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COHOMOLOGY OF GROUPS AND UNSTABLE MODULES OVER THE STEENROD ALGEBRA

HANS-WERNER HENN

ABSTRACT. The presence of Steenrod operations in the mod - p cohomology ring $H^*(BG; \mathbb{F}_p)$ of the classifying space BG of a group G allows to understand qualitative features of this ring, at least for a large class of groups including all compact Lie groups but also for many discrete groups like arithmetic groups, mapping class groups and automorphism groups of free groups.

We explain the relevant theory of unstable modules over the Steenrod algebra and indicate how this general theory can be used in the qualitative and quantitative study of group cohomology.

1. COHOMOLOGY OF GROUPS

1.1. Some fundamental structural results for cohomology groups and cohomology rings of finite groups.

For a topological group G we denote its classifying space by BG . Our groups will usually be either discrete or compact Lie so that up to homotopy equivalence there is no difference between Milnor's infinite join construction or the construction of BG as the classifying space of a (topological) category with a single object and whose morphisms are given by G . If the group is discrete then the cohomology of BG with local coefficients in a G - module M is also known as the cohomology of the group G with coefficients in M (and is usually denoted $H^*(G; M)$). In these notes the action of G on M will usually be trivial and we will always use the topologists notation $H^*(BG; M)$. We will begin by recalling some basic properties of the cohomology of finite groups.

Our first result is a finiteness result. Its proof relies heavily on elementary properties of the transfer.

Proposition 1. *Let G be a finite group. Then $H^n(BG; \mathbb{Z})$ is a finitely generated group for all n and multiplication with the order of G annihilates $H^n(BG; \mathbb{Z})$ for all $n > 0$.*

These notes, which elaborate on parts of a survey article with the title "Unstable modules over the Steenrod algebra and cohomology of groups" [H6], are a revised version of the notes for an advanced course which was held at the CRM in Bellaterra from May 27 to June 2, 1998. I would like to thank the participants for numerous discussions. Several suggestions made in these discussions have been incorporated in this version in some way or another. I would like to explicitly thank Jon Berrick for his critical reading of the first version of these notes.

Example. Let \mathbb{Z}/k denote the cyclic group of order k . Then $H^{2n}(B\mathbb{Z}/k; \mathbb{Z}) \cong \mathbb{Z}/k$ if $n > 0$ and $H^{2n+1}(B\mathbb{Z}/k; \mathbb{Z}) = 0$.

The proposition focuses interest on studying p -torsion in $H^*(BG; \mathbb{Z})$ for those primes p which divide the order of G and to the closely related cohomology ring $H^*(BG; \mathbb{F}_p)$. In fact, we will almost exclusively discuss these rings. However, the first fundamental result holds for more general coefficients.

Theorem 2 [E1], [V]. *Let G be a finite group and R be a noetherian commutative ring. Then $H^*(BG; R)$ is a finitely generated (graded commutative) R -algebra.*

One way of proving this uses an embedding of G into $U(n)$, the group of unitary $n \times n$ matrices, and knowledge of $H^*(BU(n); \mathbb{Z})$. We recall that $H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n]$ where the elements c_i are the universal Chern classes and live in degree $2i$. (The strategy to use compact Lie groups to prove theorems for finite groups will be used again in Section 3 below!) Theorem 2 is now an easy consequence of the following result.

Theorem 3. *Let G be a compact Lie group, H a closed subgroup and R a noetherian ring. Then $H^*(BH; R)$ becomes a finitely generated $H^*(BG; R)$ -module via the map induced by the inclusion of H into G .*

Outline of the proof of Theorem 3. We choose an embedding of G into $U(n)$ (for example the regular representation if G is finite). It suffices to prove the theorem in case $G = U(n)$. Consider the $H^*(-; R)$ -based Serre spectral sequence of the fibration $BH \rightarrow BU(n)$, with fibre $F = U(n)/H$. This is a spectral sequence of $H^*(BU(n); R) \cong R[c_1, \dots, c_n]$ -modules and its E_2 -term is finitely generated over this ring because the cohomology of the fibre (which is a compact manifold) is a finitely generated R -module. Because $R[c_1, \dots, c_n]$ is noetherian it follows first that all other pages of the spectral sequence consist of finitely generated $H^*(BU(n); R)$ -modules and consequently the same holds for $H^*(BH; R)$. \square

Here is another very useful consequence of this result.

Corollary 4. *Let G be a finite group and assume G contains an element of order p^k . Then there exists $n > 0$ and an element $z \in H^n(BG; \mathbb{Z})$ such that the order of each power of z is divisible by p^k .*

Proof. An element of order p^k defines an embedding ι of \mathbb{Z}/p^k into G . Now use that as a ring $H^*(B\mathbb{Z}/p^k; \mathbb{Z})$ is isomorphic to $\mathbb{Z}[y]/(p^k y)$ with y in degree 2. By Theorem 3 $H^*(B\mathbb{Z}/p^k; \mathbb{Z})$ becomes a finitely generated module over $H^*(BG; \mathbb{Z})$ via the map induced by ι and this is only possible if some power y^k of y is in the image of this map. A preimage of this power qualifies as z . \square

As a variation of this proof one could calculate the Chern classes of the regular representation of G restricted to the subgroup generated by the element of order p^k . This has the advantage of giving a concrete class z with properties as in Corollary 4. What degree is it in?

The case of the group $G = \mathbb{Z}/p \times \mathbb{Z}/p$ shows that in general there is no element in $H^*(BG; \mathbb{Z})$ whose order is equal to the order of the group. If G is an arbitrary finite group then finding the maximal order of elements of positive degree of the ring

$H^*(BG; \mathbb{Z})$ (this order is often called the *exponent* of $H^*(BG; \mathbb{Z})$) is an interesting and difficult open problem.

1.2. Quillen's F - isomorphism theorem.

Theorem 2 suggests to use the usual concepts of commutative algebra in order to study $H^*(BG; R)$. Of course, one would like to understand these concepts in terms of the group theory of G . Quillen's fundamental result [Q1] takes a substantial step in this direction if $R = \mathbb{F}_p$; it describes $H^*(BG; \mathbb{F}_p)$ up to nilpotency phenomena and determines the Krull dimension of $H^*(BG; \mathbb{F}_p)$ and its variety (or rather its set of geometric points) in a beautiful and very satisfactory way. Quillen's approach was to understand as much as possible about $H^*(BG; \mathbb{F}_p)$ by relating it via the restriction maps to the cohomology of its elementary abelian p - subgroups, i.e. groups isomorphic to $(\mathbb{Z}/p)^n$ for some natural number n . We recall that for an elementary abelian p -group E we have

$$H^*(BE; \mathbb{F}_p) \cong \begin{cases} S(E^*, 1) & \text{if } p = 2, \\ S(E^*, 2) \otimes \Lambda(E^*, 1) & \text{if } p > 2. \end{cases}$$

Here E^* is the dual of E and $S(E^*, i)$ stands for a graded symmetric algebra with E^* in degree i and similarly $\Lambda(E^*, i)$ for a graded exterior algebra with E^* in degree i .

Now consider the map $H^*(BG; \mathbb{F}_p) \longrightarrow \prod_E H^*(BE; \mathbb{F}_p)$ with E running through all elementary abelian p - subgroups of G . It is clear that elements in the target of this map must satisfy compatibility conditions (arising from inclusions and conjugations in G) to have a chance to be in its image.

More precisely Quillen considered the following category $\mathcal{A}_p(G)$: its objects are the elementary abelian p - subgroups of G and its morphisms are all group homomorphisms which can be induced by conjugation by an element in G ; i.e. for E_1 and E_2 in $\mathcal{A}_p(G)$ one defines

$$\text{mor}_{\mathcal{A}_p(G)}(E_1, E_2) = \{ \alpha \in \text{Hom}(E_1, E_2) \mid \exists g \in G \text{ such that } \alpha(e) = geg^{-1} \forall e \in E_1 \} .$$

Then $E \mapsto H^*(BE; \mathbb{F}_p)$ becomes a (covariant) functor from the opposite category $\mathcal{A}_p(G)^{op}$ to graded \mathbb{F}_p - algebras and the restriction homomorphisms induce a map $q_G : H^*(BG; \mathbb{F}_p) \longrightarrow \lim_{\mathcal{A}_p(G)^{op}} H^*(BE; \mathbb{F}_p)$. Quillen's result reads now as follows.

Theorem 5 [Q1]. *Let p be a prime and G be a finite group. Then the map*

$$q_G : H^*(BG; \mathbb{F}_p) \longrightarrow \lim_{\mathcal{A}_p(G)^{op}} H^*(BE; \mathbb{F}_p)$$

has the following properties:

- *If $x \in \text{Ker } q_G$ then x is nilpotent.*
- *For every $y \in \lim_{\mathcal{A}_p(G)^{op}} H^*(BE; \mathbb{F}_p)$ there exists a natural number n with $y^{p^n} \in \text{Im } q_G$.*

A map of graded \mathbb{F}_p - algebras satisfying the two conditions occurring in the theorem is called an F - isomorphism. We will give an outline of a proof of Quillen's Theorem in Section 3.3 below.

Quillen [Q1] showed that Theorem 5 is true for other classes of groups as well, e.g.

a) if G is a discrete group of finite virtual mod p - cohomological dimension. (It will not be important for these notes to know the precise definition of this class of groups. However, we emphasize that this class includes many “classical” groups like all (S -) arithmetic groups, e.g. $SL(n, \mathbb{Z})$, mapping class groups of Riemann surfaces, automorphism groups of free groups ..., i.e. the symmetry groups of some of the most basic objects in mathematics.)

b) if G is a compact Lie group

c) if G is a profinite group and $H^*(BG; \mathbb{F}_p)$ is replaced by the continuous mod p cohomology $H_{cts}^*(G; \mathbb{F}_p)$ and is assumed to be noetherian (this includes all p - adic analytic groups in the sense of Lazard [Lz]).

Definition 1. A topological group G is called a *Quillen - group* at the prime p if the map q_G is an F - isomorphism.

In other words, all compact Lie groups and all discrete groups of finite virtual mod p - cohomological dimension are Quillen groups. There is an obvious modification of the definition in the context of continuous cohomology of profinite groups but we will not pursue this here. For other classes of discrete Quillen groups we refer to [H2] and [H6].

Example. To construct discrete groups that are not Quillen groups takes some effort. For example, the class of Quillen groups cannot be expected to be closed with respect to colimits: the cohomology $H^*(BG; \mathbb{F}_p)$ of a colimit G of groups G_n is the limit $\lim_n H^*(BG_n; \mathbb{F}_p)$, and in this way one may get non-nilpotent elements in $H^*(BG; \mathbb{F}_p)$ which restrict trivially on any elementary abelian p - subgroup. An explicit example may be constructed as follows: according to [AC] resp. [IK] there exist finite 2 - groups F_n with elements $t_n \in H^1(BF_n; \mathbb{F}_2)$ such that $t_n^n \neq 0$, $t_n^{n+1} = 0$. Let $G_n = \prod_{k \leq n} F_k$ and $G = \text{colim}_n G_n$.

Exercise 1: Show that there is a non-nilpotent class in $H^1(BG; \mathbb{F}_2)$.

The class of Quillen groups is not well understood, in fact so far it has hardly been investigated for its own sake. It would be interesting to have a characterization of Quillen groups in more group theoretical terms.

We give some of the immediate applications of Theorem 5. The first one determines the Krull dimension of the ring $H^*(BG; \mathbb{F}_p)$ in terms of the group theory of G , more precisely in terms of $r_p(G)$, the p - rank of G , which is defined as the maximal rank of an elementary abelian p - subgroup of G .

For our purpose it is convenient to define the Krull dimension as follows: if M is a finitely generated module over a connected finitely generated \mathbb{F}_p - algebra, then the power series $\chi_M(t) = \sum_n \dim_{\mathbb{F}_p} M^n t^n$ can be shown to be the power series expansion of a rational function in t of the form

$$\frac{p(t)}{\prod_{i=1}^k (1 - t^{n_i})},$$

where $p(t)$ is a polynomial in t with integer coefficients. The Krull dimension $\delta(M)$ of M is then defined to be equal to the order of the pole of the power series $\chi_M(t)$ at $t = 1$. If G is a finite group and p is a prime we set $\chi_{G,p}(t) = \chi_{H^*(BG; \mathbb{F}_p)}(t)$ and $\delta_p(G) = \delta(H^*(BG; \mathbb{F}_p))$.

This means that the Krull dimension measures the growth of the series of numbers $\dim_{\mathbb{F}_p} M^n$. In fact, it is a nice exercise on power series of rational functions to show that the Krull dimension of an M as above is the smallest integer δ such that there exists a constant C with $\dim_{\mathbb{F}_p} M^n \leq Cn^{\delta-1}$ for all $n \geq 1$.

Example. If $V \cong (\mathbb{Z}/p)^n$ then $\chi_{V,p}(t) = \frac{1}{(1-t)^n}$, hence $\delta_p(V) = n$.

Theorem 6. *If G is any finite group then $\delta_p(G) = r_p(G)$.*

Sketch of proof. The inequality $r_p(G) \leq \delta_p(G)$ is an easy consequence of Theorem 3 applied to the inclusion of an elementary abelian p -subgroup E into G .

For the other inequality let $I = \ker q_G$ and $L = \lim_{\mathcal{A}_p(G) \circ p} H^*(BE; \mathbb{F}_p)$ so that we have an embedding $H^*(BG; \mathbb{F}_p)/I \hookrightarrow L$. Then it is easy to see that $\delta(H^*(BG; \mathbb{F}_p)/I) \leq \delta(L) \leq r_p(G)$. The ideal I is finitely generated and nilpotent hence $I^N = 0$ for some N . Moreover, I^j/I^{j+1} is a finitely generated $H^*(BG; \mathbb{F}_p)/I$ -module and from this one deduces $\delta(I^j/I^{j+1}) \leq \delta(H^*(BG)/I) \leq r_p(G)$ and finally $\delta_p(G) \leq r_p(G)$. \square

Quillen used Theorem 5 also to describe the variety of $H^*(BG; \mathbb{F}_p)$, more precisely its geometric points with values in an algebraically closed field k , in group theoretic terms and thus to lay the basis for a very interesting subsequent development in modular representation theory, namely the theory of varieties attached to modules. We will not pursue this here and refer instead to [Be] and [E2] for more information on this development.

Theorem 5 can also be used to construct interesting elements in $H^*(BG; \mathbb{F}_p)$. For example, let D be a maximal elementary abelian p -subgroup of G . If $p = 2$ let $c_D := \prod_{0 \neq x \in H^1(BE; \mathbb{F}_p)} x$, and if p is odd let $c_D := \prod_{0 \neq x \in H^1(BE; \mathbb{F}_p)} \beta x$ (where β is the Bockstein operator), be the top Dickson invariant in $H^*(BD; \mathbb{F}_p)$. This class is invariant with respect to the natural action of the general linear group $GL(D)$ on $H^*(BD; \mathbb{F}_p)$ and it restricts trivially to $H^*(BD'; \mathbb{F}_p)$ for each subgroup D' of D : hence the family of elements $(x_E) \in \prod_{E \in \mathcal{A}_p(G)} H^*(BE; \mathbb{F}_p)$, given by $x_E = c_E$ if E is conjugate to D and $x_E = 0$ otherwise, defines an element y_D in $\lim_{\mathcal{A}_p(G) \circ p} H^*(BE; \mathbb{F}_p)$, and by Theorem 5 there is an integer k and an element $z_D \in H^*(BG; \mathbb{F}_p)$ such that $q_G(z_D) = y_D^k$.

