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ENTROPY RIGIDITY OF ANOSOV FLOWS IN DIMENSION 3

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Abstract

We show that for a smooth contact Anosov flow on a closed three manifold the measure of maximal entropy is in the Lebesgue class if and only if the flow is up to finite covers conjugate to the geodesic flow of a metric of constant negative curvature on a closed surface.This shows that the ratio between the measure theoretic entropy and the topological entropy of a contact Anosov flow is strictly smaller than one on any closed three manifold which is not a Seifert bundle.

0 Introduction

Given a compact connected manifold and a flow $\varphi_t, t \in \mathbb{R}$, we can consider its set $M(\varphi)$ of invariant probability measures. There is a welknown map $\mathrm{h} : M(\varphi) \to \mathbb{R}$ which associates to each invariant probability measure its entropy. This notion is very well adapted to the study of dynamics having a strong sensitivity to initial conditions such as the Anosov flows.

In this case it is known that the maximum of h, called topological entropy, is realized by a unique probability measure ν called the Bowen-Margulis measure. A natural question is : when is ν in the Lebesgue measure class ?

This question is particularly relevant for the geodesic flows of Riemannian metrics of negative curvature. The subject matter of one of the entropy rigidity conjectures is precisely that ν is in the Lebesgue class if and only if the metric is locally symmetric.

The geodesic flows, furthermore preserve the Liouville probability measure μ, whose corresponding entropy h_μ is often called the metric entropy.

In that context this entropy rigidity conjecture admits the equivalent statement : for negatively curved compact manifolds the metric entropy and the topological entropy are equal if and only if the metric is locally symmetric. This conjecture is still open, except in dimension two, when it was solved by A. Katok.

In order to get some rigidity results for Anosov flows, one has to assume furthermore that the flow is at least C^2. Otherwise for instance, for any given C^∞- geodesic flow, there always exists a time change such that, the Bowen-Margulis measure of the new flow is in the Lebesgue class [Pa-Po]. But then the reparametrised flow is not in the C^2-class.

I don't know what can happen in higher dimensions. It seems nevertheless reasonable to conjecture that the Bowen-Margulis measure of C^2- Anosov flows on compact three manifolds are in the Lebesgue class only for a well defined list of special cases. I could not achieve this in that generality. I will furthermore assume some regularity at infinity, namely, the associated canonical one form will be supposed to be continuously differentiable.

This regularity condition is fulfilled for geodesic flows of smooth Riemannian or Finsler metrics. More generally the set of such flows consists of constant time suspensions of Anosov diffeomorphisms of the torus and of the contact Anosov flows. Before stating the theorem, let us recall some notations, definitions and facts. In what follows we only consider C^∞ Anosov flows. We could as well consider C^2 regularity.

0.1 Let M be a C^∞-closed 3-manifold. The C^∞-flow, φ_t, generated by the non-singular smooth vector field X is an Anosov flow if there exists a flow invariant splitting of the tangent bundle

$$TM = \mathbb{R}X \oplus E^s \oplus E^u \ , \tag{1}$$

a Riemann metric on M and two positive real numbers a and b such that
(i) $\forall Z \in E^s, \quad \forall t \geq 0, \ ||T\varphi_t(Z)|| \leq ae^{-bt}||Z||$
(ii) $\forall Z \in E^u, \ \forall t \geq 0 \ , \ \ ||T\varphi_{-t}(Z)|| \leq ae^{-bt}||Z||.$ (2)

If the canonical invariant one form A, given by

$$A(E^s \oplus E^u) = 0, \quad A(X) = 1 \tag{3}$$

is a C^1-contact form, that is $A \wedge dA$ is a volume form, one says that the flow is of contact type. To be more precise the generator X is the Reeb field of the one-form A, because from (3) it satisfies

$$A(X) = 1, dA(X, .) = 0 \ .$$

0.2 In general the splitting (1) is not much regular than Hölder. But in dimension three [HP1] M. Hirsh and C. Pugh have shown that the stable (unstable)distributions $\mathbb{R}X \oplus E^s$ and $\mathbb{R}X \oplus E^u$ are C^1.

