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# Mod 3 Arithmetic on Triangulated Riemann Surfaces \*

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**Abstract.** Let  $T$  be a triangulation of a Riemann surface, orientable or non-orientable and of an arbitrary genus. Suppose, a labelling of the vertices of  $T$  by three labels 0, +1, and -1 is fixed. The present paper deals with the following problem : find the number of labellings of the faces of  $T$  by two labels +1 and -1, in such a way that the sum of the labels of the faces around any vertex is equal *modulo* 3 to the given label of the vertex. If  $T$  is a planar triangulation and all labels of vertices are zeros, then the problem of existence of such a labelling of faces is equivalent, according to P. J. Heawood, to the *four-colour problem for planar triangulations*, and the corresponding counting problem is equivalent to that of counting the number of all proper four-colourings of  $T$ .

**Keywords:** Riemann surface, triangulation graph, four-colouring, Heawood vector.

**A. M. S. Subject Classification (1991):** 05 C.

*Dedicated to the memory of Percy John Heawood, 1861 - 1955, on the occasion of the first centenary of his seminal paper [17].*

## 1. Introduction.

*During his early days at Durham while he was teaching a class of divinity students mathematics, one of them asked "You have taught us to cast out nines, can you cast out devils ?" "Yes, I can," replied Heawood, "get out at once !" [14], p. 266*

Percy John Heawood has had both the privilege and stamina to almost single-handedly create the field of graph colourings and to actively participate in its developments over a

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\* Expanded version of a talk presented at the Palaiseau Conference on Combinatorics and Computer Science (Palaiseau, France, September 1997).

period of nearly sixty years (cf. [14] and [8], p. 217). His last published papers [18], [19], as his first one [16], deals with the four-colour problem, **4CP** henceforth. In fact, the papers [18], [19] pursue (albeit, one must admit, without much success) a beautiful idea from his second paper [17] :

**Theorem 1.1.** ([23], pp. 237, 260). A proper four-colouring of vertices of a planar triangulation exists *iff* its faces can be labelled by two labels +1 and -1 in such a way that the sum of the labels of the faces around any vertex is 0 *modulo* 3.

Seventy years after [17], the present author has independently discovered Heawood's theorem [3], together with, or more precisely, by way of a straightforward but elegant geometric interpretation of a *proper four-colouring* as a *folding* of a planar triangulation into a tetrahedron : see *Appendix* . (The theory of branched and folded coverings has been initiated in [24]; an attempt of an in-depth study of covering triangulations with folds can be found in [11]).

Generalizing this approach following another of Heawood's fruitful tracks, one passes from planar graphs to graphs on arbitrary Riemann surfaces [6], [16], [23], p. 261. However, discarding the planarity condition comes at a price of "casting out" the "devil" of colouring. It means that the link between  $F_3$ -arithmetic on a triangulation of a non-planar Riemann surface, on the one hand, and colourings of the vertices of this triangulation, on the other, ceases to exist (cf. Figure 1) :

**Problem 1.2.** Let  $T$  be a triangulation of a Riemann surface. Suppose, a labelling of the vertices of  $T$  by three labels 0, +1, and -1 is fixed. Find the number of labellings of the faces of  $T$  by two labels +1 and -1, such that the sum of the labels of the faces around any vertex is equal to the given vertex label *modulo* 3.

**Remark 1.3.** Our interpretation of a triangulation of a Riemann surface is most general : triangulation is any embedding of a graph into the surface which cuts it into triangles (thus, double edges are permitted : cf. Figure 1 below).

The only and, for that matter, rather curious remnant of Heawood's colouring preoccupations emerges as a generalization (Theorem 4.1 below) of Heawood's other theorem from [17] :

**Theorem 1.4.** ([23], p. 260). Let  $T$  be a planar triangulation. Then  $T$  is 3-colourable *iff* all vertices of  $T$  have even degree.

The purpose of the present paper is to introduce, prove, and investigate a counting formula for the above Problem 1.2. Several conjectures about the numbers of solutions of the system with arbitrary free terms are advanced; three of these conjectures, if true, would imply a positive answer to **4CP**. Contradictory examples, which will appear elsewhere, show that the suggested conditions of the conjectures cannot be weakened.

The counting formula is far from being effective : it represents a delicate balancing act of mutual computation of iterations. Its main advantage lies, however, in the suggestive power of its coefficients which are integer invariants of corresponding sub-triangulations. It also gives rise to two invariant polynomials, one of  $r$  (= the number of faces of  $T$ ) variables  $L(W; a, T)$ ,  $W = (w_1, \dots, w_r)$ , and the second one derived from the first one, of one variable,

$$\mathcal{L}(w; a, T) = L(W; a, T) |_{w_1=\dots=w_r=w},$$

computing at  $w = -\frac{1}{2}$  the number of solutions of Problem 1.3 (the formula (6)).

It is the present author's hope that this polynomial and other formulae introduced hereby shall prove themselves useful instruments in studying graphs of triangulations. To substantiate these expectations, intensive calculations were carried out; their results will appear elsewhere.

Our graph-theoretical terminology is that of [15], with popular notational deviations indicated, and with a couple of our own notations when [15] is silent on the subject. As to the topological prerequisites, they can be found in [10], Chapter III.