Corollary 7. *Let G be a finite group and assume there are two maximal elementary abelian p -subgroups of G which are not conjugate. Then there are non-nilpotent elements z_1 and z_2 in $H^*(BG; \mathbb{F}_p)$ such that $z_1 z_2 = 0$.*

Proof. Let D_1 and D_2 be two such subgroups and construct elements z'_1 and z'_2 as above. Then $q_G(z'_1 z'_2) = 0$, hence $z'_1 z'_2$ is nilpotent. If we replace z'_i for $i = 1, 2$ by a sufficiently large power we get the desired elements z_i . \square

For a refinement of this statement we refer to Proposition 29 below.

Examples. We finish this section with some examples illustrating Theorem 5.

a) The map q_G is not an isomorphism in general. The first example is given by the cyclic groups \mathbb{Z}/p^k , $k > 1$, in which case the map is neither injective nor surjective.

b) However, there are interesting classes of groups G for which q_G is an isomorphism or at least a monomorphism. For example, if $p = 2$, q_G is an isomorphism if $G = D_{2^n}$, the dihedral group of order 2^n , or if $G = \mathfrak{S}_n$, the symmetric groups on n letters, or $G = O_n(\mathbb{F}_q)$, the symmetry group of the bilinear form $b(x, y) = \sum_i x_i y_i$ on \mathbb{F}_q^n if \mathbb{F}_q is a field of odd order. (The general phenomenon underlying these examples is the following: the property that q_G is an isomorphism is preserved by passing from a group G to its wreath product with $\mathbb{Z}/2$, and is inherited by any group whose Sylow subgroup has this property (see [GLZ].)

c) Let D_8 be the dihedral group of order 8. It is known that $H^*(D_8; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, y_1, w_2]/(x_1 y_1)$ where the indices give the degree of the elements (cf. [AM]). The elements x_1 and y_1 are two classes as in Corollary 7, and this reflects the fact that D_8 has two non-conjugate elementary abelian 2 - subgroups of rank 2.

In general Theorem 5 says absolutely nothing about nilpotent elements. In later sections we will see that Lannes' generalization of Quillen's Theorem (cf. Theorem 19 below) provides information on nilpotent elements as well.

2. QUILLEN'S F - ISOMORPHISM THEOREM FROM THE POINT OF VIEW OF UNSTABLE MODULES

2.1. Unstable modules over the Steenrod algebra.

We will see below that Quillen's theorem can be reinterpreted as a statement on $H^*(BG; \mathbb{F}_p)$ considered as an unstable module over the Steenrod algebra. Before we get to this we review some of the basic facts on Steenrod operations and unstable modules. For details we refer to [SE] and [Sc].

Definition 2. Let p be a prime. The mod - p Steenrod algebra A_p is the following graded associative \mathbb{F}_p - algebra:

If $p = 2$ it is generated by elements Sq^i , $i = 0, 1, 2, \dots$ of degree i , with $Sq^0 = 1$ and subject to the following Adem relations:

$$\text{if } 0 < a < 2b : Sq^a Sq^b = \sum_{c=0}^{\lfloor a/2 \rfloor} \binom{b-c-1}{a-2c} Sq^{a+b-c} Sq^c .$$

If p is odd it is generated by elements P^i , $i = 0, 1, 2, \dots$ of degree $2i(p-1)$, with $P^0 = 1$, together with an element β of degree 1 with $\beta^2 = 0$, and subject to the following Adem relations:

$$\begin{aligned} \text{if } 0 < a < pb : P^a P^b &= \sum_{c=0}^{\lfloor a/p \rfloor} (-1)^{a+c} \binom{(p-1)(b-c)-1}{a-pc} P^{a+b-c} P^c , \\ \text{if } 0 < a \leq pb : P^a \beta P^b &= \sum_{c=0}^{\lfloor a/p \rfloor} (-1)^{a+c} \binom{(p-1)(b-c)}{a-pc} \beta P^{a+b-c} P^c + \\ &+ \sum_{c=0}^{\lfloor (a-1)/p \rfloor} (-1)^{a+c-1} \binom{(p-1)(b-c)-1}{a-pc-1} P^{a+b-c} \beta P^c . \end{aligned}$$

Fortunately we do not have to explicitly work with the Adem relations. What is important for us are some formal properties of the category \mathcal{U} of unstable modules (see Definition 3 below). Of course, the Adem relations are crucial in establishing these properties in the first place.

The importance of the Steenrod algebra comes from the fact that mod - p cohomology is a functor from the category \mathcal{Top} of topological spaces to the category of graded A_p - modules. In fact, more is true: the cohomology of a space is a graded A_p - module with a product structure (given by the cup-product) for which the action satisfies additional “instability axioms”, axioms which define the categories of unstable modules resp. unstable algebras over the mod - p Steenrod algebra.

Definition 3. An *unstable* A_p - module M is a graded A_p - module such that

- $Sq^i x = 0$ for all $i > |x|$, if $p = 2$;
- $\beta^e P^i x = 0$ for all $2i + e > |x|$, if p is odd and $e \in \{0, 1\}$.

Here and elsewhere in this paper $|x|$ denotes the degree of the element x .

As usual we will write \mathcal{U}_p for the (abelian) category of unstable modules and A_p - linear maps. Usually p will be fixed and understood from the context and then we will use the abbreviations *unstable module* and \mathcal{U} resp..

Definition 4. An *unstable* A_p - algebra is an unstable A_p - module K together with linear maps $\varphi : K \otimes K \longrightarrow K$, $\eta : \mathbb{F}_p \longrightarrow K$ such that

- φ and η determine a graded commutative \mathbb{F}_p - algebra structure on K .
- φ is A_p - linear. (For this to make sense we recall that A_p is a Hopf algebra, hence the category of graded A_p - modules has an associated internal tensor product. Another more explicit way of stating this is that the product in K satisfies the Cartan formula, i.e. $Sq^n(xy) = \sum_{i+j=n} Sq^i x Sq^j y$ if $p = 2$; $\beta(xy) = \beta(x)y + (-1)^{|x|} x\beta(y)$, $P^n(xy) = \sum_{i+j=n} P^i x P^j y$ if p is odd.)

- $$Sq^n x = x^2 \text{ if } p = 2 \text{ and } n = |x|,$$

$$P^n x = x^p \text{ if } p > 2 \text{ and } 2n = |x| .$$

As usual we will write \mathcal{K}_p for the category of unstable algebras over A_p ; its objects are the unstable algebras and its morphisms the degree preserving maps which are both A_p - linear and maps of graded algebras. As above we will often use the abbreviations *unstable algebra* and \mathcal{K} if p is understood from the context.

As mentioned before, the (mod - p) cohomology of a space, in particular the cohomology of the classifying space of a group, i.e. the cohomology of a group, is a natural example of an unstable algebra, the product φ being given by the cup product.

Example. Let $p = 2$ and $V = \mathbb{Z}/p^n$. Then $H^*(BV; \mathbb{F}_2) \cong \mathbb{F}_2[x_1, \dots, x_n]$ where the degree of the elements x_i is 1. The action of the Steenrod algebra is determined by the instability axioms. By the Cartan formula it suffices to determine $Sq^n x_i$ if $i > 0$ and the instability axioms give $Sq^1 x_i = x_i^2$ while $Sq^n x_i = 0$ if $n > 1$. The case of an odd prime is similar and left as an exercise.

So the action of the Steenrod algebra on $H^*(BV; \mathbb{F}_p)$ is explicitly understood. Nevertheless $H^*(BV; \mathbb{F}_p)$ remains a complicated object. To test your understanding here are some exercises.

Exercise 2: a) Show that the endomorphism ring $\text{End}_{\mathcal{U}_2}(\widetilde{H}^*(B\mathbb{Z}/2; \mathbb{F}_2))$ of the reduced homology $\widetilde{H}^*(B\mathbb{Z}/2; \mathbb{F}_2)$ is isomorphic to \mathbb{F}_2 , in other words every non-trivial endomorphism is equal to the identity. (Hint: It is enough to consider $Sq^1 x^k$ and $Sq^2 x^k$ if $x \in H^1(B\mathbb{Z}/2; \mathbb{F}_2)$ is a generator.)

b) What can you say about $\text{End}_{\mathcal{U}_p}(\widetilde{H}^*(B\mathbb{Z}/p; \mathbb{F}_p))$?

The conceptual importance of the cohomology of an elementary abelian p -group is underlined by the following deep result of Carlsson resp. Miller which in the form given here is due to Lannes and Zarati. After having struggled with Exercise 2 above you will probably appreciate the depth and beauty of this result; in any case this result was the beginning of a flourishing period of research on unstable modules and homotopy theory of classifying spaces and is also crucial for most of the results we will discuss in these notes.

Theorem 8 ([C], [M], [LZ]). *Let V be an elementary abelian p -group. Then $H^*(BV; \mathbb{F}_p)$ is an injective object in the category \mathcal{U}_p .*

Comments on the proof. The proof is quite indirect. First one considers the case that the rank of V is 1. One starts from some tautologically injective objects, the Brown Gitler modules $J(n)$ which represent the (exact) functor from \mathcal{U}_p to the category \mathcal{E}_∞ of \mathbb{F}_p -vector spaces which sends an unstable module M to $(M^n)^*$, the linear dual of the \mathbb{F}_p -vector space of homogeneous elements of M of degree n . From the Brown Gitler modules one constructs other injectives, the Carlsson modules $K(i)$ which for $p = 2$ are given as $\lim_i J(2^i)$ and represent the exact functor $\mathcal{U}_2 \rightarrow \mathcal{U}_2$, $M \mapsto (\text{colim}_k M^{2^k})^*$ where the maps in the system are given by $M^{2^k} \rightarrow M^{2^{k+1}}$, $x \mapsto Sq^{|x|}x$. The hard step is then to show that $H^*(B\mathbb{Z}/p; \mathbb{F}_p)$ is a direct summand in $K(1)$.

The case of a general V is dealt with by proving that for any injective I the tensor product $I \otimes H^*(B\mathbb{Z}/p; \mathbb{F}_p)$ is again injective. (This step uses the Carlsson modules again.) We refer to the original papers or to [Sc] for the details of the proof. \square

Warning: It is not true that the tensor product of any two injectives remains injective.

We remark that the finite groups G for which $H^*(BG; \mathbb{F}_p)$ is injective in \mathcal{U}_p are precisely those whose p -Sylow subgroup is elementary abelian [H1].

Before we can connect Theorem 5 with the theory of unstable modules we need to define nilpotent unstable modules.

Definition 5. An unstable A_p -module N is called *nilpotent* iff for each $x \in N$ there exists a natural number n such that

- $Sq^{2^n|x|} \dots Sq^{|x|}x = 0$, if $p = 2$
- $P^{p^n \frac{|x|}{2}} \dots P^{\frac{|x|}{2}}x = 0$ if $p > 2$ and x has even degree.

The justification for this terminology comes from the last property defining an unstable algebra, i.e. from the fact that in this case the condition reads $x^{p^{n+1}} = 0$.

If p is odd, one has to be a bit careful with this notion. The unstable algebra $H^*(B\mathbb{Z}/p; \mathbb{F}_p)$ has elements which are nilpotent in the classical sense (the radical, i.e. the ideal of nilpotent elements, consists precisely of the elements of odd degree) but it does not have any non-trivial nilpotent unstable submodules.

Exercise 3: Show that if $p = 2$, or if p is odd and $\beta = 0$ on K , then the ideal of nilpotent elements in an unstable algebra K is a nilpotent submodule.

The following important result of Lannes and Schwartz gives a characterization of nilpotent modules in terms of the injective objects $H^*(BV; \mathbb{F}_p)$.

Theorem 9 [LS]. *An unstable module N is nilpotent if and only if $\text{Hom}_{\mathcal{U}_p}(N, H^*(BV; \mathbb{F}_p)) = 0$ for all elementary abelian p -groups V .*

Comments on the proof. The easy part is to show that there are no non-trivial homomorphisms from a nilpotent unstable module to $H^*(BV; \mathbb{F}_p)$. If $p = 2$ this is clear because $H^*(BV; \mathbb{F}_2)$ has no non-trivial nilpotent elements and the map $x \mapsto Sq^{|x|}x$ is the squaring map and hence injective on $H^*(BV; \mathbb{F}_2)$. The situation is a bit more subtle for p odd but by making clever use of the Bockstein one can find for each element x in $H^*(BV; \mathbb{F}_p)$ an element θ in the Steenrod algebra such that θx is not nilpotent in the classical sense, and this suffices to show that any homomorphism from a nilpotent module N to $H^*(BV; \mathbb{F}_p)$ is trivial.

The converse is much harder and uses once more the Carlsson modules $K(i)$. \square

2.2. Quillen's F -isomorphism theorem from the point of view of unstable modules.

Theorem 8 and 9 clearly imply the following result.

Theorem 10. *Let p be a prime. Then G is a Quillen - group iff for every elementary abelian p -group V the map*

$$\text{Hom}_{\mathcal{U}_p}(H^*(BG; \mathbb{F}_p), H^*(BV; \mathbb{F}_p)) \longleftarrow \text{Hom}_{\mathcal{U}_p}(\lim_{\mathcal{A}_p(G)^{op}} H^*(BE; \mathbb{F}_p), H^*(BV; \mathbb{F}_p))$$

induced by q_G is an isomorphism. \square

Now assume that the category $\mathcal{A}_p(G)$ is equivalent to a finite category $\mathcal{A}'_p(G)$. We remark that this is satisfied in many interesting cases: it is clear if G is finite, and can be shown if G is compact Lie, or more generally if G is a Quillen group for which $H^*(BG; \mathbb{F}_p)$ is a finitely generated \mathbb{F}_p -algebra. This includes many discrete groups, e.g. all $\mathcal{K}_1\mathcal{F}$ -groups in the sense of Definition 8 below, in particular all (S) -arithmetic groups, mapping class groups, automorphism groups of free groups, ...

Then the inverse limit $\lim_{\mathcal{A}_p(G)^{op}} H^*(BE; \mathbb{F}_p)$ is isomorphic to the kernel of a map

$$\prod_E H^*(BE; \mathbb{F}_p) \longrightarrow \prod_{E' \longrightarrow E''} H^*(BE'; \mathbb{F}_p)$$

between finite products, indexed by the objects resp. morphisms in $\mathcal{A}'_p(G)$. Together with the injectivity of $H^*(BV; \mathbb{F}_p)$ this leads to the following isomorphism

$$\mathrm{Hom}_{\mathcal{U}_p}(\lim_{\mathcal{A}_p(G)^{op}} H^*(BE; \mathbb{F}_p), H^*(BV; \mathbb{F}_p)) \cong \mathrm{colim}_{\mathcal{A}_p(G)} \mathrm{Hom}_{\mathcal{U}_p}(H^*(BE; \mathbb{F}_p), H^*(BV; \mathbb{F}_p)) .$$

By a theorem of Adams, Gunawardena and Miller [AGM] (see also Corollary 17 below) the natural map

$$\mathbb{F}_p[\mathrm{Hom}(V, E)] \longrightarrow \mathrm{Hom}_{\mathcal{U}_p}(H^*(BE; \mathbb{F}_p), H^*(BV; \mathbb{F}_p))$$

is an isomorphism where for a set S the \mathbb{F}_p -vector space with S as a basis is denoted by $\mathbb{F}_p[S]$. Finally let $\mathrm{Rep}(V, G)$ denote the set of G -conjugacy classes of homomorphisms from V to G : then the maps $\mathrm{Hom}(V, E) \rightarrow \mathrm{Rep}(V, G), \alpha \mapsto i_E \alpha$ (with i_E denoting the inclusion of E into G) induce a bijection $\mathrm{colim}_{\mathcal{A}_p(G)} \mathrm{Hom}(V, E) \longrightarrow \mathrm{Rep}(V, G)$, and hence also an isomorphism

$$\mathrm{colim}_{\mathcal{A}_p(G)} \mathbb{F}_p[\mathrm{Hom}(V, E)] \longrightarrow \mathbb{F}_p[\mathrm{Rep}(V, G)] .$$

In other words, we obtain the following characterization of Quillen - groups.