If one furthermore assumes that the flow is of contact type, then the splitting (1) is also C^1. All theses distributions are known to be integrable. Let W^s, W^u, W^{os}, W^{ou}, denote respectively the strong stable, strong unstable, stable, unstable foliations. For all these foliations the leaves are C^∞-immersions of either $\mathbb{R}^2$ or $\mathbb{R}$ such that their jets of all orders are transversely continuous.
Let us recall the definition of the topological entropy of a flow on a compact manifold. For any $\varepsilon > 0$ and $t \geq 0$ one consider the "ball"

$$B(x, t, \varepsilon) = \{y| \ 0 \leq \tau \leq t, d(\varphi_\tau(x), \varphi_\tau(y)) < \varepsilon\} \ .$$

Then let $N(\varepsilon, t)$ be the minimal number of such open sets needed to cover the manifold. The topological entropy is then given by the formula

$$h_{\text{top}} = \lim_{\varepsilon \to 0} \overline{\lim_{\text{t} \to \infty}} \frac{1}{\text{t}} \ \text{Log N}(\varepsilon, \text{t}) \ .$$

The entropy h_m, associated to an invariant probability measure m, is less geometrical, for its construction see for instance [M]. We know the following inequality

$$h_m \leq h_{\text{top}} \ . \tag{4}$$

The equality is achieved by only one probability measure ν, known as the Bowen-Margulis measure.

0.3 results

The main result of this paper is now.

Theorem.— *Let φ_t be a C^∞-contact Anosov flow on a closed 3-manifold M^3. If its metric entropy is equal to its topological entropy, then up to finite covers, the flow φ_t is C^∞-conjugate to the geodesic flow on a closed surface of constant negative curvature.*

In a forthcoming paper we prove the existence of C^∞-contact Anosov flows on closed 3 - manifolds which are not Seifert bundles. As a by product of the above mentioned theorem, we may remark that in this case, the Bowen-Margulis measure is not in the Lebesgue measure class and the ratio metric entropy / topological entropy is strictly smaller than one.

As previously said, the case of geodesic flows on surfaces was solved by A. Katok [Ka 1]. The methods involved are of a Riemannian nature as suggested by the title of his paper "four applications of conformal equivalence to geometry and dynamics". Here the main tool is the Livsĭc homological theorem, which plays a crucial role to show that under our assumptions, the densities of the Margulis measures [M] of the strong stable leaves are in fact differentiable.

Then, with some slight modifications, we can apply a theorem of L. Green to show that the manifold is homogeneous. We prefer to give an alternative proof which produces a $C^1 - \big(SL(2,\mathbb{R}), SL(2,\mathbb{R})\big)$ structure and a C^1-conjugacy. The nice regularity theorem for the conjugacy of Anosov flows in dimension 3 of [LM] provides finally the desired C^∞-regularity.

All these techniques are well-known by experts and, in a sense, this theorem could have been proved by any of them and may be considered as a folklore theorem. Anyway, one should make it public.

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1. The Margulis Measure

A contact Anosov flow is topologically mixing (see [Ha, K]). G.A. Margulis proved that, for a topologically mixing Anosov flow, there exists a unique (up to scalars) family ν^{ou} of measures supported by the weak unstable leaves such that

$$\nu^{ou} \circ \varphi_t = e^{ht} \nu^{ou} \tag{5}$$

and invariant by the holonomy of the strong stable foliation.
Similarly, there exists a family of measures ν^s supported by the strong stable leaves such that

$$\nu^s \circ \varphi_t = e^{-ht} \nu^s \ . \tag{6}$$

One can, roughly speaking, make the weighted product of these two measures, (see [Ha, K] or [M]). I need to repeat briefly this construction. Each individual leave of any of these invariant foliations is locally a smooth submanifold. Then, any Riemannian metric g on M induces a topology on the leaves. A subset A is

open in a leaf $W^s(p)$ if $A \subset W^s(p)$ is open for the previous topology. This pair of transversal foliations W^s and W^{os} has a local product structure. Each $p \in M$ has a neighbourhood $U(p)$ which is a local product cube. $U(p) = U^{ou}(p) \times U^s(p)$ with $U^{ou}(p)$ open in $W^{ou}(p)$, and $U^s(p)$ open in $W^s(p)$. Let $\Omega \subset U(p)$ be an open set. For $q \in U^{ou}(p)$, we introduce $\mathcal{F}^\nu_\Omega(q) := \nu^s\big(\{q\} \times U^s(p) \cap \Omega\big)$.
For $z \in U^s(p)$, $A \subset U^{ou}(p)$ open, let