## 2. Historical Background and Motivations.

The  $4C$  problem has appeared first as a naive enquiry addressed by a student in London, in 1852, to his brother, also a student, who passed the question to his professor of mathematics, Augustus De Morgan. Unable to answer an apparently trivial question, De Morgan circulated it among London mathematicians. Finally, in 1878, Arthur Cayley submitted the problem to the London Mathematical Society.

Within a year, Alfred Bray Kempe has published a simple and elegant solution [20] which satisfied everybody for at least ten years. Then, Percy John Heawood pointed out a fatal flaw in Kempe's argument [16] and proposed his own and, for that matter, a very successful programme of studying  $4CP$  and related problems [16], [17] (both, Kempe and Heawood, deal with planar triangulations and colour vertices). As to Kempe's modest attempt, it became just a popular historical anecdote, - or it seemed so to the general mathematical public : cf. any introduction to graph theory published before 1970.

Kempe's approach was reductive, i. e., inductive and local. Heawood's attitude was synthetic, i. e., global and easily lending itself to generalizations to triangulations of arbitrary Riemann surfaces, as in [16], with its map colour formula [21], or, in a different way, in the present paper.

Ironically, Kempe's faulty proof gave rise to a method [9] which finally produced a computer-assisted solution of  $4CP$  [1], [2], whereas Heawood's approach, the brilliant offshoots for Riemann surfaces of non-zero genus notwithstanding [21], is still waiting its successful  $4CP$  realization.

But does it really matter ? Why should somebody bother to find a [17]-inspired solution of  $4CP$  ? We submit here three major reasons, one "negative" and two "positive", why it should be worth one's while to look for a realization in question.

First, there exists a widespread disappointment with the Kempe-inspired solution of  $4CP$ . Attested by the authors of the solution themselves, the disappointment goes deeper than they are willing to admit, and not exactly as suggested in [2], p. 1 : "*One can never rule out the chance that a short proof of the Four-Color Theorem might some day be found, perhaps by the proverbial high-school student.* ".

Indeed, the problem is not so much with the "length" or "computerization" of the proof, but with its intelligibility and instructiveness. Here is a more to the point assessment : "*Even if the Appel and Haken solution is perfectly correct ... , it is painfully obvious that it is not an adequate solution to the problem. For one thing it fails to explain why four is*

the final answer. In mathematics a decent proof is at least an explanation why the result is true.” [12], p. 6.

The authors themselves submit a tentative, very intuitive, and, to our taste, not very convincing *a posteriori* elucidation of the “number four” mystery, [12], pp. 61-62. And even the masterfully written update [22] of the original proof leaves the mystery unresolved.

Second, Heawood’s theorem (Theorem 1.1 above) lends itself to a most natural and elegant topological *a priori* explanation “why” four colours suffice [3] (cf. Appendix). Sure, there is no guarantee that such an “explanation” can be translated into an edifying proof\*. But its existence greatly encourages the search for such a proof.

Third, the generalization of Heawood’s *modulo 3* interpretation of  $4CP$  to triangulations of arbitrary Riemann surfaces promises to shed a new light into their “arithmetical” structure. In fact, thanks to [2], we know that, in the planar case, there exists a positive solution of the Problem 1.3. In what non-planar cases does it exist, too, and if not, “why”?

### 3. The Counting Formula.

As it was clear already to Heawood [17], Problem 1.3 is about the existence of solutions of a system of linear equations over the Galois field of cardinality three,  $\mathbf{F}_3$ , with a non-linear condition on the unknown variables : only solutions with non-zero components are admitted. Here is the version of Problem 1.3 which will be the formal subject of the present study :

**Problem 3.1.** Let  $T = T_{p,r}(S_g)$  be a triangulation of a Riemann surface  $S_g$  (orientable or non-orientable) of the genus  $g$ , with  $p$  vertices,  $r$  faces, and let  $M_T$  be its incidence matrix vertex-face. Suppose the vertices of  $T$  are assigned fixed labels  $a = (a_1, \dots, a_p)$  from  $\mathbf{F}_3$ . We are looking for the number  $\lambda(a, T)$  of different labellings  $x = (x_1, \dots, x_r)$  of the faces of  $T$  by the numbers  $+1$  and  $-1$ , satisfying the following system of linear equations over  $\mathbf{F}_3$  :

$$M_T \cdot x = a; \quad a \in (\mathbf{F}_3)^p = A, \quad x \in \{+1, -1\}^r = (\mathbf{F}_3^*)^r = X. \quad (1)$$

**Remark 3.2. (1)** Notice that, for any  $a$ , if  $M_T \cdot x = a$  then  $M_T \cdot (-x) = -a$ . This implies that  $\lambda(a, T) = \lambda(-a, T)$  and that  $\lambda(0, T) \geq 0$  is even. The case  $a = 0$  will be sometimes referred to as *homogeneous*.

**(2)** In particular, when  $T$  is a planar triangulation and  $a = 0$ , the number  $\frac{1}{2}\lambda(0, T)$  is equal to the number of different proper four-colourings of vertices of  $T$ , up to an arbitrary permutation of colours. This fact establishes a link between Problem 3.1 and the  $4C$  problem, of which Problem 3.1 becomes a far reaching generalization.

