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ABOUT THE ERDÖS PAIRS

Marie-Paule MULLER

Abstract

- 1) A simpler and more transparent proof is given for earlier results by B. Bollobas and J. Jin about the partitions of an interval of integers into two parts, related to a Ramsey-type problem of Erdős.
- 2) A new construction allows to improve some of their results for the problem of Erdős in the more general case of partitions into an arbitrary number of parts.

Résumé

- 1) On donne une preuve plus simple et plus transparente de résultats de B. Bollobas et J. Jin répondant à un problème de P. Erdős sur les partitions en deux parties d'un intervalle d'entiers.
- 2) Une nouvelle construction permet d'améliorer leurs résultats pour le problème d'Erdős dans le cas plus général de partitions en un nombre quelconque de parties.

AMS Subject Classification : 11 B 75 (05 A 16, 05 D 99)

Introduction.

A few years ago, Paul Erdős posed a problem about the partitions of a sequence of integers in the following terms.

Let us consider two natural integers $m < n$, and a partition into two parts A_1 and A_2 of the sequence $[m, n-1] = \{m, m+1, \dots, n-1\}$ of all the integers from m to $n-1$. We shall call it an *Erdős partition* if there are, in at least one part, some (distinct) elements whose sum is equal to n . If every partition of $[m, n]$ is an Erdős partition, (m, n) is an *Erdős pair*.

If (m, n) is an Erdős pair, then (m', n) for $m' < m$, and also (km, kn) for every positive integer k , are clearly Erdős pairs (it is enough to consider induced partitions). On the other hand, $(1, n)$ is an Erdős pair if (and only if) $n \geq 12$ (the remarks of section 1 below enable us to check this easily). So we have, for fixed $n \geq 12$, a maximal value $l_2(n)$ for m such that (m, n) is an Erdős pair, then a number $l_2 = \liminf (l_2(n)/n)$ when n tends to infinity. The problem posed by Paul Erdős in 1992 was to determine the number l_2 . In fact, it is quite easy to convince oneself that $l_2 \leq 1/4$. B. Bollobas and J. Jin [B-J] proved that $l_2 = 1/4$, giving the value of $l_2(n)$ for n large enough (however without precising a bound).

The first aim of the present note is to give a simpler and more “visual” proof of these results ⁽¹⁾. Moreover, the method which is used here allows us to determine easily the exact bound beyond

¹ Had I known about the paper of B. Bollobas et G. Jin, I would certainly not have started this work ; so, many thanks to my colleague D. Dumont for having told me about the Erdős problem before, and about the reference [B-J] afterwards !

which the general expression of $l_2(n)$ is valid.

Some sections (in particular lemma 1 and its corollary 1) are presented so that they can be applied to the more general case of partitions into p parts of $[m, n[$. For $l_p(n)$ and its inferior limit l_p defined in the obvious way, the conjecture is that $l_p = 1/2p$ (see [B-J]).

The second aim of this note is to present (in section 4) a new construction which gives support to this conjecture and allows us to improve the related results of [B-J]: we show that $l_p(n) < n/2p$ for every n , hence $l_p \leq 1/2p$.

The subject discussed here is linked to the general theme of the Ramsey-type problems (see [G-R-S]) in additive number theory. Among the classical results to which it is related, let us mention those of Davenport [D], Erdős and Heilbronn [E-H]. More recent results have been established by Bollobas, Erdős and Jin ([B-E-J], [B-J 2]).

I am very grateful to Vilmos Komornik for his encouragement and his interest in this work.

1. A few simple remarks.

Let us search for a 2-partition (two parts only) $[m, n[= A_1 \cup A_2$ which is not of Erdős. Hence we exclude the partitions for which some part certainly contains elements whose sum is n . It is natural to exclude first those for which some A_i contains *two* elements whose sum is n . We can easily “see” these partitions: let us display the numbers $m, \dots, n-1$ so that two numbers whose sum is n face each other:

if $n = 2p+1$:

		m	$m+1$...	k	...	$p-1$	p	
	$n-1$...	$n-m$	$n-m-1$...	$n-k$...	$p+2$	$p+1$

if $n = 2p+2$:

		m	$m+1$...	k	...	$p-1$	p	$p+1$
	$n-1$...	$n-m$	$n-m-1$...	$n-k$...	$p+3$	$p+2$

Thus, the partitions first excluded are those for which some complete column is contained in a part A_i . Notice that the elements in the incomplete columns on the left cannot occur in a sum equal to n , hence no matter the parts to which they belong.

