis.*

*Let  $S_0 = \mathbb{D}/G_0$ . Then  $S_0$  decomposes into two copies of a geodesic hyperbolic trirectangular quadrangle. The sides of this quadrangle are also geodesic for a natural singular flat metric.*

We have just taken  $B = e_1 e_2$ .

**1.6 Remark.** It will be useful in the sequel to reformulate the conditions on  $G_1$  and  $G_0$  in terms of fundamental domains. For  $G_1$  the conditions are that a fundamental domain in the disk be of the form given on the left of figure 3, with interior angles all equal either to  $\pi/(2n)$  or to 0. For  $G_0$  the condition is that the fundamental domain be of the form given on the left of figure 6, with two interior right angles and two angles equal either to  $\pi/n$  or to 0.

## 2. Fuchsian groups and equations in genus 1.

For the applications we will need to have a more precise description of the hyperbolic transformations  $A$  and  $B$  introduced in Lemma 1.2 and their relations with the rectangles.

**2.1 Lemma.** *Let  $\mathcal{R}$  be an equiquadrangle with interior angle  $\pi/n$ ,  $n$  even, or zero angle. Let  $\ell$  be the hyperbolic length of the horizontal median of  $\mathcal{R}$  and let  $L = \cosh(\ell/2)$ . Let*

$$L' = \sqrt{\frac{\cos(\pi/n)^2 + L^2 - 1}{L^2 - 1}} \quad \text{or} \quad L' = \frac{L}{\sqrt{L^2 - 1}} \quad \text{if the angle is zero.}$$

*Then the transformations  $A, B$  of Lemma 1.2 generating  $G_1$  are represented in  $\text{SU}(1, 1)$  by*

$$(2.1.1) \quad A = \begin{pmatrix} L & \sqrt{L^2 - 1} \\ \sqrt{L^2 - 1} & L \end{pmatrix}, \quad B = \begin{pmatrix} L' & i\sqrt{L'^2 - 1} \\ -i\sqrt{L'^2 - 1} & L' \end{pmatrix}.$$

*In the more general situation of genus 1 with a twist parameter  $t$  along the axis of  $A$  we can consider the group generated by  $A$  and  $B_1 = T \cdot B$ , where*

$$(2.1.2) \quad T = \begin{pmatrix} Tw & \sqrt{Tw^2 - 1} \\ \sqrt{Tw^2 - 1} & Tw \end{pmatrix}$$

*with  $Tw = \cosh(t \operatorname{arccosh}(L))$ .*

The proof is a simple exercise in hyperbolic trigonometry (use for example [Bu], p.454).

For the group  $G_0$  of Lemma 1.5 we have a very similar description.

**2.2 Lemma.** *Let  $G_0 = \langle A, B, e_1 \rangle$  be as in Lemma 1.5. Then  $A$  and  $B$  will have the same expression as that given in (2.1.1), but with  $n$  even or odd this time.*

*In the presence of a twist  $t$  along the axis of  $A$ ,  $A$  and  $B_1$  have the same expression as in Lemma 2.1 and  $e_1$  is just the conjugate of  $z \mapsto -z$  by a matrix of the same form as  $T$  but with  $Tw$  replaced by  $\sqrt{(Tw+1)}/2$ .*

For the special cases of quadrangles associated to the parallelograms of periods  $i$ ,  $\frac{1}{2} + \frac{i}{2}$  or  $(1 + i\sqrt{3})/2$ , we can be even more precise.

**(2.3)** For the “square” equiquadrangle we only need to take

$$(2.3.1) \quad L = \sqrt{\cos(\pi/n) + 1} \quad \text{or} \quad L = \sqrt{2} \quad \text{if the angle is 0.}$$

**(2.4)** If  $\tau = (1 + i\sqrt{3})/2$  the corresponding hyperbolic quadrangle is obtained from two copies of a hyperbolic equilateral triangle with angle  $2\pi/(3n)$ . Using the formulae in [Bu], p.454, one can compute the generators  $A$  and  $B$  for the group of the genus 1 surface. This yields:

- $A$  as in (2.1.1) with  $L = \frac{1}{2} + \cos\left(\frac{2\pi}{3n}\right)$ ;
- $B = T A T^{-1}$ ,

where

$$T = \begin{pmatrix} \exp\left(i\frac{\theta}{2}\right) & 0 \\ 0 & \exp\left(-i\frac{\theta}{2}\right) \end{pmatrix} \quad \text{with} \quad \theta = \arccos\left(\frac{2\cos(2\pi/(3n)) + 1}{2\cos(2\pi/(3n)) + 3}\right).$$

If the angle is the zero angle these matrices become

$$A = \begin{pmatrix} \frac{3}{2} & \frac{\sqrt{5}}{2} \\ \frac{\sqrt{5}}{2} & \frac{3}{2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \frac{3}{2} & \frac{(3+4i)\sqrt{5}}{10} \\ \frac{(3-4i)\sqrt{5}}{10} & \frac{3}{2} \end{pmatrix}.$$

**(2.5)** If  $\tau = \frac{1}{2} + \frac{i}{2}$ , then one has, with  $L$  as in (2.3.1),

$$A = \begin{pmatrix} L^2 & \sqrt{L^4 - 1} \\ \sqrt{L^4 - 1} & L^2 \end{pmatrix}, \quad B = \begin{pmatrix} L & e^{i\beta} \sqrt{L^2 - 1} \\ e^{-i\beta} \sqrt{L^2 - 1} & L \end{pmatrix}$$

where  $\beta = \arcsin(1/\sqrt{L^2 + 1})$ .

We will also need to associate an explicit equation for the elliptic curve defined by a rectangle and conversely.

We do this as follows, given  $\tau$  we use a variant of a classical Jacobi function

$$(2.6) \quad J_\tau(z) = -\frac{w}{K} \prod_{k=0}^{\infty} \frac{(w - \zeta^{2k})^2 (1 - \zeta^{2k+2} w)^2}{(w - \zeta^{2k+1})^2 (1 - \zeta^{2k+1} w)^2},$$

where  $\zeta = \exp(\pi i \tau)$ ,  $w = \exp(\pi i z)$  and  $K = 4 \prod_{k=1}^{\infty} \left(\frac{1 + \zeta^{2k}}{1 + \zeta^{2k-1}}\right)^4$  (for more details see for example Nehari [Ne], Chap. VI, Sec.3. See also [Bu-Si2]). We have  $J_\tau(0) = 0$ ,  $J_\tau(1) = 1$  and  $J_\tau(\tau) = \infty$ . Letting  $\mu = J_\tau(1 + \tau)$ , an equation for the elliptic curve is

$$(2.7) \quad y^2 = P(x) = x(x-1)(x-\mu).$$



For the converse choose a determination of  $\sqrt{P(x)}$  for  $x$  in the upper half plane. Extend this determination to the real line. Then

$$\tau = I_2/I_1 \quad \text{where} \quad I_1 = \int_0^1 \frac{dx}{\sqrt{P(x)}} \quad \text{and} \quad I_2 = \int_1^\mu \frac{dx}{\sqrt{P(x)}}.$$

We have used here the convention that the vertices of the rectangle get mapped to the Weierstrass point with  $x$ -coordinate 0, the midpoint of the vertical edges to the one with  $x$ -coordinate 1 and the midpoint of the horizontal edges to the point at infinity. We extend this convention to the case of parallelograms namely the vertices will be mapped to 0, the midpoint of the horizontal edges to  $\infty$  and the midpoints of the other edges to 1.

With this convention we note that

**2.8 Lemma.** *If  $\mu > 1$  corresponds to  $\tau$ , then  $\frac{\mu}{\mu-1}$  corresponds to  $-1/\tau$ ,  $1-\mu$  corresponds to  $\tau-1$  and  $1/\mu$  to  $\frac{\tau}{1+\tau}$ .*

PROOF. For the first assertion we note that the integrals along  $[-\infty, 0]$  (resp.  $[\mu, \infty]$ ) are just minus the integrals along  $[1, \mu]$  (resp.  $[0, 1]$ ). Applying the change of variable  $x \mapsto x/(x-1)$  proves the first assertion if in addition we note that the integral along  $[0, 1]$  is real while the integral along  $[1, \mu]$  is pure imaginary. For the other assertions apply the change of variables  $x \mapsto 1-x$  and  $x \mapsto 1/x$ .

**2.9 Remarks.** 1) If we are dealing with a parallelogram in place of a rectangle, then the curve will also have an equation of the form (2.7) with  $\mu$  computed using the Jacobi function (2.6). We will also of course have an analogue of Lemma 2.8 but the exact correspondence will depend on the choice of the initial parallelogram.

2) Since for a given elliptic curve there are only six possibly different values of  $\mu$ , infinitely many values of  $\tau$  will correspond to the same  $\mu$ . This is in particular the case for  $\tau$  and  $2+\tau$ .

3) For domains as in the second part of Lemma 2.1 we use a similar convention that 0 is mapped to 0, 1 is mapped to 1 and  $\tau$  is mapped to  $\infty$ . We will call  $\mu$  the image of  $1+\tau$ .

A fundamental domain for a group generated  $A$  and  $B$  (resp.  $A$  and  $T \cdot B$ ) is illustrated on the left (resp. right) of figure 7.

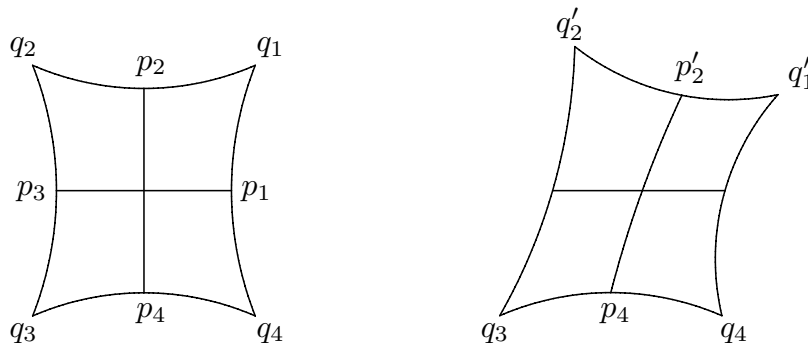


Fig. 7

We will need the following straightforward result

**2.10 Lemma.** *Let  $L, n, A, B$  and  $T$  be as in Lemma 2.1 or Lemma 2.2. Then the hyperbolic lengths of the upper and lower geodesic arcs of the fundamental domains for  $\langle A, B \rangle$  or  $\langle A, T \cdot B \rangle$  is  $\operatorname{arccosh}(L \sin(\pi/n))$ . Moreover the sum of the interior angles at  $q'_1$  and  $q'_2$  is  $2\pi/n$ .*

PROOF. This follows immediately from the fact that  $q'_1$  and  $q'_2$  are the images of  $q_1$  and  $q_2$  under  $T$ .

For the “square” equiquadrangle or the quadrangles corresponding to the period  $\frac{1}{2} + \frac{i}{2}$  or  $(1+i\sqrt{3})/2$  we have the obvious values  $\mu = 2$ ,  $\mu = 1/2$  and  $\mu = (1+i\sqrt{3})/2$  respectively.

In addition to these values there are other cases for which we can express  $\mu$  in terms of  $L$  and  $n$ . For some examples see the Appendix.

### 3. Multi-geodesic tessellation of surfaces obtained from rectangles.

Let  $R$  be an Euclidean rectangle in the complex plane. To simplify we assume the edges of  $R$  are horizontal or vertical. Assemble  $r$  copies  $R_1, \dots, R_r$  of  $R$  by pasting along sides of same length so that not only the resulting polygon is simply connected but it also remains simply connected when one removes the vertices of the rectangles. We will say that such an arrangement of rectangles is an *admissible arrangement* (the arrangement on the left of figure 8 is not admissible in this sense but the one on the right is). This restriction on the arrangement is here for purely technical reasons and could be in fact dispensed with.

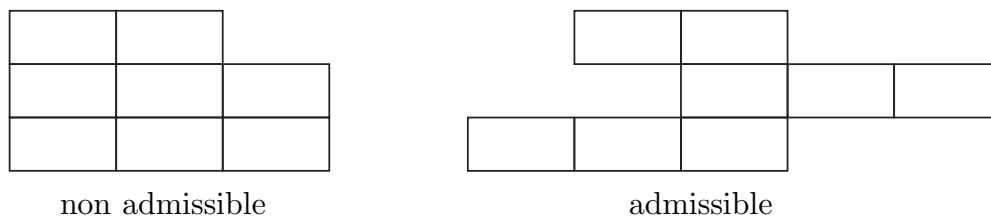


Fig. 8

Identify the remaining edges by pairs, using translations or rotations of angle  $\pi$  (half-turns), in a way that is compatible with the orientation of the rectangles, i.e. the identifications by translation can be top edge to bottom edge and right edge to left edge, and the identifications by half-turns can be top edge to top edge, left edge to left edge, right edge to right edge and bottom edge to bottom edge. We will call such surfaces *surfaces obtained from rectangles*.

We will distinguish the case when the identifications are only by translations and the case when some identifications are by half-turns. Referring to the underlying invariant differential we will call the first the *Abelian case* and the second the *quadratic case*. This denomination comes from the fact that in the first case the differential  $dz$  being invariant by translations it induces a holomorphic, or abelian, differential  $\omega$  on the surface  $S$ . In the second it is  $dz^2$  that induces a quadratic differential  $q$ . In both cases the zeros of the differential are necessarily located at images of vertices of the rectangles.

**3.1 Definition.** Let  $S$  be as above and let  $q$  be the quadratic differential induced by  $dz^2$  (in both the Abelian and quadratic case). Let  $d_1, \dots, d_k$  be the orders of the zeros of  $q$ . Let  $m_i = d_i + 2$  and let  $m$  be the least common multiple of the  $m_i$ . For a positive integer  $n$  we will say that  $\pi/n$  (or by abuse simply  $n$ ) satisfies the angle condition for the differential if  $n$  is a multiple of  $m$ . We will always consider that the zero angle satisfies the angle condition.