$$\nu_z(A) := \nu^{ou}(A \times \{z\}) \quad .$$

One may observe that $\nu(\Omega) := \int \mathcal{F}^\nu_\Omega(x) d\nu_z(x)$ is well defined and independent of z. The independence with respect to z is achieved because the family of measure ν^{ou} is invariant by the holonomy maps of the strong stable foliation. The measure ν is flow invariant, and, when normalised to be a probability measure, it is the Bowen-Margulis measure.

$$\nu = \nu^s \times \nu^{ou} \tag{7}$$

A similar construction may be performed for any pair μ^{ou}, μ^s of family of Lebesgue measures. Again, the measure

$$\mu(\Omega) := \int \mathcal{F}^\mu_\Omega(x) d\mu_z(x)$$

is well-defined and independant of z because [An] the strong stable foliation is Lebesgue absolutely continuous.

With this background material we can state the first Lemma

Lemma 1.— *If the Bowen-Margulis measure is in the Lebesgue measure class, the Margulis measures ν^s and ν^{ou} are Lebesgue measures that is, there exists densities f^s, f^{ou} such that*

$$d\nu^s = f^s d\mu^s, \; d\nu^{ou} = f^{ou} d\mu^{ou} \tag{8}$$

Proof : Let us recall that a contact flow preserves the volume form $A \wedge dA$ and the associated Liouville probability measure μ.

One may assume that the foliation W^{ou} is orientable (otherwise one can take a two-sheeted cover). Let us choose some Riemannian metric. The unique vector field ξ^u, g-unitary, tangent to the leaves of W^u and giving the orientation, is furthermore C^1 regardless to the regularity of the Anosov splitting. We have the same regularity for the unique vector field ξ^s tangent to the leaves of W^s and such that

$$dA(\xi^s, \xi^u) = 1 \tag{9}$$

The restrictions to the leaves of W^s of the one form

$$\alpha_s = dA(\cdot, \xi^u)_{|E^s} \tag{10}$$

is nowhere singular and equips the leaves of W^s with a family of Lebesgue measures μ^s. The same thing holds for

$$\omega_{ou} = A(\cdot) \wedge dA(\xi^s, \cdot)_{|\mathbf{R}X \oplus E^u} \quad .$$

with the corresponding family of measures μ^{ou}. One may easily observe that

$$\mu = \mu^s \times \mu^{ou} \quad .$$

If we assume that the Bowen-Margulis measure is absolutely continuous with respect to the Liouville measure, then by ergodicity they coincide.

Let us now only observe that ν^s is absolutely continuous with respect to μ^s (a similar argument holds for the pair ν^{ou}, μ^{ou})

For this let $A \subset U^s(p)$ with $\nu^s(A) = 0$, and set $\Omega = U^{ou}(p) \times A$.

Using the absolute continuity of the strong stable foliation, we observe that for any $q \in U^{ou}(p)$,

$$\begin{aligned} \mathcal{F}^{\mu}_{\Omega}(q) &= \mu^s\big(\{q\} \times U^s(p) \cap \Omega\big) = \mu^s\big(\{p\} \times U^s(p) \cap \Omega\big) \\ &= \mu^s(A) = 0 \end{aligned}$$

and then $0 = \mu(\Omega) = \nu(\Omega)$.

This implies that $\mathcal{F}^{\nu}_{\Omega}(q) = 0, \nu^{ou}$-almost surely and then there exists $q_0 \in U^{ou}(p)$ such that $\nu^s\big((\{q_0\} \times A) \cap \Omega\big) = 0$. And we conclude by holonomy invariance.

