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Cobordism as a Basic Topological Paradigm of Virtual Computer Animation

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Abstract. We customize here, for the purposes of precomputed and interactive deformation and animation techniques, some basic concepts and deformation methods of differential topology related to *Cobordism Theory*.

Keywords: Shape Deformation – Cobordism – Virtual, or Implicit, Animation – Manifold – Isomorphism – Isotopy – Immersion – Homotopy.

1. Introduction

This paper is a part of our programme [3-5] to bridge the gap between, on the one hand, computer animation heuristics (such as the popular *2D* deformation *morphing*, or the less known *space-time deformation method*) and, on the other hand, computer visualization artifacts built around strict mathematical methods and theorems (such as *sphere eversions* [10], or *3D* and *4D immersions of Klein bottle* [6]). The purpose of the present paper is triple.

First, and foremost, it attempts to customize, for the needs of *Computer Animation*, some methods of differential topology related to *Cobordism Theory*. *Cobordism*, interpreted here as deformation, is a more powerful and versatile tool than any other type of topological deformation acquired up to date in *Computer Graphics*. Its main, and novel, feature is its *shapewise* character, in contradistinction to known *pointwise* deformations which trace the deformation of a shape by tracing the destination of every one of its individual points. The ensuing broadening of the visualization panorama leads to the concept of *virtual, or implicit animation*, as an intrinsically defined deformation which, to become “real” animation, needs to be subsequently visualized on a “screen”, by its sinking into a visualizable *ambient space*.

Second, and in the opposite direction, the paper intends to make manifest deep mathematical intuitions hidden, all unknowingly even to their inventors, behind the facade of some animation heuristics, – with the subsequent goal to either shore up these heuristics with new rigorous technical means, or even to completely incorporate them, on the

theoretical level, into a corresponding mathematical framework.

Finally, our **third** goal is to expose some mathematical problems arising from attempts to interpret, explain, and rigorously justify needs of, and related experimental phenomena observed in, *Computer Animation*.

2. Cobordism as a Deformation

Figure 1 demonstrates that two isomorphic objects (cf. the sign \approx between two adjacent shapes) could create quite different, if not outright opposite, visual impressions. Hence, one of the primary concerns of *Topology* is the problem of recognizing when shapes (in topological parlance, *manifolds*) “looking” differently are really different or, in fact, are “the same”, or *equivalent*.

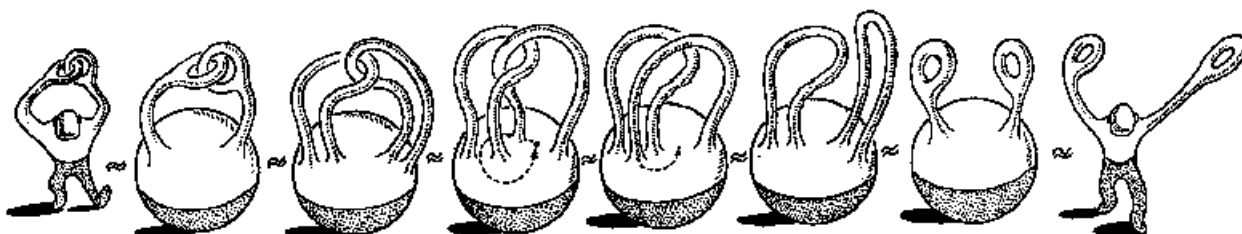


Figure 1. Visual *versus* topological equivalences; the picture is borrowed from [9].

To this end, a whole gamut of equivalence notions has been created, from *isomorphic* (*diffeomorphic* or *homeomorphic*) to *cobordant* to *homotopic*, together with corresponding sets of tools for deciding when two manifolds are homeomorphic, diffeomorphic, cobordant, etc. [7]. Some of these equivalence notions are explicitly defined deformations (*homeotopies*, *isotopies*, *immersions*) well-known in *Computer Visualization* [6]. Not so with *cobordism* which, by definition, is a manifold with two “brims” (see Figure 2 (Left) and §3), not a deformation.