We consider first the Abelian case. Let  $E$  be the genus 1 surface corresponding to the rectangle  $R$  that to fix notations we identify with  $R_1$  and let  $E^*$  be the surface obtained from  $E$  by removing the image of the vertices of the rectangle. Let  $p$  be the center of  $R_1$  and let  $h$  and  $v$  be the elements of  $\pi_1(E^*, p)$ , the fundamental group of  $E^*$ , corresponding respectively to the horizontal median and the vertical median of the rectangle oriented in the natural way. Since  $\pi_1(E^*, p)$  is isomorphic to the free group in two generators,  $h$  and  $v$  can be thought of as generating a free group.

Let  $S$  be as above and let  $S^*$  be the surface obtained by removing the vertices of the rectangles. Since the arrangement of rectangles we are starting with is admissible and in particular remains simply connected when the vertices are removed the identifications of edges define a set of generators of the fundamental group of  $S^*$ . On the other hand  $S^*$  is an unramified covering of  $E^*$ . Hence we can consider the fundamental group of  $S^*$  as a subgroup of  $\pi_1(E^*, p)$ . More precisely taking as base point the center of the rectangle  $R_1$  the fundamental group of  $S^*$  is generated by words  $w_1(h, v), \dots, w_s(h, v)$  in  $h$  and  $v$ . For example if we consider the surface obtained from three rectangles (see upper left of figure 9) with the usual identifications (by horizontal and vertical translations) then these words are  $h^2, v^2, hvh^{-1}, vhv^{-1}$  (see [Sch] and section 4 for other explicit computations of this type).

Let  $S$  and  $\omega$  be as above and let  $n$  be an integer satisfying the angle condition for  $\omega^2$ . Then there exist hyperbolic elements  $A$  and  $B$  satisfying the conditions of 1.2 generating a group  $G_1$  of signature  $(1; n)$  or  $(1; \infty)$  and such that  $\mathbb{D}/G_1 = E$ .

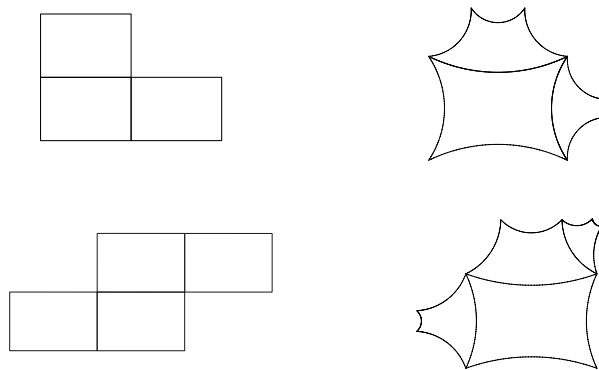


Fig. 9

**3.2 Proposition.** Let  $S$ , the Abelian differential  $\omega$ , the words  $w_1, \dots, w_s$  and  $A$  and  $B$  be as above. Let  $G$  be the group generated by  $w_1(A, B), \dots, w_s(A, B)$ . Then

- (i)  $G$  is a discrete subgroup of  $\text{Aut}(\mathbb{D})$ ;
- (ii)  $\mathbb{D}/G \cong S$  as Riemann surfaces;
- (iii) the edges of the rectangles are geodesic arcs for the hyperbolic metric induced by the Poincaré metric on  $\mathbb{D}$ ;

- (iv) *the horizontal and vertical medians of each rectangle are geodesic for the hyperbolic metric and extend to simple closed geodesics on the surface.*

PROOF. Call  $\mathcal{R}$  the equiquadrangle defined by  $A$  and  $B$ . We build a domain in  $\mathbb{D}$  starting with  $\mathcal{R}$  and using the same combinatorics as the one that defines the arrangements of rectangles, replacing everywhere the copies of the Euclidean rectangle  $R$  by copies of  $\mathcal{R}$  (see figure 9 for two examples). By construction  $G$  identifies in pairs the remaining edges of the equiquadrangles.

Assume that the signature of  $G_1 = \langle A, B \rangle$  is  $(1; n)$ . If  $x_0$  is a point of  $S$  where  $\omega$  has a zero of order  $d_i$ , the total angle at that point will be  $2(d_i + 1)\pi$  and hence corresponds to the identification of  $4(d_i + 1)$  vertices. The interior angles of the equiquadrangle are  $\pi/(2n)$  hence the sum of the hyperbolic interior angles of the vertices identified by  $G$  with  $x_0$  will be  $4(d_i + 1)\pi/(2n)$ . Since by definition  $n$  is a multiple of  $d_i + 1$  this is of the form  $2\pi/p$ . This together with the pairing convention means that we can apply Poincaré's theorem (see for example [Bea] section 9.8) and conclude that the group  $G$  is discrete and that the arrangement of equiquadrangles we have constructed is a fundamental domain for this group. If the signature is  $(1; \infty)$  then we can directly apply Poincaré's theorem (see [Bea], p.251). This proves (i). One should think of  $G$  as an orbifold fundamental group.

Next we consider the conformal equivalence  $f$  from the interior of the equiquadrangle to the rectangle, extended by continuity to the boundary minus the vertices. Using Schwarz's reflection principle we can extend this conformal map to the full arrangement of equiquadrangles. This yields a conformal equivalence from the hyperbolic arrangement to the Euclidean arrangement. By construction of  $G$  this conformal equivalence induces a conformal equivalence from  $\mathbb{D}/G - \{\text{images of vertices}\}$  to  $S^*$  which extends naturally to a biholomorphic map from  $\mathbb{D}/G$  to  $S$ . This proves (ii) and (iii).

The median (horizontal or vertical) of a rectangle in  $S$  extends to a simple closed curve which corresponds to the decomposition of  $S$  into cylinders (horizontal or vertical). By 1.2 these are locally geodesic and since they intersect orthogonally at midpoints the sides of the equiquadrangles (see 1.1) we obtain (iv).

In the quadratic case we have a very similar statement. Essentially only the initial setup is different.

By changing the arrangement if necessary we may always assume that the identifications of the form  $z \mapsto -z + c$  are between horizontal sides.

Let  $h$  be the translation that maps the left side of the rectangle  $R$  onto the right side. Let  $r_1$  be the rotation of angle  $\pi$  centered at the center of  $R$  and let  $r_2$  be the rotation of angle  $\pi$  centered at the middle of the upper side of  $R$ . Let  $S_0^*$  be the quotient of  $R - \{\text{vertices}\}$  under the identification induced by  $h$ ,  $r_1$  and  $r_2$ . We consider  $S_0^*$  as an orbifold with three cone points of order 2 and a cusp. The orbifold fundamental group of  $S_0^*$  is generated by three elements, two of order 2 and one of infinite order. We may consider these generators as being  $r_1$ ,  $r_2$  and  $h$ . Let  $S^*$  be the surface obtained from  $S$  by removing the images of the vertices of the rectangles. Identifying  $R$  with  $R_1$  we can, proceeding as in the Abelian case, write generators for the orbifold fundamental group of  $S^*$  as words in  $h$ ,  $r_1$  and  $r_2$  or better as words in  $h$ ,  $v = r_1 r_2$  and  $r_1$ . We choose these words  $w_1(h, v, r_1), \dots, w_s(h, v, r_1)$  to correspond to side pairings of the arrangement defining  $S$ .

Let  $n$  be an integer satisfying the angle condition. Choose  $A$  hyperbolic,  $e_1$  and  $e_2$  elliptic of order 2, as in section 1, such that the group  $G_0 = \langle A, e_1, e_2 \rangle$  is discrete

of signature  $(0; 2, 2, 2, n)$  or  $(0; 2, 2, 2, \infty)$  and such that  $\mathbb{D}/G_0 - \{\text{point of order } n\}$  or  $\mathbb{D}/G_0$ , depending on the signature, is conformally equivalent to  $S_0^*$ .

**3.3 Proposition.** *Let  $S$ , the quadratic differential  $q$ , the words  $w_1, \dots, w_s$ ,  $A$ ,  $e_1$  and  $e_2$  be as above.*

*Let  $B = e_1 e_2$  and let  $G$  be the group generated by  $w_1(A, B, e_1), \dots, w_s(A, B, e_1)$ . Then properties (i) to (iv) of **3.2** hold also in this case.*

The proof follows exactly the same lines as the proof of **3.2**.

We end this section by describing the hyperbolic counterpart of the natural action of  $\text{PSL}_2(\mathbb{Z})$  on the Teichmüller disk orbit of a surface obtained from rectangles.

To avoid technical difficulties we need to restrict to surfaces for which the hyperbolic metric is non singular. To be more specific we need

**3.4 Definition.** *Let  $(S, q)$  be a surface obtained from rectangles. We will say that  $(S, q)$ , or simply  $S$ , is balanced if all vertices correspond to zeros of the same order of the quadratic differential  $q$ .*

If  $(S, q)$  is balanced then choosing the hyperbolic equiquadrangle with interior angle  $\pi/(n+2)$ , where  $n$  is the order of  $q$  at the vertices, leads to a non singular hyperbolic metric on the surface.

**3.5.** *For the remainder of this section we will consider balanced surfaces with this hyperbolic metric in addition to the locally flat metric induced by the quadratic differential.*

Let  $S$  be a surface obtained from rectangles, then  $S$  has a natural decomposition into horizontal cylinders  $\mathcal{C}_1, \dots, \mathcal{C}_p$  (see for example [Hu-Le]). If a cylinder  $\mathcal{C}_i$  is formed of  $n_i$  rectangles, we will say that it is of width  $n_i$ . By Lemma **2.1** or Lemma **2.2** the horizontal medians of these cylinders are disjoint simple closed geodesics  $\gamma_i$ .

**3.6 Proposition.** *Let  $(S, q)$  be a balanced surface of genus  $g$  obtained from squares, with the hyperbolic structure defined in **3.5**. Let  $\mathcal{C}_1, \dots, \mathcal{C}_p$  be its decomposition into horizontal cylinders,  $\mathcal{C}_i$  of width  $n_i$  and let  $\gamma_1, \dots, \gamma_p$  be the corresponding simple closed geodesics. Let  $\gamma_{p+1}, \dots, \gamma_{3g-3}$  be geodesics such that  $\gamma_1, \dots, \gamma_{3g-3}$  defines a pants decomposition. Then there exist functions  $f_{p+1}, \dots, f_{3g-3}$ , defining lengths, and  $tw_1, \dots, tw_{3g-3}$ , defining twist parameters, depending only on the combinatorics of the decomposition into squares and the choice of the  $\gamma_j$ ,  $j > p$ , such that the surfaces in the  $\text{PSL}_2(\mathbb{R})$  orbit of  $(S, q)$  have Fenchel-Nielsen coordinates of the form*

$$(3.6.1) \quad \left( n_1 \ell, tw_1(\ell) + \frac{t}{n_1}, \dots, n_p \ell, tw_p(\ell) + \frac{t}{n_p}, \right. \\ \left. f_{p+1}(\ell), tw_{p+1}(\ell), \dots, f_{3g-3}(\ell), tw_{3g-3}(\ell) \right)$$

with  $\ell = 2 \operatorname{arccosh}(L)$ ,  $L$  as in **2.1** and  $t \in \mathbb{R}$ .

PROOF. We first consider the case when  $(S, q)$  is obtained from rectangles.

That the lengths of the  $\gamma_i$ ,  $1 \leq i \leq k$ , is  $n_i \ell$  follows immediately from the definition of the cylinders and the construction of the hyperbolic structure.

Let  $\gamma_j$ ,  $j > p$ , be one of the additional geodesics. Because of our convention **3.5**, Proposition **3.2** (resp. **3.3**) implies that  $\gamma_j$  is the image in  $S$  of the axis of a word  $w$  in  $A$  and  $B$  (resp.  $A$ ,  $B$  and  $e_1$ ) and the length of  $\gamma_j$  is  $2 \operatorname{arccosh}(\operatorname{trace}(w)/2)$ .

Such a word depends only on the combinatorics of the decomposition into rectangles and the choice of  $\gamma_j$ . On the other hand the coefficients of  $A$  and  $B$  only depend on  $n$ , which is fixed, and  $L = \cosh(\ell/2)$  (see Lemma **2.1**). In particular  $\operatorname{trace}(w)$ , and hence the hyperbolic length of  $\gamma_j$ , is a function of  $L$ .

Let  $\mathcal{P}_1$  and  $\mathcal{P}_2$  be two pairs of pants pasted along  $\gamma_i$ . Then the twist parameter attached to  $\gamma_i$  can be computed in terms of the lengths of boundary components of  $\mathcal{P}_1$  and  $\mathcal{P}_2$  and the length of a simple closed geodesic  $\delta_i$  contained in  $\mathcal{P}_1 \cup \mathcal{P}_2$  (see for example [Bu], Proposition **3.3.12**).

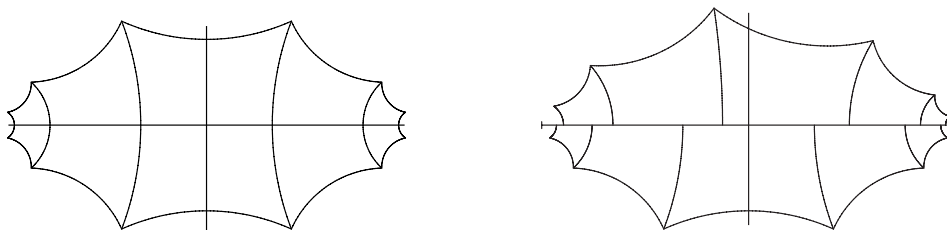
Lift this geodesic to a curve  $d_i$  in the arrangement of rectangles (which we assume as usual to have horizontal and vertical sides).

From the construction of this geodesic  $\delta_i$  (see [Bu], Chap. 3, §3) the free homotopy class of  $\delta_i$  in the surface only depends on the the combinatorics of the arrangement and the side identifications, and is independent of the choice of the specific rectangle. Hence  $\delta_i$  can be expressed as a fixed word in  $A$ ,  $B$  and  $e_1$ . The same argument as above shows that the length of  $\delta_i$  and the twist parameter for  $\gamma_i$  are functions of  $L$ .