Lemma 2.— *There exists a positive function $\lambda \in C^1(M \times \mathbb{R}; \mathbb{R})$ such that*

$$\frac{d(\mu^s \circ \varphi_t)(x)}{d\mu^s} = \lambda^{-1}(x,t) \tag{11}$$

Proof : It suffices to write the flow invariance of the unstable distribution E^u under the form

$$T\varphi_t(\xi^u) = \lambda(x,t)\xi^u\big(\varphi_t(x)\big) \ ,$$

to get the regularity of λ. The invariance of the contact form implies that for any $x \in M$ and any vector $Z^s \in T_x W^s$ one has

$$\begin{aligned} \varphi_t^* \alpha_s(Z^s) = \alpha_s\big(T\varphi_t(Z^s)\big) &= dA_{\varphi_t}(x)\Big(T\varphi_t(Z^s), \xi^u\big(\varphi_t(x)\big)\Big) \\ &= \frac{1}{\lambda(x,t)} dA_{\varphi_t}(x)\Big(T\varphi_t(Z^s), T\varphi_t(\xi^u(x))\Big) \\ &= \frac{1}{\lambda(x,t)} dA_x\big(Z^s, \xi^u(x)\big) \\ &= \frac{1}{\lambda(x,t)} \alpha_s(Z^s) \end{aligned}$$

which gives (11).

Lemma 3.— *The density of the Margulis measure ν^s satifies μ-a.e the homological equation*

$$f^s\big(\varphi_t(x)\big) = e^{-ht}\lambda(x,t) f^s(x) \quad . \tag{12}$$

We still know that the family of measures$\mu^s \circ \varphi_t, \mu^s, \nu^s \circ \varphi_t, \nu^s$ are absolutely continuous, one with respect to the other. We deduced (12) from (6), (8), (11) and

$$\frac{d\nu^s \circ \varphi_t}{d\mu^s \circ \varphi_t} = f^s(\varphi_t(x)) = \frac{d\nu^s \circ \varphi_t}{d\nu^s} . \frac{d\nu^s}{d\mu^s} \frac{d\mu^s}{d\mu^s \circ \varphi_t} \quad .$$

2. Regularity of the density.

The celebrated Livsic homological theorem immediately shows that the density f^s admits a representative b in $C^1(M,\mathbb{R})$. I just recall the statements. Given an $\mathbb{R}$-action φ over a manifold M, a $\mathbb{R}$-multiplicative cocycle is a map $C : M \times \mathbb{R} \to \mathbb{R}$ such that

$$C(x, t+\tau) = C\big(\varphi_t(x), \tau\big) \cdot C(x,t)$$

One says that C is a coboundary if there exists a boundary $b : M \to \mathbb{R}^*$ such that

$$C(x,t) = b\big(\varphi_t(x)\big)/b(x) \quad .$$

The Livsic homological theorem [Li] [LMM], now reads on.

Theorem (Livsic).— *Let φ_t be a C^2-Anosov flow, topologically transitive on a closed manifold M. For a cocycle C having a Hölder regularity, the following properties are equivalent*

i) For any periodic point of period T

$$C(x,T) = 1 \ .$$

ii) There exists a Hölder boundary b, such that

$$C(x,t) = b\big(\varphi_t(x)\big) b^{-1}(x) \ . \tag{13}$$

If furthermore φ_t preserves a Lebesgue measure m, then i) and ii) are equivalent to

iii) There exists a measurable function $f : M \to \mathbb{R}$ such that m-almost everywhere

$$C(x,t) \stackrel{m}{=} f\big(\varphi_t(x)\big) \ .f^{-1}(x)$$

And if iii) is fulfilled, and if C is C^k, $k \geq 1$, then b is C^k.

As announced, the theorem applied to the density f^s insures the existence of a positive function $b \in C^1(M, \mathbb{R}^+)$ representing it.

3. The local structure.

Let us consider the C^1-vector fields over M, $X^s = b^{-1}\xi^s \quad X^u = b\xi^u$. We have the following Lemma.