The display chosen above for the numbers $[m, n[$ also enables us to locate qualitatively three elements or more whose sum is n : we can see immediately that at most one of these elements is in the second row, and that this element must be on the right of the others:

* * *

*

When the 2-partition is not of Erdős, if two numbers u and v are in the same part A_i and if $n-u-v$ is distinct from u and v , then $n-u-v$ is not in A_i . In particular, if $u+v$ is in a column which is complete and distinct from those of u and v , then $u+v$ is also in A_i : otherwise, the element $n-u-v$ which faces it would be in A_i , and this is impossible.

These remarks allow us to prove easily that, for example, $(1, n)$ is an Erdős pair if (and only if) $n \geq 12$; or that $(2, n)$ is an Erdős pair for $n = 15$ or $n \geq 17$.

2. Flips and pincers.

Definition. Given a partition of $[m, n[$, let us call a *flip* a sequence of two consecutive numbers which do not belong to the same part.

If the partition is not of Erdős, we know that the sequence $[p, p+1, p+2]$ necessarily contains a flip, since p faces either $p+1$ or $p+2$ (according to the parity of n), and two numbers with sum n do not belong to the same part.

This remark can be generalized in the following way.

Lemma 1. For $j = 2, 3, 4, \dots$ with $j < n/m$, there is, surrounding n/j , a sequence of $j+1$ consecutive integers which contains j elements whose sum is n . With the convention that the smallest number of the sequence must figure in that sum, this smallest number is equal to the integer part of $n/j - (j-1)/2$.

Proof. Denote by q the integer part of n/j (resp. $n/j + 1/2$) if j is odd (resp. even). The sequence of length $j+1$ which is centered in $q+1/2$ (resp. q) contains j integers whose sum is n ; the element which must be omitted in the sum is determined by the class of $n \pmod{j}$. Let us specify that the last element (i.e. the $(j+1)^{\text{th}}$) of the sequence has to be omitted when n (resp. $n + j/2$) is a multiple of j , with j odd (resp. even) : thus, in that case, we have j consecutive integers whose sum is n .

Notation. Let us denote by S_j the sequence of length $j+1$ or j (according to the case) which has been described in the proof of lemma 1 : it is the shortest sequence which contains j elements with sum n .

Example ($j = 2$) : $S_2 = [p, p+1, p+2]$ if n is even, and $S_2 = [p, p+1]$ if n is odd.

Corollary 1. If $[m, n[$ is equipped with a partition which is not of Erdős, then in each sequence S_j contained in $[m, n[$ there is at least one flip.

Let us notice that asymptotically, i.e. for n large enough with the ratio n/m fixed, these sequences are contained in $[m, n[$ and are disjoint.

Definition. Given a partition of $[m, n[$, let us say that four elements $x_1 < x_2 < y_2 < y_1$ form *pincers* if :

- i) x_i and y_i belong to a same part A_i (with distinct A_1 and A_2)
- ii) $x_1 + y_1 = x_2 + y_2 \leq n-m$
- iii) the column of $x_i + y_i$ is distinct from those of x_1, y_1, x_2, y_2 .

3. "Pincers extraction" of Erdős 2-partitions.

In this section we shall only consider 2-partitions (partitions into two parts only) of $[m, n[$. In this case, it is enough to find two "well placed" flips in order to make sure that the partition is of Erdős. Moreover, the sequences which contain a flip near to n/j (cf. lemma 1) can be shortened.

Lemma 2. Given a 2-partition which is not of Erdős, if two numbers x and y are in a same part A_i and if $x+y$ is in a column which is complete and distinct from those of x and y , then $x+y$ is also in A_i .

Proof. The part A_i does not contain the element $n-x-y$, thus it contains $x+y$.

Lemma 3. If a 2-partition contains pincers, then it is of Erdős.

Proof. Consider a set of pincers (x_1, x_2, y_2, y_1) . The element $n - (x_1 + y_1)$ forms, either with x_1 and y_1 , or with x_2 and y_2 , a triple whose sum is n in its part. By condition iii), these three elements are distinct.

The following task is to obtain effectively pincers from adequate flips. In fact, two flips which are respectively situated near $n/3$ and $n/4$ will suffice (for n large enough : $n \geq 56$), essentially because " $1/3 < 1/4 + 1/3 < 2/3$ ": as the elements x_1, x_2, y_2, y_1 apt to form pincers are near $n/4$ or $n/3$, the sum $x_1 + y_1 = x_2 + y_2$ will certainly be in a *distinct* column (cf. condition iii)), in fact in a column which is further right, as $x_i + y_i$ will be between $n/3$ and $2n/3$.