Let  $\mathcal{R}_0$  be an equiquadrangle and let  $A$  and  $B$  be the hyperbolic left-right and bottom-top side pairings. For each integer  $j$  denote by  $\mathcal{R}_j$  the equiquadrangle  $A^j(\mathcal{R}_0)$ .

A cylinder  $\mathcal{C}$  of width  $n$  is the quotient of the infinite union of the  $\mathcal{R}_j$  by the identification of each  $\mathcal{R}_j$  to  $\mathcal{R}_{j+n+1}$  by  $A^{n+1}$  (see left of figure 10 where we assume that the horizontal middle geodesic is the real axis in the unit disk).

Now consider the situation with the same  $A$  but with  $B$  replaced by  $B_1 = T \cdot B$ , with  $T$  as in Lemma **2.1** and with  $Tw = \cosh(t \operatorname{arccosh}(L))$ . There are two ways we can look at this situation, one is by replacing the equiquadrangles by the domain on the right of figure 7, another is to keep the lower half of the cylinder (in  $\Im m(z) < 0$ ) fixed and shift the upper half by  $T$  (see right of figure 10). But since the length of the middle geodesic is  $n\ell$  this amounts to applying a  $\operatorname{arccosh}(Tw)/(n \operatorname{arccosh}(L)) = t/n$  twist along the middle geodesic. This proves that for  $1 \leq j \leq k$  the twist parameter for  $\gamma_j$  is now  $tw_j(\ell) + t/n_j$ .



**Fig. 10**

To end the proof let  $S$  be the surface obtained from the equiquadrangles or otherwise said from  $A$  and  $B$  and let  $S_1$  be the surface obtained from  $A$  and  $B_1 = T \cdot B$ . Let  $S'$  and  $S'_1$  be the surfaces with boundaries obtained from  $S$  and  $S_1$  respectively by cutting along  $\gamma_1, \dots, \gamma_s$ . The above construction shows that  $S'$  and  $S'_1$  are isometric. This shows that the remaining Fenchel-Nielsen coordinates, length of  $\gamma_j$  and twist along  $\gamma_j$ , remain unchanged.

We started with a decomposition into horizontal cylinders, but we would have obtained the same type of result, starting with a decomposition into vertical cylinders. This would lead to another pants decomposition and another set of Fenchel-Nielsen coordinates

$$(3.6.2) \quad \left( n'_1 \ell', tw'_1(\ell') + \frac{t'}{n'_1}, \dots, n'_{p'} \ell', tw'_{p'}(\ell') + \frac{t'}{n'_{p'}}, \right. \\ \left. g_{p'+1}(\ell'), tw'_{p'+1}(\ell'), \dots, g_{3g-3}(\ell'), tw'_{3g-3}(\ell') \right)$$

with  $\ell' = 2 \operatorname{arccosh}(L')$ ,  $L'$  as in **2.1**.

**3.7 Corollary.** *Let  $\mathcal{D}$  be a Teichmüller disk,  $\operatorname{PSL}_2(\mathbb{R})$  orbit of a surface obtained from squares. Let*

$$(n_1 \ell, t_1, \dots, n_p \ell, t_p, \ell_{p+1}, t_{p+1}, \dots, \ell_{3g-3}, t_{3g-3}) \\ (n'_1 \ell', t'_1, \dots, n'_{p'} \ell', t'_{p'}, \ell'_{p'+1}, t'_{p'+1}, \dots, \ell'_{3g-3}, t'_{3g-3})$$

be pairs of Fenchel-Nielsen coordinates, of type (3.6.1) and (3.6.2), for surfaces in  $\mathcal{D}$ .

Let  $\varphi_1$  be the composition of fractional Dehn-twists along  $\gamma_1, \dots, \gamma_k$  that consists in replacing  $t_1, \dots, t_p$  by  $t_1 + 1/n_1, \dots, t_p + 1/n_k$ . Similarly let  $\varphi_2$  be the composition of fractional Dehn-twists that consists in replacing  $t'_1, \dots, t'_{p'}$  by  $t'_1 + 1/n'_1, \dots, t'_{p'} + 1/n'_{p'}$ .

Then  $\varphi_1$  and  $\varphi_2$  generate the natural action of  $\operatorname{PSL}_2(\mathbb{Z})$  on  $\mathcal{D}$ .

PROOF. Let  $S$  be a surface in  $\mathcal{D}$  tiled by parallelograms with invariant  $\tau$ . The image of  $S$  under  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  is obtained by replacing the initial parallelograms by those with invariant  $1 + \tau$ . Let  $A$  be the hyperbolic left-right side pairing and  $B_1$  the other side pairing of the hyperbolic quadrangle associated to the parallelogram defined by  $\tau$ . Replacing  $\tau$  by  $1 + \tau$  is just a change of basis of the lattice. By construction this corresponds to replacing  $A$  and  $B_1$  by  $A$  and  $A \cdot B_1$ . But this is just taking  $t = 1$  in (3.6.1). This is the action of  $\varphi_1$ .

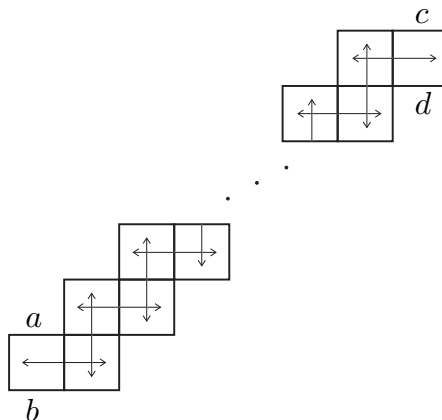
The same argument shows that  $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  acts by  $\varphi_2$  on  $\mathcal{D}$ .

**3.8 Remark.** From the form of the matrices  $A$  and  $B$  in Lemma **2.1** and the proof of Proposition **3.6** one can easily deduce that all the  $\cosh(f_j(\ell)/2)$  have algebraic expressions in  $L$ .

The same is true of the  $\cosh(t_i \ell_i / 2)$  since the exact expression given in [Bu] **3.3.11** only involves hyperbolic sines and hyperbolic cosines of lengths.

#### 4. The stairs and the escalator families.

In this section we will consider families constructed from the common admissible arrangement of rectangles illustrated in figure 11. Although these families belong to different strata we will see that the equations of the surfaces in these families are intimately related.



**Fig. 11**

The families are differentiated by the way the sides labeled  $a$ ,  $b$ ,  $c$  and  $d$  are identified. The families are as follows.

$\text{St}_1(g)$ : the number of rectangles is  $2g$  and the identification is  $a \sim b$ ,  $c \sim d$ . The surface is hyperelliptic of genus  $g$  and the Weierstrass points are the centers of the rectangles and the midpoints of the edges labeled  $a \sim b$  and  $c \sim d$ . The vertices map to two distinct points in the surface hence the surface is in the stratum  $\mathcal{H}(g-1, g-1)$ . Surfaces in this family are often called stairs (see [Sch] or [Möl]).

$\text{Esc}_1(g)$ : the number of rectangles is  $2g+2$  with  $g$  odd and the identification is  $a \sim d$ ,  $b \sim c$ . The surface is hyperelliptic of genus  $g$  and the Weierstrass points are the centers of the rectangles. The vertices of the rectangles map to four distinct points hence the surface is in the stratum  $\mathcal{H}((g-1)/2, (g-1)/2, (g-1)/2, (g-1)/2)$ . By analogy with the stairs we call this family the escalator family.

$\text{Esc}_2$ : the number of rectangles is  $2g$  with  $g$  odd and the identification is  $a \sim d$ ,  $b \sim c$ . The surface is hyperelliptic of genus  $g$  and the Weierstrass points are the centers of the rectangles and the vertices of the rectangles. The vertices of the rectangles map to two points and hence the surface is in the stratum  $\mathcal{H}(g-1, g-1)$ .

$\text{Escb}_1(g)$ : the number of rectangles is  $2g+2$  with  $g$  even and the identification is  $a \sim c$ ,  $b \sim d$ . The surface is hyperelliptic of genus  $g$  and the Weierstrass points are the centers of the rectangles. The vertices of the rectangles are mapped to four distinct points. Since the identifications  $a \sim c$  and  $b \sim d$  use a half-turn, the surface is in the stratum  $\mathcal{Q}((g-1), (g-1), (g-1), (g-1))$ . The identification used here does not really deserve to be called of escalator type but we have named it by analogy with the  $\text{Esc}_1$  family.

$\text{Escb}_2(g)$ : the number of rectangles is  $2g$  with  $g$  even and the identification is  $a \sim c$ ,  $b \sim d$ . The surface is hyperelliptic of genus  $g$  and the Weierstrass points are the centers of the rectangles and the vertices of the rectangles. The vertices of the rectangles are mapped to two points and the surface is in the stratum  $\mathcal{Q}(2g-2, 2g-2)$ .

Finally we also consider one last family based on a slightly different arrangement (see figure 12).



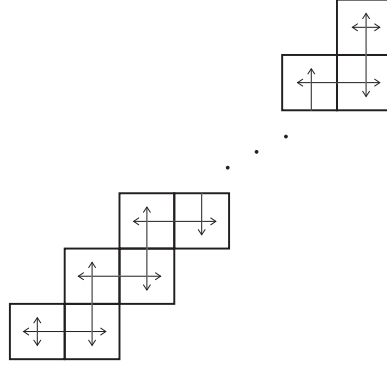


Fig. 12

$St_2(g)$  : the number of rectangles is  $2g - 1$ . The surface is hyperelliptic of genus  $g$  and the Weierstrass points are the vertices, all mapped to one point, the centers of the rectangles and the midpoints of the identified horizontal edges of the lower left rectangle and of the identified vertical edges of the upper right rectangle. The surface is in the stratum  $\mathcal{H}(2g - 2)$ .

Note that multiples of  $2g$  satisfy the angle condition for the differential on surfaces in  $St_1(g)$ ,  $Esc_2(g)$  and  $Escb_2(g)$ , while multiples of  $g + 1$  satisfy the conditions for surfaces in  $Esc_1(g)$  and  $Escb_1(g)$ . For surfaces in  $St_2(g)$  we must choose multiples of  $4g - 2$ . In particular we can replace the rectangles by equiquadrangles with angles  $\pi/(2g)$  for  $St_1(g)$ ,  $Esc_2(g)$  and  $Escb_2(g)$ , or angles  $\pi/(g + 1)$  for  $Esc_1(g)$  and  $Escb_1(g)$ , or angles  $\pi/(4g - 2)$  for  $St_2(g)$ . If we do this we are under the hypothesis of **3.5** and the hyperbolic metric will be non-singular. To compute the corresponding Fuchsian groups we only need, by Propositions **3.2** and **3.3**, to compute the words expressing the identifications in terms of the elementary horizontal and vertical identifications  $h$  and  $v$  of the rectangle and the rotation of angle  $\pi$  at the center of the rectangle. This is taken care of by

**4.1 Lemma.** *If the surface is in  $St_1(g)$  the side pairings are given by,*

$$(4.1.1) \quad \langle v, (hv)^{g-1}hvh^{-1}(hv)^{1-g}, (hv)^jh^2(hv)^{-j}, (hv)^ihv^2h^{-1}(hv)^{-i} \\ / 0 \leq j \leq g - 1, 0 \leq i \leq g - 2 \rangle$$

*If the surface is in  $St_2(g)$  they are given by,*

$$(4.1.2) \quad \langle v, (hv)^{g-1}h(hv)^{1-g}, (hv)^jh^2(hv)^{-j}, (hv)^jhv^2h^{-1}(hv)^{-j} / 0 \leq j \leq g - 2 \rangle$$

*For  $Esc_1(g)$  and  $Esc_2(g)$  the side pairings are given by,*

$$(4.1.3) \quad \langle (hv)^k, (hv)^{k-1}hv^{-1}, (hv)^jh^2(hv)^{-j}, (hv)^ihv^2h^{-1}(hv)^{-i} \\ / 0 \leq j \leq k - 1, 0 \leq i \leq k - 2 \rangle$$

*where  $k = g + 1$  for  $Esc_1(g)$  and  $k = g$  for  $Esc_2(g)$ .*

*For  $Escb_1(g)$  and  $Escb_2(g)$  they are given by,*

$$(4.1.4) \quad \langle (hv)^kr, (hv)^{k-1}hr, (hv)^jh^2(hv)^{-j}, (hv)^ihv^2h^{-1}(hv)^{-i} \\ / 0 \leq j \leq k - 1, 0 \leq i \leq k - 2 \rangle$$

where again  $k = g + 1$  for  $\text{Escb}_1(g)$  and  $k = g$  for  $\text{Escb}_2(g)$ .

For surfaces in  $\text{St}_1$  and  $\text{St}_2$  these computations have been done by G. Schmit-hüsen in [Sch]. Similar computations yield the result for the other families.

Our next objective is to compute equations for the associated algebraic curves. We do  $\text{Esc}_1(g)$  and  $\text{Escb}_1(g)$  first.

**4.2 Proposition.** *Surfaces in  $\text{Esc}_1(g)$  or  $\text{Escb}_1(g)$  correspond to algebraic curves with an equation of the form*

$$(4.2.1) \quad y^2 = x^{2g+2} + ax^{g+1} + 1$$

(with  $-2 < a < 2$  when the elementary tile is a rectangle). The Abelian differential or the quadratic differential is a scalar multiple of

$$(4.2.2) \quad \omega = \frac{x^{(g-1)/2} dx}{y} \quad \text{or} \quad q = \frac{x^{g-1} dx^2}{y^2} .$$

Conversely if  $C$  is a curve with equation (4.2.1), with  $a \neq \pm 2$ , then, if  $g$  is odd,  $(C, \omega)$  is in the  $\text{SL}_2(\mathbb{R})$  orbit of surfaces in  $\text{Esc}_1(g)$  and if  $g$  is even,  $(C, q)$  is in the  $\text{SL}_2(\mathbb{R})$  orbit of surfaces in  $\text{Escb}_1(g)$ .