Lemma 4.— *The vector fields X, X^s, X^u satisfy the Lie bracket relations*

$$[X, X^u] = -hX^u, [X, X^s] = hX^s, \ [X^s, X^u] = -X \quad . \tag{14}$$

Proof : We will denote by $\varphi_t, \varphi_t^s, \varphi_t^u$ the corresponding associated flows.
For the first Lie bracket, it is enough to observe that

$$T\varphi_t\big(X^u(x)\big) = \frac{b(x)}{b\big(\varphi_t(x)\big)}\lambda(x,t)X^u\big(\varphi_t(x)\big) = e^{ht}X^u\big(\varphi_t(x)\big) \quad ,$$

where b is the C^1 solution of (11). The argument for the second bracket is similar but with (6). The third bracket is well defined and if we use the two preceding arguments, we observe that the C^0-vector field $[X^s, X^u]$ is flow invariant due to,

$$\varphi_t^*\big([X^s, X^u](x)\big) = [\varphi_t^* X^s, \varphi_t^* X^u] = [e^{-ht}X^s, e^{ht}X^u] = [X^s, X^u] \quad .$$

Since the flow φ_t is Anosov, the invariance implies the existence of a real k such that

$$[X^s, X^u] = kX \quad .$$

The constant k is, in fact, equal to 1, due to (9).
From this point, one can use the paper of L.W. Green [Gr] to obtain a C^1-conjugacy with an algebraic flow and jump directly to the last paragraph.
In what follows, we give a direct proof of the conjugacy.

To describe the corresponding local structure, one considers a point p, the flow orbit γ crossing through p, the strong stable leave $W^s(p)$ and for any $y \in W^s(p)$ the strong unstable leave $W^u(y)$.

We choose three vector fields X_0, X_0^s, X_0^u, over the group $SL(2,\mathbb{R})$, satisfying (14). We also denote by $\psi_t, \psi_t^s, \psi_t^u$, the corresponding one parameter subgroups.

Lemma 5.— *For any point p in M, there exists $\epsilon > 0$ for which the connected open neighbourhood U of the identity in $SL(2,\mathbb{R})$, defined by*

$$U = \{g \in SL(2,\mathbb{R}), g = exp\, tX_0.exp\, t_sX^0.exp\, t_uX_0^u, for\, (t,t_u,t_s) \in]-\epsilon, +\epsilon[^3\} \ , \tag{15}$$

is such that the map

$$F : U \to M \ , \ F(g) = \varphi_{t_u}^u \circ \varphi_{t_s}^s \circ \varphi_t(p)$$

is a C^1 diffeomorphism on its image V. Furthermore, F locally conjugates the following flows

i) ψ^u and φ^u ,
ii) ψ and φ ,
iii) ψ^s and φ^s.

Proof : The map F is well defined and C^1. From the definition one has

$$TF_{Id}(X_0^{\cdot,u,s}) = (X^{\cdot,u,s})_p \quad , \tag{16}$$

and then for some ϵ, F is a local diffeomorphism by the inverse mapping theorem. Remark that any $g \in U$ may be expressed as

$$g = \psi^u_{t_u} \circ \psi^s_{t_s} \circ \psi_t(Id)$$

This shows i) because, for any $g \in U$, $\exists\ \eta(g) > 0$ such that $\forall\ t'_u \in]-\eta(g), \eta(g)[$,

$$\begin{aligned} F \circ (\psi^u_{t'_u}.(g)) &= F(\psi^u_{t'_u+t_u} \circ \psi^s_{t_s} \circ \psi_t(Id)) \\ &= \varphi^u_{t_u+t'_u} \circ \varphi^s_{t_s} \circ \varphi_t(p) \\ &= \varphi^u_{t'_u}(F(g)) \ . \end{aligned}$$

ii) Let us introduce the set $S_0 = \{g \in U, (t, t_u, t_s) \in]-\epsilon, \epsilon[^2 \times \{0\}\}$. The set $S = F(S_0)$ is a connected submanifold and by the definition of F, S $\subset W^{os}$. The same argument shows that for $g \in S_0$ there exists η such that for $t'_s \in]-\eta, \eta[$ one has

