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Abstract. We consider the one-dimensional wave equation with an indefinite sign damping and a zero order potential term. Using a shooting method, we establish the asymptotic expansion of eigenvalues and eigenvectors of the damped wave equation for a large class of coefficients. In addition, if the damping coefficient is "more positive than negative", we prove that the energy of system decays uniformly exponentially to zero. This sharp result generalizes a previous work of Freitas and Zuazua (1996).

Key words. indefinite damping, spectrum expansion, Riesz basis, exponential decay rate.

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1. Introduction. We consider the Dirichlet problem of one-dimensional wave equation with an indefinite viscous damping:

$$\begin{cases} y_{tt} - y_{xx} + 2\epsilon a y_t + by = 0, & 0 < x < 1, \\ y(0,t) = y(1,t) = 0, & \forall t > 0 \\ y(x,0) = y_0(x), \quad y_t(x,0) = z_0(x) \end{cases} \tag{1.1}$$

where the variable coefficient a is allowed to change sign. Our goal is to investigate the energy decay rate of the solution, if the coefficient a is assumed to be "more positive than negative", a conjecture put forward by Chen et al. in [2].

We know that if a is nonnegative and strictly positive on some subinterval of $[0,1]$, the energy of the solution of the equation (1.1) decays uniformly exponentially to zero as t goes to infinity (see Bardos et al. [1] and Chen et al. [2]). Recently, Freitas and Zuazua [6] established the uniform stability in the case of indefinite sign.

Now we define the linear unbounded operator A^ϵ in the space $H_0^1(0,1) \times L^2(0,1)$ as follows :

$$(1.2) \quad A^\epsilon = \begin{pmatrix} 0 & I \\ \partial_{xx} - b & 2\epsilon a \end{pmatrix}, \quad D(A^\epsilon) = \big(H^2(0,1) \cap H_0^1(0,1)\big) \times H_0^1(0,1).$$

Let γ_n be an eigenvalue of the operator $(\partial_{xx} - b)$, and v_n be the associated normalized eigenfunction in $L^2(0,1)$. In [5] Freitas proved that if $\epsilon > 0$ is small enough, the following conditions are necessary for the stability of the equation (1.1) :

$$(C_1) \qquad 0 > \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \to -\infty,$$

$$(C_2) \qquad I_n = \int_0^1 a(x) v_n^2(x) dx \geq c_0 > 0, \qquad \forall n \geq 1.$$

Later on Freitas and Zuazua [6] established the uniform energy decay rate in the case where $a \in BV(0,1)$ and $b = 0$ for $\epsilon > 0$ small enough. Their proof is based on a shooting method developed by Cox and Zuazua [4] for the wave equation. But in the absence of an adequate ansatz for the asymptotic expansion, the problem remained open in the general case, even when b is a constant.

In this work, we will establish the exponential stability in the case where $a \in BV(0,1)$ and $b \in L^1(0,1)$ for $\epsilon > 0$ small enough. We will employ a shooting method used in Rao [10] for the Rayleigh's beam equation. This approach consists in constructing, without any a priori ansatz, an explicit approximation of the characteristic equation of the underlying system. In sections 2, 3 we establish the asymptotic expansion of the eigenvalue $\lambda_{\pm n}$:

$$(1.3) \qquad \lambda_{\pm n} = -\epsilon \int_0^1 a(x) dx \pm in\pi + (i + \epsilon) O_R\Big(\frac{1}{n}\Big)$$

where $O_R(\xi) \sim \xi$ is a real number. Section 4 is devoted to a detailed study of the system of eigenvectors. We first determine the number of eigenvalues of low frequency. Next we prove that the system of eigenvectors of A^ϵ is equivalent to the usual orthonormal basis of the energy space $H_0^1(0,1) \times L^2(0,1)$.

Finally recall that the general one-dimensional damped wave equation

$$(1.4) \qquad \rho y_{tt} - (\sigma y_x)_x + 2\epsilon a y_t + by = 0$$

can be reduced into the form (1.1) by the change of x-variable and the unknown y. Therefore under suitable conditions on the coefficients ρ, σ, a, b, we can

establish the uniform stability of the general one-dimensional damped wave equation (1.4) without any difficulty.

2. A special case. In this section we consider the special case where b is a constant. From the condition (C_1), we notice that $-\gamma_n = n^2\pi^2 + b \geq \pi^2 + b > 0$ for all $n \geq 1$. For all $n \geq 1$ we define $\mu_{\pm n}$ by the equation :

$$\sqrt{\mu_{\pm n}^2 + b} + \epsilon a_0 = \pm in\pi \qquad \text{where} \quad Re\ \mu_{\pm n} < 0, \quad a_0 = \int_0^1 a(s)ds. \tag{2.1}$$

From the condition (C_2), we have $a_0 > 0$. Then $Re\ \mu_{\pm n} < 0$ implies that $Im\ \mu_n > 0$ and $Im\ \mu_{-n} < 0$. It follows that

$$\left|\mu_{\pm n} - (\epsilon a_0 \mp i\sqrt{n^2\pi^2 + b})\right| \geq \left|Im\ \mu_{\pm n} \pm \sqrt{n^2\pi^2 + b}\right| \geq \sqrt{n^2\pi^2 + b}. \tag{2.2}$$

A direct computation gives that

$$\mu_{\pm n}^2 - (\epsilon a_0 \mp i\sqrt{n^2\pi^2 + b})^2 = (\mu_{\pm n}^2 + b) - (\epsilon a_0 \mp in\pi)^2 \mp 2i\epsilon a_0(\sqrt{n^2\pi^2 + b} - n\pi).$$

Using (2.1), it follows that

$$\left|\mu_{\pm n}^2 - (\epsilon a_0 \mp i\sqrt{n^2\pi^2 + b})^2\right| = \left|\frac{2\epsilon a_0 bi}{n\pi + \sqrt{n^2\pi^2 + b}}\right| \leq \frac{2\epsilon\|a\|_\infty |b|}{n\pi}. \tag{2.3}$$

Combining (2.2) and (2.3) we get :

$$\left|\mu_{\pm n} + \epsilon a_0 \mp i\sqrt{n^2\pi^2 + b}\right| \leq \frac{2\epsilon\|a\|_\infty |b|}{n\pi\sqrt{-\gamma_1}} := \frac{\epsilon C_1}{n}. \tag{2.4}$$

In particular, for $0 < \epsilon < 1$ we have :

$$\left|\frac{\mu_{\pm n}}{n}\right| \leq \|a\|_\infty + C_1 + \sqrt{\pi^2 + |b|}. \tag{2.5}$$