Theorem 11. *Let p be a prime and let G be a group for which $\mathcal{A}_p(G)$ is equivalent to a finite category. Then G is a Quillen - group iff the natural map*

$$\mathbb{F}_p[\mathrm{Rep}(V, G)] \longrightarrow \mathrm{Hom}_{\mathcal{U}_p}(H^*(BG; \mathbb{F}_p), H^*(BV; \mathbb{F}_p))$$

is an isomorphism for all elementary abelian p -groups V .

Theorem 11 says in particular that the F -isomorphism type of $H^*(BG; \mathbb{F}_p)$ is completely captured by the sets $\mathrm{Rep}(V, G)$ resp. the vector spaces $\mathbb{F}_p[\mathrm{Rep}(V, G)]$ (with V varying), at least if $\mathcal{A}_p(G)$ is finite up to equivalence. In particular, statements on $H^*(BG; \mathbb{F}_p)$ can often be tested by the corresponding statements on these sets. For more on this point of view we refer to [HLS1].

In the next section we will see how Lannes' theory of the T -functor can be used to compute, without using Quillen's theorem, $\mathrm{Hom}_{\mathcal{U}_p}(H^*(BG; \mathbb{F}_p), H^*(BV; \mathbb{F}_p))$ as $\mathbb{F}_p[\mathrm{Rep}(V, G)]$. In fact, Lannes' results should be considered as a far reaching generalization of Theorem 11.

3. LANNES' T -FUNCTOR AND LANNES' GENERALIZATION OF QUILLEN'S THEOREM

3.1. Review of Lannes' T -functor.

The aim of this section is to introduce Lannes' T -functor and to review its main properties, at least as far as they are important for us. For details we refer to [Ln1], [Ln3] or [Sc].

Throughout this section p is a fixed prime. From now on we suppress the coefficients and write H^*BG instead of $H^*(BG; \mathbb{F}_p)$, and so on.

Given an elementary abelian p -group V , Lannes ([Ln1], [Ln3]) considered the functor $T_V : \mathcal{U} \rightarrow \mathcal{U}$ characterized by the adjunction property

$$\mathrm{Hom}_{\mathcal{U}}(T_V M, N) \cong \mathrm{Hom}_{\mathcal{U}}(M, H^*BV \otimes N).$$

One can view T_V as a ‘‘cohomological analogue’’ of the mapping space functor $X \mapsto \mathrm{map}(BV, X)$.

The existence of such a functor is quite formal: if T_V exists it is clearly right exact and preserves arbitrary direct sums, hence it is determined by its values on the free unstable modules $F(k)$ which are characterized by natural isomorphisms $\mathrm{Hom}_{\mathcal{U}}(F(k), N) \cong N^k$. The desired adjunction property of T_V forces $T_V F(n) \cong \bigoplus_k F(n-k)^{\oplus \dim H^k V}$. Conversely one can define T_V on free objects by this formula and use free presentations to define T_V on general objects.

Exercise 4: Show that $T_V B \cong B$ whenever B is a module which is bounded above, i.e. for which $B^n = 0$ if n is sufficiently large.

What makes these functors interesting is that they enjoy some magical properties which make them extremely useful. Here are the two basic properties of these functors. The first one is quite surprising, in particular in view of the analogy with the mapping space functor.

Theorem 12 [Ln1], [Ln3]. T_V is an exact functor.

Theorem 13 [Ln1], [Ln3]. T_V commutes with tensor products, i.e. the product on H^*V induces a natural transformation $\tau_{M,N} : T_V(M \otimes N) \rightarrow T_V M \otimes T_V N$ which is an isomorphism for all unstable modules M and N .

More precisely if $\gamma_M : M \rightarrow H^*BV \otimes T_V M$ denotes the adjoint of $id_{T_V M}$, then $\tau_{M,N}$ is the adjoint of the composition of the three maps

$$\gamma_M \otimes \gamma_N : M \otimes N \rightarrow H^*BV \otimes T_V M \otimes H^*BV \otimes T_V N$$

$$H^*BV \otimes T_V M \otimes H^*BV \otimes T_V N \rightarrow H^*BV \otimes H^*BV \otimes T_V M \otimes T_V N$$

$$\phi \otimes id \otimes id : H^*BV \otimes H^*BV \otimes T_V M \otimes T_V N \rightarrow H^*BV \otimes T_V M \otimes T_V N$$

where the second map interchanges the second and third tensor factor, and ϕ is the product of the unstable algebra H^*BV .

Comments on the proofs of Theorem 12 and 13. Theorem 12 is really equivalent to the following result due to Lannes and Zarati. (Recall that $J(n)$ denotes the n -th Brown Gitler module representing the functor $M \rightarrow (M^n)^*$.)

Theorem 12' [LZ]. $H^*BV \otimes J(n)$ is injective for each elementary abelian p -group V and each natural number n .

Exactness of T_V reduces the proof of Theorem 13 to a verification that $\tau_{M,N}$ is an isomorphism for $M = F(m)$ and $N = F(n)$ and by the adjointness property of

T_V this amounts to the computation of $\text{Hom}_{\mathcal{U}}(F(m) \otimes F(n), H^*BV \otimes J(k))$ for all m, n and k . \square

The following two results are derived from Theorem 12 and Theorem 13 and are also of fundamental importance.

Theorem 14 [Ln1], [Ln3]. *T_V lifts to a functor from \mathcal{K} to itself and continues to be adjoint to H^*BV , i.e. there are natural isomorphisms*

$$\text{Hom}_{\mathcal{K}}(T_V K, L) \cong \text{Hom}_{\mathcal{K}}(K, H^*BV \otimes L)$$

for all $K, L \in \mathcal{K}$.

(This is almost trivial given Theorem 13 except that one has to verify that on $T_V K$ the map $x \mapsto Sq^{|x|}x$ agrees with the squaring map if $p = 2$ resp. that $x \mapsto P^{\frac{|x|}{2}}x$ agrees with the p -th power map if p is odd.)

In order to state the next result in its proper generality we need to talk a little bit about profinite vector spaces and profinite sets.

We note that any unstable module is the colimit of its finitely generated submodules M_α , and this implies that $\text{Hom}_{\mathcal{U}}(M, N)$ is the limit of the $\text{Hom}_{\mathcal{U}}(M_\alpha, N)$. If N is of finite type (e.g. $N = H^*BV$) then $\text{Hom}_{\mathcal{U}}(M_\alpha, N)$ is a finite dimensional \mathbb{F}_p -vector space for each α and hence $\text{Hom}_{\mathcal{U}}(M, N)$ is a limit of finite \mathbb{F}_p -vector spaces, i.e. a profinite vector space. Similarly, $\text{Hom}_{\mathcal{K}}(K, L)$ is a profinite set if L is of finite type. For a profinite \mathbb{F}_p -vector space E we denote the continuous dual (the space of continuous homomorphisms from E to \mathbb{F}_p) by E' and for a profinite set S we denote the set of all continuous functions from S to \mathbb{F}_p by \mathbb{F}_p^S . We note that the category \mathcal{E}_∞ of all \mathbb{F}_p -vector spaces is equivalent to the opposite of the category \mathcal{PE} of all profinite \mathbb{F}_p -vector spaces and the equivalence is induced by taking the dual resp. the continuous dual.

Of course, if E is finite dimensional, or S is finite, the profinite topology is discrete and continuity is always guaranteed. For a first reading it will not do much harm to avoid the profinite structures by assuming appropriate finiteness conditions (cf. Corollary 16 below).

Theorem 15 [Ln1], [Ln3] (Linearization principle). *Let K be an unstable algebra. Then the natural restriction map*

$$T_V^0 K \cong \text{Hom}_{\mathcal{U}}(K, H^*BV)' \cong \mathbb{F}_p^{\text{Hom}_{\mathcal{K}}(K, H^*BV)}$$

is an isomorphism of p -Boolean algebras.

We recall that a p -Boolean algebra B is an \mathbb{F}_p -algebra with the property that $x^p = x$ for each $x \in B$. Clearly the (continuous) functions from any (profinite) set into \mathbb{F}_p form a p -Boolean algebra, and the degree 0-part of any unstable algebra does as well (because the instability axioms say $x = Sq^0 x = x^2$ resp. $x = P^0 x = x^p$ if $|x| = 0$). Theorem 15 is an easy consequence of the structure theorem for p -Boolean algebras which states that a p -Boolean algebra B is isomorphic to the algebra of \mathbb{F}_p -valued continuous functions on the spectrum $\text{spec } B$ of B , and $\text{spec } B$ can be identified with the set of algebra homomorphisms from B to \mathbb{F}_p .

Corollary 16. *Let K be an unstable algebra. Then $\mathrm{Hom}_{\mathcal{U}}(K, H^*BV)$ is finite if and only if $\mathrm{Hom}_{\mathcal{K}}(K, H^*BV)$ is finite. In this case the natural homomorphism*

$$\mathbb{F}_p[\mathrm{Hom}_{\mathcal{K}}(K, H^*BV)] \longrightarrow \mathrm{Hom}_{\mathcal{U}}(K, H^*BV)$$

*induced by the inclusion of $\mathrm{Hom}_{\mathcal{K}}(K, H^*BV)$ into $\mathrm{Hom}_{\mathcal{U}}(K, H^*BV)$ is an isomorphism. \square*

As a special case we obtain the theorem of Adams, Gunawardena and Miller (see the discussion before Theorem 11).

Corollary 17. *The natural map*

$$\mathbb{F}_p[\mathrm{Hom}(W, V)] \cong \mathbb{F}_p[\mathrm{Hom}_{\mathcal{K}}(H^*BV, H^*BW)] \longrightarrow \mathrm{Hom}_{\mathcal{U}}(H^*BV, H^*BW)$$

is an isomorphism. \square

If you haven't already succeeded before, you should now be able to give the answer to Exercise 2b.

As a final application of Theorem 14 we list the following result.

Theorem 18. *Let W be an elementary abelian p -group and let $\alpha \in \mathrm{Hom}(V, W)$. Then the maps $V \times W \rightarrow W, (v, w) \mapsto \alpha(v) + w$ induce, via adjunction, maps of unstable algebras $T_V H^*BW \rightarrow H^*BW$ such that the product of these maps gives an isomorphism of unstable algebras*

$$T_V H^*BW \longrightarrow \prod_{\alpha \in \mathrm{Hom}(V, W)} H^*BW .$$

Sketch of proof. There is a functor $U : \mathcal{U} \rightarrow \mathcal{K}$ which is left adjoint to the forgetful functor. (If M is an unstable module then the symmetric algebra SM is an unstable module with an \mathbb{F}_p -algebra structure which satisfies the Cartan formula, but the formula $Sq^{|x|}x = x^2$ resp. $P^{\frac{|x|}{2}}x = x^p$ does not hold. UM is the quotient of SM by the ideal generated by the elements $Sq^{|x|}x - x^2$ resp. $P^{\frac{|x|}{2}}x - x^p$).

One can easily check that $H^*B\mathbb{Z}/p \cong UF(1)$ where $F(1)$ is the free unstable module on a generator in degree 1. Next it is clear that T_V commutes with the functor U and this implies the computation of $T_V H^*B\mathbb{Z}/p$, and finally of $T_V H^*BW$ because T_V commutes with tensor products. \square

3.2. Lannes' generalization of Quillen's theorem.

We go back and have another look at Theorem 11. Taking $N = \mathbb{F}_p$ (considered as an unstable module concentrated in degree 0) in the adjunction formula for T_V we see that Theorem 11 is nothing but the computation of $(T_V^0 H^*BG)^*$ with T_V^0 denoting the degree 0 part of T_V . In [Ln2] Lannes computes all of $T_V H^*BG$ and this computation should be considered as a far reaching generalization of Quillen's theorem. To describe his result we need to introduce some more notation.

For a representation $\rho \in \text{Rep}(V, G)$ we choose a representative and by abuse of notation we will denote this representative again by ρ . Then we consider the homomorphism

$$c_\rho : V \times C_G(\rho) \longrightarrow G, \quad (v, g) \mapsto \rho(v)g .$$

Here $C_G(\rho)$ stands for the centralizer of $\text{Im } \rho$ in G . We obtain an induced map $c_\rho^* : H^*BG \longrightarrow H^*BV \otimes H^*BC_G(\rho)$ and an adjoint map $ad(c_\rho^*) : T_V H^*BG \longrightarrow H^*BC_G(\rho)$.

Theorem 19 [Ln2], [Ln3]. *Let G be a finite group and V be an elementary abelian p - group. Pick a representative for each element ρ in $\text{Rep}(V, G)$. Then the resulting map*

$$l_G^V : T_V H^*BG \longrightarrow \prod_{\rho \in \text{Rep}(V, G)} H^*BC_G(\rho)$$

with components $ad(c_\rho^)$ is an isomorphism in \mathcal{K} .*

As in the case of Theorem 5 this result also holds for other classes of groups, in particular it holds for all classes of groups listed after Theorem 5. In the case of groups of finite virtual mod - p cohomological dimension and in the case of compact Lie groups this was shown in [Ln2] and for profinite groups in [H4]. Still another class was considered in the appendix of [H2].

Definition 6. We call a topological group G a *Lannes - group* at the prime p if the map l_G^V is an isomorphism for all elementary abelian p - groups V (with modifications as in Section 1 if G is a profinite group).

Clearly, a Lannes - group for which $\mathcal{A}_p(G)$ is equivalent to a finite category is also a Quillen - group. The class of Lannes groups is, just as the class of Quillen groups, not well understood, and it would be nice to have a characterization of this class of groups in more group theoretical terms.

Theorem 19 has various applications to the study of group cohomology which do not involve the Steenrod algebra and which refine Quillen's results (cf. Sections 4, 5 and 6 below). Philosophically, Theorem 19 emphasizes the rôle of centralizers of elementary abelian p - subgroups as a second deeper level of information on H^*BG , the first level being given by the category of elementary abelian p - subgroups of G . We will elaborate on this philosophy in the next two sections.

Outline of a proof of Theorem 19 (for G compact Lie). The first step in the proof is to generalize the result to be proved. So we consider G - CW - complexes X and the functor $L_1 = L_1^G$ from the category of G - CW - complexes to graded \mathbb{F}_p - vector spaces, say, which assigns to X the graded vector space $T_V H_G^* X$ where $H_G^* X$ denotes the equivariant cohomology $H_G^*(X)$, i.e. the cohomology of the space $EG \times_G X$. (Here EG denotes as usual the total space of the universal principal G - bundle over BG ; so if G is discrete, EG is just the universal cover of BG .) We will usually suppress G from the notation if the group is known from the context. Note that if X is a point then $L_1(X)$ is just the source of the map l_G^V .