Moreover the equation of the elliptic curve corresponding to one elementary rectangle (or more generally elementary parallelogram) is

$$(4.2.3) \quad y^2 = x^4 + ax^2 + 1$$

with the same  $a$  as in (4.2.1) and we have

$$a = \frac{2\mu - 4}{\mu} ,$$

where  $\mu$  is the invariant of the rectangle as defined in section 2.

PROOF. We first note that, from the combinatorial structure of the identifications and the fact that the angle at the vertices of the equiquadrangles, defining the non-singular hyperbolic metric, is  $\pi/(g+1)$  one can easily check that the surface has an automorphism  $f_g$  of order  $g+1$ , the fixed points of which are the four images of the vertices.

Now assume that  $g$  is odd and hence the surface  $S$  is in  $\text{Esc}_1(g)$ . The quotient of  $S$  under  $f_g^2$  is a surface  $E$  in  $\text{Esc}_1(1)$  obtained from four rectangles. The quotient map is ramified precisely at the vertices of the rectangles, i.e. the fixed points of  $f_g$ .

To reconstruct this situation, we first note that in this case  $f_1$  is an involution and we can always assume that it is induced by  $x \mapsto -x$ . The four vertices will then be points above  $x = 0$  and  $x = \infty$ . Label the rectangles from 1 to 4 starting with the lower left. The choice we have just made implies that if the  $x$  coordinates of the center of rectangles 1 and 2 are  $x_1$  and  $x_2$  then the centers of rectangles 3 and 4 are  $-x_1$  and  $-x_2$ . We can still make one choice, so we choose the midpoints of the horizontal edges of rectangles 2 and 3 to have  $x$ -coordinate 1. The midpoints of the horizontal edges of rectangles 1 and 4 will then correspond to  $-1$ . But this implies that the involution obtained by rotating the arrangement of rectangles by

angle  $\pi$  will be induced by  $x \mapsto 1/x$ . In which case we have  $x_2 = -1/x_1$ . Hence an equation of the form

$$(4.3) \quad y^2 = (x^2 - x_1^2)(x^2 - 1/x_1^2) = x^4 + a x^2 + 1$$

Now we have four involutions induced by half-turn around:

- (1) the vertices; this is  $f_1$  which is  $(x, y) \mapsto (-x, y)$ ;
- (2) the midpoints of horizontal sides; this is  $(x, y) \mapsto (1/x, y/x^2)$ ;
- (3) the centers of the rectangles; this is  $(x, y) \mapsto (x, -y)$ ;
- (4) the midpoints of vertical edges.

Since (4) is the composition of (1), (2) and (3), it is  $(x, y) \mapsto (-1/x, -y/x^2)$ , and hence its fixed points are the points with  $x$ -coordinate  $\pm i$ .

To end the description of  $E$  we note that for rectangles we have real structures obtained by taking reflections in the horizontal or vertical medians of the rectangles. The first must fix the points above  $\pm i$  and the second the points above  $\pm 1$ . This implies that the first is  $(x, y) \mapsto (1/\bar{x}, -\bar{y}/\bar{x}^2)$  and the second is  $(x, y) \mapsto (1/\bar{x}, \bar{y}/\bar{x}^2)$ . This in turn implies that  $|x_1| = 1$  and hence  $-2 < a < 2$ .

Now consider the genus  $g$  curve with equation

$$(4.4) \quad y^2 = x^{2g+2} + a x^{g+1} + 1,$$

$g = 2n + 1$  odd. This curve has an obvious automorphism of order  $g + 1$  defined by  $\varphi : (x, y) \mapsto (\zeta x, y)$ , where  $\zeta$  is a primitive  $(g + 1)$ -th root of unity. The fixed points of  $\varphi$  are the points above  $x = 0$  and the points at infinity. The quotient of this curve under  $\varphi^2$  is the curve with equation (4.3). The quotient morphism is  $(x, y) \mapsto (x^{n+1}, y)$  which is precisely ramified at the points above  $x = 0$  and the points at infinity. This proves (4.2.1) for surfaces in  $\text{Esc}_1(g)$ .

If  $g$  is even and  $S$  is in  $\text{Escb}_1(g)$  we consider the surface  $S'$  in  $\text{Esc}_1(2g + 1)$  obtained from the same rectangles. Then  $S'$  is a double cover of  $S$  and the covering is precisely ramified at the vertices of the rectangles. But we know from the above that  $S'$  has an equation of the form  $y^2 = x^{4g+4} + a x^{2g+2} + 1$ . From this it is not hard to deduce that  $S$  has again an equation of the form (4.2.1).

Since the differentials vanish at the points above  $x = 0$  and the points at infinity, we also get (4.2.2).

The third assertion follows from the fact that the image in moduli space of the set of surfaces in  $\text{Esc}_1(g)$  (resp.  $\text{Escb}_1(g)$ ) is an algebraic curve.

To end the proof we only need to note that the surface  $E$  obtained from four rectangles is obviously isomorphic to the one obtained from one rectangle and that the curve defined by (4.3) is isomorphic to

$$y^2 = x(x-1) \left( x - \frac{4}{2-a} \right)$$

the isomorphism being induced by

$$x \mapsto \frac{2x_1(x-x_1)}{(1+x_1^2)(x_1x-1)}.$$

If the rectangles are in fact squares we have  $\mu = 2$  hence,

**4.5 Corollary.** *Let  $S$  be a surface of genus  $g$  in  $\text{Esc}_1$  or  $\text{Escb}_1$ . If  $S$  is tiled by squares, then an equation for  $S$  is*

$$y^2 = x^{2g+2} + 1 .$$

Form **4.2** we are going to deduce the other cases. We do  $\text{Esc}_2(g)$  and  $\text{Escb}_2(g)$  first.

**4.6 Proposition.** *Let  $S$  be a surface of odd, respectively even, genus  $g$  in  $\text{Esc}_2$ , respectively  $\text{Escb}_2$ . Then the corresponding algebraic curve has an equation of the form*

$$(4.6.1) \quad y^2 = x(x^{2g} + ax^g + 1) .$$

*The Abelian, resp. quadratic, differential is a scalar multiple of*

$$(4.6.2) \quad \omega = \frac{x^{(g-1)/2} dx}{y} \quad \text{resp.} \quad q = \frac{x^{g-1} dx^2}{y^2} .$$

*Conversely if  $C$  is a curve with equation (4.6.1), with  $a \neq \pm 2$ , then, if  $g$  is odd,  $(C, \omega)$  is in the  $\text{SL}_2(\mathbb{R})$  orbit of surfaces in  $\text{Esc}_2$  and if  $g$  is even,  $(C, q)$  is in the  $\text{SL}_2(\mathbb{R})$  orbit of surfaces in  $\text{Escb}_2$ .*

*Moreover the equation of the elliptic curve corresponding to one elementary rectangle (or more generally elementary parallelogram) is*

$$(4.6.3) \quad y^2 = x^4 + ax^2 + 1$$

*with the same  $a$  as in (4.6.1) and this parameter only depends on the elementary parallelogram used in the construction.*

*In particular if  $S$  is tiled by squares, then an equation for  $S$  is*

$$(4.6.4) \quad y^2 = x(x^{2g} + 1) .$$

**PROOF.** Let  $S$  be in  $\text{Esc}_2(g)$  or in  $\text{Escb}_2(g)$  depending on the parity of  $g$ . Consider the surface  $S'$  in  $\text{Esc}_1(2g-1)$  obtained from the same rectangles. Then  $S'$  is an unramified double cover of  $S$ . More precisely let  $h$  be  $f_{2g-1}^g$  composed with the hyperelliptic involution. Then  $S$  is the quotient  $S'/h$ . From **4.2** we know that  $S'$  has an equation of the form (4.2.1) and that  $h$  is  $(x, y) \mapsto (-x, -y)$ .

From this it is immediate to deduce (4.6.1) the quotient map being  $(x, y) \mapsto (x^2, xy)$ . The rest easily follows from Proposition **4.2** and its proof.

For  $\text{St}_1(g)$  and  $\text{St}_2(g)$  the expressions we find are not so nice, but they can nevertheless be deduced from **4.2** and **4.6**.

Let  $S'$  be a surface in  $\text{Esc}_1(2g-1)$ . The involution induced by rotation of the arrangement of rectangles has 4 fixed points, but if we compose this involution with the hyperelliptic involution we obtain an involution with no fixed points. The quotient  $S$  of  $S'$  under this last involution is in  $\text{St}_1(g)$  and obtained from the same rectangles. We note also that all surfaces in  $\text{St}_1(g)$  are obtained in this way.

On the other hand, by Proposition **4.2**,  $S'$  has an equation of the form  $y^2 = x^{4g} + ax^{2g} + 1$  and the involution we are considering is  $(x, y) \mapsto (1/x, y/x^{2g})$ . An equation for the quotient is not as easy to express as in previous cases but we can do the following. Let  $x_1, 1/x_1, \dots, x_{2g}, 1/x_{2g}$  be the roots of  $x^{4g} + ax^{2g} + 1$ . Let  $t_k = i \frac{1+x_k}{1-x_k}$ . Then  $y^2 = \prod (x^2 - t_k^2)$  is also an equation for  $S$  and the involution  $(x, y) \mapsto (1/x, y/x^{2g})$  is now  $(x, y) \mapsto (-x, -y)$ . Hence

**4.7 Proposition.** *Let  $S$  be in  $St_1(g)$  and let  $t_i$  be as above. Then an equation for  $S$  is*

$$y^2 = x \prod (x - t_k^2) .$$

*The differential defining the locally flat metric on  $S$  is*

$$\frac{(x+1)^{g-1} dx}{y} .$$

For surfaces in  $St_2(g)$  we can use the same argument but starting with a surface  $S'$  in  $Esc_2(2g-1)$ . Let  $x_1, 1/x_1, \dots, x_{2g-1}, 1/x_{2g-1}$  be the roots of  $x^{4g-2} + a x^{2g-1} + 1$  and let  $t_k = i \frac{1+x_k}{1-x_k}$ .

**4.8 Proposition.** *Let  $S$  be in  $St_2(g)$  and let the  $t_k$  be as above, then an equation for  $S$  is*

$$y^2 = x(x+1) \prod (x - t_k^2) .$$

*The differential defining the locally flat metric on  $S$  is*

$$\frac{(x+1)^{g-1} dx}{y} .$$

If the surfaces are tiled by squares, i.e. if we have  $a = 0$  in the equations for  $S'$  in  $Esc_1(2g-1)$  or in  $Esc_2(2g-1)$ , a tedious but elementary computation shows that we have

**4.9 Corollary.** *If  $S$  is the surface in  $St_1(g)$  tiled by squares then  $S$  has for equation*

$$y^2 = x \left( \sum_k (-1)^k \binom{4g}{2k} x^k \right) .$$

*If  $S$  is the surface in  $St_2(g)$  tiled by squares then  $S$  has for equation*

$$y^2 = x(x+1) \left( \sum_k (-1)^{k+1} \binom{4g-2}{2k} x^k \right) .$$

## 5. Balanced genus 2 surfaces tiled by four rectangles.

In the last section we have seen relations between different surfaces tiled by rectangles. The aim of this section is to explore in more detail the consequences of Proposition 3.6 and Corollary 3.7 in the case of balanced surfaces of genus 2 tiled by four rectangles. We will also exhibit the action of other fractional Dehn twists exchanging the different families.

**5.1 Proposition.** *There are exactly four  $PSL_2(\mathbb{R})$  orbits of balanced surfaces tiled by four rectangles. These are the orbits of the surfaces described in figure 13 (where identifications are indicated by numbers and top-top or bottom-bottom identifications are by half-turns while all others are by translations).*

PROOF. Consider a balanced surface tiled by 4 rectangles. We leave aside the case when the surface is a torus. The angles at the vertices of the rectangles add up to

$4 \cdot 2\pi = 8\pi$ . Thus the vertices cannot be identified to a single point on the surface: this would be a zero of order 6 for the quadratic differential, but the multiplicities of the zeros of a quadratic differential add up to  $4g-4$ , a multiple of 4. The vertices can be identified to two points on the surface, each of angle  $4\pi$ , corresponding to two zeros of order 2 for the quadratic differential (possibly the square of an Abelian differential with two zeros of order 1). The case when the vertices are identified to four points on the surface has been ruled out (they would be points of angle  $2\pi$  and the surface would be a torus).

Let us therefore enumerate all (connected) surfaces obtained from 4 rectangles, the vertices of the rectangles being identified to 2 points of angle  $4\pi$ . Decomposing the surface into horizontal cylinders, we find one of the four possible situations: all 4 rectangles are lined up horizontally and form a single horizontal cylinder of width 4 rectangles; they form 2 cylinders of widths 2 and 2 or of widths 1 and 3; or they form three cylinders of widths 1, 1, and 2. There cannot be 4 horizontal cylinders of widths 1: if all rectangles have their left and right sides glued together then the surface is a torus.

Now consider the four possible cylinder decompositions in turn.

**Case 1:** one cylinder of width 4. Consider the 4 rectangles lined up horizontally, forming a wide rectangle; the surface results from the identification of the left and right sides of this wide rectangle, and of pairwise identifications of the horizontal rectangle edges. Label the bottom sides of the rectangles 1, 2, 3, 4, from left to right, and likewise their top sides 5, 6, 7, 8.

Gluing 1-2, 1-4, 2-3, 3-4, 5-6, 5-8, 6-7, 7-8 are forbidden, they would yield a cone point of angle  $\pi$ . Gluing 1-5 and 2-6 simultaneously is also forbidden, it would yield a point of angle  $2\pi$ . Avoiding such gluings, let us enumerate the possible surfaces we can obtain by gluing pairs of sides.

(1-3, 2-4, 5-7, 6-8), (1-3, 2-5, 4-7, 6-8), (1-3, 2-6, 4-8, 5-7), (1-3, 2-7, 4-5, 6-8) and (1-3, 2-8, 4-6, 5-7) give five different surfaces in  $\mathcal{Q}(2, 2)$ .

(1-5, 2-8, 3-7, 4-8) and (1-6, 2-5, 3-8, 4-7) give two different surfaces in  $\mathcal{H}(1, 1)$ .