However, any cobordism can be interpreted as an *implicit deformation*, since it can be always supplied by a suitable *Morse function* [7] which cuts it into “slices” (cf. Figure 2 (Left)), in such a way that :

- (i) the two outermost slices are the brims themselves;
- (ii) the Morse function interpreted as time, the cobordism becomes a *reversible deformation* (i. e., a 1D–dimensional collection of slices) of one of its brims into the second one.

Cobordism surpasses in its versatility other types of topological deformations. Thus, for example, a circle is *cobordant*, but neither *isotopic*, nor *homotopic* to two disjoint circles : cf. Figure 2 (Left). In this case, the deformation transforms the circle into the

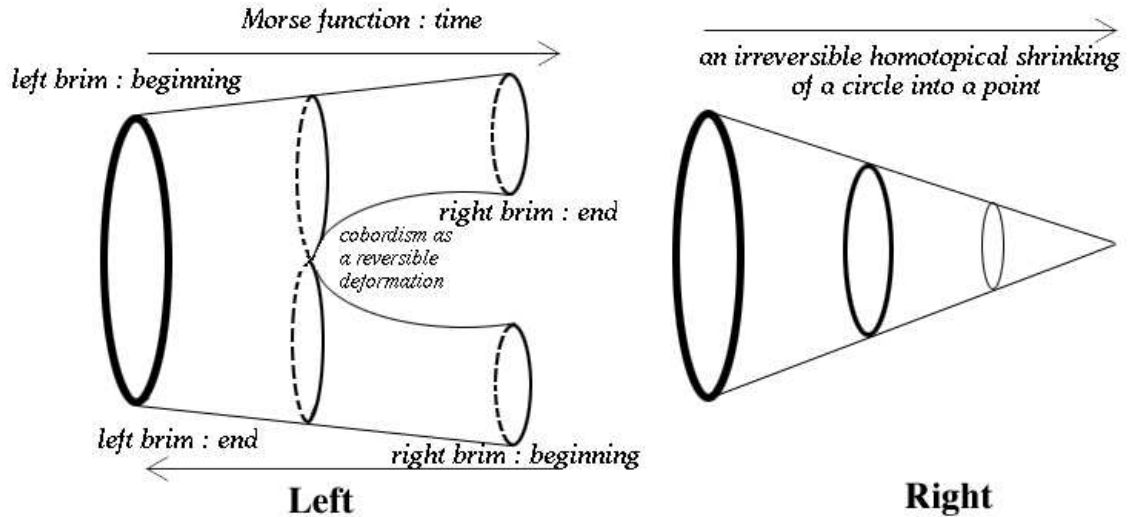


Figure 2. Left: A circle is cobordant to two circles.
Right: A circle is *homotopic* but neither *isotopic*, nor *cobordant* to a point.

figure of eight, and then separates two circles by splitting their joint into two points. It is this splitting that is iso- and homotopically unavailable.

The versatility of cobordism as deformation is a by-product of its another attractive and important quality. By contrast with other known topological deformations which could be described as *pointwise*, cobordism is a *shapewise*, or *pointless*, deformation which does not trace the trajectories of individual points of shapes : see for details §§3, 7.

Finally, in the most important for *Computer Graphics* interval of dimensions, $1 \leq dim \leq 4$, the cobordism deformation technique has the advantage of *theoretical predictability* : given two manifolds of the same dimension, M_0 and M_1 , one can, at least theoretically, decide whether there exists a cobordism deforming M_0 into M_1 . An efficient construction of such a cobordism is another matter : see for details §§5, 7.

We work here with only *smooth cobordisms* which generate deformations without, or with only a finite number of *isolated non-degenerated singularities*. This restriction is important : *folding* and *shrinking* (Figure 2 (Right) : a circle is *homotopic* but neither *isotopic*, nor *cobordant* to a point) are, for example, out of reach for smooth cobordisms. And so are all deformations producing shapes with singular curves, as in the case of Figure 3, where a torus detaches itself from a plane through a circle (see for details §6).