**(3)** Notice also that

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\* As Kenneth I. Appel and Wolfgang Haken rightly point out, “*But it is also conceivable that no such proof is possible. In this case a new and interesting type of theorem has appeared, one which has no proof of the traditional type.*” [1]

$$\sum_{\forall a \in A} \lambda(a, T) = \#X = 2^r. \quad (2)$$

Our main result will be a counting formula for  $\lambda(a, T)$ , Theorem 3.4 below. To describe it, we need few standard notations.

**Notations 3.3. (1)** Let  $T = T_{p,r}(S_g) = \{V, E, F\}$  be a triangulation of a Riemann surface, as above, with, respectively, sets of vertices  $V$ , edges  $E$ , and faces  $F$ ,

$$V = \{v_1, \dots, v_p\}, E = \{e_1, \dots, e_q\}, F = \{f_1, \dots, f_r\}.$$

Euler's theorem for graphs on surfaces states that  $p - q + r = 2 - 2g$ , which in the case of a triangulation implies  $2q = 3r$  and

$$r = 2p + 4g - 4 \geq 2p.$$

**(2)** If a vertex  $v \in V$  belongs, or is *adjacent* to a face  $f \in F$ , we shall write  $v \in f$ . The number  $\alpha(v)$  of the faces adjacent to a vertex  $v$  is called the *degree* of  $v$  (instead of degree, some use the term *valence* [21]). For a subset  $V'$  of vertices,  $\emptyset \subseteq V' \subseteq V$ , we define its *broad neighbourhood complex*, or *shell* for short,  $\bar{t}(V')$  (respectively, its *narrow neighbourhood complex*, or *core* for short,  $\bar{\bar{t}}(V')$ ), as the sub-triangulation of  $T$  generated by the sub-sets of faces ( $\bar{f}(V')$  and  $\bar{\bar{f}}(V')$ , respectively) which meet at least one (respectively, two) vertices of  $V'$ .

Since  $\bar{f}(\emptyset) = \bar{\bar{f}}(\emptyset) = \emptyset$ , we assume that  $\bar{t}(\emptyset) = \bar{\bar{t}}(\emptyset) = \emptyset$ , too. The terms *broad* and *narrow* concur obviously with the inclusions :

$$\bar{f}(V') \supseteq \bar{\bar{f}}(V'); \bar{t}(V') \supseteq \bar{\bar{t}}(V').$$

We shall also use the notations

$$r' = \alpha(V') = \#\bar{f}(V'), s' = \beta(V') = \#\bar{\bar{f}}(V'),$$

*with  $r' \geq s'$  and  $\alpha(V) = \beta(V) = r = \#F$ .*

The last  $\alpha$ -notation is compatible with the above degree notation :

$$\alpha(v) = \alpha(\{v\}) = \#\bar{f}(\{v\}).$$

**(3)** We use arithmetical operations over both the ring  $\mathbf{Z}$  of integers and the Galois field  $\mathbf{F}_3$ . When it does not lead to confusion, we use the numbers 0, +1, and -1 as elements of  $\mathbf{F}_3$  and the standard arithmetical notations for operations over both integers and the "numbers" 0, +1, and -1. Otherwise, the qualifier (*mod* 3) is added.

**Theorem 3.4.** Let  $T$  be a triangulation of a Riemann surface, as above; let  $\omega = -\frac{1}{2}$ . Then the number  $\lambda(a, T)$  of solutions of the system (1) is given by the following counting formula :

$$\begin{aligned}
\lambda(a, T) &= 2^r \cdot 3^{-p+1} \cdot \sum_{\emptyset \subseteq V' \subseteq V \setminus v_p} \omega^{r'} \cdot \theta(a|_{V'}, \bar{t}(V')) \\
&= 2^r \cdot 3^{-p+1} \cdot \left( 1 + \sum_{v \in V \setminus v_p} \omega^{\alpha(v)} \cdot \theta_1(a_v) \right. \\
&\quad \left. + \sum_{v, v' \in V \setminus v_p} \omega^{\alpha(\{v, v'\})} \cdot \theta_2(a_v, a'_v; \bar{t}(\{v, v'\})) + \dots \right),
\end{aligned} \tag{3}$$

where  $\theta(a|_{V'}, \bar{t}(V'))$  are integer functions depending only on the restriction  $a|_{V'}$  of the vector  $a$  to  $V'$  and on the core  $\bar{t}(V')$ , i. e., on the *narrow neighbourhood complex* of  $V'$  (and not on its *broad neighbourhood complex*  $\bar{t}(V')$ ). (The exclusion  $V \setminus v_p$  in the above and below, (5), summations is explained below, Theorem 4.1, Assumption 4.3).

**Note 3.5.** (1) The functions  $\theta(\dots)$  are defined below, Definition 5.3(2), (11). They are calculated iteratively, from smaller subsets of vertices (and the corresponding sub-triangulations) to bigger ones. The proof of Theorem 3.4 will be given below, §5. A far-reaching combinatorial generalization of the above formula and of the narrow dependency property has been announced in [7] and will be proved elsewhere.