If $\rho : V \longrightarrow G$ is a homomorphism let X^ρ denote the subspace fixed by the image of ρ , i.e. $X^\rho := \{x \in X \mid \rho(v)x = x \ \forall v \in V\}$. The centralizer $C_G(\rho)$ acts in a

natural way on X^ρ so we can consider $H_{C_G(\rho)}^*(X^\rho)$ and the functor $L_2 = L_2^G$ (again G will usually be suppressed) from the category of G - CW - complexes to graded \mathbb{F}_p - vector spaces, say, which assigns to X the product $\prod_{\rho \in \text{Rep}(V, G)} H_{C_G(\rho)}^*(X^\rho)$. If X is a point this is just the target of the map l_G^V .

Next one defines a natural transformation $l_{G, X}^V : L_1(X) \rightarrow L_2(X)$ which for X a point specializes to l_G^V ; its ρ -th component (for $\rho \in \text{Rep}(V, G)$) is adjoint to a map $H_G^*(X) \rightarrow H^*BV \otimes H_{C_G(\rho)}^*(X^\rho)$ which is defined as follows: the inclusion $X^\rho \subset X$ and the homomorphism $V \times C_G(\rho) \rightarrow G, (v, g) \mapsto \rho(v)g$ induce a map $BV \times (EC_G(\rho) \times_{C_G(\rho)} X^\rho) \rightarrow EG \times_G X$ and the induced map in cohomology is the required map. Now the generalization of Theorem 19 reads as follows:

Theorem 19'. *Suppose G is a compact Lie group. Then the map $l_{G, X}^V : L_1(X) \rightarrow L_2(X)$ is an isomorphism for each finite G - CW - complex X .*

Outline of the proof of Theorem 19'. The proof uses the same strategy as Quillen's original proof of Theorem 5. The functors L_1 and L_2 are invariant with respect to G - homotopy equivalences and associate to a homotopy pushout Mayer-Vietoris sequences (for L_1 this uses the exactness of T_V). They both turn finite coproducts into finite products and therefore $l_{G, X}^V : L_1(X) \rightarrow L_2(X)$ is an isomorphism if $l_{G, G/K}^V : L_1(G/K) \rightarrow L_2(G/K)$ is an isomorphism for each G - orbit G/K which appears in X .

The crucial step is now to reduce to the case where all isotropy subgroups are elementary abelian (cf. Section 6 in [Q1]). To this end we consider an embedding of G into a unitary group and consider the resulting G - space $F := U(n)/S$ where S is the elementary abelian subgroup of $U(n)$ generated by all diagonal matrices of order p . Then one considers the following coequalizer diagram of G - spaces

$$X \times F \times F \begin{array}{c} \xrightarrow{p_2} \\ \xrightarrow[p_1]{} \end{array} X \times F \xrightarrow{p} X$$

with $p_1(x, f, f') = (x, f)$, $p_2(x, f, f') = (x, f')$ and $p(x, f) = x$. The crucial observation by Quillen was that after applying cohomology one obtains an equalizer diagram of \mathbb{F}_p - algebras (even unstable algebras)

$$(*) \quad H_G^*(X) \xrightarrow{p} H_G^*(X \times F) \begin{array}{c} \xrightarrow{p_2^*} \\ \xrightarrow[p_1^*]{} \end{array} H_G^*(X \times F \times F)$$

The point of all this is that the isotropy subgroups for the G - action on F are all elementary abelian (they are conjugate in $U(n)$ to subgroups of S !) and hence the isotropy groups for the G - action on $X \times F$ and $X \times F \times F$ are also elementary abelian.

Now exactness of T_V implies that we have an equalizer diagram

$$L_1(X) \xrightarrow{p} L_1(X \times F) \begin{array}{c} \xrightarrow{p_2^*} \\ \xrightarrow[p_1^*]{} \end{array} L_1(X \times F \times F)$$

and a modification of Quillen's argument shows that

$$L_2(X) \xrightarrow{p} L_2(X \times F) \xrightarrow[p_1^*]{p_2^*} L_2(X \times F \times F)$$

is also an equalizer diagram (in fact a product of equalizer diagrams, one for each $\rho \in \text{Rep}(V, G)$). Therefore it suffices to show the theorem for the G -spaces $X \times F$ resp. $X \times F \times F$, i.e. it is now enough to check the theorem for G -orbits G/E where E is an elementary abelian p -subgroup of G .

Next one checks that for $i = 1, 2$ there are natural "Shapiro type" isomorphisms $L_i^G(G/E) \cong L_i^E(pt)$ where pt denotes a one point space. For $i = 1$ this is an immediate consequence of the usual Shapiro isomorphism $H_G^*(G/E) \cong H_E^*(pt)$; for $i = 2$ it is an interesting exercise (see also Lemma 2.8 in [H3]). Furthermore, one verifies that with respect to these isomorphisms the map $l_{G,G/E}^V$ identifies with $l_{V,pt}^E$. Therefore it suffices to verify Theorem 19 in the case $G = E$ and this is guaranteed by Theorem 18. \square

3.3 An outline of a proof of Quillen's Theorem.

The proof of Theorem 19 given above suggests the following strategy for a proof of Quillen's original theorem (for compact Lie groups) which is independent of the theory of unstable modules, and in fact, is nothing but a streamlined version of Quillen's original proof.

First one looks for a generalization of Theorem 5. For this we need to introduce certain functors from $\mathcal{A}_p(G)^{op}$ to the category $\mathcal{E}ns$ of sets. The category of such functors will also be denoted by $\mathcal{A}_p(G) - \mathcal{E}ns$; its objects will be called $\mathcal{A}_p(G)$ -sets.

The functor $E \mapsto H^*BE$ will be simply denoted by H^* .

If X is a G -space then the centralizer $C_G(E)$ acts on the E -fixed points X^E and, if π_0 denotes the set of path components, we can form the set $\pi_0(X^E)/C_G(E)$. Then it is easy to check that the assignment $E \mapsto \pi_0(X^E)/C_G(E)$ extends canonically to a functor $F_X : \mathcal{A}_p(G)^{op} \rightarrow \mathcal{E}ns$. If we want to emphasize the group as well we will denote this functor by $F_{G,X}$.

For a G -space X we obtain a map $q_{G,X}$ from H_G^*X to the set of natural transformations $\text{Hom}_{\mathcal{A}_p(G) - \mathcal{E}ns}(F_X, H^*)$ of $\mathcal{A}_p(G)$ -sets as follows: For $c \in \pi_0(X^E)$ we choose a point x_c in the component of c . The inclusions $\{x_c\} \subset X^E \subset X$ on the space level and $E \subset G$ on the group level induce a map $c^* : H_G^*X \rightarrow H^*BE$ which one can easily check to be independent of the choice of x_c . If $u \in H_G^*X$ then the E -th component of $q_{G,X}(u)$ maps c to $c^*(u)$; it is straightforward to check that $q_{G,X}(u)$ is a natural transformation from F_X to H^* . Furthermore the target of $q_{G,X}$ is a graded \mathbb{F}_p -algebra (because H^* takes values in graded \mathbb{F}_p -algebras) and the map $q_{G,X}$ is a homomorphism of graded \mathbb{F}_p -algebras.

Here is the generalization of Theorem 5.

Theorem 5'. *Assume G is a compact Lie group and X is a finite G - CW - complex. Then the natural map*

$$q_{G,X} : H_G^* X \longrightarrow \text{Hom}_{\mathcal{A}_p(G)-\varepsilon_{ns}}(F_X, H^*)$$

is an F - isomorphism.

If X is a point then the functor F_X is the constant functor with value a point; in this case the target is nothing but $\lim_{\mathcal{A}_p(G)^{op}} H^*$ and $q_{G,X}$ is just Quillen's map q_G .

Sketch of proof of Theorem 5'. Abbreviate the source of $q_{G,X}$ by $Q_1(X)$ and its target by $Q_2(X)$. Then both Q_1 and Q_2 are functors from G - spaces to graded \mathbb{F}_p - algebras which are invariant with respect to G - homotopy equivalences and turn finite coproducts into products. Furthermore Q_2 turns a homotopy pushout into a pullback of graded \mathbb{F}_p - algebras; Q_1 associates to a homotopy pushout a Mayer-Vietoris sequence and this means (cf. Proposition 3.2 in [Q1]) that up to F - isomorphism it also turns a homotopy pushout into a pullback. (Note that it is precisely at this point where we have to give up the hope for a genuine isomorphism in Quillen's Theorem.)

Consequently, we may concentrate on the case of an orbit $X = G/K$. Using Quillen's trick (cf. Section 6 in [Q1]) of embedding G into a unitary group $U(n)$ and working with the projection $X \times F \longrightarrow X$, one reduces the case of a general G - space to the case of an orbit $X = G/E$ with E elementary abelian.

Then one identifies $q_{G,G/E}$ with $q_{E,pt}$ as above and finally one checks that the map $q_{E,pt}$ is a (genuine!) isomorphism. \square

4. APPROXIMATIONS UP TO HIGHER NILPOTENCY

4.1. Localizations in abelian categories.

We consider Quillen's Theorem as a first order approximation to H^*BG and we will discuss other approximations in this and the following section. The philosophy behind these approximations is best understood from the point of view of localizations in abelian categories. We try to explain this in a non-technical way and therefore we deliberately remain a bit vague. For more details we refer to Gabriel [Ga].

So suppose \mathcal{C} is a (small) abelian category and \mathcal{I} is a set of injective objects. Consider the following subcategory $\mathcal{N}(\mathcal{I})$ of \mathcal{C} : An object N belongs to $\mathcal{N}(\mathcal{I})$ iff $\text{Hom}_{\mathcal{C}}(N, I) = 0$ for all $I \in \mathcal{I}$. Then $\mathcal{N}(\mathcal{I})$ is a *Serre subcategory*, i.e. if

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

is an exact sequence in the category \mathcal{C} , then M_2 is in $\mathcal{N}(\mathcal{I})$ if and only if M_1 and M_3 are.

In such a situation one might want to systematically ignore objects in $\mathcal{N}(\mathcal{I})$, e.g. one might like to regard those morphisms as isomorphisms for which the kernel and cokernel are in $\mathcal{N}(\mathcal{I})$. Technically this can be achieved by passing to the quotient category $\mathcal{C}/\mathcal{N}(\mathcal{I})$; this is an abelian category which has the same objects as \mathcal{C} but one changes the morphisms in such a way that the objects in $\mathcal{N}(\mathcal{I})$ become isomorphic to the trivial object, but no others. (Can you make this precise?)

Often there is an alternative to passing to this quotient category, called *localization away from $\mathcal{N}(\mathcal{I})$* . The category \mathcal{C} is called a localizing subcategory if there is a functor $L : \mathcal{C} \rightarrow \mathcal{C}$ and a natural transformation $\lambda : id_{\mathcal{C}} \rightarrow L$ such that the following statements hold for every object C of \mathcal{C} :

- The kernel and cokernel of the map $\lambda : C \rightarrow LC$ are in $\mathcal{N}(\mathcal{I})$.
- The object LC is $\mathcal{N}(\mathcal{I})$ - closed, i.e. whenever $f : A \rightarrow B$ is a morphism in \mathcal{C} with $\text{Ker } f$ and $\text{Coker } f$ in $\mathcal{N}(\mathcal{I})$, then the induced map $\text{Hom}_{\mathcal{U}}(f, C)$ is an isomorphism.

If localization away from $\mathcal{N}(\mathcal{I})$ exists it is clearly unique up to natural equivalence and it picks for each object C a particularly nice object, namely LC , among all those objects which in the quotient category become isomorphic to C .

Injectivity of $I \in \mathcal{I}$ implies that λ induces an isomorphism $\text{Hom}_{\mathcal{C}}(LC, I) \cong \text{Hom}_{\mathcal{C}}(C, I)$. This shows that LC depends on the functor $F_C : \mathcal{I} \rightarrow \mathcal{Ab}, I \mapsto \text{Hom}_{\mathcal{C}}(C, I)$ (where \mathcal{I} is considered as the full subcategory with object set \mathcal{I} and \mathcal{Ab} denotes the category of abelian groups). In fact, in favourable cases LC can even be reconstructed from this functor. For example, if \mathcal{I} is a finite category (i.e. the set of objects and the set of morphisms is finite) and the functor F_C takes values in finite abelian groups, then LC can be described as follows:

Let $\text{Hom}_+(F_C(I), I)$ denote additive maps from the abelian group $F_C(I)$ to I . (In case $F_C(I)$ is finite one can make sense out of this in an arbitrary abelian category. Can you explain how?) There is a natural “evaluation map” $\varepsilon : C \rightarrow \prod_{I \in \mathcal{I}} \text{Hom}_+(F_C(I), I)$ and it is easy to check that the kernel of this map is the largest subobject of C which is in $\mathcal{N}(\mathcal{I})$. Furthermore LC can be described as a subobject of the product and ε induces the localization map λ . More precisely, the map ε is easily seen to be equalized by the two natural maps

$$\prod_I \text{Hom}_+(F_C(I), I) \rightrightarrows \prod_{I_1 \rightarrow I_2} \text{Hom}_+(F_C(I_1), I_2)$$

where the products extend over all objects resp. morphisms in \mathcal{I} . Now it is not hard to check that the equalizer of these two maps is $\mathcal{N}(\mathcal{I})$ - closed and the map from C to the equalizer qualifies as localization away from $\mathcal{N}(\mathcal{I})$.

As long as the products above make sense the map from C to the equalizer is always a good candidate for the localization, even if the finiteness assumptions on \mathcal{I} and F_C are not satisfied. In particular the equalizer is always $\mathcal{N}(\mathcal{I})$ - closed (cf. Exercise 5 below). However, it may be hard to compute the group of homomorphisms out of the equalizer (because maps out of an arbitrary product are usually difficult to understand), and hence it may be hard to verify that λ has kernel and cokernel in $\mathcal{N}(\mathcal{I})$. Nevertheless the philosophy still works in some interesting cases where these finiteness assumptions are not satisfied.

Localization away from $\mathcal{N}(\mathcal{I})$ is particularly interesting if the category $\mathcal{N}(\mathcal{I})$ has interesting properties and if LC (i.e. the functor F_C) is computable. In the case of the category of unstable modules (and variations thereof) the functor F_{H^*BG} is often “computable” via Lannes’ Theorem, and hence LH^*BG is often accessible. In fact, at least philosophically the results in Section 4.2 and Section 5 should be looked at from this point of view.

Example. Consider the category \mathcal{U} . Pick a skeleton \mathcal{E}_s of the category \mathcal{E} of elementary abelian p -groups and consider $\mathcal{I} = \{H^*BV | V \in \mathcal{E}_s\}$. By Theorem 9 we see that in this case $\mathcal{N}(\mathcal{I})$ is the full subcategory $\mathcal{N}il$ of nilpotent modules. Localization away from $\mathcal{N}il$ has been studied in [HLS1], [HLS2].