Cobordisms with singularities [14], which are the subject of our forthcoming paper [5], are powerful enough to imitate deformations with isolated singular sub-manifolds (see for details §§6, 7). Still, even such cobordisms cannot be held accountable for all “meaningful” piecewise-smooth deformations emerging in *Computer Animation*.

The truth is, a “realistic” animation scenario might involve such complicated self-intersections, foldings, and other “wild” singularities, that it becomes impossible to formally and exhaustively treat it with any available to date topological methods, cobordisms included (see for details §7).

3. Theory of Cobordisms without Singularities at Glance : Basic Definitions

To be more (topologically) specific, *manifold* X will mean throughout the paper a *compact, smooth, orientable or non-orientable* n -dimensional (nD , for short; $n \geq 1$) manifold, *with boundaries*, $\partial X = Y \neq \emptyset$, $\dim Y = \dim X - 1$, or *without boundaries (closed manifold)*, $\partial X = \emptyset$.

Thus, the left (respectively, right) brim of the cobordism of Figure 2 (Left) is a compact, smooth, closed orientable $1D$ manifold with one (respectively, two) component(s). Smoothness of a manifold is permitted to be disrupted by a finite number of *isolated non-degenerate singularities*.

Two smooth closed manifolds $M_0 = M_0^n$, $M_1 = M_1^n$ of the same dimension n are *cobordant*, if their disjoint union is diffeomorphic to the boundary of a compact smooth manifold $M = M^{n+1}$, called *cobordism* :

$$M_0 \cup M_1 \cong \partial M.$$

If M_0 , M_1 are cobordant, then there exists on M a *Morse function* f , such that

$$f : M \rightarrow [0, 1], f(M_0) = 0, f(M_1) = 1.$$

Since (by definition of Morse function) f has neither *degenerate singularities*, nor *critical multiple values*, all “slices”

$$M_t = f^{-1}(t), 0 \leq t \leq 1,$$

of the manifold M are smooth (sub-)manifolds, with the possible exception of a finite number of slices M_{t_k} with a single non-degenerate singularity corresponding to a critical value t , $0 < t_1 < \dots < t_r < 1$.

Defintion : Cobordism as a Shapewise, or Implicit, Deformation.

(1) We interpret the one-parameter family

$$\{M_t, 0 \leq t \leq 1\}$$

as a shapewise (or pointless) deformation of M_0 into M_1 which does not, and has no standard intrinsic means to, trace the trajectories of individual points of shapes.

(2) Compare this deformation to pointwise deformation of, say, isotopy [7]. A shape-wise deformation achieves the similarity of neighbouring shapes M_t and $M_{t+\epsilon}$ not because of the proximity in $M_{t+\epsilon}$ of images of proximate points in M_t , but because of the proximity of two shapes in the ambient manifold M of the cobordism.

4. Virtual, or Implicit, Animation Paradigm

Traditionally, a computer animation scenario is understood as a continuous, temporally arranged collection of deformations of a (typically, but not necessary, $2D$) shape

“floating” in an ambient (typically, 3D) space. In the light of the cobordism deformation techniques, this vision lends itself to the following far reaching generalization :

Virtual, or Implicit, Animation Paradigm : Pulling a Source into a Target along a Deformation Scene.

(1) **The topological formalism and problem.** Given two intrinsically or extrinsically defined manifolds S and T (called *source* and *target*), and a manifold M , (called *deformation scene*), with two sub-manifolds $S_r, T_r \subset M$ (called *source* and *target receivers*), find

(i) immersions

$$g : S \implies S_r; h : T \implies T_r;$$

(ii) and a deformation *deform*, carrying (or, if one prefers, pulling) the embedded source $S' = g(S) \subseteq S_r$ into the embedded target $T' = h(T) \subseteq T_r$ through (respectively, along) M .