(2) The main idea behind the formula (3) is to re-construct the number  $\lambda(a, T)$  from numbers of partial solutions of the system (1) : first, one evaluates the numbers of solutions of a void system of equation, then of a one equation of the system, then of two equations, etc. Finally, the functions  $\theta(\dots)$  are reconstructed (see §5) with the help of Möbius inversion formula [13]. (Compare this including-excluding procedure with that, almost sixty years old, of H. Whitney [27].)

(3) It is the narrow dependency property that makes the claim (3) nontrivial, its proof rather involved, and the functions  $\theta(\dots)$  so interesting invariants of respective sub-triangulations. In fact, it is the inclusion-exclusion procedure which “distils” the narrow dependency property from its broad version : cf. Theorem 5.5(2) and Remark 5.6.

(4) To give the first inkling of the structure of the summation formula (3), let us compute its lowest terms. The first summand in (3) is the  $\emptyset$ -approximation of  $\lambda(a, T)$ . To calculate it, denote by  $A_0$  the hyperplane of the vector space  $A$  (1) cut off by the equation

$$a_1 + \dots + a_p = 0.$$

According to Theorem 4.1 below, the system (1) has no solutions if  $a \notin A_0$ . Thus, according to (2),

$$\lambda_0(a, T) = \frac{\#X}{\#A_0} = 2^r \cdot 3^{-p+1}$$

is the average number of solutions when  $a$  is varied and  $T$  is fixed (in contradistinction to the way, in a similar context, the average is defined in [26]).

(5) Restricting ourselves to  $\emptyset$ - and *one-vertex*-approximations, we take into account the impact of vertices and their neighbourhoods on  $\lambda(a, T)$ , but disregard the influence of edges and more complicated sub-graphs of  $T$ , etc.

(6) Proceeding in this way, we calculate, for example :

$$\begin{aligned}
\theta_0 &= \theta(a |_{\emptyset, \bar{t}(\emptyset)}) = 1; \\
\theta_1(\xi) &= \theta_1(a_j) |_{a_j=\xi} = \theta(a |_{\{v_j\}, \bar{t}(\{v_j\})}) = 2 - 3 \cdot \xi^2; \\
\theta_2(\xi, \eta) &= \dots = \begin{cases} \theta_1(\xi) \cdot \theta_1(\eta) + 3 \cdot \theta_1(\xi - \eta), & \text{if } \{v_j, v_k\} \text{ is an edge in } T; \\ \theta_1(\xi) \cdot \theta_1(\eta), & \text{otherwise.} \end{cases}
\end{aligned} \tag{4}$$

The above formula for  $\theta_2(\xi, \eta)$  is “weakly multiplicative”. This property of functions  $\theta$  is the subject of Theorem 5.5(3).

(7) Note that the above “small” coefficients  $\theta(\dots)$  are positive when  $a = 0$  (which, in particular, is the case of  $4CP$  for planar triangulations). This phenomenon, which can be verified for  $\#V' = 3, 4$  as well, remains an enigma : cf. Conjecture 6.4 below.

**Definition 3.6.** Let  $T$  be a triangulation of a Riemann surface, as above. Assign to each face  $f_j$  a formal variable  $w_j, 1 \leq j \leq r$ . Proceeding from the formula (3), define the polynomial

$$L(W; a, T) = \sum_{\emptyset \subseteq V' \subseteq V \setminus v_p} \left( \theta(a |_{V'}, \bar{t}(V')) \cdot \prod_{f_j \in \bar{f}(V')} w_j \right), \tag{5}$$

We also define

$$\mathcal{L}(w; a, T) = L(W; a, T) |_{w_1=\dots=w_r=w}$$

so that, obviously,

$$\lambda(a, T) = 2^r \cdot 3^{-p+1} \cdot \mathcal{L}(-\frac{1}{2}; a, T). \tag{6}$$

#### 4. Rank of the Incidence Matrix.

Let  $M_T$  be the incidence vertex-face matrix of a triangulation  $T$  of a Riemann surface  $S$ , as above, Problem 3.1. Attempting to solve the system (1), one has first to eliminate one trivial obstruction :

**Theorem 4.1.** (In the planar case,  $g = 0$ , [17]; for Riemann surfaces of arbitrary genus  $g \geq 0$ , [6]).

(1) Since each face of  $M_T$  is a triangle, all but three entries of any column of  $M_T$  are zeros. This means that the sum of all rows of  $M_T$  is zero in  $\mathbf{F}_3^r$ ,

$$\mathbf{1}_p \times M_T = \mathbf{0}_r$$

which implies :

(a) the system (1) has no solutions if the free-term vector  $a$  does not belong to  $A_0$  (cf. Note 3.5(4)) :



$$(\mathbf{1}_p, a) \neq 0 \implies \forall x \in X. M_T \cdot x \neq a;$$

(b) the inequality holds :

$$\text{rank}_{\mathbf{F}_3}(M_T) \leq p - 1.$$

(2) If  $\text{rank}_{\mathbf{F}_3}(M_T) = p - 1$ , then, according to the inequality (Euler's formula, cf. Notations 3.3(1))

$$r = 2p + 4g - 4 \geq 2p,$$

any  $((p - 1) \times r)$ -sub-matrix of  $M_T$  has the rank  $p - 1$ .