Exercise 5: Assume \mathcal{C} is an abelian category and \mathcal{I} is a set of injectives in \mathcal{C} . Show that

- a) I is $\mathcal{N}(\mathcal{I})$ -closed for every $I \in \mathcal{I}$.
- b) A product of $\mathcal{N}(\mathcal{I})$ -closed objects is $\mathcal{N}(\mathcal{I})$ -closed. (This holds even for arbitrary products as long as they exist).
- c) If M_1 and M_2 are $\mathcal{N}(\mathcal{I})$ -closed and if M is the kernel of a map $M_1 \rightarrow M_2$ then M is $\mathcal{N}(\mathcal{I})$ -closed.
- d) Let $\mathcal{C} = \mathcal{U}$, $\mathcal{N}(\mathcal{I}) = \mathcal{N}il$ and assume G is a Quillen group. Conclude that up to natural equivalence Quillen’s map q_G qualifies as localization away from $\mathcal{N}il$.

4.2 Localization away from n -nilpotent modules.

In this section we will explain some of the main results of [HLS3].

Given the discussion above it is natural to consider localization functors on \mathcal{U} based on other sets of injectives. Let $n \in \mathbb{N}$, $n > 0$. In this section we will consider the set $\mathcal{I}_n := \{H^*BV \otimes J(k) | V \in \mathcal{E}_s, k = 0, \dots, n-1\}$. The values $F_M(H^*BV \otimes J(k))$ of the functor F_M are in this case given by the dual of the vector space $T_V^k(M)$, in particular they have a beautiful interpretation if G is a Lannes group and $M = H^*BG$. Moreover the category $\mathcal{N}(\mathcal{I}_n)$ also has a beautiful characterization given as follows.

Definition 7 - Proposition 20. *Let n be a natural number. An unstable module N is called n -nilpotent if the following equivalent conditions hold*

- (1) N is in $\mathcal{N}(\mathcal{I}_n)$.
- (2) $T_V M$ is $(n-1)$ -connected, i.e. $T_V^k M = 0$ for $k < n$.
- (3) Every finitely generated submodule of M has a finite filtration by unstable modules such that the successive quotients in this filtration are n -fold suspensions of unstable modules.

The full subcategories of all n -nilpotent modules is usually denoted $\mathcal{N}il_n$ instead of $\mathcal{N}(\mathcal{I}_n)$.

It is obvious that (1) and (2) are equivalent by the defining property of the Brown Gitler modules $J(n)$. For the equivalence of (1) and (3) we refer to [Sc].

Note that Theorem 9 says that an unstable module is 1-nilpotent in the sense of Definition 7 if and only if it is nilpotent in the sense of Definition 5.

There is a characterization of n - nilpotent modules in terms of Steenrod operations (cf. [H6]) but the characterization given here is usually more useful and in any case more conceptual. We emphasize that the concept relies crucially on the presence of Steenrod operations and the instability condition (it is the instability axiom which prevents a general unstable module from being a suspension!).

The subcategories $\mathcal{N}il_n$ turn out to be localizing subcategories. The corresponding localization functors and natural transformations will be denoted by L_n and λ_n . We will now describe the localization of H^*BG away from $\mathcal{N}il_n$. The point of view in the following discussion is a bit different from the discussion in Section 4.1 in so far as we do not construct L_n and λ_n from the corresponding functor

$$F_{H^*BG} : \mathcal{I}_n \rightarrow \mathcal{A}b, I \mapsto \text{Hom}_{\mathcal{U}}(H^*BG, I)$$

precisely as suggested above. Instead, we process the information carried by this functor in a way which is more adequate from the point of view of group cohomology. The resulting description is equivalent to the one suggested in Section 4.1 (even though \mathcal{I}_n does not satisfy the strong finiteness assumptions), but it is fair to say that the translation from one description to the other is subtle.

As in Section 4.1 we start by considering the following natural map which is induced by evaluation:

$$\varepsilon_n : M \longrightarrow \prod_{(V, k < n)} \text{Hom}_+(\text{Hom}_{\mathcal{U}}(M, H^*BV \otimes J(k)), H^*BV \otimes J(k)) .$$

Here the product ranges over all elementary abelian p - groups V and all $k < n$. The injectivity of the objects $H^*BV \otimes J(k)$ implies that the kernel K of this map satisfies property (1) of Definition 7, hence K is n - nilpotent, and in fact, K is the largest n - nilpotent submodule of M . By the defining property of T_V it is clear that ε_n factors through the map $M \longrightarrow \prod_V H^*BV \otimes T_V M$ whose components are the adjoints of the identity of $T_V M$, and even through

$$M \longrightarrow \prod_V H^*BV \otimes (T_V M)^{<n}$$

where $(-)^{<n}$ denotes the quotient by the submodule generated by elements of degree at least n . We denote this last map by $\bar{\lambda}_n$. Furthermore, it is clear that the obvious map

$$\prod_V H^*BV \otimes (T_V M)^{<n} \longrightarrow \prod_{(V, k < n)} \text{Hom}_+(\text{Hom}_{\mathcal{U}}(M, H^*BV \otimes J(k)), H^*BV \otimes J(k))$$

is injective, hence the kernel of ε_n is equal to the kernel of $\bar{\lambda}_n$.

Now consider the case $M = H^*BG$ where G is a Lannes group. By Theorem 19 the map $\bar{\lambda}_n$ takes the following form

$$\bar{\lambda}_n : H^*BG \longrightarrow \prod_V \prod_{\text{Rep}(V, G)} H^*BV \otimes (H^*BC_G(\rho))^{<n} .$$

The components of this map are induced by the maps $c_\rho : V \times C_G(\rho) \rightarrow G$ and these can be factored as $V \times C_G(\rho) \rightarrow \text{Im } \rho \times C_G(\rho) \rightarrow G$ with $H^*B \text{Im } \rho \otimes H^*BC_G(\rho) \rightarrow H^*BV \otimes H^*BC_G(\rho)$ injective. Therefore we have proved the following result.

Theorem 21 [HLS3]. *Let p be a prime and G be a Lannes group. Then the kernel of the map*

$$\widetilde{\lambda}_n : H^*BG \longrightarrow \prod_{E \in \mathcal{A}_p(G)} H^*BE \otimes (H^*BC_G(E))^{<n}$$

*whose components are induced by the maps $c_E : E \times C_G(E) \rightarrow G, (e, g) \mapsto eg$ is the largest n - nilpotent submodule of H^*BG . \square*

Note that if $n = 1$ the map $\widetilde{\lambda}_n$ agrees with the map $H^*BG \rightarrow \prod_E H^*BE$ considered in Quillen's theorem. The localization $L_n H^*BG$ can now, just as in the case $n = 1$, be described by compatibility conditions inside this product.

In case $n = 1$ the compatibility conditions are completely determined by the category $\mathcal{A}_p(G)$. In the general case there are additional conditions; roughly speaking they arise from varying the groups E and the fact that the two homomorphisms

$$E \times E \times C_G(E) \longrightarrow E \times C_G(E)$$

given by

$$(e_1, e_2, g) \mapsto (e_1 e_2, g) \text{ resp. } (e_1, e_2, g) \mapsto (e_1, e_2 g)$$

are equalized by the homomorphism

$$E \times C_G(E) \rightarrow C_G(E), (e, g) \mapsto eg .$$

(We will denote the former two homomorphisms by $m_E \times id$ resp. $id \times c_E$ where we have taken the liberty to use the symbol c_E again although here its target need not agree with G .) More generally (and this includes the phenomenon of varying the groups), if $\alpha : E_1 \rightarrow E_2, e \mapsto heh^{-1}$ is a morphism in the Quillen category, then we have two homomorphisms

$$E_1 \times E_1 \times C_G(E_2) \rightarrow E_1 \times C_G(E_1), (e_1, e_2, g) \mapsto (e_1 e_2, h^{-1} g h)$$

$$E_1 \times E_1 \times C_G(E_2) \rightarrow E_2 \times C_G(E_2), (e_1, e_2, g) \mapsto (he_1 h^{-1}, he_2 h^{-1} g)$$

whose compositions with the maps

$$c_{E_i} : E_i \times C_G(E_i) \rightarrow G, (e, g) \mapsto eg, i = 1, 2$$

agree up to conjugacy in G . Hence the induced maps

$$H^*BG \longrightarrow H^*BE_i \otimes (H^*BC_G(E_i))^{<n}$$

are equalized by the two maps to

$$H^*BE_1 \otimes (H^*BE_1 \otimes H^*BC_G(E_2))^{<n} .$$

Theorem 22 [HLS3]. *Let p be a prime and G be a Lannes - group for which $\mathcal{A}_p(G)$ is equivalent to a finite category. The map $\widetilde{\lambda}_n$ induces a map from H^*BG to the equalizer*

$$Eq : \prod_E H^*BE \otimes (H^*BC_G(E))^{<n} \rightrightarrows \prod_{E_1 \rightarrow E_2} H^*BE_1 \otimes (H^*BE_1 \otimes H^*BC_G(E_2))^{<n}$$

and this induced map can be identified with λ_n , the localization away from $\mathcal{N}il_n$. (The products in this diagram extend over the objects resp. the morphisms in Quillen's category.)

We repeat (see Section 2.2) that the finiteness condition on the category is implied by the assumption that H^*BG is a Lannes group with H^*BG a finitely generated \mathbb{F}_p - algebra. Roughly speaking, Theorem 22 says that $L_n H^*BG$ is determined by $\mathcal{A}_p(G)$ and the mod - p cohomology of the groups $C_G(E)$ in degrees less than n .

Outline of the proof. There are two things to be checked: that the equalizer is $\mathcal{N}il_n$ - closed, and that the map induced by λ_n has kernel and cokernel in $\mathcal{N}il_n$.

In fact, it is not hard to show that $H^*V \otimes F$ is $\mathcal{N}il_n$ - closed whenever F is an unstable module which is concentrated in degrees less than n . Then Exercise 5 allows to conclude that the equalizer is $\mathcal{N}il_n$ - closed.

To show that $\text{Ker } \lambda_n$ and $\text{Coker } \lambda_n$ are in $\mathcal{N}il_n$ it suffices to show that $\text{Hom}_{\mathcal{U}}(\lambda_n, I)$ is an isomorphism for all $I \in \mathcal{I}_n$, in other words that $T_V(\lambda_n)$ is an isomorphism in degrees $< n$ for all elementary abelian V . Because of the magical properties of Lannes' T - functor this turns out to be manageable. \square

The proof that we just outlined is much more direct than the proof given in [HLS3] where (a version) of Theorem 22 was presented as a special case of the general study of localizations away from $\mathcal{N}il_n$. In fact, the determined reader should be able to supply the details of the proof. All he needs to know about Lannes' functor has been provided in Section 3.

In the following special case the compatibility conditions simplify drastically.

Corollary 23 [HLS3]. *Let G be a Lannes - group for which $\mathcal{A}_p(G)$ is equivalent to a finite category. Assume that all elements of G of order p are central (and hence generate a central elementary abelian p - subgroup D). Then the map c_D induces a homomorphism of unstable algebras*

$$H^*BG \longrightarrow Eq : H^*BD \otimes (H^*BG)^{<n} \rightrightarrows H^*BD \otimes (H^*BD \otimes H^*BG)^{<n}$$

which can be identified with the localization map λ_n away from $\mathcal{N}il_n$. (Again Eq stands for equalizer and the two maps in the equalizer diagram are induced by the maps $m_D \times id$ resp. $id \times c_D$.)

Exercise 6: Derive Corollary 23 from Theorem 22.

One of the interests in these higher localizations is that they give a sequence of approximations which become better and better as n grows. In fact, the following result shows that this sequence stabilizes to a genuine isomorphism after a finite number of steps.

Theorem 24 [HLS3]. *Assume G is a Lannes group such that H^*BG is finitely generated as an \mathbb{F}_p - algebra.*

a) *There exist natural numbers $d_0(G)$ and $d_1(G)$ such that λ_n (and hence $\widetilde{\lambda}_n$) is a monomorphism for $n > d_0(G)$ and λ_n is an isomorphism for $n > d_1(G)$.*

b) *Let G be a finite group or a compact Lie group and let U be a unitary group such that G embeds into U . Then we have the following inequalities:*

- $d_0(G) \leq \dim U - \dim G$
- $d_1(G) \leq 2 \dim U - \dim G$. (Here $\dim G$ etc. refer to the dimension of G considered as a manifold.)

Comment on the proof. Part (a) is a general fact about localizations of unstable modules which happen to be unstable algebras and are finitely generated as \mathbb{F}_p - algebras.

For the proof of Part (b) one uses once more Quillen's trick (i.e. the equalizer diagram (*) from the proof of Theorem 19'. This reduces the problem to finding bounds for d_0 and d_1 in the cases

$$H_G^*(U(n)/S) \cong H_S^*(G \setminus U(n))$$

and

$$H_G^*(U(n)/S \times U(n)/S) \cong H_{S \times S}^*(G \setminus U(n) \times U(n)) .$$

Therefore one is lead to study d_0 and d_1 in the case of equivariant cohomology of an elementary abelian group acting on a smooth manifold. This case can be handled by using a result of Duflot on the structure of such equivariant cohomology rings [Du1]. \square

So the invariants $d_0(G)$ and $d_1(G)$ are not merely theoretical constants but they can be effectively bounded above. The bounds given in the theorem are very crude in general. For refinements we refer the reader to sections II.2 and II.3 of [HLS3]. We remark that the precise computation of the invariants $d_0(G)$ and $d_1(G)$ appears to be a difficult problem in general, and further investigation of these invariants should be worthwhile.

There are also bounds for many discrete groups based on extensions of Theorem 24 to equivariant cohomology (see [HLS3]) but the bounds are less simple to state.

Exercise 7: Use the known computations of H^*BG (cf. [AM]) together with the description of λ_n given in Corollary 23 to determine the invariants d_0 and d_1 for the groups \mathbb{Z}/p^n , S^1 and S^3 at any prime, and for Q_8 at the prime 2.

In principle $L_n H^*BG$ (and thus H^*BG) can be computed from knowing the category $\mathcal{A}_p(G)$ and the low dimensional cohomology of the spaces $BC_G(E)$. The examples in Exercise 7 suggest that this may not be a particularly efficient way to compute the cohomology of a finite group, at least not in cases which are accessible by traditional methods. The situation might change, however, if the groups get sufficiently complicated. In any case, if G is a finite group, there is a precise algorithm starting from a well defined finite set of data. This algorithm has not been seriously tested yet but one can hope that with increasing computer power this may become a feasible method for computing the cohomology of a finite group.

The invariants d_0 and d_1 behave well with respect to certain group theoretic constructions like the wreath product construction. This can sometimes be used to keep the bounds for these invariants much smaller than those given by Theorem 24. For example, in [HLS3] we show that if $p = 2$ and if q is a prime power with $q \equiv 3 \pmod{4}$ then $d_0(GL(n, \mathbb{F}_q)) = 0$, $d_1(GL(n, \mathbb{F}_q)) \leq 2$. From this we were able to recover Quillen's computation of the mod - 2 cohomology of the groups $GL(n, \mathbb{F}_q)$ [Q2].

As an immediate further consequence of Theorem 21 and Theorem 24 we get some information on the nilpotency-height of the ideal of nilpotent elements in H^*BG (in terms of the invariant $d_0(G)$). For $p = 2$ this ideal agrees with the kernel $N(G)$ of the restriction map $H^*BG \longrightarrow \prod_{E \in \mathcal{A}_2(G)} H^*BE$, for odd primes it is closely related to $N(G)$.