Next for all $n \geq 1$, we define the region $G_{\pm n}$ as :

$$G_{\pm n} = \{|\lambda - \mu_{\pm n}| \leq \epsilon/n\}. \tag{2.6}$$

Let $\lambda \in G_{\pm n}$. Then using (2.1) and (2.5), we find that

$$\begin{aligned}
|\lambda^2 + b| &\geq |\mu_{\pm n}^2 + b| - |\lambda + \mu_{\pm n}||\lambda - \mu_{\pm n}| \\
&\geq (n\pi - \epsilon|a_0|)^2 - \frac{\epsilon}{n}\left(2|\mu_n| + \frac{\epsilon}{n}\right) \\
&\geq \frac{\pi^2}{4} - 2\epsilon(1 + \|a\|_\infty + C_1 + \sqrt{\pi^2 + |b|}) > \frac{\pi^2}{6}
\end{aligned}$$

provided that $\epsilon > 0$ is small enough :

$$\epsilon < \epsilon_1 := \frac{\pi^2}{30(1 + \|a\|_\infty + C_1 + \sqrt{\pi^2 + |b|})} < 1. \tag{2.7}$$

Hence we can choose $0 < Arg(\lambda^2 + b) < 2\pi$ for $\lambda \in G_n$ and $-2\pi < Arg(\lambda^2 + b) < 0$ for $\lambda \in G_{-n}$ such that the function $\lambda \to \sqrt{\lambda^2 + b}$ is analytic in each region $G_{\pm n}$.

As in Cox and Zuazua [4], we consider the initial value problem :

$$\begin{cases} \lambda^2 y - y_{xx} + 2\epsilon a\lambda y + by = 0, & 0 < x < 1, \\ y(0) = 0, \qquad y_x(0) = 1. \end{cases} \tag{2.8}$$

From the classical theory of ordinary differential equations we know that the problem (2.8) has a unique solution which is analytic with respect to the parameters ϵ, λ (Coddington and Levinson [3]), and $\lambda_{\pm n}$ is an eigenvalue of the operator A^ϵ if and only if $\lambda_{\pm n}$ is a zero of the function $\lambda \to y(1,t)$. Furthermore the eigenvalue $\lambda_{\pm n}$ is geometrically simple and the algebraic multiplicity of $\lambda_{\pm n}$ is the order of $\lambda_{\pm n}$ as a zero of the function $\lambda \to y(1,\lambda)$ (Neimark [9]).

Following an idea used in Rao [10], we will construct an explicit approximation of the solution y of the initial value problem (2.8). In fact if a, b are constants, the frequencies of the problem (2.8) are given by

$$\tau = \pm\sqrt{\lambda^2 + 2\epsilon a\lambda + b} = \pm\sqrt{\lambda^2 + b} \pm \epsilon a + O\Big(\frac{\epsilon}{|\lambda|}\Big).$$

This suggests us to approach the solution y of (2.8) by

$$v(x,\lambda) = \frac{1}{\sqrt{\lambda^2 + b} + \epsilon a(0)} \sinh\Big(\sqrt{\lambda^2 + b}\ x + \int_0^x \epsilon a(s)ds\Big). \tag{2.9}$$

By virtue of (2.6), we see that the denominator $\sqrt{\lambda^2 + b} + \epsilon a(0)$ in (2.9) doesn't vanish in the region $G_{\pm n}$ for $0 < \epsilon < \epsilon_1$. We notice that v satisfies the initial value conditions of (2.8). We will justify that v is indeed a good approximation of the problem (2.8).

Lemma 2.1. Assume that $a \in L^\infty(0,1)$ and b is a constant. Then there exists a constant $C_2 > 0$, depending only on $\|a\|_\infty$ and b, such that for all $n \geq 1$ and $0 < \epsilon < \epsilon_1$ the solution y of the initial value problem (2.8) satisfies the following estimation:

$$|y| \leq \frac{C_2}{\sqrt{|\lambda^2 + b|}}, \quad \forall \lambda \in G_{\pm n}. \tag{2.10}$$

Proof Let $\lambda \in G_{\pm n}$. Then we have $\lambda = \mu_{\pm n} + \epsilon r_n e^{i\theta}$ with $0 \le r_n \le 1/n$. By virtue of the choice of $Arg(\lambda^2+b)$, we see that $Im\ \sqrt{\lambda^2+b} > 0$ for $\lambda \in G_n$ and $Im\ \sqrt{\lambda^2+b} < 0$ for $\lambda \in G_{-n}$. Therefore we obtain that

$$\left|\sqrt{\lambda^2+b} - (\epsilon a_0 \mp in\pi)\right| \ge \left|Im\ \sqrt{\lambda^2+b} \pm n\pi)\right| \ge n\pi. \tag{2.11}$$

Using (2.1) we have

$$\lambda^2+b = \mu_{\pm n}^2 + 2\epsilon\mu_{\pm n} r_n e^{i\theta} + \epsilon^2 r_n^2 e^{i2\theta} + b = (\epsilon a_0 \mp in\pi)^2 + 2\epsilon\mu_{\pm n} r_n e^{i\theta} + \epsilon^2 r_n^2 e^{i2\theta}.$$

Then using (2.5), we deduce that

$$\begin{aligned} &\left|\lambda^2 + b - (\epsilon a_0 \mp in\pi)^2\right| \\ &\le 2\epsilon\big(r_n^2 + r_n|\mu_n|\big) \le 2\epsilon\big(1 + \|a\|_\infty + C_1 + \sqrt{\pi^2+|b|}\big) := 2\epsilon C_3. \end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12), we have

$$\left|\sqrt{\lambda^2+b} + (\epsilon a_0 \mp in\pi)\right| \le \frac{2\epsilon C_3}{n\pi} \le \epsilon C_3. \tag{2.13}$$

It follows that from (2.13) that

$$\left|Re\sqrt{\lambda^2+b}\right| \le \epsilon(\|a\|_\infty + C_3) \le 2\epsilon C_3 \le 1 \tag{2.14}$$

where the last inequality is due to (2.7). Next using (2.5) and (2.13) we obtain that

$$\frac{|\lambda|}{\sqrt{|\lambda^2+b|}} \le \frac{|\mu_{\pm n}| + \epsilon}{n\pi - 2\epsilon C_3} \le \frac{|\mu_{\pm n}/n| + 1}{\pi - 2\epsilon C_3} \le \frac{C_3}{\pi - 2\epsilon C_3} \le \frac{C_3}{2} \tag{2.15}$$

where the last inequality is due to (2.7).

Now setting

$$z(x) = \frac{1}{\sqrt{\lambda^2+b}} \sinh\sqrt{\lambda^2+b}\ x,$$

then from the variation of constants formula we have

$$y(x) = z(x) - 2\epsilon\lambda \int_0^x a(s)y(s)z(x-s)ds.$$

It comes from (2.14) and (2.15) that

$$|y(x)| \le \frac{\cosh 1}{\sqrt{|\lambda^2+b|}} + C_3 \cosh 1 \int_0^x |a(s)||y(s)|ds.$$

Applying Gronwall's inequality, we obtain that

$$|y(x)| \le \frac{\cosh 1}{\sqrt{|\lambda^2+b|}} \exp\Big(C_3 \cosh 1 \int_0^1 |a(s)|ds\Big) := \frac{C_2}{\sqrt{|\lambda^2+b|}}.$$

The proof is complete.