(3) The inequality holds :

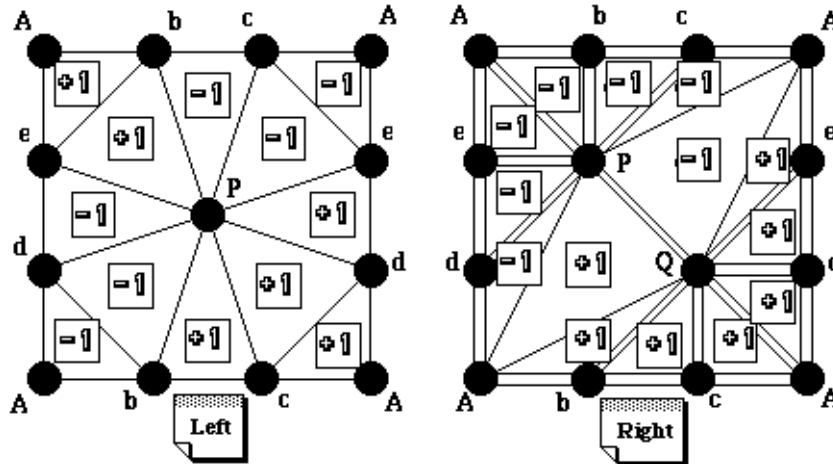
$$\text{rank}_{\mathbf{F}_3}(M_T) \geq p - 2.$$

(4) The following two properties are equivalent :

(i)  $\text{rank}_{\mathbf{F}_3}(M_T) = p - 2$ ,

(ii) the triangulation  $T$  has a proper three-colouring of its vertices; such a colouring is unique up to a permutation of the colours.

**Note 4.2.** In the planar case,  $g = 0$ , the above two properties are equivalent to the third one : a proper three-colouring exists iff all vertices of  $T$  have even degrees, see Theorem 1.4 above. Two examples of triangulations of a torus, Figure 1, with all vertices of even degrees, and with the chromatic numbers 5 and 7, respectively, demonstrate that in the case  $g > 0$  the “if” condition ceases to be true.



**Figure 1.** *Left.* Triangulation of a torus with six vertices,  $A, P, b, c, d, e$ , all of even degree (respectively, 4, 8, 6, 6, 6, 6), which is not three-colourable (cf. Theorem 1.4). Albeit there exists (as displayed on the diagramme above) a  $\pm 1$  labelling of faces of the triangulation with sums around vertices equal to 0 modulo 3, it is not four- but five-colourable (cf. Theorem 1.1).

*Right.* Triangulation of a torus with seven vertices,  $A, P, Q, b, c, d, e$ , all of even degree (respectively, 10, 8, 8, 4, 4, 4, 4), with a  $\pm 1$  labelling of faces having the above 0 *modulo* 3 property. The sub-graph of this triangulation with seven vertices and double-lined edges is the complete graph  $K_7$ ; thus, the triangulation is seven-colourable.

**Assumption 4.3.** In what follows, we assume (without much loss of generality, according to Theorem 4.1(4)) that

$$\text{rank}_{\mathbf{F}_3}(M_T) = p - 1.$$

Thus, one of the equations of the system (1) can be disregarded, and according to Theorem 4.1(2), it can be any of the equations; in what follows, it will be the equation corresponding to the vertex  $v_p$ . This fact accounts for the exclusion  $V \setminus v_p$  in the summations (3) and (5).

## 5. Partial Counting Functions and Formulae.

Henceforth, the triangulation  $T$  is fixed and does not appear in the notations of objects associated with it. Removing from  $T$  the vertex  $v_p$ , one gets the (sub-)triangulation  $T^0$  with the set of vertices  $V^0 = V \setminus v_p$ . Replace now the system (1) with its truncated version,

$$M^0 \cdot x = a; \quad a \in \mathbf{F}_3^{p-1} = A^0, \quad x \in \{+1, -1\}^r = (\mathbf{F}_3^*)^r = X, \quad (7)$$

with the  $((p-1) \times r)$  matrix  $M^0$  being the sub-matrix of the incidence vertex-face matrix  $M_T$ , as above, whose last row (corresponding to  $v_p$ ) is omitted.