Corollary 25 [HLS3]. *Let G be a Lannes - group for which H^*BG is a finitely generated algebra. Then the n - th power of the ideal $N(G)$ is trivial if $n > d_0(G)$.*

Proof. If x is in $N(G)$ then x restricts trivially to H^*BE for all $E \in \mathcal{A}_p(G)$ and hence, if \tilde{H}^* denotes the ideal of elements of positive degree, $c_E^*(x) \in H^*BE \otimes (\tilde{H}^*BC_G(E))^{<n}$. Consequently, if x_1, \dots, x_n are in $N(G)$ then $c_E^*(x_1 \dots x_n) = 0$ and we are done by Theorem 24. \square

It is not hard to produce (say by considering cartesian products) for each $n \in \mathbb{N}$ a finite p - group F_n such that $N(F_n)^n \neq 0$. For $p = 2$ even more is true as we have already remarked in Section 1: by ([AC], [IK]) there exist finite 2 - groups F_n with elements $t_n \in H^*(BF_n; \mathbb{F}_2)$ such that $t_n^n \neq 0$, $t_n^{n+1} = 0$.

5. COMMUTATIVE ALGEBRA OF UNSTABLE H^*BG - MODULES; APPROXIMATIONS OF H^*BG UP TO TORSION MODULES

Throughout this section p is a fixed prime and G is a Lannes - group with finitely generated mod - p cohomology ring H^*BG . In this section we discuss results of [H2]. These results can be understood from the point of view developed in Section 4.1 but this would lead us too far afield, so we will only describe the results and try to explain their relevance for the study of group cohomology.

We will discuss in particular how much information about H^*BG is carried by the cohomology of various centralizers of elementary abelian p - subgroups of G . The centralizers will be indexed by suitable collections \mathcal{O} of elementary abelian p - subgroups of G . These collections will have the following two properties:

- If $E \in \mathcal{O}$ and E' is conjugate to E then $E' \in \mathcal{O}$
- If $E \in \mathcal{O}$ and $E \subset E'$ then $E' \in \mathcal{O}$.

We call such collections *open*. The justification for this terminology comes from the fact that the conjugacy classes of elementary abelian p - subgroups E are in 1 - 1 correspondence with prime ideals $\mathfrak{p}_E \subset H^*BG$ which are invariant with respect to the reduced power operations P^i if $p > 2$ and with respect to all Sq^i if $p = 2$

[Q1]; the prime ideal \mathfrak{p}_E is given as the radical of the kernel of the restriction map $H^*BG \rightarrow H^*BE$. A homogeneous ideal which is invariant with respect to the Sq^i resp. P^i -operations will be called an *invariant ideal*.

Via this correspondence open collections correspond to intersections of the invariant part of the prime ideal spectrum of the graded ring H^*BG with subsets which are open in the Zariski topology. Furthermore, if \mathfrak{a} is an ideal in H^*BG then it is clear that the collection

$$\mathcal{O}(\mathfrak{a}) := \{E \in \mathcal{A}_p(G) \mid \mathfrak{a} \not\subset \mathfrak{p}_E\}$$

forms an open collection, and this collection depends only on the radical of the ideal \mathfrak{a} . Conversely, if an open collection \mathcal{O} is given, it determines a radical invariant ideal

$$\mathfrak{a}(\mathcal{O}) = \bigcap_{E \notin \mathcal{O}} \mathfrak{p}_E$$

and the open subset in the Zariski topology is given as the set of prime ideals not containing \mathfrak{a} . The assignments $\mathfrak{a} \mapsto \mathcal{O}(\mathfrak{a})$ and $\mathcal{O} \mapsto \mathfrak{a}(\mathcal{O})$ establish bijections between the radical invariant ideals of H^*BG and open collections in $\mathcal{A}_p(G)$ (cf. [H2]).

For example, if $\mathfrak{a} = \tilde{H}^*BG$ is the ideal consisting of the elements of positive degree in H^*G then $\mathcal{O}(\mathfrak{a})$ is precisely the collection of all non-trivial elementary abelian p -subgroups; if G is a compact Lie group, this is an immediate consequence of Theorem 3 in section 1, in the general case one can deduce it from Proposition 29 below.

For an open collection \mathcal{O} we will denote the full subcategory of $\mathcal{A}_p(G)$ with object set \mathcal{O} also by \mathcal{O} . The assignment $E \mapsto H^*BC_G(E)$ extends to a functor from \mathcal{O} to unstable modules with an H^*BG -module structure (which for $H^*BC_G(E)$ is given via the restriction homomorphism).

Here is a preliminary version of the main result of this section. A stronger result in which we will also establish the link with suitable localizations will be given below.

Theorem 26 [H2]. *Let G be a Lannes - group such that H^*BG is a finitely generated \mathbb{F}_p -algebra. Let $\mathfrak{a} \subset H^*BG$ be a radical invariant ideal and let $\mathcal{O}(\mathfrak{a})$ be the associated open collection of elementary abelian p -subgroups. Then the restriction maps induce a map*

$$\rho : H^*BG \longrightarrow \lim_{\mathcal{O}(\mathfrak{a})} H^*BC_G(E)$$

whose kernel and cokernel are \mathfrak{a} -torsion, i.e. they are annihilated by some large power of \mathfrak{a} .

Note that from a computational point of view the two properties defining an open collection are reasonable in the following sense: If $E \subset E'$ then the centralizer $C_G(E')$ will be contained in $C_G(E)$ and will usually be easier to understand than $C_G(E)$, both group theoretically and cohomologically. (In fact, $H^*BC_G(E')$ is determined by $H^*BC_G(E)$ via Lannes' Theorem applied to $H^*BC_G(E)$). Can you

make this precise?) In order to apply the theorem in concrete cases one would try to compute as many $H^*BC_G(E)$ as possible; the minimal E 's in that collection define an open collection \mathcal{O} of elementary abelian p -subgroups (all those which contain one of the minimal ones) and chances are that $H^*BC_G(E)$ can be computed for all $E \in \mathcal{O}$ and with some luck $\lim_{\mathcal{O}(\mathfrak{a})} H^*BC_G(E)$ can be computed as well.

We discuss some special cases of Theorem 26.

1) Suppose E is a maximal elementary abelian p -subgroup. Then the collection of all elementary abelian p -subgroups which are conjugate to E form an open collection $\mathcal{O}(\mathfrak{a})$. We denote the corresponding radical invariant ideal \mathfrak{a} by $\mathfrak{a}(E)$. This ideal consists of all elements which restrict nilpotently to E' whenever E' is not conjugate to E . If we abbreviate $\text{Aut}_{\mathcal{A}_p(G)}(E)$ by A then the inverse limit over the category \mathcal{O} can be identified with the invariants $(H^*BC_G(E))^A \cong (H^*BC_G(E))^{N_G(E)}$ where $N_G(E)$ denotes the normalizer of E in G . Therefore Theorem 26 yields the following result.

Corollary 27 (cf. [Du2]). *Suppose G is a Lannes group such that H^*BG is a finitely generated \mathbb{F}_p -algebra. Let E be a maximal elementary abelian p -subgroup. Then the kernel and cokernel of the map*

$$\rho : H^*BG \longrightarrow (H^*BC_G(E))^{N_G(E)}$$

are torsion with respect to $\mathfrak{a}(E)$. \square

We have seen in Section 1 that there exists an element $z_E \in H^*BG$ which restricts trivially to all elementary abelian p -subgroups which are not conjugate to E and to a power of the top Dickson invariant on all elementary abelian p -subgroups which are conjugate to E . So z_E belongs to \mathfrak{a} but not to \mathfrak{p}_E , and therefore \mathfrak{a} -torsion modules are killed by localization at \mathfrak{p}_E . Now localization at \mathfrak{p}_E is exact and hence we derive the result of Duflot that $(H^*BG)_{\mathfrak{p}_E}$, the localization of H^*BG at the minimal invariant prime ideal \mathfrak{p}_E (given as radical of the kernel of restriction to a maximal elementary abelian p -subgroup E) is isomorphic to $((H^*BC_G(E))_{\mathfrak{p}_E})^{N_G(E)}$, the invariants of the localization of $H^*BC_G(E)$ with respect to the action of $N_G(E)$. Corollary 27 can in this case be considered as a strengthening of Duflot's result in so far as it gives more precise information about the kernel and cokernel of the map ρ .

2) At the other extreme we consider the collection of all non-trivial elementary abelian p -subgroups of G . The corresponding category will be denoted $\mathcal{A}_p^*(G)$. As remarked above the corresponding ideal \mathfrak{a} is just the augmentation ideal \tilde{H}^*BG . It is clear that in this case a finitely generated torsion module is nothing but a finite module. If G is a compact Lie group, then by Theorem 3 the cohomology of the centralizers $C_G(E)$ are finitely generated H^*BG -modules for all E . In fact, this remains true if G is any Lannes group for which H^*BG is a finitely generated \mathbb{F}_p -algebra [DW3], [H7]. It follows that the target of ρ is finitely generated as H^*BG -module. Therefore the kernel and cokernel of ρ are finitely generated \mathfrak{a} -torsion modules, hence finite; in particular ρ is an isomorphism in all sufficiently large degrees. In other words Theorem 26 specializes to the following result.

Corollary 28. *Suppose G is a Lannes group such that H^*BG is a finitely generated \mathbb{F}_p -algebra. Then the kernel and cokernel of the map*

$$\rho : H^*BG \longrightarrow \lim_{\mathcal{A}_p^*(G)} H^*BC_G(E)$$

are finite. \square

In the case of a compact Lie group more can be said: if G contains an element of order p then according to an important theorem of Jackowski and McClure [JM] ρ is an isomorphism (see Theorem 32 below).

3) In the general case we can use Theorem 26 to control the *size* (by which we mean the Krull dimension, or equivalently the order of the pole of the Poincaré series at $t = 1$) of the kernel of the map ρ ; in fact, the size of this kernel can be bounded by the size of H^*BG/\mathfrak{a} and this can in turn be bounded by the maximal rank of an $E \notin \mathcal{O}$. Furthermore, the theorem also gives a tool to construct elements in H^*BG : for this one can take any element $z \in \lim_{\mathcal{O}(\mathfrak{a})} H^*BC_G(E)$; if x_1, \dots, x_k are in \mathfrak{a} and k is large enough then $x_1 \cdots x_k z$ is in the image of the map. Non-trivial elements x_i can often be constructed as characteristic classes and bounds for k are also often available, e.g. via the methods of part II of [HLS3]. The existence of non-trivial classes $x_i \in \mathfrak{a}$ is also guaranteed by the following result.

Proposition 29 [CaH]. *Assume G is a Quillen - group such that $\mathcal{A}_p(G)$ is equivalent to a finite category. Let $\mathcal{O} \subset \mathcal{A}_p(G)$ be an open collection. Then there is an element $x \in H^*BG$ which restricts nilpotently to H^*BE iff $E \notin \mathcal{O}$.*

Examples. a) In [H2] we used Theorem 26 to give an upper bound for the size of the kernel K_n of the restriction map from $H^*(BGL(n, \mathbb{Z}[1/2]); \mathbb{F}_2)$ to $H^*(BD_n; \mathbb{F}_2)$ (with D_n denoting the subgroup of diagonal matrices, i.e. $D_n \cong (\mathbb{Z}[1/2]^\times)^n \cong \mathbb{Z}^n \times (\mathbb{Z}/2)^n$). This kernel is non-trivial for n large, e.g. if $n = 32$ [Dw2]. If $\sigma(K_n)$ denotes this size, and if n_0 is the minimal n with K_n non-trivial, then Theorem 26 can be used to show $\sigma(K_n) \leq n - n_0 + 1$ for $n \geq n_0$. In fact, we show even more in [H2], namely $\sigma(K_n) = n - n_0 + 1$ for $n \geq n_0$.

b) In joint work with F. Cohen and Y. Xia we will investigate certain mapping class groups and will produce new classes in their cohomology by the method outlined above.

Theorem 26 is contained in the following more general result. To state it we need to introduce more notation.

First we need to extend the context in which we are working. We consider the category $H^*BG - \mathcal{U}$ of unstable H^*BG -modules. Its objects are unstable modules M over the Steenrod algebra equipped with the structure of a graded H^*BG -module such that the structure map $H^*BG \otimes M \rightarrow M$ is A_p -linear, i.e. it obeys the Cartan formula. Morphisms are degree preserving homomorphisms which are both A_p -linear and H^*BG -linear. The full subcategory whose objects are finitely generated as H^*BG -modules is denoted by $H^*BG_{fg} - \mathcal{U}$. Natural objects of $H^*BG - \mathcal{U}$ are given by H^*BG , $H^*BC_G(E)$ or more generally by the cohomology $H_G^*X := H^*(EG \times_G X)$ of the Borel construction of a G -space X .

If X is a finite G - CW - complex and G is a suitable discrete group (e.g. if G is finite) then H_G^*X is even in $H^*BG_{fg} - \mathcal{U}$.

For an invariant ideal \mathfrak{a} and an unstable H^*BG - module M we consider

$$F_{\mathfrak{a}}M := \{x \in M \mid \mathfrak{a}^n x = 0 \text{ for some } n \in \mathbb{N}\} .$$

It is straightforward to check (using the Cartan formula) that $F_{\mathfrak{a}}$ defines a functor from $H^*BG - \mathcal{U}$ resp. $H^*BG_{fg} - \mathcal{U}$ to itself. This functor is left exact, and as the category $H^*BG_{fg} - \mathcal{U}$ is abelian with enough injectives [H2], $F_{\mathfrak{a}}$ has right derived functors denoted by $R^i F_{\mathfrak{a}}$. If $\mathfrak{a} = \widetilde{H}^*BG$ and M is in $H^*BG_{fg} - \mathcal{U}$ then it is clear that $F_{\mathfrak{a}}M$ is the largest finite subobject of M ; in this case we denote the functor simply by F and its derived functors by $R^i F$. For $M \in H^*BG_{fg} - \mathcal{U}$ all the modules $R^i FM$ are finite.

Here is the generalization of Theorem 26.

Theorem 30 [H2]. *Let G be a Lannes - group with H^*BG a finitely generated algebra and $\mathfrak{a} \subset H^*BG$ an invariant ideal with associated open collection $\mathcal{O}(\mathfrak{a})$. Denote the full subcategory of $\mathcal{A}_p(G)$ with $\mathcal{O}(\mathfrak{a})$ as object set again by $\mathcal{O}(\mathfrak{a})$.*

a) *Then there is an exact sequence*

$$0 \longrightarrow F_{\mathfrak{a}}H^*BG \longrightarrow H^*BG \xrightarrow{\rho} \lim_{\mathcal{O}(\mathfrak{a})} H^*BC_G(E) \longrightarrow R^1 F_{\mathfrak{a}}H^*BG \longrightarrow 0$$

in which the components of ρ are induced by the restriction maps. In particular, the kernel and cokernel of ρ are unstable finitely generated \mathfrak{a} - torsion modules. Furthermore, the map ρ is localization away from the subcategory of unstable finitely generated \mathfrak{a} - torsion modules.

b) *If \lim^i denotes the i - th right derived functor of \lim then there are natural isomorphisms*

$$\lim_{\mathcal{O}(\mathfrak{a})}^i H^*BC_G(E) \cong R^{i+1} F_{\mathfrak{a}}H^*BG$$

*for all $i > 0$. In particular, $\lim_{\mathcal{O}(\mathfrak{a})}^i H^*BC_G(E)$ is an unstable finitely generated \mathfrak{a} - torsion module for all $i > 0$.*

Comment on the proof. This result relies heavily on two results. Firstly, the category $H^*BG_{fg} - \mathcal{U}$ has enough injectives, and secondly these injectives are explicit enough (they are closely related to the injectives $H^*BV \otimes J(n)$ in the category \mathcal{U}) that for them one can check the theorem directly. The remainder of the proof is then standard homological algebra. \square

The following special case of this theorem needs to be emphasized. In it $\mathcal{A}_p^*(G)$ denotes the full subcategory of $\mathcal{A}_p(G)$ consisting of all non-trivial elementary abelian p - subgroups of G .