$$F(\psi^s_{t'_s}(g)) = \varphi^s_{t'_s}(F(g)) \ .$$

This in turn may be rephrased by introducing $f = F|_{S_0}$. The restrictions $\overline{X}^s_0$ of the vector field X^s_0 to S_0 is tangent to S_0 and $\overline{X}^s$ the restriction of the vector field X^s to S is also tangent to S. Then we have $f_*(\overline{X}^s_0) = \overline{X}^s$.
Furthermore there exist two functions $a, b \in C^1(S, \mathbb{R})$ such that with the notations $\overline{X} = X|_S, \overline{X}_0 = X_{0|_S}, f_*(\overline{X}_0) = a\overline{X} + b\overline{X}^s$.
We may observe that in the set $\gamma = F(\gamma_0)$ with $\gamma_0 = \{g \in U,\ (t, t_u, t_s) \in]-\epsilon, \epsilon[\times\{0\}^2\}$ one has $a|_\gamma = 1$, $b|_\gamma = 0$. Using the invariance of the Lie bracket by diffeomorphisms, we may write

$$-h\overline{X}^s = f^*(-h\overline{X}^s_0) = f^*[\overline{X}_0, \overline{X}^s_0] = [a\overline{X} + b\overline{X}^s, \overline{X}^s] \quad .$$

For which we may deduce the pair of equations,

$$-da(\overline{X}^s) = 0,\ ha\ + db(\overline{X}^s) = +h \quad .$$

This shows that in S

$$a\Big(\varphi^s_{t^s}\big(\varphi_t(p)\big)\Big) = a\big(\varphi_t(p)\big) = 1$$

and then, for the same reason, we get $b \equiv 0$ in S. So we have shown that

$$f_*(\overline{X}_0) = \overline{X} \quad . \tag{17}$$

A similar argument will help us to show that (17) may be extended to U. Again by construction, there exists on $F(U)$ three C^1-real functions a, b, c such that

$$F_*(X^u_0) = X^u, \quad F_*(X_0) = aX + bX^s + cX^u$$

and furthermore along the piece of stable leave S, we have

$$a_{|S} = 1, \quad b_{|S} = 0, \quad c_{|S} = 0$$

Using again the Lie bracket relation

$$F_*([X_0, X_0^u]) = [F_*(X_0), X^u]$$

we immediately deduce

$$-da(X^u) \cdot X - db(X^u) \cdot X^s - dc(X^u)X^u + ahX^u - bX = hX^u \qquad (18)$$

This implies $b \equiv 0$, and then $a \equiv 1$, and finally $c \equiv 0$.

iii) It remains to check that $F_*(X_0^s) = X^s$.
We still known it in S. Again, on U there exists three real C^1-functions $\tilde{a}, \tilde{b}, \tilde{c}$ such that

$$F_*(X_0^s) = \tilde{a} \cdot X^s + \tilde{b} \cdot X^u + \tilde{c}X$$

with

$$\tilde{a}_{|S} = 1, \quad \tilde{b}_{|S} = 0, \quad \tilde{c}_{|S} = 0$$

Using our previous result $F_*(X_0) = X$. We have,

$$F_*[X_0^s, X_0^u] = F_*(-X_0) = -X = [\tilde{a}X^s + \tilde{b}X^u + \tilde{c}X, X^u]$$

which gives

$$\begin{aligned} d\tilde{a}(X^u) &= 0 \\ d\tilde{b}(X^u) &= -\tilde{c}h \\ d\tilde{c}(X^u) &= 1 - \tilde{a} \end{aligned}$$

Whose unique solution satisfying these initial conditions is

$$\tilde{a} \equiv 1, \quad \tilde{c} \equiv 0, \quad \tilde{b} \equiv 0 \quad .$$

From Lemma 5 we can derive some information about the local structure.

Proposition 6.— *Let M^3 be a smooth 3-manifold carrying three C^1-vectors fields X, X^u, X^s which satisfy the Lie bracket relations (14) then M^3 admits a C^1-$(SL(2,\mathbb{R}), SL(2,\mathbb{R}))$ structure.*

Proof : We have observed (Lemma 5) that for any point p one can produce a C^1-chart, valued in $SL(2,\mathbb{R})$.