Theorem 2.1 Assume that $a \in BV(0,1)$ and b is a constant. Then there exists constant $C_0 > 0$, depending only on a, b, such that for all $n \ge 1$ and $0 < \epsilon < \epsilon_1$ the following estimations hold for the solution y of (2.8) :

$$\Big|y(x,\lambda) - \frac{1}{\sqrt{\lambda^2+b}} \sinh\Big(\sqrt{\lambda^2+b}\; x + \int_0^x \epsilon a(s)ds\Big)\Big| \le \frac{C_0\epsilon}{|\lambda^2+b|}, \tag{2.16}$$

for all $\lambda \in G_{\pm n}$.

Proof Putting :

$$Lv = \lambda^2 v - v_{xx} + 2\epsilon a\lambda v + bv,$$

then a straight forward computation gives that

$$\begin{aligned} Lv = &- \frac{\epsilon}{\sqrt{\lambda^2+b}+\epsilon a(0)}\Big\{a' \cosh\Big(\sqrt{\lambda^2+b}\; x + \int_0^x \epsilon a(s)ds\Big) \\ &+ \Big(\epsilon a^2 + \frac{2ab}{\lambda+\sqrt{\lambda^2+b}}\Big) \sinh\Big(\sqrt{\lambda^2+b}\; x + \int_0^x \epsilon a(s)ds\Big)\Big\}. \end{aligned} \tag{2.17}$$

Since $\lambda \in G_{\pm n}$, from (2.4), (2.7) and (2.13) we deduce easily that

$$\Big|Re\Big(\sqrt{\lambda^2+b}\; x + \int_0^x \epsilon a(s)ds\Big)\Big| \le 3\epsilon C_3 < 1, \tag{2.18}$$

$$2\big|\sqrt{\lambda^2+b} - \epsilon a(0)\big| \ge \sqrt{|\lambda^2+b|} + n\pi - 3\epsilon C_3 \ge \sqrt{|\lambda^2+b|}, \tag{2.19}$$

$$|\lambda + \sqrt{\lambda^2+b}| \ge n\pi + \sqrt{n^2\pi^2+b} - 2\epsilon C_3 \ge 2. \tag{2.20}$$

Inserting (2.18), (2.19) and (2.20) into (2.17) gives

$$|L(v)| \le \frac{2\epsilon}{\sqrt{|\lambda^2+b|}}\Big(|a|^2 + |a||b| + |a'|\Big)\cosh 1. \tag{2.21}$$

Applying the variation of constants formula and using (2.10), (2.21), we obtain that

$$\begin{aligned} |v(x,\lambda) - y(x,\lambda)| &\le \int_0^x |Lv(s,\lambda)||y(x-s,\lambda)|ds \\ &\le \frac{2\epsilon C_2 \cosh 1}{|\lambda^2+b|} \int_0^1 \Big(|a|^2 + |a||b| + |a'|\Big)ds := \frac{\epsilon C_0'}{|\lambda^2+b|}. \end{aligned} \tag{2.22}$$

Finally inserting the estimation :

$$(2.23)\quad \Big|v(x,\lambda)-\frac{1}{\sqrt{\lambda^2+b}}\sinh\Big(\sqrt{\lambda^2+b}\;x+\int_0^x \epsilon a(s)ds\Big)\Big|\le \frac{2\epsilon\|a\|_\infty\cosh 1}{|\lambda^2+b|}$$

into (2.22) gives (2.16) with $C_0 := C_0' + 2\|a\|_\infty \cosh 1$. The proof is thus complete.

Now let C_0 be the constant appearing in the estimation (2.16). For all $n \ge 1$ and $0 < \epsilon < \epsilon_1$ we define the region $\Gamma_{\pm n}$ as follows:

$$(2.24)\qquad \Gamma_{\pm n} = \{|\lambda - \mu_{\pm n}| \le \epsilon\rho_n\}, \qquad \rho_n = 2C_0\sqrt{\frac{1}{n^2\pi^2+b}}.$$

Assume that

$$(2.25)\qquad \epsilon < \epsilon_2 := \min\Big\{\epsilon_1,\quad \frac{\epsilon_1}{2C_0}\sqrt{\pi^2-|b|}\Big\}.$$

Then we have the inclusion: $\Gamma_{\pm n} \subset G_{\pm n}$ for all $n \ge 1$.

Lemma 2.2 Assume that $a \in BV(0,1)$ and b is a constant. Then the following estimation holds for $\epsilon > 0$ small enough:

$$(2.26)\qquad \Big|\sinh\Big(\sqrt{\lambda^2+b}+\epsilon a_0\Big)\Big| > \frac{C_0\epsilon}{\sqrt{|\lambda^2+b|}}, \qquad \lambda \in \partial\Gamma_{\pm n}.$$

Proof We first assume that $0 < \epsilon < \epsilon_2$. Let $\lambda \in \partial\Gamma_{\pm n}$, then $\lambda = \mu_{\pm n} + \epsilon\rho_n e^{i\theta}$. Since $Im\sqrt{\lambda^2+b} > 0$ for $\lambda \in \Gamma_n$ and $Im\ \sqrt{\lambda^2+b} < 0$ for $\lambda \in \Gamma_{-n}$, then have

$$(2.27)\qquad \begin{aligned}&\Big|\sqrt{\lambda^2+b}-\epsilon a_0 \pm in\pi + \frac{2\epsilon C_0 e^{i\theta}}{n\pi}\Big|\\ \ge&\Big|Im\ \sqrt{\lambda^2+b}\pm n\pi\Big| - \frac{2\epsilon C_0}{\pi}\\ \ge& n\pi - \frac{2\epsilon C_0}{\pi} \ge \frac{n\pi}{2}.\end{aligned}$$

On the other hand, using (2.1) we get

$$(2.28)\qquad \begin{aligned}&(\lambda^2+b) - \Big(\epsilon a_0 \mp in\pi - \frac{2\epsilon C_0 e^{i\theta}}{n\pi}\Big)^2\\ =&\mu_{\pm n}^2 + 2\epsilon\mu_{\pm n}\rho_n e^{i\theta} + \epsilon^2\rho_n^2 e^{2i\theta} + b - (\epsilon a_0 \mp in\pi)^2\\ &+ \frac{2\epsilon C_0 e^{i\theta}}{n\pi}(\epsilon a_0 \mp in\pi) - \Big(\frac{2\epsilon C_0 e^{i\theta}}{n\pi}\Big)^2\\ =&2\epsilon e^{i\theta}\big(\mu_n\rho_n \mp i2C_0\big) + \epsilon^2\rho_n^2 e^{i2\theta} + 4\frac{\epsilon^2 a_0 C_0 e^{i\theta}}{n\pi} - \Big(\frac{2\epsilon C_0 e^{i\theta}}{n\pi}\Big)^2.\end{aligned}$$