As it was suggested above (Note 3.5(2)), along with the system (7), one needs to study all its sub-systems :

**Definition 5.1.** For any  $a \in A^0$  and any subset  $V'$  of  $V^0$ ,  $\emptyset \subseteq V' \subseteq V^0$ , define the functions  $\lambda(a, V')$  and  $\bar{\lambda}(a, V')$  as follows :

- (i) either  $V' = \emptyset$ , and then  $\lambda(a, \emptyset) = \#X = 2^r$  ;
- (ii) or

$$V' \neq \emptyset, V' = \{v_{i_1}, \dots, v_{i_{p'}}\}, 1 \leq i_1 < i_2 < \dots < i_{p'} \leq p - 1,$$

and then (Notations 3.3)

$$\begin{aligned} a|_{V'} &= \{a_{i_1}, \dots, a_{i_{p'}}\}; \\ M^0[V'] &= (m_{i_k, j})_{\substack{1 \leq j \leq r \\ 1 \leq k \leq p'}}; \\ \Lambda(a, V') &= \{x \in X \mid M^0[V'] \cdot x = a|_{V'}\}; \\ \lambda(a, V') &= \#\Lambda(a, V'); \\ \bar{\lambda}(a, V') &= 2^{r-r'} \cdot \lambda(a, V'), \text{ where } r' = \#\bar{f}(V'). \end{aligned} \quad (8)$$

**Lemma 5.2.** The function  $\bar{\lambda}(a, V')$  has integer values and substantially depends only on (this means : is completely determined by) the sub-vector  $a|_{V'}$  and the shell  $\bar{t}(V')$  of  $V'$  in  $T^0$ .

*Proof.* The values  $x_j$  corresponding to those faces  $f_j$  which are not adjacent to one of the vertices of  $V'$ , do not contribute to the summation and can be chosen arbitrary. The number of such faces is  $r - r'$ .

Instead of  $\bar{\lambda}(a, V')$ , we shall sometimes use the explicit notations  $\bar{\lambda}(a|_{V'}, \bar{t}(V'))$ .

**Definition 5.3. (1)** For any  $a \in A^0$  and any  $V', \emptyset \subseteq V' \subseteq V^0$ , define the function

$$\zeta(a, V') = \zeta(a|_{V'}, \bar{t}(V'))$$

inductively, as follows :

$$\begin{aligned} \zeta(a, \emptyset) &= 1; \\ \forall V', \quad \emptyset \neq V' \subseteq V^0. \\ \zeta(a, V') &= 2^{-r'} \cdot 3^{p'} \cdot \bar{\lambda}(a|_{V'}, \bar{t}(V')) - \sum_{\emptyset \subseteq V'' \subsetneq V'} \zeta(a|_{V''}, \bar{t}(V'')). \end{aligned} \tag{9}$$

**(2)** For any  $a \in A_0$  and any  $V', \emptyset \subseteq V' \subseteq V^0$ , define the function  $\theta(a, V')$ , as follows :

$$\begin{aligned} \theta(a, V') &= \theta(a|_{V'}, \bar{t}(V')) = (-2)^{-r'} \cdot \zeta(a|_{V'}, \bar{t}(V')); \\ \zeta(a, V') &= \omega^{r'} \cdot \theta(a, V'), \text{ where } \omega = -\frac{1}{2}. \end{aligned} \tag{10}$$

**Lemma 5.4. (1)** According to (9), for any  $V', \emptyset \subseteq V' \subseteq V^0$ , one has, tautologically,

$$\bar{\lambda}(a, V') = 2^{r'} \cdot 3^{-p'} \cdot \sum_{\emptyset \subseteq V'' \subseteq V'} \zeta(a, V'').$$

**(2)** Applying Möbius inversion formula [13], we have :

$$\zeta(a, V'') = \sum_{\emptyset \subseteq V'' \subseteq V'} (-1)^{p' - p''} \cdot 2^{-r'} \cdot 3^{p'} \cdot \bar{\lambda}(a, V''). \tag{11}$$

**(3)** Thus (cf. (8)),

$$\begin{aligned} \bar{\lambda}(a, V') &= 2^{r'} \cdot 3^{-p'} \cdot \sum_{\emptyset \subseteq V'' \subseteq V'} \omega^{r''} \cdot \theta(a, V''), \\ \lambda(a, V') &= 2^r \cdot 3^{-p'} \cdot \sum_{\emptyset \subseteq V'' \subseteq V'} \omega^{r''} \cdot \theta(a, V''). \end{aligned}$$

**Theorem 5.5. (1)** The functions  $\theta(\dots)$  defined in (10) are integer-valued.

**(2)** These functions substantially depend only on the sub-vector  $a|_{V'}$  and the core  $\bar{t}(V')$  of  $V'$  in  $T^0$  (and not on the shell, as in (9), (10)) :

$$\theta(a, V') = \theta(a|_{V'}, \bar{t}(V')).$$

(3) The functions  $\bar{\lambda}(\dots)$  (8) and  $\theta(\dots)$  (10) are weakly multiplicative, in the following sense :

$$\begin{aligned} \forall V', V'' (\emptyset \subseteq V', V'' \subseteq V^0). \quad \bar{t}(V') \cap \bar{t}(V'') = \emptyset \implies \\ \bar{\lambda}(a, V' \cup V'') = \bar{\lambda}(a, V') \cdot \bar{\lambda}(V''), \\ \theta(a, V' \cup V'') = \theta(a, V') \cdot \theta(V''). \end{aligned}$$

(The low part of the formula for  $\theta_2$  in (4) is a special case of the general formula for  $\theta(a, V' \cup V'')$ , when  $\#V' = \#V'' = 1$  and  $\#(V' \cup V'') = 2$ .)