Corollary 31 [H2]. *Let G be a Lannes - group with H^*BG a finitely generated algebra.*

a) *Then there is an exact sequence*

$$0 \longrightarrow FH^*BG \longrightarrow H^*BG \xrightarrow{\rho} \lim_{\mathcal{A}_p^*(G)} H^*BC_G(E) \longrightarrow R^1 FH^*BG \longrightarrow 0$$

in which the components of ρ are induced by the restriction maps. In particular, the kernel and cokernel of ρ are finite. Furthermore, the map ρ is localization away from the subcategory of unstable finite H^*BG - modules.

b) There are natural isomorphisms

$$\lim_{\mathcal{A}_p^*(G)}^i H^*BC_G(E) \cong R^{i+1}FH^*BG$$

for all $i > 0$. In particular, $\lim_{\mathcal{A}_p^*(G)}^i H^*BC_G(E)$ is finite for all $i > 0$.

We emphasize that part (b) of this result is really relevant for the study of cohomology of groups. This will become clear in Section 6 below.

We close this section with some comments on and consequences of Theorem 30 and Corollary 31.

The results in [H2] are even more general; there is a version which holds for each unstable module $M \in H^*BG_{fg} - \mathcal{U}$ (in fact, H^*BG can even be replaced by an unstable algebra K as long as it is finitely generated as an \mathbb{F}_p - algebra). In this generality the cohomology of the centralizers is replaced by certain “components” of $T_V M$; if $M = K = H^*G$ these components are given via Theorem 19 by the cohomology of the spaces $BC_G(E)$. We refer the reader to [H2] for more details. Here we just want to remark that these “components” serve in the world of unstable H^*BG - modules as a replacement of the sheaf associated to a module over a commutative ring in the classical context. In particular, in the world of unstable modules and unstable algebras the functor $E \mapsto H^*BC_G(E)$ defined on $\mathcal{A}_p(G)$ plays the rôle of the structure sheaf of the ring H^*BG , and this yields a conceptual explanation why the cohomology of the centralizers plays such a crucial role in understanding H^*BG . This point of view is also supported by the formal analogy of Theorem 30 with results of Grothendieck on local cohomology in the classical situation of modules over a noetherian commutative ring R [Gr].

If G is a finite group (or a compact Lie group) which contains an element of order p then one can show [H2] that R^iFH^*BG is trivial for all $i \geq 0$ and therefore the map ρ in Corollary 31 is an isomorphism and the higher limits $\lim_{\mathcal{A}_p^*(G)}^i H^*BC_G(E)$ vanish for all $i > 0$. Thus one recovers a result by Jackowski and McClure [JM] resp. by Dwyer and Wilkerson [DW1] (see Theorem 32 below) which plays a central role in the homotopy theory of classifying spaces and p - compact groups ([DW2], [JMO]).

Examples. a) If G is discrete or profinite then ρ need no longer be an isomorphism. In [H4] the “profinite analogue” of Corollary 31 was used to compute the continuous mod - p cohomology in large dimensions of $GL(p-1, \mathbb{Z}_p)$ and the $(p-1)$ - st “Morava stabilizer groups”, i.e. the (p) -Sylow subgroups of the group of units in the maximal order of the division algebras over \mathbb{Q}_p whose dimension is $(p-1)^2$ and whose Hasse invariant is $\frac{1}{p-1}$. For $p = 3$ additional arguments allowed to extend the calculation in large dimensions to one in all dimensions and in those cases ρ turned out not to be an isomorphism.

b) The case of $H^*(BSL(3, \mathbb{Z}[1/2]); \mathbb{F}_2)$ was considered in [H3] and [H5]: Corollary 31 was used in [H3] to carry out the calculation in large dimensions, and building

on this the calculation was completed in [H5] (see Section 6 below); in this case the complete calculation showed that ρ happens to be an isomorphism again.

Remark. Theorem 30 implies that the restriction map

$$\text{res}_{\mathcal{O}(\mathfrak{a})} : H^*BG \longrightarrow \prod_{E \in \mathcal{O}(\mathfrak{a})} H^*BC_G(E)$$

is injective if and only if $F_{\mathfrak{a}}H^*BG = 0$. In particular, if there exists a non-trivial element in \mathfrak{a} which is not a zero-divisor then clearly $\text{res}_{\mathcal{O}(\mathfrak{a})}$ is injective. This was first proved in [HLS3]. The following “converse” was proved in [CaH]: if $\text{res}_{\mathcal{O}(\mathfrak{a})}$ is injective then there exists an element $x \in \mathfrak{a}$ which is not a zero-divisor; in fact, any element provided by Proposition 29 has this property.

We finish this section by remarking that Corollary 31 is somewhat reminiscent of results on Farrell cohomology of groups of finite virtual cohomological dimension. We will come back to this in the next section.

6. THE CENTRALIZER SPECTRAL SEQUENCE

As before p is a fixed prime. In this section the coefficients need not necessarily be equal to \mathbb{F}_p so we will no longer skip them from the notation.

The higher limits of Theorem 30 and Corollary 31 have also geometric significance; they occur as E_2 - term in a homotopy colimit spectral sequence. The reader uncomfortable with homotopy colimits is encouraged to consult [JMO] for an introduction to the concept. In this section homotopy colimits will be primarily discussed to the extent that they produce interesting spectral sequences; they can be defined for functors F from a small category \mathcal{C} to topological spaces and give a recipe to glue together the spaces $F(c)$, $c \in \mathcal{C}$, in a homotopy invariant way encoded by the category \mathcal{C} . Our first example is the functor $E \mapsto EG \times_G (G/C_G(E))$ from $\mathcal{A}_p(G)^{op}$ to spaces. Note that if E is the trivial subgroup this functor takes value BG .

First we recall the following result of Jackowski and McClure which “reconstructs” BG from the spaces $EG \times_G (G/C_G(E)) \simeq BC_G(E)$ with E non-trivial.

Theorem 32 [JM]. *Let G be a compact Lie group (which may be finite but is assumed to contain an element of order p). Then the natural map*

$$\pi : \text{hocolim}_{\mathcal{A}_p(G)^{op}} EG \times_G G/C_G(E) \longrightarrow BG$$

induces an isomorphism in homology with coefficients in the p - local integers $\mathbb{Z}_{(p)}$.

Starting from the result of Jackowski and McClure the following generalization was obtained in [H3]. To state it we need to introduce some notation. Let G be a (topological) group and X a G - space. If H is a subgroup of G then X^H denotes the subspace of X fixed by all of H ; the p - singular locus X_s is defined to be $X_s := \bigcup_g X^{\langle g \rangle}$ where g runs through all elements of G of order p and $\langle g \rangle$ denotes the subgroup generated by g . The G - space X gives rise to a functor $E \mapsto EG \times_{C_G(E)} X^E$ from $\mathcal{A}_p(G)^{op}$ to spaces.

Theorem 33 [H3]. *Assume G is a discrete group or a compact Lie group and X is any G - CW - complex. If G is discrete assume that the isotropy group of each $x \in X$ is finite. Then the natural map*

$$\pi : \text{hocolim}_{\mathcal{A}_p^*(G)^{op}} EG \times_{C_G(E)} X^E \longrightarrow EG \times_G X_s$$

induces an isomorphism with coefficients in $\mathbb{Z}_{(p)}$.

Outline of the proof. We regard this result as a comparison theorem on functors from G - spaces to graded \mathbb{F}_p - vector spaces, say. So let F_1^G be the functor which associates to a G - space X the cohomology of $\text{hocolim}_{\mathcal{A}_p^*(G)^{op}} EG \times_{C_G(E)} X^E$, and F_2^G the functor which associates to X the cohomology of $EG \times_G X_s$. We want to show that the natural transformation π^* between these functors is an isomorphism for certain G - spaces X . These functors are G - homotopy invariant, they turn arbitrary coproducts into products and associate to homotopy pushouts Mayer-Vietoris sequences. Therefore it suffices to show that π^* is an isomorphism for each orbit G/K which appears in these spaces. The crucial observation is now that both functors satisfy ‘‘Shapiro type’’ isomorphisms $F_i^G(G/K) \cong F_i^K(pt)$, and then we are done by Theorem 32. \square

Of primary interest for us is the resulting homotopy colimit spectral sequence [BK].

Corollary 34 [H3]. *Assume G is a discrete group or a compact Lie group and X is a G - CW - complex. If G is discrete assume that the isotropy group of each $x \in X$ is finite. Then for every $\mathbb{Z}_{(p)}$ - module M there exists a first quadrant cohomological spectral sequence*

$$E_2^{s,t} = \lim_{\mathcal{A}_p^*(G)}^s H_{C_G(E)}^t(X^E; M) \implies H_G^{s+t}(X_s; M) . \quad \square$$

There are strong vanishing results for this spectral sequence in case $M = \mathbb{F}_p$. In fact, in [JM] Theorem 32 was proved by showing via Mackey functor techniques that in this case the spectral sequence is concentrated on the vertical edge and the natural map $H^*(BG; \mathbb{F}_p) \longrightarrow \lim_{\mathcal{A}_p^*(G)} H^*(BC_G(E); \mathbb{F}_p)$ is an isomorphism. Dwyer and Wilkerson [DW1] gave a proof of this using techniques from the theory of unstable modules and in [H2] still another such proof was given (cf. the discussion after Corollary 31 above). In the general case the results of [H2] (cf. Theorem 30 above) imply:

Theorem 35 [H3]. *Assume G is a discrete group or a compact Lie group and X is a G - CW - complex with finitely many equivariant cells. If G is discrete assume that the isotropy group of each $x \in X$ is finite. If coefficients are in \mathbb{F}_p then $E_2^{s,*}$ is a finite graded \mathbb{F}_p - vector space for each $s > 0$, in particular $E_2^{s,t} = 0$ if $s > 0$ and t is large.*

An additional vanishing result has been proved by Oliver. (From the point of view of the discussion after Corollary 31 above this result corresponds to the vanishing of local cohomology above the Krull dimension of the ring.)

Theorem 36 [O]. *Let G be any group and let F be any functor from $\mathcal{A}_p^*(G)$ to $\mathbb{Z}_{(p)}$ - modules. Then $\lim_{\mathcal{A}_p^*(G)}^s F = 0$ for all $s \geq r_p(G)$, the p - rank of G .*

As a consequence, under the assumptions of Theorem 35 and if $r_p(G)$ is finite then the spectral sequence is essentially concentrated on the vertical edge and it almost collapses. In particular, the edge homomorphism $H_G^*(X_s; \mathbb{F}_p) \longrightarrow \lim_{\mathcal{A}_p^*(G)} H_{C_G(E)}^*(X^E; \mathbb{F}_p)$ is an isomorphism in large dimensions.

These results have interesting consequences for computations of group cohomology. First we introduce certain classes of groups by slightly modifying a definition due to Kropholler [K].

Definition 8. A discrete group G is said to be an $\mathcal{H}_1\mathcal{F}$ - group iff there exists a G - CW - complex X with the following properties:

- X is finite dimensional and mod - p acyclic,
- the isotropy group G_x of each $x \in X$ is a finite group.

In this case we call X an $\mathcal{H}_1\mathcal{F}$ - complex for G .

• If in addition $G \backslash X$ is compact, then G is said to be an $\mathcal{K}_1\mathcal{F}$ - group and X a $\mathcal{K}_1\mathcal{F}$ - complex for G .

In the appendix of [H2] we proved that a $\mathcal{K}_1\mathcal{F}$ - group G is always a Lannes group and H^*BG is always a finitely generated \mathbb{F}_p - algebra. Extensions to certain $\mathcal{H}_1\mathcal{F}$ - groups were discussed in the appendix of [H6].

As in [K] one can define $\mathcal{H}_\alpha\mathcal{F}$ - groups for every ordinal α but we will refrain from doing so because we are only interested in the case $\alpha = 1$. All groups G of finite virtual cohomological dimension are $\mathcal{H}_1\mathcal{F}$ - groups and very often they are even $\mathcal{K}_1\mathcal{F}$ - groups, e.g. if G is an $(S -)$ arithmetic group [Se], a mapping class group [Ha], an outer automorphism group of a free group [CV] or a word hyperbolic group in the sense of Gromov [GH].

If X is a mod - p acyclic G - CW - complex then $H_G^*(X; \mathbb{F}_p) \cong H^*(BG; \mathbb{F}_p)$, and if X is also finite dimensional then Smith theory implies that the fixed point sets X^E are also mod - p acyclic and therefore $H_{C_G(E)}^*(X^E; \mathbb{F}_p) \cong H^*(BC_G(E); \mathbb{F}_p)$. Furthermore $H_G^*(X; \mathbb{F}_p) \cong H_G^*(X_s; \mathbb{F}_p)$ if s exceeds the dimension of X ; more precisely, the relative group $H_G^*(X, X_s; \mathbb{F}_p)$ is naturally isomorphic to $H^*(G \backslash (X, X_s); \mathbb{F}_p)$ and therefore vanishes if s exceeds the dimension of X . Thus we have an almost collapsing “centralizer spectral sequence” which converges not to $H^*(BG; \mathbb{F}_p)$ but to the closely related object $H_G^*(X_s; \mathbb{F}_p)$. More precisely we have the following result.

Corollary 37 [H3]. *a) Assume G is an $\mathcal{H}_1\mathcal{F}$ - group, X an $\mathcal{H}_1\mathcal{F}$ - complex for G and M is a $\mathbb{Z}_{(p)}$ - module. Then there exists a first quadrant cohomological spectral sequence*

$$E_2^{s,t} = \lim_{\mathcal{A}_p^*(G)}^s H^t(BC_G(E); M) \implies H_G^{s+t}(X_s; M) .$$

with $E_2^{s,*} = 0$ for $s \geq r_p(G)$.

b) If G is a $\mathcal{K}_1\mathcal{F}$ - group, X a $\mathcal{K}_1\mathcal{F}$ - complex for G and $M = \mathbb{F}_p$ then $E_2^{s,}$ is a finite graded \mathbb{F}_p - vector space for each $s > 0$ and $E_2^{s,*} = 0$ for $s \geq r_p(G)$. Furthermore, both the edge homomorphism*

$$H_G^*(X_s; \mathbb{F}_p) \longrightarrow \lim_{\mathcal{A}_p^*(G)} H^*(BC_G(E); \mathbb{F}_p)$$

and the map

$$\rho : H^*(BG; \mathbb{F}_p) \longrightarrow \lim_{\mathcal{A}_p^*(G)} H^*(BC_G(E); \mathbb{F}_p)$$

are isomorphisms in large dimensions.

The corollary suggests a program for computing the mod - p cohomology of a discrete $\mathcal{K}_1\mathcal{F}$ - group consisting of the following steps (each of which can be quite difficult though).