For the 2-partitions, the following lemma gives a sequence which is shorter than S_4 , so it will allow us to situate more precisely a flip near $n/4$.

Notation. Denote by V the integer part of $(n-3)/4$.

lemme 4. Assume that $n \geq 11$. Given a 2-partition of $[V, n[$, if the sequence $[V, V+1, V+2, V+3]$ is contained in a part, then the partition is of Erdős. In the case $n \not\equiv 2 \pmod{4}$, it is enough for $[V, V+1, V+2]$ to be contained in a part.

Proof. In fact, V is the integer part of $(p-1)/2$ (let us recall that p , which is the smallest element of S_2 , is itself the integer part of $(n-1)/2$). Hence, following lemma 1, $[V, V+1, V+2]$ contains two elements whose sum is p . As V must be one of these two elements (either $p = V+(V+1)$, or $p = V+(V+2)$), we can also find in $[V, V+1, V+2]$ two elements whose sum is $p+1$.

Suppose now that the partition is not of Erdős, the described sequence being contained in a part. If n is odd, we have $p+1 = n-p$; as p and $p+1$ are in the same part by lemma 2, we arrive to a contradiction. If n is even, we have $p+2 = n-p$; as $p+2$ is equal either to $(V+1)+(V+2)$ or to $(V+1)+(V+3)$ (according to the parity of p), lemma 2 leads once again to a contradiction.

The bound $n \geq 11$ traduces the condition that the column of p must be disjoint from the

considered sequence, in order to apply lemma 2 .

Remark 1 . As it is formulated, this proof may also be viewed as the starting point of a recursive process which could be useful for the study of the 3-partitions (of course, lemma 2 must be replaced by the corresponding lemma).

Remark 2 . In case $n \not\equiv 2 \pmod{4}$, the sequence $[V, V+1, V+2]$ of lemma 4 corresponds to S_4 deprived of its first and last elements : in this case, we see that the integer part of $(n-6)/4$ is equal to $V-1$. But in case $n \equiv 2 \pmod{4}$, lemma 4 does not improve the result of lemma 1 : the integer parts of $(n-6)/4$ and $(n-3)/4$ are equal, thus $[V, V+1, V+2, V+3] = S_4$ in this case.

Notation. Denote by S'_4 the sequence of length 3 or 4 (according to the case) which has been described in lemma 4 .

Definition. A flip $[x, x+1]$ is said to be of *type 12* if x is an element of the part A_1 and $x+1$ of A_2 . More generally, the *type* of a given sequence is the corresponding sequence of the indices of the parts which contain the given elements.

The following lemma may be seen as a complement to lemma 4 . Picking out a few other cases which force the 2-partition to be of Erdős, it will allow us to precise the values of n such that the pair (V, n) is of Erdős.

Lemma 5. Consider the sequence of the first four or five (according to the case) elements of $[V, n[$, with $n > 18$. If its type falls within one of the following cases, then the 2-partition is of Erdős :

for $n \equiv 0 \pmod{4}$, type 11#1

for $n \equiv 1 \pmod{4}$, type 11#1, 1#11 or 11##1

for $n \equiv 2 \pmod{4}$, type 11##1 or 1#1#1

where # is a generic character for 1 or 2 .

Proof. Similar to the proof of lemma 4 . Apply lemma 2 , after having noticed that two elements which face each other (either p and $n-p$, or $p-1$ and $n-p+1$, according to the case) can be obtained as sums of elements belonging to the same part.

Theorem. For $n \geq 56$, every 2-partition of $[m, n[$ with $m \leq V$ is of Erdős.

($n = 55$, $m = V$) There is a 2-partition of $[13, 55[$ which is not of Erdős.

For every n , there is a 2-partition of $[V+1, n[$ which is not of Erdős.

Proof of the theorem. Let us suppose that the partition induced on $[V, n[$ is not of Erdős.

By lemma 4 and corollary 1, there is at least one flip in each of the sequences S'_4 and S_3 . We shall distinguish two cases, depending on their types.

Let U be the smallest element of S_3 . Recall that V is the integer part of $(n-3)/4$, and that U is the integer part of $n/3 - 1$ (lemma 1) .

Case 1. The sequences S'_4 and S_3 contain respectively flips with different types.

First, let us make sure that S'_4 and S_3 are disjoint (provided that $n \geq 42$). If $n \not\equiv 2 \pmod{4}$, we see immediately that $V+3 \leq U$. If $n \equiv 2 \pmod{4}$, then $V = (n-6)/4$, hence $V+4 \leq U$.