The remaining possible gluings, starting with 1-7 or 1-8, would yield surfaces already listed (one can see that by cutting the leftmost square and pasting it to the right, or cutting the rightmost square and pasting it to the left).

**Case 2:** two cylinders of widths 2 and 2.

The two cylinders have to be glued one to the other, so let us number the rectangles  $R_1, R_2, R_3, R_4$ , and suppose we start with  $R_2$  glued to the right of  $R_1$ ,  $R_3$  glued on top of  $R_2$ ,  $R_4$  glued to the right of  $R_3$ . Label 1, 2, 4 the bottom sides of  $R_1, R_2, R_4$ , and 3, 5, 6 the top sides of  $R_1, R_3, R_4$ .

Gluing 1-2, 5-6 are forbidden, they would yield a cone point of angle  $\pi$ . Gluing 1-5 and 2-6 simultaneously is also forbidden, it would yield a point of angle  $2\pi$ .

This leaves the following possible gluings.

(1-3, 2-5, 4-6) and (1-3, 2-6, 4-5) give surfaces in  $\mathcal{H}(1, 1)$ , (1-4, 2-5, 3-6), (1-4, 2-6, 3-5) give surfaces in  $\mathcal{Q}(2, 2)$ . (1-5, 2-3, 4-6) and (1-5, 2-4, 3-6) give the surfaces already listed as (1-3, 2-6, 4-5) and (1-4, 2-6, 3-5) respectively. (1-6, 2-3, 4-5) gives a surface in  $\mathcal{H}(1, 1)$ , and (1-6, 2-4, 3-5) a surface in  $\mathcal{Q}(2, 2)$ .

**Case 3:** two cylinders of widths 3 and 1.

Again let us number the rectangles  $R_1, R_2, R_3, R_4$ , and without loss of generality start with  $R_2$  glued to the right of  $R_1$ ,  $R_3$  glued to the right of  $R_2$ ,  $R_4$  glued on top of  $R_1$ . Label 1, 2, 3 the bottom sides of  $R_1, R_2, R_3$ , and 4, 5, 6 the top sides of  $R_4, R_2, R_3$ .

The following gluings are possible, and give rise to different surfaces.

(1–4, 2–6, 3–5), (1–5, 2–6, 3–4), (1–6, 2–5, 3–4).

**Case 4:** three cylinders of widths 1, 1 and 2.

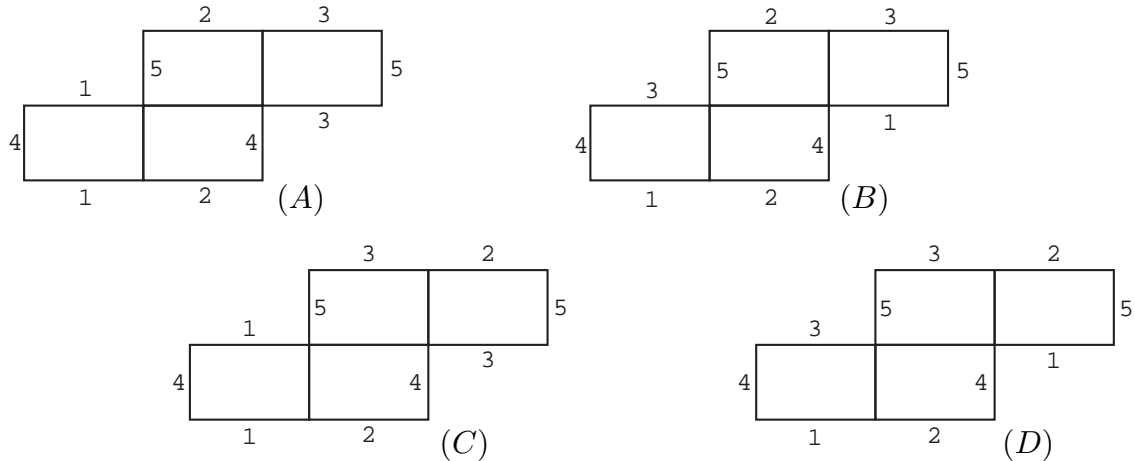
Without loss of generality we can start with  $R_2$  glued to the right of  $R_1$ ,  $R_3$  glued below  $R_1$ , and  $R_4$  glued either on top of  $R_1$  (sub-case 1) or on top of  $R_2$  (sub-case 2).

In sub-case 1, label 1 and 2 the bottom sides of  $R_4$  and  $R_2$ , 3 and 4 the top sides of  $R_3$  and  $R_2$ . The possible gluings are (1–2, 3–4) and (1–4, 2–3).

In sub-case 2, label 1 and 2 the bottom sides of  $R_4$  and  $R_2$ , 3 and 4 the top sides of  $R_1$  and  $R_3$ . The possible gluings are (1–2, 3–4) and (1–3, 2–4).

The gluings (1–3, 2–4) of sub-cases 1 and 2 yield the same surface, so we get three different surfaces from case 4.

Our case study evidenced 19 different balanced surfaces tiled with 4 rectangles; one can compute their  $SL(2, \mathbb{Z})$ -orbits and see that these 19 surfaces fall into 4 different orbits, representatives of which are presented in figure 13 (the detailed description of the 19 different cases is the object of the rest of this section).



**Fig. 13**

To give a full description of these families we need a more detailed description of the geometry of the hyperbolic quadrangles we will consider.

Let  $L$  be as in Lemma 2.1, then since the angle is  $\pi/4$  we have  $L' = \sqrt{\frac{2L^2 - 1}{2L^2 - 2}}$ . If we take  $A$  and  $B$  as in (2.1.1), with this value of  $L'$ , and  $T$  as in (2.1.2), then  $A$  and  $T \cdot B$  generate a Fuchsian group of signature (1; 2). Moreover all Fuchsian groups of signature (1; 2) are conjugate to one of this form. Note also that in this context the elliptic element  $e_1$  of order two at the center of symmetry of the quadrangle is the conjugate of  $z \mapsto -z$  by a matrix of the same form as  $T$  but with  $Tw$  replaced by  $\sqrt{(Tw + 1)/2}$ .

Call  $L_1$  (resp.  $L_2$ ) the hyperbolic cosine of the hyperbolic distance between  $p_1$  and  $p_3$  (resp.  $p_2$  and  $p_4$ ) in figure 7 (left), call  $L_3$  (resp.  $L_4$ ) the hyperbolic cosine of the hyperbolic distance between  $q_1$  and  $q_2$  (resp.  $q_1$  and  $q_4$ ) (figure 7 left) and finally call  $L'_2$  (resp.  $L'_4$ ) the hyperbolic cosine of the hyperbolic distance between  $p_3$  and  $p'_2$  (resp.  $q_4$  and  $q'_1$ ) (figure 7 right).

**5.2 Lemma.** *We have the following relations*

- (i)  $L_2 = \frac{L_1 + 1}{L_1 - 1}$ ;
- (ii)  $L_3 = 2L_1 + 1$ ;
- (iii)  $L_4 = 2L_2 + 1$ ;
- (iv)  $L'_2 = 4Tw^2(L_2 + 1) - 1 = Tw^2 \frac{2L_1}{L_1 - 1} - 1$  (with  $Tw$  as in (2.1.2));
- (v)  $L'_4 = 2L'_2 + 1$ .

PROOF. The first three assertions are immediate consequence of the relations between the side lengths of a trirectangular quadrangle with remaining angle  $\pi/4$  (see [Bu], p.454). For (iv) we note that the trace of  $T \cdot B$  is  $L'' = 2TwL'$  ( $L'$  as above). But  $L_2 = 2L'^2 - 1$  and  $L'_2 = 2L''^2 - 1$ , combined with (i) this yields (iv). For (v) we note that  $q'_1$  is the image of  $q_4$  under the action  $T \cdot B$  and from this the distance between  $q_4$  and  $q'_1$  can easily be computed and the relation checked.

**5.3 Proposition.** *Let  $S_A, S_B, S_C$  or  $S_D$  be in the  $SL_2(\mathbb{R})$  orbit of one of the surfaces of type (A), (B), (C) or (D) of 5.1. Then Fenchel-Nielsen coordinates of these surfaces are of the form*

- (i)  $(\ell, tw, \ell, tw, \ell', 0)$  for  $S_A$ ;
- (ii)  $(\ell, tw, \ell, tw, \ell', \frac{1}{2})$  for  $S_B$ ;
- (iii)  $(\ell, tw + \frac{1}{2}, \ell, tw, \ell', 0)$  for  $S_C$ ;
- (iv)  $(\ell, tw + \frac{1}{2}, \ell, tw, \ell', \frac{1}{2})$  for  $S_D$ ,

where  $\cosh(\ell'/2) = 2 \cosh(\ell/2) + 1$ .

PROOF. We first note that the surfaces are balanced and that in all cases the quadratic differential defining the locally flat metric has two zeros of order 2. Hence replacing the rectangles by an equiquadrangle with interior angle  $\pi/4$  (left of figure 7) leads to a smooth hyperbolic surface. Moreover since the angles at the vertices are  $\pi/4$  the union of the arcs labeled 1 and 3 in figure 13 (all cases) forms a simple closed hyperbolic geodesic that we will call  $\gamma_3$ .

The medians of the horizontal cylinders are also simple closed hyperbolic geodesics  $\gamma_1$  (for the lower cylinder) and  $\gamma_2$  (for the upper cylinder). Since the  $\gamma_i$  do not intersect they define pants decomposition of the surfaces.

To obtain Fenchel-Nielsen coordinates we start with case (A) for equiquadrangles. In this case reflection along  $\gamma_1$  clearly fixes point-wise  $\gamma_2$  and  $\gamma_3$ , from this it follows that the twist parameters are all zero in this case hence Fenchel-Nielsen coordinates of the form  $(\ell, 0, \ell, 0, \ell', 0)$ . The assertion that  $\cosh(\ell'/2) = 2 \cosh(\ell/2) + 1$  immediately follows from 5.2 (ii).

We have here only considered the equiquadrangle case to use the fact that the  $\gamma_i$  are multigeodesic, but clearly, using 5.2 (iv), we can replace the rectangles by more general quadrangles of the form illustrated on the right of figure 7, this yields Fenchel-Nielsen coordinates of the form  $(\ell, tw, \ell, tw, \ell', 0)$  with of course again  $\cosh(\ell'/2) = 2 \cosh(\ell/2) + 1$ . That this describes the full  $SL_2(\mathbb{R})$  orbit follows from Proposition 3.6 (for this case see also [Si2] section 3).

The claim for surface of type (B) can also be deduced from [Si2] section 3, but to cover also the relation between type (C) and (D) we are going to deduce this from a more general result.

**5.4 Lemma.** *Let  $S$  be a surface in  $Esc_2(g)$ , if  $g$  is odd, or  $Escb_2(g)$ , if  $g$  is even. If  $S$  is in  $Esc_2(g)$  then the union of the arcs labeled  $a \sim d$  and  $b \sim c$  (see figure 11)*



forms a simple closed hyperbolic geodesic  $\gamma$ . If  $S$  is in  $\text{Escb}_2(g)$  then the union of the arcs labeled  $a \sim c$  and  $b \sim d$  forms a simple closed hyperbolic geodesic that we will also denote by  $\gamma$ .

Let  $S'$  in  $\text{St}_1(g)$  be obtained from the same rectangles. In the same way  $\{a \sim b\} \cup \{c \sim d\}$  defines a simple closed geodesic  $\gamma'$  in  $S'$ .

Then  $S'$  is obtained from  $S$  by applying a half-Dehn twist along the geodesic  $\gamma$ . Conversely  $S$  is obtained from  $S'$  by applying a half-Dehn twist along the geodesic  $\gamma'$ .

PROOF. Label consecutively the rectangles of figure 11 from 1 to  $2g$  starting with the rectangle on the lower left hand side. Note also that both for  $S$  and  $S'$  the vertices lie in two orbits.

In all cases label  $\circ$  those in the orbit of the upper left corner of rectangle 1 and label  $\bullet$  those in the other orbit.

Since the angle at the vertices is  $\pi/(2g)$  the combinatorics of the rectangles at these points is described in figure 14. In the upper figure the geodesic  $\gamma$  corresponds to the horizontal line passing through  $\circ$  and  $\bullet$ , while in the lower figure the horizontal line represents  $\gamma'$ . This proves the assertions on  $\gamma$  and  $\gamma'$ .

Since  $\circ$  and  $\bullet$  split  $\gamma$  into two arcs of equal hyperbolic lengths it follows from the definition of a half-Dehn twist that we pass from the upper part of figure 14 to the lower part by performing a half-Dehn twist along  $\gamma$ . But the upper part gives the combinatorics for  $\text{Esc}_2$  or  $\text{Escb}_2$  while the lower gives the combinatorics for  $\text{St}_1$ . This proves Lemma 5.4.

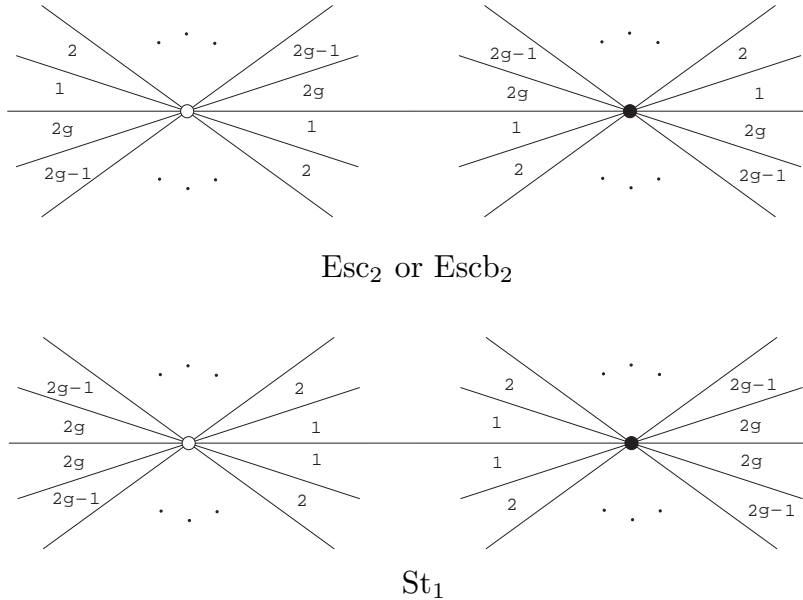


Fig. 14

For simplicity we have formulated 5.4 for rectangles and equiquadrangles, but obviously, although the geodesics may not conicide, the statement can be generalized for parallelograms by Proposition 3.6.