From (2.4), we deduce that

$$|\mu_n\rho_n \mp i2C_0| \le 2\epsilon C_0(|a_0| + C_1)\sqrt{\frac{1}{n^2\pi^2 + b}}. \tag{2.29}$$

Combining (2.28) and (2.29), it follows that

$$\begin{aligned}&\Big|(\lambda^2 + b) - \Big(\epsilon a_0 \mp in\pi - \frac{2\epsilon C_0 e^{i\theta}}{n\pi}\Big)^2\Big| \\ &\le 4C_0^2\Big(\frac{1}{\pi^2}+\frac{1}{|\gamma_1|}\Big)\epsilon^2 + 4C_0\Big(\frac{|a_0|}{\pi} + \frac{|a_0|}{\sqrt{|\gamma_1|}} + \frac{C_1}{\sqrt{|\gamma_1|}}\Big)\epsilon^2 := C_4\epsilon^2\end{aligned} \tag{2.30}$$

which together with (2.27) gives

$$\Big|\sqrt{\lambda^2 + b} + \epsilon a_0 \mp in\pi - \frac{2\epsilon C_0 e^{i\theta}}{n\pi}\Big| \le \frac{2C_4\epsilon^2}{n\pi}. \tag{2.31}$$

Now using (2.31) and the inequality :

$$\big|\sinh(\alpha + i\beta)\big| \ge \frac{2}{\pi}\sqrt{\alpha^2 + \beta^2}, \quad \forall \alpha, \beta \in R, \quad |\beta| \le \frac{\pi}{2}. \tag{2.32}$$

we find that

$$\big|\sinh(\sqrt{\lambda^2 + b} + \epsilon a_0)\big| \ge \frac{4}{\pi}\Big(\frac{\epsilon C_0}{n\pi} - \frac{C_4\epsilon^2}{n\pi}\Big). \tag{2.33}$$

Once again from (2.31), we get :

$$\sqrt{|\lambda^2 + b|} \ge n\pi - \epsilon(\|a\|_\infty + C_0 + C_4) \tag{2.34}$$

which combined to (2.33) gives (2.26), if $\epsilon > 0$ satisfies :

$$\epsilon \le \epsilon_3 := \min\Big\{\epsilon_2, \quad \frac{(4-\pi)C_0}{\pi\big(4C_4 + C_0(\|a\|_\infty + C_0 + C_4)\big)}\Big\}. \tag{2.35}$$

The proof is thus complete.

Theorem 2.2 Assume that $a \in BV(0,1)$ and b is a constant. Then for $0 < \epsilon < \epsilon_3$ the operator A^ϵ admits a unique eigenvalue in $G_{\pm n}$ for each $n \ge 1$. Moreover, we have the asymptotic expansion

$$\big|\lambda_{\pm n} + \epsilon a_0 \mp i\sqrt{n^2\pi^2 + b}\big| \le \epsilon\Big(\frac{C_1}{n} + \frac{C_0}{\sqrt{n^2\pi^2 + b}}\Big). \tag{2.36}$$

Proof From (2.16) and (2.26), we deduce that

$$\left|y(1,\lambda) - \frac{1}{\sqrt{\lambda^2+b}}\sinh\left(\sqrt{\lambda^2+b}+\epsilon a_0\right)\right| < \left|\frac{1}{\sqrt{\lambda^2+b}}\sinh\left(\sqrt{\lambda^2+b}+\epsilon a_0\right)\right|$$

for any $\lambda \in \partial\Gamma_{\pm n}$. Via Rouché's theorem, there exists one simple zero $\lambda_{\pm n} \in \Gamma_{\pm n}$ of the function $\lambda \to y(1,\lambda)$. In particular, we get

$$|\lambda_{\pm n} - \mu_{\pm n}| \leq \frac{2C_0\epsilon}{\sqrt{n^2\pi^2+b}}, \qquad \forall n \geq 1. \tag{2.37}$$

which together with (2.4) implies (2.36). The proof is thus complete.

Remark. Let $\lambda_{\pm n}$ be an eigenvalue of the operator A^ϵ and $u_{\pm n} = (y_{\pm n}, \lambda_{\pm n} y_{\pm n})$ be the associated eigenvector. We should prove (but we leave it to Theorem 4.3) that the system of eigenvectors $(u_{\pm n})_{n\geq 1}$ forms a Riesz basis in the space $H_0^1(0,1) \times L^2(0,1)$.

3. High frequencies in the general case. In this section we will consider the general case. Assume that $\epsilon > 0$ is small enough. Then from the condition (C_1), we see that the eigenvalues $\lambda_{\pm n}$ are complex. Moreover we have the expession :

$$Re\lambda_{\pm n} = \frac{-\epsilon\int_0^1 a|y_{\pm n}|^2 dx}{\int_0^1 |y_{\pm n}|^2 dx}. \tag{3.1}$$

We will prove that the real part of the eigenvalues of high frequency can be uniformly localized to the left of the imaginary axis of the complex plan.

We will assume that $\|a\|_\infty \neq 0$ (the contrary case is trivial). On the other hand, in order to clarify the independence of the various constants $C's$ appearing in the estimations on the parameter ϵ, we assume that $0 < \epsilon < 1$. Therefore we consider the equation (2.8) only for $|Re\lambda| \leq 2\|a\|_\infty$. We will construct an explicit approximation of the solution y of the initial value problem (2.8). We use the same idea as in the previous case. But this time we neglect the zero order potential by and we approach the solution y of (2.8) by $\widetilde{v}$:

$$\widetilde{v} = \frac{1}{\lambda + \epsilon a(0)}\sinh\left(\lambda x + \epsilon\int_0^x a(s)ds\right). \tag{3.2}$$

Theorem 3.1 Assume that $a \in BV(0,1)$, $b \in L^1(0,1)$. Let $|\lambda| \geq 2\|a\|_\infty$ and $|Re\lambda| \leq 2\|a\|_\infty$. Then there exists constant $C_5 > 0$, depending only on

a and b, such that the solution y of equation (2.8) satisfies the following estimations :

$$\left|y(x,\lambda) - \frac{1}{\lambda}\sinh\left(\lambda x + \epsilon\int_0^x a(s)ds\right)\right| \leq \frac{C_5}{|\lambda|^2}, \tag{3.3}$$

$$\left|y_x(x,\lambda) - \cosh\left(\lambda x + \epsilon\int_0^x a(s)ds\right)\right| \leq \frac{C_5}{|\lambda|}. \tag{3.4}$$

Proof Let y be the solution of (2.8). We first prove that there exists C_6, depending only on a and b, such that the following estimations hold:

$$|y| \leq \frac{C_6}{|\lambda|}, \qquad |y_x| \leq C_6. \tag{3.5}$$

In fact using the variation of constants formula, we have

$$y = \frac{1}{\lambda}\sinh\lambda x - \int_0^x \frac{2a\epsilon\lambda + b}{\lambda}\sinh\lambda(x-s)y(s)ds.$$

It follows that

$$|y| \leq \frac{\cosh(2\|a\|_\infty)}{|\lambda|} + \cosh(2\|a\|_\infty)\int_0^x \Big(2|a| + \frac{|b|}{2\|a\|_\infty}\Big)|y(s)|ds.$$

Using Gronwall's inequality, we obtain

$$|y| \leq \frac{\cosh(2\|a\|_\infty)}{|\lambda|}\exp\Big(\cosh(2\|a\|_\infty)\int_0^1 \Big(2|a| + \frac{|b|}{2\|a\|_\infty}\Big)ds\Big) := C_6.$$

This implies that

$$|y_x| \leq \cosh(2\|a\|_\infty) + C_5\cosh(2\|a\|_\infty)\int_0^x \Big(2|a| + \frac{|b|}{2\|a\|_\infty}\Big)ds := C_6'.$$

Taking C_6 as the maximum of C_6 and C_6' we obtain the estimations (3.5).