Hence, the two flips provide four elements $x_1 < x_2 < y_2 < y_1$ which apply for forming pincers. It only remains to verify condition iii), in order for $n - (x_i + y_i)$ to be distinct from x_1, x_2, y_2, y_1 . In fact, we shall prove that $x_i + y_i$ is too small for its column to reach y_1 (equivalently, $x_i + y_i$ is on the right of y_1), which is expressed by the inequality: $x_1 + 2.y_1 < n$. The most "unfavourable" situation is the one in which x_1 and y_1 are as large as possible, the flips being situated at the ends of the sequences: let us say S'_4 of type 1112 or 112, together with S_3 of type 2221 or 221, according to the class of $n \pmod{12}$. The pincers condition iii) is expressed by an inequation involving V, U and n . Now, the expressions of V and U as functions of n provide a bound which is a sufficient condition: $n \geq 47$ is valid no matter the class of $n \pmod{12}$. Let us summarize this little study in the following table:

type of S'_4	type of S_3	pincers condition iii)	sufficient condition obtained
1112	221 ($n \equiv 0 \pmod{3}$)	$V+2U+6 < n$	$(n-3)/4 + 2n/3+4 < n : n > 39$
112	"	$V+2U+5 < n$	" "
112	2221 ($n \not\equiv 0 \pmod{3}$)	$V+2U+7 < n$	$(n-3)/4 + 2(n-1)/3+5 < n : n > 43$
1112 ($n \equiv 2 \pmod{4}$)	"	$V+2U+8 < n$	$(n-6)/4 + 2(n-1)/3+6 < n : n > 46$

Case 2. The sequences S'_4 et S_3 do not contain flips of distinct types.

In this case, there is one (unique) flip in each sequence; let us say, of type 12. The fact that the two sequences are disjoint does no longer ensure the existence of pincers: it is enough to concatenate S'_4 of type 1112 with S_3 of type 1222 in order to be convinced of that.

First, notice that S'_4 and S_3 are not contiguous as soon as $n \geq 54$ (we have $V+4 \leq U$, or $V+5 \leq U$, according to the length of S'_4).

i) If there is an element of type 2 between the two flips, then the flip of S'_4 gives us the first two elements of pincers whose fourth element is, at most, equal to U : therefore, the pincers condition iii) is evidently fulfilled.

Here are some examples: [1112]2[1222], [1122][1222], [112]221.. [1112].

ii) If there is no element of type 2 between the two flips and if S'_4 and S_3 are not contiguous, then the configuration is of type [112]1... [1..2]. Four elements are likely to form pincers (of type 2,1,1,2, now). The most "unfavourable" situation is the one in which the flip of S_3 is at the end of the sequence: the extremal elements of the pincers are the last elements of S'_4 and S_3 . It is enough to adapt the study which has been made in case 1. It may be noticed that the configuration [1112]1... (for $n \equiv 2 \pmod{4}$) has not even to be studied anymore: it is of Erdős by lemma 5.

type of S'_4	type of S_3	pincers condition iii)	sufficient condition obtained
112	112 ($n \equiv 0 \pmod{3}$)	$V+2U+6 < n$	$(n-3)/4 + 2n/3+4 < n : n > 39$
112	1112 ($n \not\equiv 0 \pmod{3}$)	$V+2U+8 < n$	$(n-3)/4 + 2(n-1)/3+6 < n : n > 55$

Finally, taking into account all the preceding considerations, we obtain the bound $n \geq 56$, which is valid in any case.

The previous analysis may also be exploited in order to construct a non-Erdős partition of

[13, 55[. As a consequence of both case 2 above and lemma 2, the partition is (and must be) of the following type :

	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
type:	1	1	2	1	1	1	1	2	2	2	2	2	2	2	1
...	42	41	40	39	38	37	36	35	34	33	32	31	30	29	28
type: ...	2	2	1	2	2	2	2	1	1	1	1	1	1	1	2

It is easy to give a 2-partition of $[V+1, n[$ which is not of Erdős. But a more general construction will be presented in the next section.

4. Non-Erdős p-partitions.

The following construction allows us to obtain very simply, for every integer n and whatever the number of parts p , a non-Erdős p -partition of $[m, n[$, provided that $m \geq n / 2p$.

Notice that the example for $p=3$ and n divisible by 60 of [B-J] is a product of a different process (besides it has to be slightly modified in order to be conclusive).

We begin with the study of p -partitions of an interval $[m/n, 1[$ of *real numbers*.

Proposition 1. There exists a non-Erdős p -partition of the real interval $[1/2p, 1[$: none of the p parts contain a finite subfamily whose sum is equal to 1.