Since  $S_A$  is in  $\text{St}_1(2)$  and  $S_B$  in  $\text{Escb}_1(2)$  the statement of 5.3 for type (B) follows.

The same argument also shows that one passes from  $S_C$  to  $S_D$  by a half-Dehn twist along  $\gamma_3$ .

To end the proof we note that  $S_C$  (resp.  $S_D$ ) is obtained from  $S_A$  (resp.  $S_B$ ) by applying a half-Dehn twist along  $\gamma_2$ , as can be immediately checked by looking at the identifications.

We end this section by computing the equations for the different families.

**Case (A).** This family is of course  $\text{St}_1(2)$  and equations can be recovered by applying the results of section 2. But to highlight the links between the different cases we are going to use a slightly different approach.

We use the fact that the surface has a non hyperelliptic involution induced by a rotation of angle  $\pi$ . Hence we can look for an equation of the form

$$(5.5) \quad y^2 = (x^2 - a^2)(x^2 - 1)(x^2 - b^2) .$$

If we label the rectangles  $R_1$  to  $R_4$  starting with the lower left one, the existence of the additional involution imposes that the midpoints of the horizontal edges of rectangles  $R_2$  and  $R_3$  have  $x$ -coordinate 0 or  $\infty$ . We choose 0. Since composing this involution with the hyperelliptic one fixes the vertices, these will have  $x$ -coordinate  $\infty$ . We normalize further by choosing the  $x$ -coordinate of the Weierstrass point at the center of rectangle  $R_1$  to be  $-1$  which forces the center of the rectangle  $R_4$  to have  $x$ -coordinate 1. Finally we choose the Weierstrass points  $(-b, 0)$  and  $(b, 0)$  to be the midpoints of the horizontal edges of rectangles  $R_1$  and  $R_4$  respectively. With this fixed, a map from the surface to the genus 1 surface tiled by one rectangle is induced by the map

$$(5.6) \quad f : x \mapsto \frac{x^2(x^2 - b^2)}{1 - b^2} .$$

But now  $a$  and  $-a$  are simply solutions of  $f(x) = 1$  distinct from  $\pm 1$ . Hence

$$(5.7) \quad a = \sqrt{b^2 - 1} .$$

To complete the description of the genus 1 quotient (or alternatively recover  $b$  in terms of the genus 1 quotient) we only need to compute for which values of  $\lambda$  the equation  $x^2(x^2 - b^2) - \lambda(1 - b^2) = 0$  has a double root. This yields

$$(5.8) \quad \lambda = \frac{b^4}{4(b^2 - 1)} .$$

This does not conform to our convention on the  $\mu$  invariant for genus 1 but we easily find,

$$(5.9) \quad \mu = \frac{\lambda}{\lambda - 1} = \frac{b^4}{(b^2 - 2)^2} = \frac{(a^2 + 1)^2}{(a^2 - 1)^2} .$$

Summarizing, we have an equation for the algebraic curve, of the form

$$(5.10) \quad y^2 = (x^2 - a^2)(x^2 - 1)(x^2 - a^2 - 1)$$

and a degree 4 map, ramified at the points at infinity,

$$(5.11) \quad (x, y) \mapsto \left( \frac{(2x^2 - a^2 - 1)^2}{(a^2 - 1)^2}, y \frac{4x(2x^2 - a^2 - 1)}{(a^2 - 1)^3} \right)$$

onto the genus 1 curve defined by

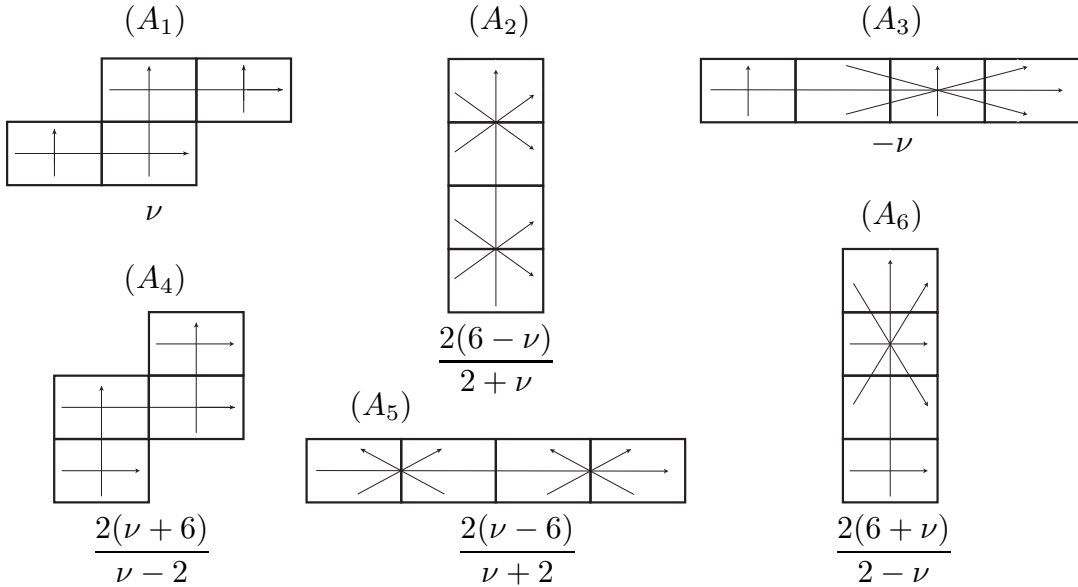
$$(5.12) \quad y^2 = x(x-1) \left( x - \frac{(a^2+1)^2}{(a^2-1)^2} \right).$$

To obtain the values of  $a$  for the other combinations of rectangles in the same  $SL_2(\mathbb{Z})$ -orbit we note that the difference between cases  $(A_1)$  to  $(A_6)$  (see figure 15) is in the repartition of the Weierstrass points among vertices, centers of rectangles, horizontal edges and vertical edges. These are  $(0, 4, 2, 0)$   $(A_1)$ ,  $(0, 0, 2, 4)$   $(A_2)$ ,  $(0, 2, 4, 0)$   $(A_3)$ ,  $(0, 4, 0, 2)$   $(A_4)$ ,  $(0, 0, 4, 2)$   $(A_5)$  or  $(0, 2, 0, 4)$   $(A_6)$  (see figure 15). We can then proceed as above.

But a better solution here is to replace, in equation (5.9),  $\mu$  by  $1 - \mu$ ,  $1/\mu$  and so forth. For this, note that one passes from  $(A_1)$  to  $(A_2)$  by replacing the rectangle defined by  $\tau$  by the parallelogram defined by  $\tau + 1$ , hence by **2.8** and **2.9**,  $\mu$  by  $1 - \mu$ . From the geometric point of view this is just one of the cases of Corollary **3.7** and can be described as applying half twists along  $\gamma_1$  and  $\gamma_2$ .

To pass from  $(A_1)$  to  $(A_3)$  we replace  $\tau$  by  $\tau/(1 - \tau)$ , and hence  $\mu$  by  $1/\mu$ . From there the other transformations needed are clear, one passes from  $(A_2)$  to  $(A_6)$  (resp. from  $(A_3)$  to  $(A_5)$ ) by a quarter Dehn-twist along the median of the vertical cylinder (resp. horizontal cylinder) and from  $(A_4)$  to  $(A_5)$  by two vertical half-twists.

The different values one obtains are best expressed in terms of  $\nu = a^2 + 1/a^2$  which is a modular invariant (see [Si2] section **3**). The values one obtains are given in figure 15.



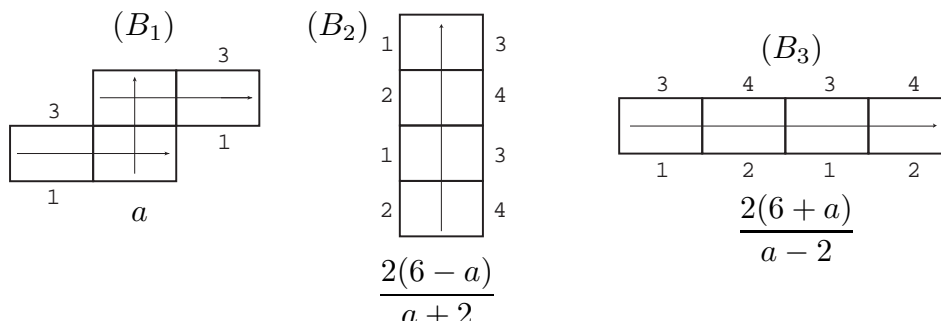
**Fig. 15**

**Case (B).** This family is  $Escb_2(2)$  and we have an equation of the form  $y^2 = x(x^4 + ax^2 + 1)$  (see section **4**) with

$$(5.13) \quad a = \frac{2\mu - 4}{\mu}.$$

There are three surfaces  $(B_1)$ ,  $(B_2)$  and  $(B_3)$  in the  $SL_2(\mathbb{Z})$  orbit (see figure 16). One passes from  $(B_1)$  to  $(B_2)$  by two half-Dehn twists along the medians of the

horizontal cylinders or as above by replacing  $\mu$  by  $1 - \mu$ . In the same way one passes from  $(B_1)$  to  $(B_3)$  by replacing  $\mu$  by  $1/\mu$ . This yields the different values indicated in figure 16.



**Fig. 16**

There are of course obvious similarities between case (A) and case (B). These are explored in detail in [Si2] section 3.

**5.14 Remark.** Here again we can be far more general. For exactly the same reasons, the transformations

$$a \mapsto \frac{2(6-a)}{a+2} \quad \text{and} \quad a \mapsto \frac{2(6+a)}{a-2}$$

correspond to replacing in any of the escalator families  $\tau$  by  $1+\tau$  or  $1/\tau$  respectively. These in turn correspond to applying half-Dehn twists along the horizontal cylinders or the vertical cylinders.

**Case (C).** We again label the rectangles in figure 13,  $R_1$  to  $R_4$ , starting with the upper left. We normalize so that the  $x$ -coordinate of the vertices of the rectangles are the points at infinity. We normalize further so that the center of rectangle  $R_1$  has  $x$ -coordinate 1 and the midpoint of the horizontal edge between rectangles  $R_2$  and  $R_3$  has  $x$ -coordinate 0. Note that since this is not a Weierstrass point the midpoint of the lower edge of rectangle  $R_2$  (arc labeled 2 in figure 13) will also have  $x$ -coordinate 0. We will call  $a$ ,  $b$  and  $c$  the  $x$ -coordinate of, respectively, the midpoint of the horizontal edges of rectangle  $R_1$ , the center of rectangle  $R_2$  and the midpoint of the upper horizontal edge of rectangle  $R_3$  (arc labeled 3 in figure 13). This is also the lower edge of rectangle  $R_4$ . Finally we will call  $d_1$  and  $d_2$  the  $x$ -coordinates of the midpoints of the vertical edges of rectangles  $R_3$  and  $R_4$ . The Weierstrass points are the points with  $x$ -coordinates 1,  $a$ ,  $b$ ,  $c$ ,  $d_1$  and  $d_2$ .

With this notation we can choose for the map  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  that induces the covering map from the surface to the genus 1 curve obtained from one rectangle, the map

$$(5.15) \quad f(x) = \frac{x^2(x-a)(x-c)}{(1-a)(1-c)}.$$

This map sends 1,  $b$ , and two points with the same  $x$ -coordinate, to 1. Again this means that the equation  $f(x) = 1$  must have a double root (different from 1). This imposes conditions on  $a$  and  $c$ , namely

$$(5.16) \quad a = \frac{-(3t^2 + 4t + 2)(t + 2)}{t(t^2 + 2t + 2)} \quad c = \frac{-(t^2 + 4t + 6)(t + 1)}{t^2 + 2t + 2}.$$

From this one recovers

$$(5.17) \quad b = -\frac{t^2 + 3t + 2}{t}.$$

Let  $\lambda = f(d_i)$ . In addition to the  $d_i$  we again have two points with the same  $x$ -coordinate. Hence again  $f(x) = \lambda$  must have a double root (with of course  $\lambda \neq 1$ ). This imposes

$$(5.18) \quad \lambda = -\frac{(t^2 - 2)^2(3t^2 + 4t + 2)^3(t^2 + 4t + 6)^3}{1024t^3(t^2 + 2t + 2)^2(t + 2)^3(t + 1)^3}.$$

This defines the genus 1 quotient and from this on can easily compute the  $d_i$ . We find

$$(5.19) \quad d_1 = \frac{(t^2 - 2 + (t^2 + 4t + 2)i\sqrt{2})(t^2 - 2)}{4t(t^2 + 2t + 2)},$$

$$d_2 = \frac{(t^2 - 2 - (t^2 + 4t + 2)i\sqrt{2})(t^2 - 2)}{4t(t^2 + 2t + 2)}.$$

To obtain a full map we must introduce the points  $p$  and  $q$  which are mapped under  $f$  of (5.15), to  $f(d_1) = f(d_2)$  and 1 respectively. We have

$$p = -\frac{(t^2 + 4t + 6)(3t^2 + 4t + 2)}{4t(t^2 + 2t + 2)} \quad \text{and} \quad q = -\frac{2(t^2 + 3t + 2)}{t^2 + 2t + 2}.$$

With this a full map to the genus 1 curve is

$$(x, y) \mapsto \left( \frac{x^2(x - a)(x - c)}{(1 - a)(1 - c)}, y \frac{x(x - p)(x - q)}{\sqrt{(1 - a)^3(1 - c)^3}} \right).$$

This does not comply with our convention of section **2** but to recover  $\mu$  in terms of  $\lambda$  we only need to set

$$(5.20) \quad \mu = \frac{\lambda}{\lambda - 1}.$$

We will use this later.