On the other hand, a straight forward computation gives that

$$L\widetilde{v} = -\frac{\epsilon a'}{\lambda + \epsilon a(0)}\cosh\left(\lambda x + \epsilon\int_0^x a(s)ds\right) + \frac{b - \epsilon^2 a^2}{\lambda + \epsilon a(0)}\sinh\left(\lambda x + \epsilon\int_0^x a(s)ds\right).$$

It follows that

$$|L\widetilde{v}| \leq \frac{2\cosh(3\|a\|_\infty)}{|\lambda|}\big(|a'| + |a|^2 + |b|\big). \tag{3.6}$$

Once again using the variation of constants formula, we have

$$\widetilde{v}(x,\lambda) - y(x,\lambda) = \int_0^x L\widetilde{v}(s,\lambda)y(x-s,\lambda)ds. \tag{3.7}$$

From the estimations (3.5)-(3.6) comes that

$$\begin{aligned} |\widetilde{v}(x,\lambda) - y(x,\lambda)| &\le \int_0^1 |L\widetilde{v}(s)||y(x-s,\lambda)|ds \\ &\le \frac{2C_5\cosh(3\|a\|_\infty)}{|\lambda|^2}\int_0^1 \big(|a'| + |a|^2 + |b|\big)ds := \frac{C_5'(T_a, \|a\|_2, \|b\|_1)}{|\lambda|^2} \end{aligned} \tag{3.8}$$

where T_a denotes the total variation of the function a.

Differentiating the equation (3.7) gives

$$\widetilde{v}_x(x,\lambda) - y_x(x,\lambda) = \int_0^x L\widetilde{v}(s,\lambda)y_x(x-s,\lambda)ds.$$

Using (3.5) and (3.6), it follows that

$$\begin{aligned} &|\widetilde{v}_x(x,\lambda) - y_x(x,\lambda)| \\ &\le \int_0^1 |L\widetilde{v}(s,\lambda)||y_x(x-s,\lambda)|ds \le \frac{C_5'(T_a, \|a\|_2, \|b\|_1)}{|\lambda|}. \end{aligned} \tag{3.9}$$

Finally using the explicit expression (3.2) into (3.8)-(3.9) gives the estimations (3.3)-(3.4) with the constant $C_5 = C_5' + 2\cosh(3\|a\|_\infty)$. The proof is complete.

Now let C_5 be the constant appearing in the estimations (3.3)-(3.4) and N be an integer. We define the regions $\Gamma_{\pm n}$:

$$\begin{cases} \Gamma_{\pm n} = \{|\lambda + \epsilon a_0 \mp in\pi| \le C_5/n\}, & n > N, \\ \Gamma_N = \{-\|a\|_\infty \le Re\lambda \le \|a\|_\infty, \quad -N\pi \le Im\lambda \le N\pi\}. \end{cases} \tag{3.10}$$

Lemma 3.1 There exists an integer N, depending only on a and b, such that for all $n > N$ the following estimation holds

$$|\sinh(\lambda + \epsilon a_0)| > \frac{C_5}{|\lambda|}, \qquad \lambda \in \Gamma_{\pm n}. \tag{3.11}$$

Proof Let $\lambda \in \Gamma_{\pm n}$. Then $\lambda + \epsilon a_0 = \pm in\pi + C_5 e^{i\theta}/n$. Applying (2.32) we obtain that

$$\big|\sinh(\lambda + \epsilon a_0)\big| = \big|\sinh(\pm in\pi + C_5 e^{i\theta}/n)\big| = \big|\sinh(C_5 e^{i\theta}/n)\big| \ge \frac{2C_5}{\pi n}.$$

Since $|\lambda| \geq n\pi - |a_0| - C_5$ for $\lambda \in \Gamma_{\pm n}$, we get the estimation (3.11) provided that

$$n > \max\Big\{\frac{3(C_5 + \|a\|_\infty)}{\pi}, \quad \frac{C_5}{\|a\|_\infty}\Big\}. \tag{3.12}$$

Here we add the second term in (3.12) to guarantee that $\Gamma_{\pm n}$ is included in the strip $\{|Re\lambda| \leq 2\|a\|_\infty, \quad |\lambda| \geq 2\|a\|_\infty\}$.

Theorem 3.2 Assume that $a \in BV(0,1)$ and $b \in L^1(0,1)$. Then for $n > N$ the operator A^ϵ admits a unique eigenvalue in each $\Gamma_{\pm n}$, and a finite number of eigenvalues in the region Γ_N. Moreover, we have the asymptotic expansion:

$$\lambda_{\pm n} = -\epsilon a_0 \pm in\pi + (\epsilon + i)O_R\Big(\frac{1}{n}\Big) \tag{3.13}$$

where $O_R(\xi) \sim \xi$ denotes one real number.

Proof We first notice that the spectrum of A^ϵ is discrete, therefore there exists at most a finite number of eigenvalues in the compact region Γ_N.

Next combining (3.3) and (3.11) we obtain that

$$\Big|y(1,\lambda) - \frac{1}{\lambda}\sinh(\lambda + \epsilon a_0)\Big| < \Big|\frac{1}{\lambda}\sinh(\lambda + \epsilon a_0)\Big|$$

for any $\lambda \in \partial\Gamma_{\pm n}$. Via Rouché's theorem, we deduce that there exists one and only one eigenvalue $\lambda_{\pm n} \in \Gamma_{\pm n}$. In particular we have the following rough estimation:

$$\Big|\lambda_{\pm n} + \epsilon a_0 \mp in\pi\Big| \leq \frac{C_5}{n}. \tag{3.14}$$