Let us check that the change of charts is given by left multiplication in $SL(2,\mathbb{R})$. Consider two such charts $(U_1, F_1), (U_2, F_2)$ as before. If $F_1(U_1) \cap F_2(U_2) \neq \emptyset$ then the C^1-diffeomorphism

$$F_{12} = F_2^{-1}F_1 : F_1^{-1}(F_1(U_1) \cap F_2(U_2)) \to F_2^{-1}(F_1(U_1)) \cap F_2(U_2))$$

is such that by Lemma 5 i), ii), iii)

$$F_{12*}X_0 = X_0,\ F_{12*}X_0^u = X_0^u,\ F_{12*}X_0^s = X_0^s\ .$$

Furthermore choose $x \in F_1(u_1) \cap F_2(u_2)$, $x = F_1(g_1) = F_2(g_2)$ then $F_{12}(g_1) = g_2$ and on $SL(2,\mathbb{R})$ the map $g_1 g_2^{-1} F$ admits a fixed point and preserves the left invariant vector fields, hence it is the identity on its domain.

Global structure and conjugacy. We first produce a global C^1-conjugacy. Let us recall that to give a $C^1 - (SL(2,\mathbb{R}), SL(2,\mathbb{R}))$ structure is equivalent to the existence of a C^1-vector field on $M \times SL(2,\mathbb{R})$, transverse to the fibers. The choice of a base point and of a leaf determines a developping map $D : \widetilde{M} \to SL(2,\mathbb{R})$. We equip the manifold M with the Riemannian metric g for which the vector fields $X^{\cdot,s,u}$ form an orthonormal frame, and $SL(2,\mathbb{R})$ with the left invariant Riemannian metric g_o associated to the triple $X_0^{\cdot,s,u}$. By that construction the developping map D is a local isometry from the geodesically complete manifold $(\widetilde{M}, \widetilde{g})$, (where $\widetilde{g}$ is the lift of g to the universal cover), to $(SL(2,\mathbb{R}), g_0)$. Then D is a Riemannian covering, hence D can be lifted to a global C^1-diffomorphism $\widetilde{D} : \widetilde{M} \to SL(2,\mathbb{R})$, such that the following diagram is commutative

$$\begin{array}{ccc} & & \widetilde{SL(2,\mathbf{R})} \\ & \widetilde{D} \nearrow & \Big\downarrow \pi \\ \widetilde{M} & \underset{D}{\longrightarrow} & SL(2,\mathbf{R}) \end{array}$$

and $T\widetilde{D}(\widetilde{X}^{\cdot,u,s}) = X_0^{\cdot,u,s}$.

Let $\gamma \in \pi_1(M)$ and $\sigma : \pi_1(M) \to Diff(\widetilde{M})$ be a representation of the fundamental group. The diffeomorphism $\widetilde{D}\sigma(\gamma)\widetilde{D}^{-1} : \widetilde{SL(2,\mathbb{R})} \to \widetilde{SL(2,\mathbb{R})}$ obviously preserves the left invariant vector fields. We then obtain a group morphism $\rho : \pi_1(M) \to \widetilde{SL(2,\mathbb{R})}$ given by

$$\rho(\gamma) = \widetilde{D}\sigma(\gamma)\widetilde{D}^{-1}\ .$$

Whose image Γ is a discrete subgroup. The C^1- conjugacy F is provided by the diagram

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{D}} & \widetilde{SL(2,\mathbf{R})} \\ {\scriptstyle P_1}\Big\downarrow & & \Big\downarrow{\scriptstyle P_2} \\ M & \xrightarrow{F} & \Gamma \setminus \widetilde{SL(2,\mathbf{R})} \end{array}$$

because for any $\dot{x} \in P_1^{-1}\{x\}$, $F(x) = P_2 \circ \widetilde{D}(\dot{x}) = P_2 \circ \widetilde{D}(\sigma(\gamma)\dot{x})$

Now we can use the regularity result of $[LM]$ which states that a C^1-conjugacy of two C^∞-Anosov flows on closed C^∞-3-manifolds is in fact a C^∞-conjugacy.
To conclude the proof of the theorem, one may use the classification of homogeneous Seifert bundles of type $\Gamma\backslash SL(2,\mathbb{R})$, where $\Gamma \subset \widetilde{SL(2,\mathbb{R})}$ is discrete and cocompact which has been established by $[R.V]$.