A cobordism-related solution of this problem in low (one to four) dimensions is discussed in §5. See below *Claim : Virtual Animations are Cobordisms*.

(2) **Visually explicit deformations.** The collection B of all shapes created by *deform* on its way from S' to T' , these two shapes including, is called *deformation body*. The deformation is *visually explicit* if the source, target, and deformation scene are defined as manifolds in an ambient “screen” space R_n . An heuristically created sample of a visually explicit deformation [1] is discussed below, §6.

(3) **The visualization stage of the paradigm.** Suppose the mathematical part of the problem has been successfully resolved. If the constructed deformation is not *visually explicit*, one needs to meaningfully visualize $S, T, M, S_r, T_r, g, h, deform, B$, by sinking them into an ambient “screen” space, with the whole gamut of options, from pure theoretical observations to computer implementation.

Claim : Virtual Animations are Cobordisms. Any virtual animation carrying a shape S into a shape T , can be viewed as a cobordism supplied with a Morse function induced by the animation. And *vice versa*, a pair (*cobordism, Morse function*), with the brims of the cobordism “cut off”, respectively, by the values 0 and 1 of the Morse function, gives rise to a virtual animation carrying the shape of the 0-brim into the shape of the 1-brim.

5. Theory of Cobordisms without Singularities at Glance : Problem of Cobordants

In the formal framework of *Cobordism Theory*, realization of the above *Virtual Animation Paradigm* becomes

Problem of Cobordants. Given two shapes (compact, smooth, closed manifolds) M_0 and M_1 , determine whether they are cobordant. If the answer is positive, find the cobordism, its Morse function, and the corresponding shapewise deformation.

If $dim(M_0) = dim(M_1) \geq 5$, no general answer to the problem is known, apart from a beautiful reduction of this “differentiable” problem to the open combinatorial problem of

computing homotopy groups of the manifold [7], [11].

Fortunately for us, visualizing mostly one- to four-dimensional shapes [2], the solution of the problem of cobordants in the dimensions below 5 is at a much more advanced stage.

Problem of Cobordants in Low Dimensions.

$dim = 1$. 1D manifolds without boundaries are finite collections of disjoint circles, the number of circles (components of the manifold) being the only and full characteristics (or *invariant*) of a 1D manifold [7]. Why two different 1D manifolds are cobordant, and how to build their cobordism, with an associated Morse function, can be easily surmised from Figure 2 (Left), where a circle is demonstrated to be cobordant to two circles.

$dim = 2$. A compact, smooth, closed surface is either orientable, – and then diffeomorphic to the sphere S_2 with a finite number of *handles*, or non-orientable, – and then diffeomorphic to the projective plane RP^2 with a finite number of handles [7]. It is not difficult to see, extending the device of Figure 2 (Left) to the two-dimensional case, why any two orientable (respectively, non-orientable) surfaces are cobordant, and how to effectively build a cobordism and an associated Morse function which attaches (or, in the opposite direction, removes) a new handle. On the other hand, RP^2 is not cobordant to S_2 , and any deformation of RP^2 into S_2 , and vice versa, necessarily passes by a stage with a “wild” singularity.

$dim = 3$. Using complicated and tricky geometry, Vladimir Rokhlin has proved in 1951 that any two 3D compact, smooth, closed manifolds are cobordant [7], [12]. The problem of effective procedure for this and the following, $dim = 4$, existence theorem is discussed below, §7.

$dim = 4$. *Orientable case* (Rokhlin, 1952 [8], [13]). There exists an infinite series of 4D compact, smooth, closed, orientable manifolds without boundaries which are mutually not cobordant [8], with any other 4D manifold of this type being cobordant to some manifold from the series.