Putting

$$\xi(x) = \int_0^x a(s)ds - x\int_0^1 a(s)ds \tag{3.15}$$

then using (3.3), (3.4) and (3.14), we find easily

$$\Big|y_{\pm n}(x) - \frac{1}{\lambda_{\pm n}}\sinh\big(\epsilon\xi(x) \pm in\pi x\big)\Big| \leq \frac{C_6}{n^2}, \tag{3.16}$$

$$\Big|y_{\pm nx}(x) - \cosh\big(\epsilon\xi(x) \pm in\pi x\big)\Big| \leq \frac{C_6}{n}, \quad \forall n > N \tag{3.17}$$

where we have put

$$C_6 = \big(1 + \cosh(3\|a\|_\infty)\big) C_5.$$

From (3.16) we see that the leading term of the eigenfunction $y_{\pm n}$ doesn't depend on the coefficient b. By virtue of the expression (3.1), we see that $Re\lambda_{\pm n}$ doesn't depend on b either. We first calculate

$$2\int_0^1 a|\sinh(\epsilon\xi(x) \pm in\pi x)|^2 dx \tag{3.18}$$
$$= a_0 \int_0^1 \cosh(2\epsilon\xi(x))dx + \int_0^1 a(x)\cos 2n\pi x dx,$$

$$2\int_0^1 |\sinh(\epsilon\xi(x) \pm in\pi x)|^2 dx = \int_0^1 \cosh(2\epsilon\xi(x))dx. \tag{3.19}$$

From (3.16) we can write

$$\Big||y_{\pm n}(x)|^2 - \frac{1}{|\lambda_{\pm n}|^2}\big|\sinh\big(\epsilon\xi(x) + in\pi x\big)\big|^2\Big| \le \frac{C_7}{n^3} \tag{3.20}$$

with

$$c_7 = \frac{C_6^2 + 2C_6\cosh(2\|a\|_\infty)}{n^3}.$$

Inserting (2.18) -(3.20) into (3.1) gives that

$$\big|Re\lambda_{\pm n} + \epsilon a_0\big| \le \epsilon \frac{\Big|\int_0^1 a(x)\cos 2n\pi x dx\Big| + \frac{2\|a_0 - a\|_\infty C_7|\lambda_{\pm n}|^2}{n^3}}{\int_0^1 \cosh(2\epsilon\xi(x))dx - \frac{2C_7|\lambda_{\pm n}|^2}{n^3}}. \tag{3.21}$$

Since

$$\begin{cases} \displaystyle\int_0^1 \cosh(2\epsilon\xi(x))dx \ge 1, \\ \displaystyle\Big|\int_0^1 a(x)\cos 2n\pi x dx\Big| \le \frac{T_a}{n}, \\ \displaystyle\frac{n\pi}{2} \le |\lambda_{\pm n}| \le 2n\pi. \end{cases} \tag{3.22}$$

Then inserting (3.22) into (3.21), we obtain that

$$\big|Re\lambda_{\pm n} + \epsilon a_0\big| \le \frac{2\epsilon}{n}\big(T_a + 16\pi^2 C_7\|a\|_\infty\big) := \frac{C_8\epsilon}{n}, \tag{3.23}$$

provided that

$$n > 16\pi^2 C_7. \tag{3.24}$$

The proof is thus complete.

4. Low frequencies and root system. Let us denote by $\lambda_{\pm k}$ the eigenvalues of A contained in the region Γ_N with the algebraic multiplicity $m_{\pm k} \geq 1$. For ϵ small enough, the eigenvalues are complex and appear in conjugate pairs. Thus we have $m_k = m_{-k}$. We choose λ_k such that $Im\ \lambda_k > 0$, and λ_{-k} such that $Im\ \lambda_{-k} < 0$. Accordingly, we denote by $(u_{\pm k,j})_{j=1}^{m_{\pm k}}$ an arbitrary orthonormal basis of the eigenspace $Ker(A - \lambda_{\pm k})^{m_{\pm k}}$.

Let $n > N$, we denote by $u_{\pm n} = (y_{\pm n}, \lambda_{\pm n} y_{\pm n})$ the eigenvectors of high frequency. On constructing the corresponding biorthogonal system as in Cox-Zuazua [4], it is easily seen that the system of eigenvectors

$$(u_{\pm k,j})_{1 \leq k \leq K, 1 \leq j \leq m_{\pm k}} \cup (u_{\pm n})_{n > N} \tag{4.1}$$

is linearly independent in the energy space $H_0^1(0,1) \times L^2(0,1)$. We arrange the eigenvectors of low frequency in the following way :

$$u_{\pm k,j} = \widetilde{u}_{\pm n}, \qquad n = j + \sum_{k=1}^{k-1} m_k. \tag{4.2}$$

Here we have used the symbol $\widetilde{u}_{\pm n}$ to avoid possible ambiguity with the eigenvectors $u_{\pm n}$ of the high frequency. Setting $\widetilde{N} = \sum_{k=1}^{K} m_k$, we write the system of eigenvectors in the following form:

$$(\widetilde{u}_{\pm n})_{1 \leq n \leq \widetilde{N}} \cup (u_{\pm n})_{n > N} \tag{4.3}$$

Theorem 4.1 The operator A^0 has exactly $2N$ eigenvalues in the region Γ_N.

Proof In the case $\epsilon = 0$, A^0 is skew adjoint and has a compact resolvent. The corresponding system of eigenvectors $(\widetilde{u}^0_{\pm n})_{1 \leq n \leq \widetilde{N}} \cup (u^0_{\pm n})_{n > N}$ is complete, orthogonal and almost normal, therefore forms a Riesz basis in the energy space $H^1(0,1) \times L^2(0,1)$. Let us denote by $e_{\pm n}$ the usual orthonormal basis of $H_0^1(0,1) \times L^2(0,1)$:

$$e_{\pm n} = \begin{pmatrix} \frac{1}{in\pi} \sinh in\pi x \\ \pm \sinh in\pi x \end{pmatrix}, \qquad n \geq 1. \tag{4.4}$$

Assume that $\widetilde{N} < N$, from (3.16)-(3.17) it follows that

$$\sum_{1\le n\le \widetilde{N}} \|\widetilde{u}^0_{\pm n} - e_{\pm n}\|^2_{H^1_0(0,1)\times L^2(0,1)} + \sum_{n>N} \|u^0_{\pm n} - e_{\pm n}\|^2_{H^1_0(0,1)\times L^2(0,1)} < +\infty.$$

Thanks to Bari's theorem, we find that the subsystem $(e_{\pm n})_{1\le n\le \widetilde{N}} \cup (e_{\pm n})_{n>N}$, which is quadratically close to the Riesz basis $(\widetilde{u}^0_{\pm n})_{1\le n\le \widetilde{N}} \cup (u^0_{\pm n})_{n>N}$, would also be a Riesz basis in the energy space $H^1_0(0,1)\times L^2(0,1)$. This contradicts the linearly independence of the system $(e_{\pm n})_{n\ge 1}$. Changing the role of $u^0_{\pm n}$ and $e_{\pm n}$ we obtain that $\widetilde{N} = N$. Now there is no more ambiguity, we can replace $\widetilde{u}^0_{\pm n}$ by $u^0_{\pm n}$ in (4.2). The proof is thus complete.