Conclusion. It is not clear whether the C^1-regularity of the canonical form is really needed. It seems that one may conjecture that if φ_t is a C^∞-Anosov flow on a closed 3-manifold whose Bowen Margulis measure is in the Lebesgue class, then up to finite covers the flow is C^∞conjugate to an algebraic model, that is, a suspension of a hyperbolic Linear automorphism of a 2-torus or the geodesic flow of a hyperbolic closed surface.

References

[An] Anosov, D.V. 1967. *Geodesic flowson compact manifolds of negative curvature*, Trudy, Math. Inst. Steklov., 90, Proc. Steklov Inst. Math., p. 209.

[Ba] T. Barbot *géométrie transverse des flots d'Anosov.* Thèse (1993). Ecole Normale Sup. Lyon.

[G1] E. Ghys *Flots d'Anosov dont les feuilletages stables sont différentiables* Ann. Sci. Ec. Norm. Sup. 20 (1987) 251-270.

[GK] V. Guillemin, D. Kazhdan *Some inverse spectral results for negatively curved 2-manifolds ; on the cohomology of certain dynamical systems* topology 19 (1980) 291-299 ; 301-312.

[Gr] L.W. Green *Remarks on uniformly expanding homocycle parametrizations.* J. Differential geometry 13 (1978), 263-271.

[Ha-K] B. Hasselblatt, A. Katok *Introduction to the modern theory of dynamical systems.* Encyclopedia of Mathematics 54 (1995) Cambridge University Press.

[HP1] M. Hirsch, C. Pugh *Stable manifolds and hyperbolic sets* Proc. Symp. in pure Math. vol. 14, Amer. Math. Soc. Providence, R.I. (1970) 133-164.

[HP2] M. Hirsch, C. Pugh *Smoothness of horocycle foliations* J. Differential Geom. 10 (1978) 5, 225-238.

[H.K.] S. Hurder, A. Katok *Differentiability, Rigidity and Godbillon-Vey Classes for Anosov Flows* Pub. I.H.E.S. (1990) n° 72 5-61.

[J] Jean-Lin Journé *On a Regularity Problem Occuring in Connection with Anosov Diffeomorphisms* Commun. Math. Phys. 106 (1986) 345-351.

[Ka1] A. Katok *Four Applications of Conformal Equivalence to Geometry and Dynamics* Erg. Th. and Dynamical Systems, 8 (1988) 139-152.

[Ka2] A. Katok *Entropy and closed geodesics* Erg. Th. and Dynamical Systems, 2 (1982) 339-367.

[Le1] F. Ledrappier *Harmonic measures and Bowen-Margulis measures*, Isr. J. Math. 71 (1990), n° 3, 275-287.

[Le2] F. Ledrappier *A renewal theorem for the distance in negative curvature* Proc. Sympos. Pure Math., Amer. Math. Soc., Providence RI, 1995.

[Li] A.M. Livsic *Cosmology of Dynamical Systems* Math. USSR Izvestija vol. 6 (1972) n° 6 1278-1301.

[LM] R. de Lallave, R. Moriyôn *Invariants for Smooth Conjugacy of Hyperbolic Dynamical Systems IV* Commun. Math. Phys. 116 (1988) 185-192.

[LMM] R. de Lallave, J. Marco, R. Moriyôn *Canonical perturbation theory of Anosov systems, and regularity results for Livsic cohomology equation* Annals of Math. vol 123 n° 3 (1986) 537-612.

[LS] A. Livsic, J. Sinai *On invariant measures compatible with the smooth structure for transitive Y-systems* Soviet Mat. Doklady vol 13 (1972) n° 6 1656-1659.

[M] G.A. Margulis Functional Analysis Application 4 54-64 *certain measures associated with U-flows on compact manifolds* (1970).

[Mo] G.D. Mostow *Intersections of discrete subgroups with Cartan subgroups* K. Indian Math. Soc. 34 (1970) 203-214.

[P1] J. Plante *Anosov flows* Amer. J. Math. 94 (1972) 729-755.

[Ra] M.S. Rughunathan *Discrete subgroups of Lie groups* Springer Berlin-Heidelberg-New-York (1972).

[RV] F. Raymond, T Vasquez *3-manifolds whose universal coverings are Lie groups* Topology and its Applications, Vol 12, (1981), 161-179.