- Determine the (category of) elementary abelian p - subgroups of G and the centralizers of these subgroups.
- Compute the mod - p cohomology of (the classifying spaces of) the centralizers (as a functor on $\mathcal{A}_p^*(G)$).
- Compute $H_G^*(X_s; \mathbb{F}_p)$ from the (almost collapsing) centralizer spectral sequence. (Note that the input into this spectral sequence is independent of the choice of X !)
- Find a suitable $\mathcal{K}_1\mathcal{F}$ - complex X , compute $H^*(G \backslash (X, X_s); \mathbb{F}_p)$ and evaluate the long exact mod - p cohomology sequence of the pair $EG \times_G (X, X_s)$. (This part is vital for getting at the low-dimensional cohomology of G .)

Example. The group $G = SL(3, \mathbb{Z}[1/2])$ is known to be a $\mathcal{K}_1\mathcal{F}$ - group. For $M = \mathbb{F}_2$ this group provides a particularly favourable example for this program and was dealt with in [H3] and [H5].

The first three steps do not involve the space X explicitly and are relatively easy. They were carried out in [H3]; over $\mathbb{Z}[1/2]$ all elementary abelian 2 - subgroups in $SL(n, \mathbb{Z}[1/2])$ are diagonalizable, the centralizers in case $n = 3$ are small and their cohomology was known. The 2 - rank is 2, so the spectral sequence has only two columns ($s = 0, 1$) and degenerates into a short exact sequence

$$0 \longrightarrow \Sigma \lim_{\mathcal{A}_p^*(G)}^1 H^*(BC_G(E); \mathbb{F}_2) \longrightarrow H_G^*(X_s; \mathbb{F}_2) \longrightarrow \lim_{\mathcal{A}_p^*(G)} H^*(BC_G(E); \mathbb{F}_2) \longrightarrow 0 .$$

Here Σ denotes the suspension functor and appears here in order to make all maps degree preserving. The inverse limit turns out to be isomorphic to $\mathbb{F}_2[w_2, w_3] \otimes \Lambda(e_3, e_5)$, the tensor product of a polynomial algebra on the universal Stiefel Whitney classes w_2 and w_3 (coming from the embedding of $\mathbb{Z}[1/2] \subset \mathbb{R}$) and an exterior algebra on classes e_3 and e_5 of dimension 3 and 5 respectively. The \lim^1 - term is non-trivial; it is isomorphic to $\Sigma^3\mathbb{F}_2$.

The last step of the program was carried out in [H5]; this was quite involved and required a careful analysis of the action of $SL(3, \mathbb{Z}[1/2])$ on a suitable space X ; such a space is given by the product $X_2 \times X_\infty$ where X_2 denotes the Bruhat-Tits-building for $SL(3, \mathbb{Q}_2)$ and X_∞ the symmetric space $SL(3, \mathbb{R})/SL(3, \mathbb{Z})$. The relative cohomology turned out to compensate the contribution coming from the \lim^1 - term and the final result can be stated as follows.

Theorem 38 [H5]. *The map ρ induces an isomorphism*

$$H^*(BSL(3, \mathbb{Z}[1/2]); \mathbb{F}_2) \cong \mathbb{F}_2[w_2, w_3] \otimes \Lambda(e_3, e_5) .$$

Furthermore the restriction map to the diagonal matrices in $SL(3, \mathbb{Z}[1/2])$ is injective.

Example. Another interesting application of Theorem 33 (which does not strictly follow the program outlined above) occurs in joint work in progress with J. Lannes. We study the mod - 2 cohomology of the orthogonal group (with respect to the standard bilinear form) $O(n, \mathbb{Z}[1/2])$ and show that the reduction map to $O(n, \mathbb{F}_3)$ induces an isomorphism in mod - 2 cohomology for $n \leq 14$. The groups $O(n, \mathbb{Z}[1/2])$ belong to the class $\mathcal{K}_1\mathcal{F}$ and in the range $n \leq 14$ we show that $X = X_s$ for a suitable “building” which serves as a $\mathcal{K}_1\mathcal{F}$ - complex for $O(n, \mathbb{Z}[1/2])$. This together with Theorem 33 enables us to prove by induction that the reduction map induces an isomorphism for $n \leq 14$.

Similarly, if $O'(n, \mathbb{Z}[1/2])$ and $O'(n, \mathbb{F}_3)$ denote the orthogonal groups for the bilinear form $(x, y) \mapsto \sum_{i < n} x_i y_i + 2x_n y_n$, then the reduction map induces an isomorphism for $n \leq 13$ but not for $n = 14$. This can be used to show that the restriction map in mod - 2 cohomology from $GL(n, \mathbb{Z}[1/2])$ to its subgroup D_n of diagonal matrices is not injective for $n \geq 14$ thus improving on Dwyer’s result [Dw2] which required $n \geq 32$ (see Example (a) after Proposition 29).

The “centralizer spectral sequence” has certain similarities with a “normalizer spectral sequence” in Farrell cohomology for groups of finite virtual cohomological dimension. Recall that above the virtual cohomological dimension Farrell cohomology (with any coefficients) agrees for such groups with ordinary cohomology. Furthermore there is a spectral sequence whose E_1 - term is given by the Farrell cohomology of intersections of normalizers of elementary abelian p - subgroups of G and which converges to the Farrell cohomology of G [Br]. The relation between the two constructions leading to these spectral sequences has been investigated in work of [S1] and more recently [Dw1]. The input in both spectral sequences is different; in the case of the centralizer spectral sequence it has a homological nature (namely derived functors of \lim) and we believe that it is therefore easier to handle. The example of $SL(3, \mathbb{Z}[1/2])$ and $p = 2$ gives some evidence to this claim.

Given that the two spectral sequences abut to the same object (at least in large dimensions) one starts wondering whether they might agree from some stage on. There is a bit of experimental evidence in this direction. E.g. in the case of $SL(3, \mathbb{Z}[1/2])$ the two spectral sequences agree in large total degree from E_2 on. Results by Chun Nip Lee [Le] on the edge homomorphism of the spectral sequence give some more evidence.

We summarize by expressing our belief that the “centralizer spectral sequence” is more effective for cohomology with trivial coefficients \mathbb{F}_p . Its E_2 - term is more conceptual and its abutment is more geometrical (the cohomology of the Borel construction $EG \times_G X_s$ vs. the Farrell cohomology whose geometrical interpretation in low dimension is less clear). In fact, this geometric information was quite useful in computing the low dimensional cohomology of $SL(3, \mathbb{Z}[1/2])$ [H5].

In the case of a coefficient module M with non-trivial G - action the centralizer spectral sequence still converges to $H^*(BG; M)$ in sufficiently large dimensions [Dw3]. However, it is not clear whether there are still vanishing results for the E_2 -term in the spirit of Theorem 35.

REFERENCES

- [AGM] J. F. Adams, J. H. C. Gunawardena and H. R. Miller, *The Segal conjecture for elementary abelian p -groups*, *Topology* **24** (1985), 435–460.
- [AM] A. Adem and R. J. Milgram, *Cohomology of finite groups*, Grundlehren der mathematischen Wissenschaften 309, Springer Verlag, 1994.
- [AC] G. S. Avrunin and J. F. Carlson, *Nilpotency degree of cohomology rings in characteristic 2*, *Proc. AMS* **118** (1993), 339–343.
- [Be] D. Benson, *Representations and cohomology II: Cohomology of groups and modules*, Cambridge studies in advanced mathematics 31, Cambridge University Press, 1991.
- [BK] A. K. Bousfield and D. M. Kan, *Homotopy limits, completions and localizations*, Lecture Notes in Math. 304 Springer, 1972.
- [Br] K. Brown, *Groups of virtually finite dimension*, London Math. Soc. LNS 36, Homological Group Theory (edited by C.T.C. Wall), Cambridge University Press, 1979, pp. 27–70.
- [CaH] J. F. Carlson and H.-W. Henn, *Cohomological detection and regular elements in group cohomology*, *Proc. Amer. Math. Soc.* **124** (1996), 665–670.
- [C] G. Carlsson, *G. B. Segal's Burnside ring conjecture for $(\mathbb{Z}/2)^k$* , *Topology* **22** (1985), 83–103.
- [CV] M. Culler and K. Vogtmann, *Moduli of graphs and automorphisms of free groups*, *Invent. math.* **84** (1986), 91–119.
- [Du1] J. Duflo, *Smooth toral actions*, *Topology* **284** (1983), 253–265.
- [Du2] J. Duflo, *Localization of equivariant cohomology rings*, *Trans. Amer. Math. Soc.* **284** (1984), 91–105.
- [Dw1] W.G. Dwyer, *Sharp homology decompositions for classifying spaces of finite groups*, Preprint 1996.
- [Dw2] W.G. Dwyer, *Exotic cohomology for $GL_n(\mathbb{Z}[1/2])$* , Preprint 1996.
- [Dw3] W.G. Dwyer, *private communication*.
- [DW1] W. Dwyer and C. Wilkerson, *A cohomology decomposition theorem*, *Topology* **31** (1992), 433–443.
- [DW2] W. G. Dwyer and C. W. Wilkerson, *Homotopy fixed point methods for Lie groups and finite loop spaces*, *Annals of Mathematics* **139** (1994), 395–442.
- [DW3] W. Dwyer and C. Wilkerson, *Smith theory and the functor T* , *Comm. Math. Helv.* **66** (1991), 1–17.
- [E1] L. Evens, *The cohomology ring of a finite group*, *Trans. AMS* **101** (1961), 224–239.
- [E2] L. Evens, *The cohomology of groups*, Oxford Mathematical Monographs, Clarendon Press, 1991.
- [Ga] P. Gabriel, *Des catégories abéliennes*, *Bull. Soc. Math. Fr.* **90** (1962), 323–448.
- [Gr] A. Grothendieck (notes by R. Hartshorne), *Local cohomology*, Springer Lecture Notes in Math. 41, 1967.
- [GH] E. Ghys and P. de la Harpe, *Sur les Groupes Hyperboliques d'après Mikhail Gromov*, Progress in Mathematics 83, Birkhäuser, 1990.
- [GLZ] J. Gunawardena, J. Lannes and S. Zarati, *Cohomologie des groupes symétriques et application de Quillen*, *Advances in Homotopy Theory*, London Math. Soc., Lect. Notes Series **139**, 61–68.
- [Ha] J. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, *Invent. math.* **84** (1986), 157–176.
- [H1] H.-W. Henn, *Classifying spaces with injective mod - p cohomology*, *Comm. Math. Helv.* **64** (1989), 200–206.
- [H2] H.-W. Henn, *Commutative algebra of unstable K - modules, Lannes' T - functor and equivariant mod - p - cohomology*, *Journal für die reine und angewandte Mathematik* **478** (1996), 189–215.
- [H3] H.-W. Henn, *Centralizers of elementary abelian p - subgroups, the Borel construction of the singular locus and applications to the cohomology of discrete groups*, *Topology* **36** (1997), 271–286.
- [H4] H.-W. Henn, *Centralizers of elementary abelian p - subgroups and mod - p cohomology of profinite groups*, *Duke Mathematical Journal* **91** (1998), 561–585.
- [H5] H.-W. Henn, *The cohomology of $SL(3, \mathbb{Z}[1/2])$* , Heft Nr. **136** Forschergruppe Topologie und nichtkommutative Geometrie Univ. Heidelberg, Mai 1996 (to appear in K - theory).

- [H6] H.-W. Henn, *Unstable modules over the Steenrod algebra and cohomology of groups*, Group Representations: Cohomology, group actions and topology. Proceedings of Symposia in Pure Mathematics **63**, 277-300.
- [H7] H.-W. Henn, *Finiteness Properties of injective resolutions of certain unstable modules over the Steenrod algebra and applications*, Math. Ann. **291** (1991), 191-203.
- [HLS1] H.-W. Henn, J. Lannes and L. Schwartz, *Analytic functors, unstable algebras and cohomology of classifying spaces*, Contemp. Mathematics **96** (1989), 197-220.
- [HLS2] H.-W. Henn, J. Lannes and L. Schwartz, *The categories of unstable modules and unstable algebras modulo nilpotent objects*, American Journal of Math. **115** (1993), 1053-1106.
- [HLS3] H.-W. Henn, J. Lannes and L. Schwartz, *Localization of unstable A - modules and equivariant mod - p cohomology*, Math. Ann. **301** (1995), 23-68.
- [IK] K. Inoue and A. Kono, *Nilpotency of a kernel of the Quillen map*, J. Math. Kyoto Univ. **33** (1993), 1047-1055.
- [JM] S. Jackowski and J. McClure, *Homotopy decomposition of classifying spaces via elementary abelian subgroups*, Topology **31** (1992), 113-132.
- [JMO] S. Jackowski, J. McClure and R. Oliver, *Homotopy theory of classifying spaces of compact Lie groups*, Algebraic topology and its applications, Springer-Verlag, 1994, pp. 81-123.
- [K] P. Kropholler, *On groups of type $(FP)_{\infty}$* , Journal of pure and applied algebra **90** (1993), 55-67.
- [Ln1] J. Lannes, *Sur la cohomologie modulo p des p -groupes abéliens élémentaires*, Proceedings of the Durham Symposium on Homotopy theory 1985, Cambridge University Press 1987.
- [Ln2] J. Lannes, *Cohomology of groups and function spaces*, Preprint 1986.
- [Ln3] J. Lannes, *Sur les espaces fonctionnels dont la source est le classifiant d'un p -groupe abélien élémentaire*, Publ. Math. IHES **75** (1992), 135-244.
- [LS] J. Lannes and L. Schwartz, *Sur la structure des A - modules instables injectifs*, Topology **28** (1989), 153-169.
- [LZ] J. Lannes and S. Zarati, *Sur les \mathcal{U} -injectifs*, Ann. Scient. Ec. Norm. Sup. **19** (1986), 303-333.
- [Lz] M. Lazard, *Groupes analytiques p - adiques*, Publ. Math. IHES **26** (1965).
- [Le] C. N. Lee, *Farrell cohomology and centralizers of elementary abelian p - subgroups*, Preprint.
- [M] H. Miller, *The Sullivan Conjecture on maps from classifying spaces*, Ann. of Math. **120** (1984), 39-87.
- [O] R. Oliver, *Higher limits via Steinberg representations*, Comm. in Algebra **22** (1994), 1381-1393.
- [Q1] D. Quillen, *The spectrum of an equivariant cohomology ring I, II*, Ann. of Math. **94** (1971), 549-572, 573-602.
- [Q2] D. Quillen, *On the cohomology and K - theory of the general linear groups over a finite field*, Ann. of Math. **96** (1972), 552-586.
- [Sc] L. Schwartz, *Unstable modules over the Steenrod algebra and Sullivan's fixed point set conjecture*, University of Chicago Press, 1994.
- [Se] J. P. Serre, *Arithmetic groups*, London Math. Soc. LNS 36, Homological Group Theory (edited by C.T.C. Wall), Cambridge University Press, 1979, pp. 105-136.
- [Sl] J. Slomińska, *Homotopy colimits on E - I -categories*, Algebraic Topology, Poznań 1989 Proceedings, Springer Lecture Notes in Math. 1474, 1991.
- [SE] N. Steenrod and D. B. A. Epstein, *Cohomology operations*, Ann. of Math. Studies 50, Princeton University Press, 1962.
- [V] B. B. Venkov, *Cohomology algebras for some classifying space*, Dokl. Akad. Nauk SSSR **127** (1959), 943-944.