Theorem 4.2 Assume that $a \in BV(0,1)$ and $b \in L^1(0,1)$. Then for $\epsilon > 0$ small enough the operator A^ϵ admits exactly $2N$ eigenvalues in the region Γ_N.

Proof We first write

$$(4.5)\quad \left(A^\epsilon\right)^{-1} = \begin{pmatrix} 0 & (\partial_{xx} - b)^{-1} \\ I & 0 \end{pmatrix} + \epsilon \begin{pmatrix} 2(\partial_{xx} - b)^{-1} a & 0 \\ 0 & 0 \end{pmatrix} := (A^0)^{-1} + \epsilon B.$$

Next for $\delta > 0$ small enough we define the region $D_{\pm k}$ as

$$(4.6)\qquad D_{\pm k} = \{|\lambda - \lambda^0_{\pm k}| \le \delta\}.$$

Since A^{-1} is compact and skew adjoint, then the eigenvalues $\lambda^0_{\pm k}$ are normal points of $(A^0)^{-1}$ and the contour $\partial D_{\pm k}$ consists of regular points of $(A^0)^{-1}$. Applying Theorem I.3.1 in Gohberg and Krein [7], we conclude that there exists $\epsilon_4 > 0$ such that for $0 < \epsilon < \epsilon_4$ the operator $\left(A^\epsilon\right)^{-1}$ has the same number of eigenvalues than $(A^0)^{-1}$ in each region $D_{\pm k}$:

$$(4.7)\qquad \lambda_{\pm k,j} \in D_{\pm k}, \qquad k = 1, 2, \cdots, K; \quad j = 1, 2, \cdots, m_{\pm k}.$$

Since A^ϵ is an analytic family (see Kato [8]), arranging the root vectors $u_{\pm k,j}$ in the same way as in (4.2), we find that

$$(4.8)\qquad \lim_{\epsilon\to 0} u_{\pm n} = u^0_{\pm n}, \qquad \forall n \ge 1.$$

The proof is thus complete.

Now we introduce a subspace :

$$(4.9)\qquad V = \{y \in H^1(0,1) : \quad y(0) = 0\},$$

and a linear application L :

$$L(y,z) = (y_x, z), \qquad (y,z) \in V \times L^2(0,1). \tag{4.10}$$

Then we verify easily that L is an isomorphism from $V \times L^2(0,1)$ onto $L^2(0,1) \times L^2(0,1)$.

Setting :

$$e_0 = u_0 = \begin{pmatrix} x \\ 0 \end{pmatrix}, \tag{4.11}$$

the extended system $(e_{\pm n})_{n\geq 1}$ becomes an orthonormal basis of $V \times L^2(0,1)$, and the extended system $(u_{\pm n})_{n\geq 1}$ remains linearly independent in the space $V \times L^2(0,1)$.

Putting

$$B^\epsilon = \begin{pmatrix} \cosh \epsilon\xi(x) & \sinh \epsilon\xi(x) \\ \sinh \epsilon\xi(x) & \cosh \epsilon\xi(x) \end{pmatrix}, \ \Phi^\epsilon_{\pm n} = \begin{pmatrix} \cosh\left(\epsilon\xi(x) \pm in\pi x\right) \\ \sinh\left(\epsilon\xi(x) \pm in\pi x\right) \end{pmatrix} \tag{4.12}$$

we have

$$B^\epsilon \Phi^0_{\pm n} = \Phi^\epsilon_{\pm n}, \qquad Le_{\pm n} = \Phi^0_{\pm n}, \qquad \forall n \geq 0. \tag{4.13}$$

Since the matrix B^ϵ has a bounded inverse, we see that B^ϵ is a bounded invertible linear operator of $L^2(0,1) \times L^2(0,1)$. Moreover, we verify easily that

$$\|B^\epsilon\| \leq 2\cosh(2\|a\|_\infty), \qquad \|(B^\epsilon)^{-1}\| \leq 2\cosh(2\|a\|_\infty). \tag{4.14}$$

Theorem 4.3 Assume that $a \in BV(0,1)$ and $b \in L^1(0,1)$. There exists a linear bounded invertible operator S^ϵ of $V \times L^2(0,1)$ such that

$$S^\epsilon e_{\pm n} = u_{\pm n}, \qquad \forall n \geq 0. \tag{4.15}$$

Moreover we can find $\epsilon_5 > 0$, depending only on a and b, such that the following estimations hold

$$\sup_{0<\epsilon<\epsilon_5} \|S^\epsilon\| < +\infty, \qquad \sup_{0<\epsilon<\epsilon_5} \|\left(S^\epsilon\right)^{-1}\| < +\infty. \tag{4.16}$$

Proof We first assume that $0 < \epsilon < \epsilon_4$. Then using (3.16)-(3.17), we calculate the high frequency:

$$\sum_{n>N} \|Lu_{\pm n} - \Phi^{\epsilon}_{\pm n}\|^2_{L^2(,0,1)\times L^2(0,1)} \tag{4.17}$$
$$\leq \sum_{n>N} \frac{C_6^2(1+4\pi^2)}{n^2} < \frac{C_6^2(1+4\pi^2)\pi^2}{6}.$$

By virtue of the choice of eigenvectors $u_{\pm n}$ of low frequency and (3.17) we have :

$$\sum_{0\leq n\leq N} \|Lu_{\pm n} - \Phi^{\epsilon}_{\pm n}\|^2_{L^2(,0,1)\times L^2(0,1)} \leq (2N+1)\big(1+\cosh(2\|a\|_\infty)\big). \tag{4.18}$$

Combining (4.17) and (4.18) we find a constant C_9, depending only on a and b, such that

$$\sum_{n\geq 1} \|Lu_{\pm n} - \Phi^{\epsilon}_{\pm n}\|^2_{L^2(,0,1)\times L^2(0,1)} \leq C_9. \tag{4.19}$$

Now defining the linear operator R^ϵ as follows :

$$R^\epsilon e_{\pm n} = L^{-1}B^\epsilon \Phi^0_{\pm n} - u_{\pm n}, \qquad \forall n \geq 0, \tag{4.20}$$

from (4.19) we deduce that

$$\|R^\epsilon\|^2 \leq \|L^{-1}\|^2 \sum_{n\geq 1} \|\Phi^{\epsilon}_{\pm n} - Lu_{\pm n}\|^2_{L^2(0,1)\times L^2(0,1)} \leq C_9\|L^{-1}\|^2. \tag{4.21}$$

Therefore R^ϵ is a Hilbert-Schmidt operator of the space $V \times L^2(0,1)$. Now setting

$$S^\epsilon = L^{-1}B^\epsilon L - R^\epsilon, \tag{4.22}$$

then from (4.13) and (4.20) it follows that

$$S^\epsilon e_{\pm n} = L^{-1}B^\epsilon L e_{\pm n} - R^\epsilon e_{\pm n} = L^{-1}B^\epsilon \Phi^0_{\pm n} - R^\epsilon e_{\pm n} = u_{\pm n}, \qquad n \geq 0.$$

Moreover using (4.14) and (4.22) we get easily that

$$\|S^\epsilon\| \leq 2\|L^{-1}\|\|L\|\cosh(2\|a\|_\infty) + C_9\|L^{-1}\|^2, \qquad 0 < \epsilon < \epsilon_4.$$

This gives the first estimation of (4.16).