(4) Let

$$\begin{aligned} \emptyset \subseteq V' \subseteq V'' \subseteq V^0, \\ V'' = \{v_{i_1}, \dots, v_{i_{p''}}\}, 1 \leq i_1 < i_2 < \dots < i_{p''} \leq p-1. \end{aligned}$$

Then

$$\begin{aligned} \sum_{\alpha=(\alpha_1, \dots, \alpha_{p''}) \in \mathbf{F}_3^{p''}} \bar{\lambda}(a \mid_{a_{i_1}=\alpha_1, \dots, a_{i_{p''}}=\alpha_{p''}}, V') = 2^d \cdot \bar{\lambda}(a, V' \setminus V''), \\ d = \#(\bar{f}(V'') \setminus \bar{f}(V' \setminus V'')). \end{aligned}$$

(5) Let

$$\begin{aligned} \emptyset \subseteq V', V'' \subseteq V^0, \quad V' \cap V'' = \emptyset, \\ V'' = \{v_{i_1}, \dots, v_{i_{p''}}\}, 1 \leq i_1 < i_2 < \dots < i_{p''} \leq p-1. \end{aligned}$$

Then

$$\sum_{\alpha=(\alpha_1, \dots, \alpha_{p''}) \in \mathbf{F}_3^{p''}} \bar{\lambda}(a \mid_{a_{i_1}=\alpha_1, \dots, a_{i_{p''}}=\alpha_{p''}}, V') = 3^{p''} \cdot \bar{\lambda}(a, V')$$

(6) Respectively,

$$\begin{aligned} \forall k, 1 \leq k \leq p'. \quad \sum_{\alpha \in \mathbf{F}_3} \theta(a, V') \mid_{a_{i_k}=\alpha} \\ = \theta(a, V') \mid_{a_{i_k}=0} + \theta(a, V') \mid_{a_{i_k}=1} + \theta(a, V') \mid_{a_{i_k}=-1} = 0. \end{aligned}$$

**Remark 5.6.** Whereas the claims (1), (3), (5) of the above theorem can be proved relatively easily, as simple formal corollaries of Definitions 5.1, 5.3, the proofs of the statements (4), (6) are less evident, and that of the statement (2) is definitely more involved. (It will be given elsewhere.) This is because the property of “narrow dependency” is neither true for the functions  $\bar{\lambda}(\dots)$ , nor can it be imposed in a straightforward manner. In fact, it is the inclusion-exclusion procedure (9), (10) which “distills” the narrow dependency property from its broad brand of Lemma 5.2; cf. Note 3.5(3).

## 6. Enquiries and Conjectures.

Calculations, both theoretical and based on examples, with respect to the Problem 1.2, suggest several conjectures concerning the system (1) and the counting function  $\lambda(a, T)$ . Three of these conjectures, if true, imply the positive answer to *4CP*. The logic of justification of these conjectures and the insights involved therein are rather intricate, and are somewhat similar to those explicated in [12] with respect to a version of the proof [2] of *4CP*.

**Conjecture 6.1 : Weak Homogeneity Conjecture.** In the homogeneous case,  $a = 0$ , the counting function  $\lambda$  is positive :

$$\forall S \forall T. \lambda(0, T) > 0$$

**Conjecture 6.2 : Strong Homogeneity Conjecture.** The counting function  $\lambda(a, T)$  attains its maximum in the homogeneous case,  $a = 0$  :

$$\forall S \forall T \forall a \in \mathbf{F}_3^p. \lambda(0, T) \geq \lambda(a, T).$$

**Conjecture 6.3 : “Big Triangulations” Conjecture.**

(1) **Strong (or Global) Version.** If the triangulation  $T$  is “big enough”, then the system (1) has a solution for any  $a$  :

$$\exists P > 0. \forall p > P. \forall S \forall T = T_{p,r}(S) \forall a \in \mathbf{F}_3^p. \lambda(a, T) > 0.$$

(2) **Weak (or Local) Version.** If, for a given Riemann surface  $S_g$ , the triangulation  $T = T_{p,r}(S_g)$  is “big enough”, then the system (1) has a solution for any  $a$  :

$$\forall S. \exists P = P(S) > 0. \forall p > P. \forall T = T_{p,r}(S). \forall a \in \mathbf{F}_3^p. \lambda(a, T) > 0.$$

**Conjecture 6.4 : Local Homogeneity Conjecture.** (Cf. the formulae (4).) The functions  $\theta$  are positive in the homogeneous case :

$$\forall S \forall T \forall V' \subseteq (V \setminus v_p). \theta = \theta(0 |_{V'}, \bar{t}(V')) > 0.$$

**Mock Conjecture 6.5.** The above Local Homogeneity Conjecture implies the Weak Homogeneity Conjecture (Conjecture 6.1) and is “its real reason and explanation”.