Proof. The parts A_1, \dots, A_p are constructed in order for the interior of each of these parts to be stable by addition. Moreover, the parts to which the boundary points will be attributed must be chosen adequately. The choice which is made below will allow to improve (very slightly) the bound m when the proposition will be applied to the p -partitions of the integers $[m, n[$.

Let us define the following intervals, where i ($1 \leq i \leq p$) is a counter for the parts, and k ($k \geq 0$) is a counter for the connected components of each part :

- for $k = 0$,

$$J_i^0 = \begin{cases} \left[\frac{1}{2^p}, \frac{1}{2^{p-1}} \right] & \text{if } i = 1 \\ \left] \frac{1}{2^{p-i+1}}, \frac{1}{2^{p-i}} \right[& \text{if } 2 \leq i \leq p \end{cases}$$

- for $k \geq 1$ with $2^k \leq 2^{p-i}$,

$$J_i^k = \begin{cases} \left] \frac{2^k}{2^{p-i+1}}, \frac{2^k}{2^{p-i}} \right] & \text{if } \frac{2^k}{2^{p-i}} \leq \frac{1}{2} \\ \left] \frac{2^k}{2^{p-i+1}}, \frac{2^k}{2^{p-i}} \right[& \text{if } \frac{2^k}{2^{p-i+1}} = \frac{1}{2} \\ \left[\frac{2^k}{2^{p-i+1}}, \frac{2^k}{2^{p-i}} \right[& \text{if } \frac{2^k}{2^{p-i+1}} > \frac{1}{2} \end{cases}$$

Let A_i be the union of the J_i^k ($k \geq 0$). In short, if we leave aside the boundary points, a partition of $]1/2p, 1/p[$ is extended homothetically to $]1/2p, 1[$.

An elementary checking may (and has to) be made : $1/2$ is a boundary-point.

i) There are a value $k = K$, and a value $i = I$, such that

$$\frac{2^K}{2^{p-I+1}} = \frac{1}{2}$$

Indeed, let us write p to the base 2 : $p = b_K b_{K-1} \dots b_1 b_0$ (where $b_K = 1$) .

We have : $2^K \leq p \leq 1 + 2 + \dots + 2^K = 2^{K+1} - 1$, equivalently : $p < 2^{K+1} \leq 2p$, from where we obtain the desired value of i (with $1 \leq i \leq p$).

Of course, 1 is then a boundary-point too, and the partition is well-defined.

ii) Clearly, for elements taken in J_i^0 and not all of them identical, the sum is different from 1 .

iii) From i), ii) and from the choice which has been made at the boundary-points of the intervals J_i^k , we deduce that the sum of elements of A_i , all of them being distinct, is different from 1 . The only case where the checking is not trivial is the one with all the elements taking the form

$$\frac{2^k}{2^{p-i}}$$

(i.e. ends of intervals). Denoting by k_i the largest integer such that

$$\frac{2^{k_i}}{2^{p-i}} \leq \frac{1}{2}$$

(in fact, $k_i = K$ or $k_i = K-1$) , the sum of the given elements is at most

$$\frac{1}{2^{p-i}} \left(2^{k_i+1} - 1 \right) \leq \frac{1}{2^{p-i}} (2^{p-i} - 1) < 1$$

Notice that the case where all the elements are at the beginning of intervals is immediate

$$\left(\frac{1}{2^p} + \frac{2^k}{2^p} \neq 1 \text{ as } 1+2^k \text{ is odd} \right)$$

As an evident corollary of proposition 1 , we obtain

Proposition 2. If $m/n \geq 1/2p$, then there exists a non-Erdős partition into p parts of the interval of natural numbers $[m, n[$.

For some values of $n \bmod{2p}$, this bound obtained for m can be slightly improved (one unit less), as a result of the discretisation : among the reals, the only elements which are to be taken into account are the multiples of $1/n$, and they have to be situated in relation to $1/2p$.

For $p = 2$ for example :

- if $n = 0 \bmod{4}$ or $n = 3 \bmod{4}$, the bound which has been given by proposition 2 is optimal : the condition $m \geq n/4$ is equivalent to $m \geq V+1$ (cf. §3.)

- if $n = 1 \bmod{4}$, the smallest element of the interval J_1^0 is $(n+3)/4$, that is $V+2$ in the considered case ; it is straightforward to check that the partition remains non-Erdős if the

preceding integer $(n-1)/4$ is joined to A_1 .

- if $n = 2 \pmod{4}$, the smallest element of the interval J_1^0 is $V+2$ too . As before, the preceding integer, $(n-2)/4$, can be joined to A_1 .

Hence we have, for every value n , a non-Erdős 2-partition of $[V+1, n[$.

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