The solution we have found is of course far from optimal since solving (5.18) for a specific value of  $\lambda$  yields in general 16 solutions in  $t$ . On the other hand these solutions come in groups of 4. Namely if  $t$  is a solution then so are,

$$(5.21) \quad t, \frac{2}{t}, -\frac{t+2}{t+1}, -\frac{2(t+1)}{t+2}.$$

Moreover these 4 solutions yield isomorphic curves, since replacing  $t$  by  $\frac{2}{t}$  leaves  $b$  fixed, exchanges  $a$  and  $c$  and exchanges the  $d_i$ , while replacing  $t$  by  $-\frac{t+2}{t+1}$  replaces  $b$  by  $1/b$ ,  $a$  by  $c/b$ ,  $c$  by  $a/b$  and the  $d_i$  by  $d_i/b$ . In other words these depend on choices in the above computations.

We can use relation (5.21) to simplify the equation (5.18). If we let  $w$  be a root of

$$f_\lambda = (\lambda - 1)x^4 - (6\lambda + 2)x^3 + 12\lambda x^2 - (8\lambda - 2)x + 1 ,$$

or equivalently

$$(5.22) \quad f_\mu = x^4 + (2 - 8\mu)x^3 + 12\mu x^2 - (2 + 6\mu)x + \mu - 1 ,$$

then solutions of the form (5.21) will be

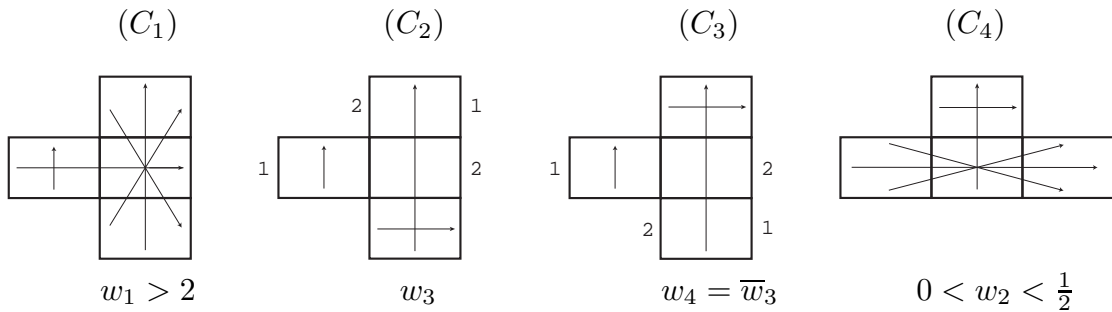
$$(5.23) \quad \frac{\sqrt{2}(u + 1 + \sqrt{2})}{u - 1 - \sqrt{2}}, \quad \text{where} \quad \left( \frac{u^2 + 1}{u^2 - 1} \right)^2 = w .$$

On the other hand we cannot improve further. The reason is that for the four surfaces in the  $\text{SL}_2(\mathbb{Z})$ -orbit we have the same repartition of Weierstrass points. Namely, with the same convention as before, it is  $(0, 2, 2, 2)$ .

To differentiate the cases we must take a closer look at equation (5.22). If  $\mu$  is real  $> 1$  then,  $f_\mu = 0$  will have two real roots,  $w_1 > 2$  and  $0 < w_2 < 1/2$ , and two complex conjugate roots  $w_3$  and  $w_4$ . Computing  $a, b, c, d_1$  and  $d_2$  in terms of these roots we find that  $a, b$  and  $c$  will be real for  $w_1$  while  $d_1$  and  $d_2$  are complex conjugate, for  $w_2$  on the other hand  $b, d_1$  and  $d_2$  will be real and  $a$  and  $c$  complex conjugate. Comparing with the real structures induced by reflection along the horizontal axis or the vertical axis in cases  $(C_1)$  and  $(C_4)$  we conclude that  $w_1$  corresponds to  $(C_1)$  and  $w_2$  corresponds to  $(C_4)$  (recall that  $d_1$  and  $d_2$  are on vertical edges while  $a$  and  $c$  are on horizontal edges).

In the absence of additional information we can not distinguish between  $(C_2)$  and  $(C_3)$  which are mirror images of each other and correspond to the two complex conjugate roots  $w_3$  and  $w_4$ .

Finally we note that we pass from  $(C_1)$  to  $(C_2)$  (resp.  $(C_3)$ ) by a third of a Dehn twist (resp. minus a third of a twist) along the geodesic median of the vertical cylinder of width 3 (and a full twist along the cylinder of width 1). We pass from  $(C_2)$  to  $(C_4)$  by a third of a Dehn-twist along the geodesic median of the horizontal cylinder of width 3.



**Fig. 17**

**Case (D).** Label as before the rectangles in figure 13  $R_1$  to  $R_4$  starting with the lower left. The surface in this case has two non-hyperelliptic involutions with centers at the midpoints of the vertical edges of rectangles  $R_1$  and  $R_2$  for the first and at the centers of the rectangles  $R_3$  and  $R_4$  for the second.

We normalize so that the  $x$ -coordinates of the vertices are  $\pm i$ , the centers of rectangles  $R_3$  and  $R_4$  are the points at infinity and the midpoints of the vertical edges of rectangles  $R_1$  and  $R_2$  have  $x$ -coordinate 0. We denote  $\pm a$  the centers of rectangles  $R_1$  and  $R_2$  and  $\pm b$  the midpoints of the vertical edges of rectangles  $R_3$  and  $R_4$ .

These choices yield an equation of the curve in the form

$$(5.24) \quad y^2 = (x^2 - a^2)(x^2 + 1)(x^2 - b^2) .$$

A map from the curve to the genus 1 curve defined by one rectangle is induced by

$$(5.25) \quad f(x) = \frac{x^2(x^2 - a^2)}{(x^2 + 1)^2} .$$

Since by construction we have  $f(b) = f(-b) = 1$  we find

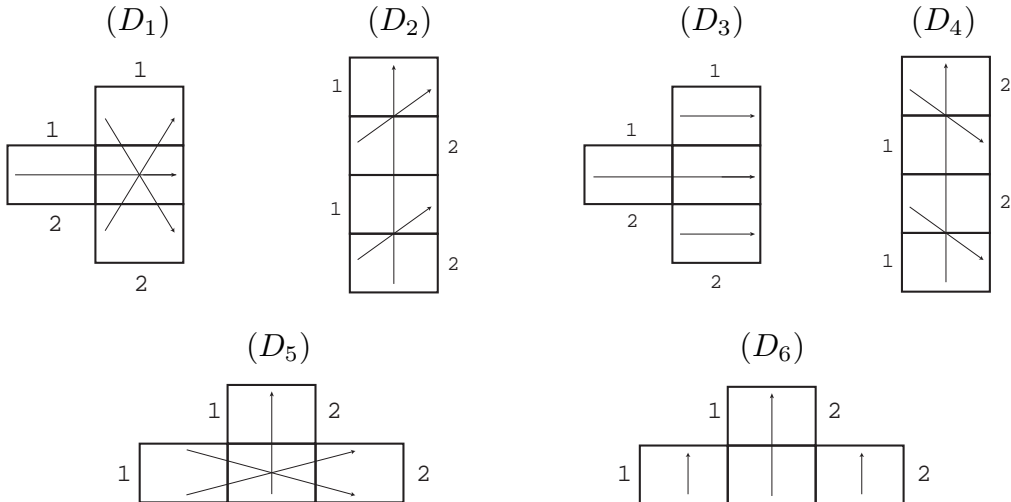
$$(5.26) \quad b = \frac{\pm i}{\sqrt{a^2 + 2}} .$$

To find the complete equation of the genus 1 curve, we look for double roots of the equation  $f(x) = \lambda$ . This yields

$$(5.27) \quad \lambda = 1 - \mu = -\frac{a^4}{4(a^2 + 1)} ,$$

where  $\mu$  conforms with the convention of section 2.

There are six different configurations in an  $\mathrm{SL}_2(\mathbb{Z})$  orbit (see figure 18) and to differentiate them we again look at the possible real structures and repartition of Weierstrass points among vertices, centers and horizontal and vertical edges. These are  $(2, 2, 0, 2)$  for  $D_1$  and  $D_3$ ,  $(2, 0, 2, 2)$  for  $D_2$  and  $D_4$  and  $(2, 2, 2, 0)$  for  $D_5$  and  $D_6$  (see figure 18).



**Fig. 18**

Hence the surfaces in this family come in three pairs. In the first  $(D_1), (D_3)$  the two surfaces are real, but the first has only one real component, while the second

has three (for the real structure induced by reflection along the vertical median of the cylinder of width 4). For  $\mu > 1$  equation (5.27) has 2 real roots and two pure imaginary roots

$$(5.28) \quad \pm \sqrt{2\mu - 2 + 2\sqrt{\mu^2 - \mu}} \quad \text{and} \quad \pm \sqrt{2\mu - 2 - 2\sqrt{\mu^2 - \mu}} .$$

Because of (5.24) and (5.26) this implies that the first corresponds to  $(D_1)$  while the second corresponds to  $(D_3)$ . The relation between  $(D_5)$  and  $(D_6)$  is similar and to find the corresponding value of  $a$  we only need, by Lemma 2.8, to replace  $\mu$  by  $\mu/(\mu - 1)$  in (5.28). The first value corresponds to  $(D_5)$ , the second to  $(D_6)$ .

To pass from  $(D_1)$  to  $(D_2)$  by a quarter Dehn-twist along the geodesic median of the vertical cylinder of width 4 and similarly from  $(D_2)$  to  $(D_3)$ , from  $(D_3)$  to  $(D_4)$  and from  $(D_4)$  to  $(D_1)$ . Hence the values of  $a$  for  $(D_2)$  and  $(D_4)$  are obtained by replacing  $\mu$  by  $1/\mu$  in (5.28) (see 2.8).

Finally note that we pass from  $(D_2)$  to  $(D_5)$  by a quarter Dehn-twist along the geodesic median of the horizontal cylinder of width 4.

We note here that combining (5.24), (5.25) and (5.28), we obtain equations

$$(5.29) \quad \begin{aligned} y^2 &= (x^2 + 1) \left( x^2 - 2 \left( \mu - 1 + \sqrt{\mu^2 - \mu} \right) \right) \left( x^2 + \frac{\mu - \sqrt{\mu^2 - \mu}}{2\mu} \right) \quad \text{for } (D_1) \\ y^2 &= (x^2 + 1) \left( x^2 - 2 \left( \mu - 1 - \sqrt{\mu^2 - \mu} \right) \right) \left( x^2 + \frac{\mu + \sqrt{\mu^2 - \mu}}{2\mu} \right) \quad \text{for } (D_3) \end{aligned}$$

with similar relations between  $(D_2)$  and  $(D_4)$  and between  $(D_5)$  and  $(D_6)$ .

## 6. Remarks on curves defined over number fields.

An immediate consequence of (5.29) is that if  $\mu$  is a general squarefree integer, then  $(D_1)$  and  $(D_3)$  are Galois conjugate in  $\mathbb{Q}[\sqrt{\mu}]$ . The interesting point here is that one passes from  $(D_1)$  to  $(D_3)$  by a fractional Dehn-twist. There seems to be many more instances where such a phenomenon occurs. Simple examples arise for a real quadratic number field  $K = \mathbb{Q}[\sqrt{d}]$  if  $\mu$  is a unit of norm 1 or if  $\mu = \frac{1}{2} + b\sqrt{d}$  since in the first case  $\mu$  and  $1/\mu$  are conjugate in  $K|\mathbb{Q}$  while in the second  $\mu$  and  $1 - \mu$  are conjugate. Examples of the first are  $\mu = 9 + 4\sqrt{5}$  and  $\mu = 97 + 56\sqrt{3}$  while  $\mu = (2 + \sqrt{5})/4$  and  $\mu = (12 + 7\sqrt{3})/24$  are examples for the second (see tables in Appendix).

The phenomenon is not restricted to quadratic extensions. Forgetting the reference to  $\mu$  the relation between three surfaces in the same  $\text{PSL}_2(\mathbb{Z})$  orbit of case (B) can be expressed as follows: Let  $a_1$ ,  $a_2$  and  $a_3$  be the three corresponding parameters of the form (5.13), then the  $a_i$  are the roots of an equation of the form

$$(6.1) \quad x^3 - \alpha x^2 - 36x + 4\alpha$$

and again one passes from one to the others by fractional Dehn-twists.

Similarly the six elements in a same  $\text{PSL}_2(\mathbb{Z})$  orbit in case (A) are obtained by letting the parameter  $\nu$  be one of the roots of

$$(6.2) \quad x^6 - (\alpha + 72)x^4 + (8\alpha + 1296)x^2 - 16\alpha .$$



For case (C), rewriting (5.24), the four surfaces are obtained by letting  $w$  be one of the roots of

$$(6.3) \quad x^4 + (2 - 8\alpha)x^3 + 12\alpha x^2 - (2 + 6\alpha)x + \alpha - 1.$$

The situation is a little less satisfactory here since we were not able to express directly equations for these surfaces in terms of the parameter  $w$ . On the other hand one can show, although the computations are too long to be presented here, that the coordinates of the isomorphy classes of the surfaces in the Igusa moduli space (see [Ig]) can be expressed in terms of the parameter  $w$ .