## 7. Appendix.

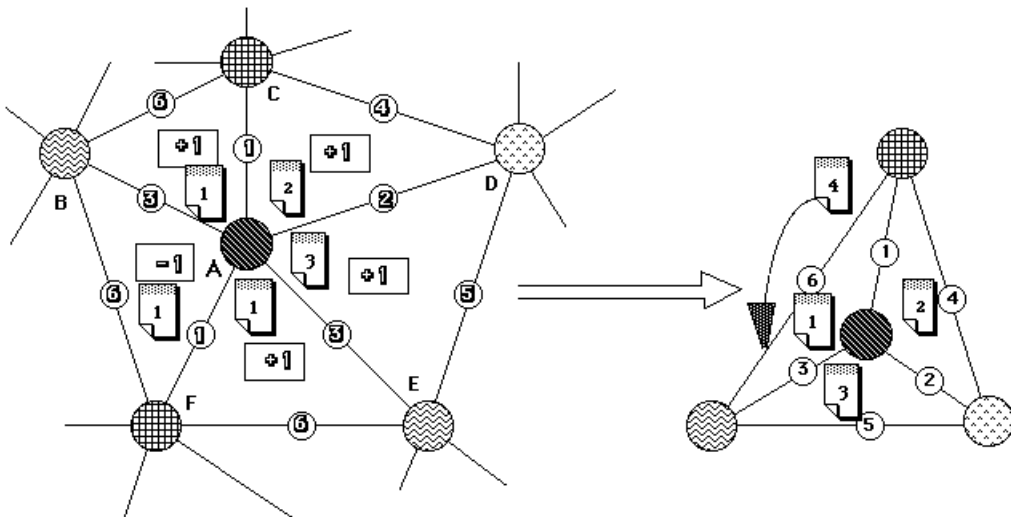
Heawood’s theorem (Theorem 1.1 above) has an elegant geometric interpretation *via* a folding of a planar triangulation into a tetrahedron. This “folding” interpretation [3], [4] lends itself naturally to a generalization to a  $n$ -dimensional case [5] :

(1) **Construction 1 : 4C-tetrahedron.** Identify the four colours with four vertices of a 3D simplex; call it the four-colour (4C-) tetrahedron. Then interpret four colouring

of vertices of a planar triangulation as a mapping of the set of these vertices into the set of four vertices of the  $4C$ -tetrahedron.

(2) **Construction 2 : Extending a Proper Four Colouring to the Tetra-Colouring.** A four colouring of a triangulation is proper *iff* the above mapping can be naturally extended to the mapping of the set of edges of the triangulation into the set of edges of the  $4C$ -tetrahedron, and then to the mapping of the respective sets of faces. Call such mapping a tetra-colouring of the triangulation.

(3) **Heawood's Theorem : "Only If".** Given a planar triangulation, fix orientations on it and on the  $4C$ -tetrahedron. Given a proper four-colouring of the triangulation, construct its tetra-colouring extension and assign to each of the faces of the triangulation the label  $+1$  or  $-1$ , according to whether the tetra-colouring preserves or not the chosen orientations. We call this labelling a *Heawood vector* over the triangulation. Then the sum of the labels of the faces around any vertex is equal to the *local multiplicity of the mapping* [28] times 3 (= the number of faces around a vertex in the tetrahedron) : cf. Figure 2. Thus, this sum is  $0 \pmod{3}$  !



**Figure 2.** A proper four-colouring of a sub-triangulation (on the left), with six vertices A –F. Four colours are identified with four vertices of the  $4C$ -tetrahedron (on the right). All edges and faces of the tetrahedron have their numerical labels, respectively, 1–6 and  $\boxed{1}$  –  $\boxed{4}$  (including the sole invisible face labelled  $\boxed{4}$  ). The colouring of vertices of the sub-triangulation induces its mapping on the upper side of the tetrahedron, with the corresponding labellings of its edges and faces. The foldings occur along the edges AB, AF, and AC. The local multiplicity of the mapping at A is equal to  $+1$ .

(4) **Heawood's Theorem : "If".** As above, for a given planar triangulation and the  $4C$ -tetrahedron, fix orientations. Suppose a labelling of the faces of the triangulation by  $+1$  and  $-1$  is given, with the property that the sum of the labels of the faces around any vertex is  $0 \pmod{3}$ , ( $0 \pmod{3}$  property for short). Choose a pair of faces, one from the triangulation, another from the  $4C$ -tetrahedron, and map the first face on the second one

in such a manner that the orientation of the face would be preserved if the face is labelled by  $+1$ , and would be inverted otherwise. Then the  $0 \bmod 3$  property guarantees that the mapping can be extended in a unique way on all faces of the triangulation defining its tetra-colouring.

**(5) Corollary.** Suppose a labelling of the faces of the triangulation by  $+1$  and  $-1$  is given, satisfying the above  $0 \bmod 3$  property. Then the sum of the labels of all faces is equal to the global multiplicity of the corresponding tetra-colouring times 4 (= the number of faces in the tetrahedron); thus, this sum is  $0 \bmod 4$  [3].

**(6) Generalization to the  $n$ -dimensional case.** Heawood vector can be defined for any proper  $(n + 2)$ -colouring,  $n \geq 2$ , of a graph corresponding to a  $n$ -dimensional simplicial complex  $T$ , and the corresponding congruences  $\bmod(n + 3 - \dim(s))$  can be conjectured and proved for any simplex  $s$  of  $T$ ,  $-1 \leq \dim(s) \leq n + 1$ . Here too, as in the two-dimensional case (Remark 1.3), we work within the most general interpretation of a  $n$ -dimensional simplicial complex [6].

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