On the other hand, since the system $(u_{\pm n})_{n\geq 1}$ is linearly independent, using Fredholm's Alternative we show easily that S^ϵ has a bounded inverse $(S^\epsilon)^{-1}$ in $V \times L^2(0,1)$. If the second estimation of (4.16) fails, there would exist a sequence $u^\epsilon \in V \times L^2(0,1)$ such that

$$\|u^\epsilon\|_{V\times L^2(0,1)} = 1, \qquad L^{-1}B^\epsilon L u^\epsilon - R^\epsilon u^\epsilon \to 0, \quad \text{in} \quad V \times L^2(0,1). \tag{4.23}$$

Let R^0 denote the limit operator of R^ϵ:

$$R^0 e_{\pm n} = L^{-1}\Phi^0_{\pm n} - u^0_{\pm n} = e^0_{\pm n} - u^0_{\pm n}, \qquad \forall n \geq 0. \tag{4.24}$$

Then using (4.19) and Lebesgue's convergence theorem, we can prove easily that

$$\lim_{\epsilon\to 0^+} \|R^\epsilon - R^0\|^2 = 0. \tag{4.25}$$

Since $B^\epsilon \to I, R^\epsilon \to R^0$ for the uniform topology of $V \times L^2(0,1)$ as $\epsilon \to 0^+$, there exists a subsequence, still indexed by u^ϵ, which converges to u strongly in $V \times L^2(0,1)$. Then passing to the limit in (4.23) we get

$$\|u\|_{V\times L^2(0,1)} = 1, \qquad u - R^0 u = 0. \tag{4.26}$$

Then indeed let $u = \sum_{n\geq 0} \alpha_{\pm n} e_{\pm n}$. Using (4.24) we deduce easily that

$$\sum_{n\geq 0} \alpha_{\pm n} u^0_{\pm n} = 0. \tag{4.27}$$

Since $(u^0_{\pm n})$ is a Riesz Basis, it follows that $\alpha_{\pm n} = 0$ for all $n \geq 0$. This contradicts $\|u\|_{V\times L^2(0,1)} = 1$. The proof is thus complete.

Theorem 4.4 Assume that the conditions $(C_1), (C_2)$ hold. Let $a \in BV(0,1)$ and $b \in L^1(0,1)$. Then there exist positive constants $\epsilon_0 > 0, \omega, M > 0$ depending only on a, b such that for all $0 < \epsilon < \epsilon_0$, the solution y of the system (1.1) satisfies the following estimation :

$$\|y(t)\|^2_{H^1_0(0,1)} + \|y_t(t)\|^2_{L^2(0,1)} \leq M e^{-\omega\epsilon t}\Big(\|y_0\|^2_{H^1_0(0,1)} + \|y_1\|^2_{L^2(0,1)}\Big), \tag{4.28}$$

for all $t \geq 0$.

Proof From the condition (C_2) and the asymptotic expansion (3.16), we deduce that $I_n \to a_0 \geq c_0 > 0$. For the validity of Theorems 4.2, 4.3, we first assume that ϵ satisfies

$$\epsilon < \epsilon_6 := \min\{\epsilon_4, \quad \epsilon_5\}.$$

Using the expansion (3.23) and taking into account the conditions (3.12) and (3.24), we find that the eigenvalues of high frequency are to the left of the imaginary axis:

$$Re\lambda_{\pm n} \leq -\frac{c_0\epsilon}{2} \tag{4.29}$$

provided that

$$n > N_0 := \max\Big\{16\pi^2 C_7, \quad \frac{2C_8}{a_0}, \quad \frac{3(C_5+\|a\|_\infty)}{\pi}, \quad \frac{C_5}{\|a\|_\infty}\Big\}. \tag{4.30}$$

But from the condition $\lambda'_{\pm n}(0) = -I_n \leq -c_0$, we can find $0 < \epsilon_0 \leq \epsilon_6$ such that for all $0 < \epsilon < \epsilon_0$ the remaining $2N_0$ eigenvalues of low frequency satisfy also (4.29).

Now let $(y_0, y_1) \in H_0^1(0,1) \times L^2(0,1)$ such that

$$(y_0, y_1) = \sum_{1\leq k\leq K}\sum_{1\leq j\leq m_k} \alpha_{\pm k,j} u_{\pm k,j} + \sum_{n>N} \alpha_{\pm n} u_{\pm n}. \tag{4.31}$$

Then the solution of the problem (1.1) is given by

$$\begin{aligned}&(y(t), y_t(t))\\ &= \sum_{1\leq k\leq K} e^{\lambda_{\pm k}t} \sum_{1\leq j\leq m_k} \alpha_{\pm k,j} \sum_{1\leq l\leq j} \frac{t^{j-l}}{(j-l)!} u_{\pm k,l} + \sum_{n>N} e^{\lambda_{\pm n}t}\alpha_{\pm n}u_{\pm n}.\end{aligned}$$

Using (4.15) and (4.29), we get

$$\begin{aligned}&\|y(t)\|^2_{H_0^1(0,1)} + \|y_t(t)\|^2_{L^2(0,1)}\\ &\leq \|S^\epsilon\|^2 \exp(-c_0\epsilon t)\Big\{\sum_{1\leq k\leq K}\sum_{1\leq j\leq m_k} |\alpha_{\pm k,j}|^2 \sum_{1\leq l\leq j}\Big(\frac{t^{j-l}}{(j-l)!}\Big)^2 + \sum_{n>N}|\alpha_{\pm n}|^2\Big\}.\end{aligned} \tag{4.32}$$

Recalling from Theorem 4.2 that the number of eigenvalues of low frequency is exactly $2N$, then the multiplicity $m_k \leq N$. Hence we deduce that there exists a constant C_N depending only on N such that

$$\sum_{1\leq l\leq j}\Big(\frac{t^{j-l}}{(j-l)!}\Big)^2 \leq \sum_{1\leq l\leq N}\Big(\frac{t^{N-l}}{(N-l)!}\Big)^2 \leq C_N(1+t^{2N}). \tag{4.33}$$

It follows that

$$\begin{aligned}&\|y(t)\|^2_{H_0^1(0,1)} + \|y_t(t)\|^2_{L^2(0,1)}\\ &\leq C_N\|S^\epsilon\|^2\|(S^\epsilon)^{-1}\|^2(1+t^{2N})\exp(-c_0\epsilon t)\Big(\|y_0\|^2_{H_0^1(0,1)} + \|y_1\|^2_{L^2(0,1)}\Big)\end{aligned} \tag{4.34}$$

which together with (4.16) gives (4.28) with $\omega < c_0$. The proof is thus complete.

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