6. Space-Time Deformations and Cobordisms

Two known heuristic deformation techniques, *morphing* and *space-time deformation method*), incorporate, unknowingly to their inventors, some features of the above deformation scheme. We illustrate here, on the example of the space-time deformation method, or *STDM* [1], why and how these popular heuristic animation models might benefit from an “infusion” of cobordisms-related methods.

The central idea of *STDM* is to see as the principal object of study and manipulation not a shape by itself but its image in the space-time or, in our terminology, §4, its deformation body. Take first the space-time image of a motionless shape M sinked into an ambient space R^n , $n = dim(M) + 2$. This image is the cylinder $M \times I$, with I being the time interval, $I = [0, 1]$. Start to deform $M \times I$ in R^n , in one or another acceptable way. The time cross-sections of the deformed cylinder generate a continuous transformation of M .

One immediately recognizes in the *STDM* strategy some formal components of the

above definitions of cobordism, §3, and virtual animation, §4. Since *STDM* does not care about singularities (cf. the “flying” torus of Figure 3), its adequate interpretation would need cobordisms with singularities.

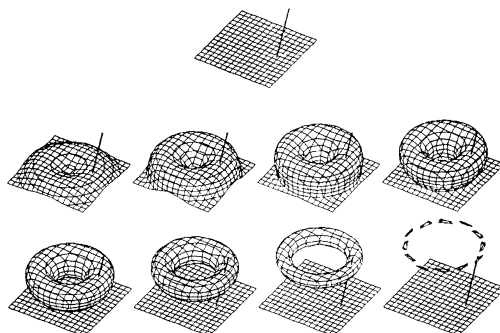


Figure 3. A *STDM* deformation with a singular circle; the picture is borrowed from [1].

On the other hand, from a theoretical point of view, *STDM*, being a particular case of what is called above *visually explicit deformation* is considerably weaker than the cobordism deformation method. As a result, *STDM* lacks *theoretical predictability* of cobordism deformations, §§3, 7. The last drawback is typical for other heuristic techniques as well. It considerably restricts the purposefulness of the search for new non-trivial artifacts : in many interesting cases, such artifacts are “stumbled upon” almost accidentally.

7. Open Mathematical Tool-Box

In this section, we list open problems suggested by the above exposition of the cobordism deformation method.

Problem 1. (Cf. the remark at the end of §2.) *Is there a more general than cobordism topological concept of deformation, suitable for the needs of Computer Visualization ?* The question is neither intuitively straightforward, nor easy. We address it our forthcoming paper [5].

Problem 2. (Cf. §3, Definition of spacewise deformations.) *Is there a “natural” parametrization of a cobordism transforming it into a pointwise deformation ?* Such a parametrization should handle splitting of points and of sub-manifolds, as it is clear from Figure 2 (Left) and the related remark in §2.

Problem 3. (Cf. the remarks at the end of §2 and §7.) *Is there an extension of Smooth Cobordism Theory to manifolds with “wild” singularities, suitable for the needs of Computer Visualization ?* The answer is yes : cobordisms with singularities are studied since the late sixties [14]. Unfortunately, the theory in question, as it stands now, is built in a too abstract setting to be easily exposed and applied in *Computer Visualization* [5].

Problem 4. (Cf. §5, Problem of Cobordants in Low Dimensions, $dim = 2$, and §6.) *In particular, is there a 2D theory of cobordisms with singularities which could effectively*

simulate deformation of surfaces with “wild” singularities ? The answer is yes : see for details our forthcoming paper [5].

Problem 5. (Cf. §5, Problem of Cobordants in Low Dimensions, $dim = 3, 4$.) *Is there an algorithmic procedure which, for a given pair of 3D (respectively, 4D) manifolds, can effectively decide whether they are cobordant or not ?* The difficulty of the problem is apparent from the pioneering work [9]. There is a hope that Rokhlin’s pure geometric and semi-constructive proof [12] can be made fully constructive. This direction is especially promising for *Computer Visualization*, since any progress here depends on, and seems likely to considerably improve, our techniques of modelling and manipulating 3D and 4D manifolds.

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