### Appendix: Values of $\mu$ for some equiquadrangles.

The values given in the tables that follow were computed numerically using a variant of the method described in [Bu-Si2] and should be considered as conjectured values. This being said most of these values are in fact exact and can be deduced from the known exact uniformization of certain curves (see for example [Ai-Si], [Bu-Si1], [Bu-Si2], [Si1] or [Si2]). In fact these exact cases often come from surfaces at the intersection of two families. For example the curve defined by  $y^2 = x^6 + 1$  is in  $\text{Escb}_1(2)$  and obtained from squares. But it also has an automorphism of order 4 and hence is also in  $\text{Escb}_2(2)$ , the corresponding equation is  $y^2 = x(x^4 - \frac{10}{3}x^2 + 1)$  and from this we obtain for angle  $\pi/4$  that  $L = \sqrt{2}$  yields  $\mu = \frac{4}{3}$ . In a similar fashion the curve with equation  $y^2 = x(x^4 + 1)$  is isomorphic to the one with equation  $y^2 = x^6 - 5i\sqrt{2}x^3 + 1$  and from this we can deduce that the surface defined by the trace triple (see below)  $(3 + 2\sqrt{2}, 4 + 2\sqrt{2}, 4 + 2\sqrt{2})$  yields  $\mu = \frac{1}{2} - \frac{5\sqrt{2}}{4}i$ . The computations are generally more involved but the method is the same. We give one last example: surfaces with Fenchel-Nielsen coordinates  $(\ell, \frac{1}{2}, \ell, \frac{1}{2}, \ell, \frac{1}{2})$  admit a second pants decomposition with coordinates  $(\ell', \frac{1}{2}, \ell, 0, \ell', \frac{1}{2})$  (see [Si2]) where  $L' = \cosh(\ell'/2) = \frac{3L-1}{2(L-1)}$  if  $L = \cosh(\ell/2)$ . We have  $\ell' = \ell$  for  $L = (5 + \sqrt{17})/4$  and from this information it is possible to compute the  $\mu$  for  $(1 + \sqrt{17})/4 = \cosh(\ell/4)$  in the  $\pi/3$  table.

In the first two tables  $L$  denotes the hyperbolic cosine of the hyperbolic half-length of the horizontal median of the equiquadrangle with interior angle  $\pi/n$  and  $\mu$  is the invariant defined in (2.7) or **2.9 3**. All the corresponding surfaces are with twist parameter zero.

We complete this list with a few examples with nonzero twist parameter. These are best described in terms of a trace triple and we use the convention of [Ac-Na-Ro], that is we describe them with a triple  $(x^2, y^2, z^2)$ , where  $x = \text{trace}(A)$ ,  $y = \text{trace}(B)$  and  $z = \text{trace}(A \cdot B^{-1})$ ,  $A$  and  $B$  being the generators of the group. For the surfaces with zero twist the corresponding trace triple is  $(4L^2, 4L'^2, 4L^2L'^2)$ ,  $L'$  as in Lemma **2.1**.

It should be noted that the examples presented in this appendix all have arithmetic Fuchsian groups (see tables in [Ta] and [Ac-Na-Ro]). On the other hand for general  $n$ , and in particular large enough  $n$ , the groups described in **(2.3)**, **(2.4)** and **(2.5)** are not arithmetic Fuchsian groups. They are however subgroups of triangle groups, and this is in accordance with a conjecture of Chudnovsky and Chudnovsky ([Ch-Ch] section 7) that if a curve defined over  $\overline{\mathbb{Q}}$  has a Fuchsian group  $G$  in  $\text{PSL}_2(\mathbb{R} \cap \overline{\mathbb{Q}})$  then  $G$  is either an arithmetic group or a subgroup of a triangle group.

For angle $\pi/3$		For angle $\pi/4$	
$L$	$\mu$	$L$	$\mu$
$\frac{\sqrt{2+\sqrt{2}}}{2}$	$\frac{837+1107\sqrt{2}}{2401}$	$\frac{\sqrt{10+2\sqrt{17}}}{2}$	$\frac{1151-217\sqrt{17}}{256}$
$\frac{\sqrt{7+\sqrt{17}}}{2}$	$\frac{23+\sqrt{17}}{27}$	$\frac{\sqrt{2+\sqrt{6}}}{2}$	$\frac{12+7\sqrt{3}}{24}$
$\frac{\sqrt{6+2\sqrt{3}}}{2}$	$27-15\sqrt{3}$	$\sqrt{3}$	$\frac{128}{125}$
$\sqrt{2}$	$\frac{27}{25}$	$\frac{1+\sqrt{5}}{2}$	$\frac{2+\sqrt{5}}{4}$
$\frac{\sqrt{5+\sqrt{5}}}{2}$	$\frac{32}{27}$	$\frac{\sqrt{5+\sqrt{17}}}{2}$	$\frac{897-217\sqrt{17}}{2}$
$\frac{\sqrt{4+2\sqrt{2}}}{2}$	$\frac{1564+1107\sqrt{2}}{2401}$	$\sqrt{2}$	$\frac{4}{3}$
$\frac{1+\sqrt{17}}{4}$	$\frac{621+27\sqrt{17}}{512}$	$\frac{\sqrt{18+2\sqrt{33}}}{4}$	$\frac{283+21\sqrt{33}}{256}$
$\frac{\sqrt{6}}{2}$	$2$	$\frac{\sqrt{4+2\sqrt{2}}}{2}$	$2$
$\frac{\sqrt{14+2\sqrt{17}}}{4}$	$-108+27\sqrt{17}$	$\frac{\sqrt{3+\sqrt{11}}}{4}$	$\frac{9+7\sqrt{33}}{18}$
$\frac{\sqrt{4+\sqrt{2}}}{2}$	$\frac{58+41\sqrt{2}}{27}$	$\frac{\sqrt{6}}{2}$	$4$
$\frac{\sqrt{2+\sqrt{10}}}{4}$	$\frac{32}{5}$	$\frac{\sqrt{14+2\sqrt{17}}}{4}$	$\frac{1151+217\sqrt{17}}{256}$
$\frac{\sqrt{5}}{2}$	$\frac{27}{2}$	$\frac{\sqrt{2+\sqrt{10}}}{4}$	$9+4\sqrt{5}$
$\frac{\sqrt{3+\sqrt{3}}}{2}$	$27+15\sqrt{3}$	$\frac{\sqrt{5}}{2}$	$\frac{128}{3}$
$\frac{\sqrt{10+2\sqrt{17}}}{4}$	$109+27\sqrt{17}$	$\frac{\sqrt{3+\sqrt{3}}}{2}$	$97+56\sqrt{3}$
$\frac{\sqrt{3+\sqrt{2}}}{2}$	$\frac{1566+1107\sqrt{2}}{2}$	$\frac{\sqrt{10+2\sqrt{17}}}{4}$	$\frac{897+217\sqrt{17}}{2}$
<b>For angle <math>\pi/5</math></b>			
$\frac{\sqrt{8+2\sqrt{5}}}{2}$	$\frac{65+29\sqrt{5}}{125}$	$\frac{\sqrt{2+\sqrt{10}}}{4}$	$\frac{1621+725\sqrt{5}}{121}$
$\frac{\sqrt{5+\sqrt{5}}}{2}$	$2$		

For angle $\pi/6$		For angle $\pi/8$	
$L$	$\mu$	$L$	$\mu$
$\frac{\sqrt{3} + \sqrt{7}}{2}$	$128 - 48\sqrt{7}$	$\sqrt{2 + \sqrt{2}}$	$\frac{-4 + 8\sqrt{2}}{7}$
2	$\frac{81}{80}$	$\frac{\sqrt{6 + 2\sqrt{2}}}{2}$	$\frac{3 + 2\sqrt{2}}{4}$
$\frac{3\sqrt{2} + \sqrt{10}}{4}$	$\frac{4096 - 1216\sqrt{10}}{243}$	$\sqrt{\cos(\pi/8) + 1}$	2
$\frac{\sqrt{6 + 2\sqrt{7}}}{2}$	$\frac{512 - 160\sqrt{7}}{81}$	$\frac{\sqrt{4 + 2\sqrt{2}}}{2}$	$\frac{11 + 8\sqrt{2}}{7}$
$\frac{\sqrt{10}}{2}$	$\frac{32}{27}$	$\frac{\sqrt{4 + \sqrt{2}}}{2}$	$12 + 8\sqrt{2}$
$\sqrt{2}$	$\frac{81}{49}$	<b>For angle 0</b>	
$\frac{1 + \sqrt{3}}{2}$	2	$\sqrt{5}$	$\frac{125 - 55\sqrt{5}}{2}$
$\frac{\sqrt{7}}{2}$	$\frac{81}{32}$	$\sqrt{3}$	$\frac{9}{8}$
$\frac{\sqrt{6}}{2}$	$\frac{32}{5}$	$\sqrt{2}$	2
$\frac{\sqrt{3 + \sqrt{7}}}{2}$	$\frac{512 + 160\sqrt{7}}{81}$	$\frac{\sqrt{6}}{2}$	9
$\frac{\sqrt{2} + \sqrt{10}}{4}$	$\frac{4096 + 1216\sqrt{10}}{243}$	$\frac{\sqrt{5}}{2}$	$\frac{125 + 55\sqrt{5}}{2}$
$\frac{\sqrt{5}}{2}$	81		
$\frac{\sqrt{3} + \sqrt{7}}{4}$	$128 + 48\sqrt{7}$		

For surfaces with a nonzero twist parameter, in terms of the trace triple  $(x^2, y^2, z^2)$ .

For  $\pi/3$  (i.e. sum of the interior angles equal to  $4\pi/3$ ).

$$(3 + \sqrt{7}, 4 + \sqrt{7}, 5 + \sqrt{7}), \quad \mu = \frac{59 + 17\sqrt{7}}{54}(1 + i).$$

$$(3 + 2\sqrt{2}, 4 + 2\sqrt{2}, 4 + 2\sqrt{2}), \quad \mu = \frac{1}{2} - \frac{5\sqrt{2}}{4}i.$$

$$(9, 6, 6), \quad \mu = 1/2$$

$$(1 + 4\rho + 4\rho^2, 1 + 4\rho + 4\rho^2, 1 + 4\rho + 4\rho^2), \quad \rho = \cos(2\pi/9), \quad \mu = (1 - \sqrt{3}i)/2.$$

For  $\pi/4$  (i.e. sum of the interior angles equal to  $\pi$ ).

$$(9, 7, 7), \mu = \frac{1}{2} - \frac{13\sqrt{7}}{98}i.$$

$$(4 + 2\sqrt{3}, 4 + 2\sqrt{3}, 4 + 2\sqrt{3}), \mu = (1 - \sqrt{3}i)/2.$$

$$(3/2 + \sqrt{2}, 4 + 2\sqrt{2}, 4 + 2\sqrt{2}), \mu = 1/2.$$

$$(3 + \sqrt{6}, 5 + 2\sqrt{6}, 6 + 2\sqrt{6}), \mu = (5 + 2\sqrt{6})(1 - \sqrt{3}i)/2.$$

For  $\pi/6$  (i.e. sum of the interior angles equal to  $2\pi/3$ ).

$$(9, 8, 8), \mu = \frac{1}{2} - \frac{7\sqrt{2}}{16}i.$$

$$(7/2 + 3\sqrt{5}/2, 5 + 2\sqrt{5}, 5 + 2\sqrt{5}), \mu = \frac{1}{2} - \frac{79\sqrt{5}}{80}i.$$

$$(3 + 2\sqrt{2}, 6 + 4\sqrt{2}, 6 + 4\sqrt{2}), \mu = \frac{1}{2} - \frac{11\sqrt{2}}{11}i.$$

$$(7 + 4\sqrt{3}, 4 + 2\sqrt{3}, 4 + 2\sqrt{3}), \mu = 1/2.$$

$$(\rho^2, \rho^2, \rho^2), \rho = 1 + 2\cos(\pi/9), \mu = (1 - \sqrt{3}i)/2.$$

#### REFERENCES

- [Ac-Na-Ro] P. Ackermann, M. Näätänen and G. Rosenberger, *The arithmetic Fuchsian groups with signature  $(0; 2, 2, 2, q)$* , Research and Exposition in Math. **27** (2003), 1–9.
- [Ai-Si] A. Aigon and R. Silhol, *Hyperbolic hexagons and algebraic curves in genus 3*, J. London Math. Soc. **66** (2002), 671–690.
- [Bea] A. Beardon, *The Geometry of Discrete Groups*, Springer G.T.M. 91, Berlin Heidelberg New York, 1991.
- [Bu] P. Buser, *Geometry and Spectra of Compact Riemann Surfaces*, Birkhäuser, Boston Basel Berlin, 1992.
- [Bu-Se] P. Buser and M. Seppälä, *Real structures of Teichmüller spaces, Dehn twists, and moduli spaces of real curves*, Math. Z. **232** (1999), 547–558.
- [Bu-Si1] P. Buser and R. Silhol, *Geodesics, periods and equations of real hyperelliptic curves*, Duke Math. J. **108** (2001), 211–250.
- [Bu-Si2] P. Buser and R. Silhol, *Some remarks on the uniformizing function in genus 2*, Geometriae Dedicata **115** (2005), 121–133.
- [Ch-Ch] D.V. Chudnovsky and G.V. Chudnovsky, *Computer algebra in the service of mathematical physics and number theory*, Computers and Mathematics, Proceedings Stanford 1986, Dekker 1990, 109–232.
- [Hu-Le] P. Hubert and S. Lelièvre, *Prime arithmetic Teichmüller discs in  $\mathcal{H}(2)$* , Israel J. of Math. **151** (2006), 281–321.
- [Ig] J. I. Igusa, *Arithmetic variety of moduli for genus two*, Ann. of Math. **72** (1960), 612–649.
- [Ku-Nä] T. Kuusalo and M. Näätänen, *Geometric uniformization in genus 2*, Annales Acad. Sci. Fennicae **20** (1995), 401–418.
- [Mc] C. T. McMullen, *Billiards and Teichmüller curves on Hilbert modular surfaces*, J. Amer. Math. Soc. **16** (2003), 857–885.
- [Mö] M. Möller, *Teichmüller curves, Galois actions and  $\widehat{GT}$ -relations*, Math. Nachr. **278** (2005), 1061–1077.
- [Ne] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York Toronto London, 1952.
- [Sch] G. Schmithüsen, *Examples for Veech groups of origamis*, Contemporary Math. **397** (2006), 193–206.
- [Si1] R. Silhol, *Hyperbolic Lego and equations of algebraic curves*, Contemporary Math. **311** (2002), 313–334.
- [Si2] R. Silhol, *On some one parameter families of genus 2 algebraic curves and half twists*, to appear in Commentarii Math. Helvetici **82** (2007).
- [Ta] K. Takeuchi, *Arithmetic Fuchsian groups with signature  $(1; e)$* , J. Math. Soc. Japan **35** (1983), 381–407.